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DIFFEOMORPHISM CLASSIFICATION OF FINITE GROUP ACTIONS ON DISKS

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In this paper we discuss the diffeomorphism classification of finite group actions on disks. We answer the question when an action on a space M can be extended to an action on a disk such that the action is free away from M. Let the singular set consist of the points with nontrivial isotropy group. We show (under some dimension assumptions) that disks with diffeomorphic neighborhoods of the singular set can be imbedded into each other. As a consequence we find a classification of group actions on disks in terms of the neighborhood of the singular set and an element in the Whitehead group of G.

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finite group actions on disks
equivariant diffeomorphism classification

1. Introduction and history

In this note we want to discuss finite group actions of disks. The group is always denoted by G. We concentrate on two problems.

Q1: Suppose M is a G space. When can we extend the action on M to an action on a disk?

Q2: How can we classify G actions on disks up to diffeomorphism?

An answer to Question 1 is given in Theorem 1. The theorem is set up such that it is helpful in answering Question 2 within a dimension range, called the Gap hypothesis.

Both problems have a longer history. We give only sample references here. A complete answer to Q1 has been given by Oliver in the G CW category [9]. For the differentiable category (and the G PL category) there are answers in [7] for $G=\mathbb{Z}_n$, and for a general finite group results are stated in [1, § VI]. For Q2 we find answers in [7] for $G=\mathbb{Z}_n$, and for any finite G for semilinear actions in [15].

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In 1979 independent progress was reported by Assadi [2, announced in 1] and Dovermann and Rothenberg [5]. There was considerable overlap. The solution discussed here extends both results and it was proposed in Spring 1980. The approach here is strongly influenced by Assadi's viewpoint, but the methods for solving them are taken from [18]. We apply a mild extension of [18], a special case of surgery theory as in [4]. The idea of completed bundle data comes from [11]. Assadi's results (in part with W. Browder) suggest the general answer to both questions considered here, except the bundle theoretic part, but the methods used here confirm these answers in more generality and with less work. A discussion about the general problem of classifying actions on disks will be given at the end of the next section.

This paper is written requiring a minimum of references, basically only Hirsch's Lemma and Section 4 of [18]; and all details are carried out. So all tools required were available for a considerable time. I want to thank the referee and Reinhard Schultz. Their suggestions led to improvements, made the paper more self contained, and let to including many details.

2. Results

We rephrase Q1 to fit it into our approach. Suppose M is a G space such that $G_x \neq 1$ for all $x \in M$.

Q1′: When does G act smoothly on a disk D of dimension n such that $M \subset D$, the action on D extends the action on M, and the action on $D-M$ is free.

A positive answer to Q1′ will imply that every component of M^H (H a nontrivial subgroup of G) is a smooth manifold and as such it has a dimension. So we suppose that M is a stratified space, and that the strata have well defined dimensions. The maximum of these dimensions is denoted by $d(M)$.

Suppose M is again a finite G CW complex, and $\tilde{H}_*(M^P, \mathbb{Z}_p) = 0$ for all p groups P, p is a prime. Construct a finite G CW complex V containing M such that G acts freely on $V-M$, and $\tilde{H}_*(V)$ is torsion free and in only one dimension, say m. By [12] $\tilde{H}_m(V)$ is projective and we set $o(M) = (-1)^m[\tilde{H}_m(V)]$, [ ] denote the class in $\tilde{K}_0(\mathbb{Z}[G])$. V is constructed by freely attaching cells to kill off low dimensional homotopy groups and $o(M)$ is well defined. To see this let V' be another complex as above. We can assume that its nonvanishing homology is in dimension m too. Construct W like V but such that it contains $V \cup V'$ and only $\tilde{H}_{m+1}(W)$ is nonzero, and projective. As G acts freely on $W-V$, $o(W) = o(V)$; use the sequence for (W, V). Symmetrically $o(W) = o(V')$, hence $o(V) = o(V')$.

A similar invariant for studying more complicated problems has been introduced in [9, 10, 1]. Those invariants were introduced to study Q1′ in the homotopy category but allowing prescribed isotropy groups in $D-M$.

Completion is a functorial procedure which replaces any space, vector bundle, or map in the equivariant category by a corresponding element in the non-equivariant category. For example a G space M becomes $\hat{M} = E \times_G M$ where E is a contractible G space with free G action. Note that $BG = \hat{*} = E/G$. The terminology 'completion' is motivated by the fact that $(RG)^{\wedge} = \hat{K}_G(*) = K(BG)$.

Theorem 1. *Suppose $d(M) < \frac{1}{2}n$ and $n \geq 6$. Then* Q1′ *has a positive answer if and only if there exists a smooth connected compact G manifold U of dimension n with the following properties*:

(i) *the G action on U extends the action on M, $G_x = 1$ for $x \in U - M$, ∂U is non empty, and $\tilde{H}_*(M^P, Z_p) = 0$ for all p groups P, p is a prime.*

(ii) $o(M) = 0$.

(iii) *The stable completed tangent bundle $\hat{T}U$ of U is the pullback of a bundle over BG.*

Theorem 1′. *The conclusion of Theorem* 1 *also holds if $d(M) \leq \frac{1}{2}n$ and $\frac{1}{2}n$-dimensional components of M^H are simply connected, $H \subseteq G$.*

The reader finds a sufficient assumption for a positive answer to Q1′ in [1, § VI], and a necessary and sufficient condition if only semifree actions are considered. See [1, § VI, 2.5 and 2.6]. Following some inspiration from Assadi we can improve Theorem 1 to:

Theorem 1″. *The conclusion of Theorem* 1 *holds if the assumption $d(M) < \frac{1}{2}n$ is replaced by $H_i(M) = 0$ for $i > r$ where $r < \min\{n - d(M), \frac{1}{2}n\}$.*

The classification of action on disks uses Theorem 1 and a trick from the non-equivariant setting, see e.g. [8, p. 108]. To present this we restate Question 2.

Q3: Suppose D_1 and D_2 are two G disks. When can D_1 be equivariantly inbedded into D_2 such that $G_x = 1$ for $x \in D_2 - D_1$?

Theorem 2. *Suppose D_1 and D_2 are two G disks which satisfy the dimension assumptions as they are made for U in theorem* 1 *or* 1′. *Then* Q3 *has an affirmative answer if and only if there exist smooth neighborhoods U_i of D_i^s, $i = 1, 2$, such that U_1 is G diffeomorphic to U_2.*

Here $D^s = \{x \in D \mid G_x \neq 1\}$ is called the singular set. The imbedding $D_1 \to D_2$ can be chosen to extend a given diffeomorphism $U_1 \to U_2$.

To obtain the classification result we make this definition. With the notation as in Theorem 2 we say that neighborhoods U_1 and U_2 of M are equivalent if they contain smaller neighborhoods U_1' and U_2' of M which are diffeomorphic. This is an equivalence relation for neighborhoods.

With the assumptions as in Theorem 2 we have:

Theorem 3. *G diffeomorphism classes of disks are classified by equivalence classes of neighborhoods of the singular set and an element in* Wh(G). *Each element in* Wh(G) *can be realized.*

In the semifree case the neighborhoods can be chosen to be the normal bundle of M in D, in which case the classification is given by the fixed point set, the normal bundle, and an element in the Whitehead group. This was the classification used in [1, 2, 5].

Modulo an indeterminancy the diffeomorphism classification of actions on spheres can be reduced to the classification of disks. This is standard and also described in [5]. It is done as follows.

If S is a homotopy sphere on which G acts smoothly such that S^H is a homotopy sphere for all $H \subseteq G$, S is called *semilinear.* Assume S^G is nonempty and connected. Then removing a linear disk about $x \in S^G$ turns S into a semilinear disk. For such a semilinear disk we have a generalized Whitehead torsion invariant defined (see e.g. [6] or [13], it is denoted by $\tau(S)$. The family of G oriented diffeomorphism classes of n-dimensional semilinear G spheres, where $\dim S^G \geq 1$, with a fixed tangent representation at a fixed point, form an abelian group C_α under connected sum. Here α is a G module which is the tangent representation at a fixed point to these spheres. Those spheres for which τ vanishes form a subgroup C_α^+. If A_α is the set of G diffeomorphism classes of G spheres of dimension n with connected nonempty fixed point set and tangent representation α, then C_α^+ acts on A_α via connected sum. In fact, S_1 and S_2 in A_α are in the same orbit if and only if D_1 and D_2 are diffeomorphic. Here D_1 is S_1 without the interior of a linear disk around a fixed point. In brief, if $S_1, S_2 \in A_\alpha$, then $S_1 \cong S_2 \# S$ for some $S \in C_\alpha^+$ if and only if $S_1 - \mathring{D} \cong S_2 - \mathring{D}$. Here $\mathring{D}$ is the interior of a linear disk around a fixed point. As $S_i - \mathring{D}$ is a disk the classification of spheres is reduced.

The groups C_α^+ and C_α have been subject of several studies. See e.g. [14, 15, 3, 16, 17] et al. These groups and their action on A_α are not yet understood in general.

As we use Wall's approach to surgery theory the bundle data play a central role. Specifically in answering Q1′ it turns out that we can work with bundle data (Theorem 1(iii)) which necessarily exist and which are powerful enough to approach Q1′ in general. This distinguishes the problem for disks from others. The exploitation of this fact allows us (in comparison to Assadi's work) to give a necessary and sufficient condition in Theorem 1 and hence classify all actions in a given dimension range. So one of the main points of this article is to point out and apply this idea of good bundle data. It is also this aspect which we want to restrict our attention to, and where our main contribution lies.

The approach we took above is valid as long as we try to leave M (the anticipated singular set of the action on a disk) unchanged. One might try to apply above procedure inductively (induction over the partially ordered set of subgroups of G).

Let U be as in Theorem 1. Surgery on U^H inside of U, $H \subset G$ would require stronger bundle data, namely unstable normal bundle data (compare [4] in particular § 4). The only problem is that these are very difficult to use together with completed bundle data as in Theorem 1(iii). In brief, we don't know of bundle data which necessarily exist and which are sufficient to do surgery also on fixed sets for groups H which are nontrivial.

3. Proofs

We start out with explaining the connection between our bundle data and surgery. Suppose U is a G manifold. Let ξ be a bundle over BG and $\pi_G : \hat{U} \to BG$ induced from $U \to *$. Assume a stable vectorbundle isomorphism $b : \hat{T}U \to \pi_G^* \xi$. We say that (ξ, b) or just b is *stable completed bundle data* for U. (This concept was introduced in [11].)

Surgery Lemma. *Stable completed bundle data are appropriate for equivariant surgery in the free part.*

Being *appropriate for surgery in the free part* means the following. Suppose we are given a homotopy class in $\pi_*(U)$, $\pi_*(U, \partial U)$ or $\pi_*(\partial U)$. Then the bundle data allows us to choose a regular homotopy class such that the immersed sphere has a trivial normal bundle. So, assuming that there is an imbedding in this regular homotopy class, we do surgery on this class (together with its translates under G). For the manifold resulting from this surgery it is required that we again have bundle data, i.e. in our case stable completed bundle data.

Proof of the Surgery Lemma. (For a definition of surgery in this context see [18] or [4, Definition 4.4].) Suppose we are given G manifold pair $(U, \partial U)$ with stable completed bundle data (ξ, b). Consider a class $\mu \in \pi_k(\partial U)$ represented by $\iota' : S^k \to \partial U$. Extend ι' equivariantly to a map $i : G \times S^k \times D^{n-k-1} \to \partial U$ where G acts trivially on $S^k \times D^{n-k-1}$. Here $\dim U = n$. Set $\mathbb{D}_0 = S^k \times D^{n-k-1}$ and $\mathbb{D} = D^{k+1} \times D^{n-k-1}$. Let $\pi_G : U \to BG$ be the map induced by $U \to *$. We find the commutative diagram of stable bundle isomorphisms:

$$\begin{array}{ccc} (i^*TU)^{\wedge} & \xrightarrow{\hat{\imath}^* b} & \hat{\imath}^* \pi_G^* \xi \\ \uparrow & & \uparrow \\ i^*TU|_{\mathbb{D}_0} & \xrightarrow{B} & \hat{\imath}^* \pi_G^* \xi|_{\mathbb{D}_0} \end{array}$$

The bundles in the top row are over $E \times_G G \times \mathbb{D}_0$, the ones in the bottom row are over $\mathbb{D}_0$. The vertical maps are obtained from including $\mathbb{D}_0 \to (G \times \mathbb{D}_0)\hat{}$ which is a homotopy equivalence (denote the homotopy inverse by j), and B is obtained by restriction. Consider also this diagram of stable vector bundle isomorphisms:

$$
\begin{array}{ccc}
T\mathbb{D} & \xrightarrow{\;\Omega\;} & \mathbb{D} \times \xi_x \\
\uparrow & & \uparrow \\
T\mathbb{D}|_{\mathbb{D}_0} & \xrightarrow{\;\Lambda\;} & \mathbb{D}_0 \times \xi_x \\
{\scriptstyle \mathrm{d}i|_{\mathbb{D}_0}} \uparrow & & \uparrow {\scriptstyle =} \\
i^*TU|_{\mathbb{D}_0} & \xrightarrow{\;B\;} & i^*\pi_G^*\xi|_{\mathbb{D}_0}, \; x \in \operatorname{Im} i(U).
\end{array}
$$

obtained as follows. Choose an isomorphism Ω between the trivial bundles $T\mathbb{D}$ and $\mathbb{D} \times \xi_x$. By restriction Ω induces Λ. Choose the immersion i such that the differential di restricted to $\mathbb{D}_0$ makes the bottom diagram commute. A homotopy of i changes the bottom row only by a homotopy, and i exists by Hirsch's Lemma [18, p. 10]; first find $i|_{\mathbb{D}_0}$, then $i = G \times i|_{\mathbb{D}_0}$. Assume that this regular homotopy class of immersions can be represented by an imbedding; this will always be possible in our applications. We attach $\mathbb{D}$ along $\mathbb{D}_0$ and set $U' = U \cup_i G \times \mathbb{D}$. As $\mathbb{D}_0 \to (G \times \mathbb{D}_0)\hat{}$ was a homotopy equivalence and B was obtained by restricting $\hat{i}^*b$ we have that j^*B and $\hat{i}^*b$ are homotopic bundle isomorphisms. After a homotopy of b we can assume that $\hat{i}^*b = j^*B$. As $T(U') = TU \cup_{\mathrm{d}i} G \times T\mathbb{D}$, the bundle isomorphism B extends to an isomorphism

$$\psi : T(U')|_{G\times\mathbb{D}} \to \bar{\pi}_G^*\xi|_{G\times\mathbb{D}};$$

$\bar{\pi}_G : (U')\hat{} \to BG$ is induced by $U' \to *$ and extends π_G, and Ω extends B. As $\hat{i}^*b = j^*B$, $\hat{\psi} : (TU'|_{G\times D})\hat{} \to \bar{\pi}_G\xi|_{(G\times D)\hat{}}$ extends $\hat{i}^*b$ and $(\xi, b \cup_{\hat{i}^*b} \hat{\psi})$ provides us with new bundle data. So we extended the bundle data over U' which proves the lemma—in case we do surgery on a class $\mu \in \pi_k(\partial U)$. The case $\mu \in \pi_k(U, \partial U)$ is trivial as bundle data are obtained by restriction. But the imbedding still has to be chosen in accordance with Hirsch's Lemma. If $\mu \in \pi_k(U)$ the argument is similar to the above one, but we attach a handle to $U \times I$.

Proof of Theorem 1. *Necessity.* For part (i) we can choose $U = D$. The homological assumptions on M follow from Smith theory. We proved that $o(M) = o(D) = 0$. Condition (iii) is an immediate consequence of the fibration $D \to \hat{D} \to BG$, it gives rise to an exact sequence in K-theory.

Let us make a few remarks before we show sufficiently in Theorem 1. Let the notation be as in this theorem. Let $\mu \in \pi_*(\partial U)$ be a class we want to kill by surgery.

To do so we attach a handle together with its translates under G to ∂U along μ. As in the Surgery Lemma we do this by using the bundle data to choose a good regular homotopy class of imbeddings; the bundle data will then extend over the resulting manifold. Let $\mu \in \pi_*(U, \partial U)$. To kill such a class by surgery means to delete a handle from U, so we also delete its translates.

We shall prove:

Lemma 4. *The disk D required to show sufficiently in Theorem 1 can be constructed by attaching and subtracting handles to U along ∂U in the free part.*

Proof (and proof of sufficiency in Theorem 1). In the following steps we do surgery below the middle dimension, so the spheres we want to kill can be represented by imbedded spheres in the correct regular homotopy class, such that they do not meet their translates under the action of G (use a general position argument). First of all kill $\pi_0(\partial U)$, $\pi_1(U, \partial U)$, $\pi_1(\partial U)$. Now we can replace homotopy groups by homology groups. Continue killing $H_*(\partial U)$ and $H_*(U, \partial U)$ as far as it is possible by surgery below the middle dimension.

Now we have to distinguish two cases.

(a) Suppose $\dim U = 2m$. By now we reduced the long exact homology sequence to

$$0 \to H_{m+1}(u, \partial U) \to H_m(\partial U) \to H_m(U) \to H_m(U, \partial U) \xrightarrow{\partial} H_{m-1}(\partial U) \to H_{m-1}(U) \to 0.$$

We do additional surgery steps to make ∂ a surjection. This is done by representing classes of $H_{m-1}(U)$ in $H_{m-1}(\partial U)$ and applying surgery to them. This means that also the two extreme terms in the above sequence vanish. By [12, 6.1] $H_m(U)$ is a projective $\mathbb{Z}[G]$ module, hence stably free by assumption (ii) in Theorem 1. After some additional surgeries on trivial classes in $H_{m-1}(\partial U)$ we can assume that $H_m(U)$ is free. By Poincare Duality and the universal coefficient theorem $H_m(U) \cong H_m(U, \partial U)$. Again $H_m(U, \partial U) \cong \pi_m(U, \partial U)$ so a basis can be represented by classes $\mu_i : (D^m, S^{m-1}) \to (U, \partial U)$. By Wall's argument [18, p. 39–41] these classes (together with their translates under G) can be used for surgery and that makes the above sequence collapse.

(b) Suppose $\dim U = 2m + 1$. By surgery below the middle dimension the sequence collapsed to

$$0 \to H_{m+1}(U, \partial U) \to H_m(\partial U) \to H_m(U) \to 0$$

Wall proposes two ways to kill these homology groups. The one he carries out in detail first attaches handles to ∂U (surgeries on classes in $\pi_m(\partial U)$) to obtain U' and after that he kills $H_m(U')$. We want to avoid doing surgery on classes in $H_*(U)$. As explained in [18, p. 41] the terms in the above sequence can be killed by surgery on classes in $H_{m+1}(U, \partial U)$ but the argument requires that $(U, \partial U)$ is 2-connected. This is an assumption which is satisfied after we did surgery below the middle dimension.

We should remind the reader that we use Wall's proof without his references to preferred basis; this is only a simplification of the argument there and produces a disk for us. His arguments that homotopy classes in the middle dimension can be killed need one trivial modification; namely the first step is to represent classes by immersion which miss M, and this is possible by our dimension assumption.

Proof of Theorem 1′. There is only one additional case we have to discuss, namely if $\dim U = 2m$ and there exists an m-dimensional component for M^H for some $H \subset G$. To carry out the above proof we have to assure that a homotopy class which we want to kill by surgery (only those in $\pi_m(U, \partial U)$ matter) misses U^H. Let $\mu : (D^m, S^{m-1}) \to (U, \partial U)$ represent this class. Wall explains how a piping argument can be used to eliminate intersections of $\mu(D^m)$ and N, where N is a 1-connected submanifold of U which intersects ∂U, see [18, p. 39] and the references there. In the application there $N = \mu'(D^m)$ where μ' is another class in $\pi_m(U, \partial U)$. So the proof of Theorem 1 can also be applied in this case.

Proof of Theorem 1″. The proof uses a trick I learnt from Assadi. In the surgery procedure we start out with a U which is G homotopy equivalent to M. This can be done by picking a smooth regular neighborhood U_0 of M in M in U. So U_0 also satisfies the assumptions in our theorem. By surgery in dimensions smaller than r (r as in the theorem) we produce U_1 the surgery result from U_0. For this U_1 we have a single homology group $H_r(U_1)$ which we have to kill. So we have a sequence

$$0 \to H_{n-r}(U_1, \partial U_1) \to H_{n-r-1}(\partial U_1) \to 0 \cdots 0 \to H_r(\partial U_1) \to H_r(U_1) \to 0$$

(if $r < \frac{1}{2}(n-1)$, otherwise we apply Theorem 1). So $H_r(U_1)$ is killed by surgery on classes $H_r(\partial U_1)$.

In the next proof we will apply this lemma:

Lemma 5. *Suppose D is a smooth disk and U is a G submanifold of D of codimension zero. Suppose (ξ, b) is completed bundle data for U. Then this bundle data is obtained by restriction from completed bundle data $(\xi, \bar{b})$ for D.*

Remark. Note that the bundle ξ over BG is the same for both sets of bundle data. In fact, up to isomorphism, there is exactly one bundle which pulls back to $\hat{T}D$, and there is exactly one bundle which pulls back to $\hat{T}U$.

Proof. By assumption we have the stable isomorphism $b : \hat{\pi}_U^* \xi \to \hat{T}U$ where $\pi_U : U \to *$. Extend $\hat{\pi}_U^* \xi$ and b to $\bar{\xi}_0$ and $\bar{b} : \xi_0 \to \hat{T}D$ over $\hat{D}$. Let $\hat{\pi}_D : \hat{D} \to BG$ be the homotopy equivalence induced by $\pi_D : D \to *$ and ρ an inverse. Set $\bar{\xi} = \rho^* \xi_0$ so $\rho^*(\bar{b}) : \bar{\xi} \to \rho^*(\hat{T}D)$. Apply $\hat{\pi}_D^*$, then we find

$$\begin{array}{ccc}
\hat{\pi}_D^*\rho^*(\bar{b}):\hat{\pi}_D^*\bar{\xi} & \longrightarrow & \hat{\pi}_D^*\rho^*\hat{T}D \\
\Big\downarrow = & \circlearrowright & \Big\downarrow = \\
\bar{b}:\xi_0 & \longrightarrow & \hat{T}D \\
\Big\uparrow & & \Big\uparrow \\
b:\hat{\pi}_U{}^*\xi & \longrightarrow & \hat{T}U
\end{array}$$

The top row defines bundle data $(\bar{\xi}, \hat{\pi}_D^*\rho^*(\bar{b}))$ for D, whilst the center row restricted to $\hat{U}$ gives the bundle data (ξ, b) in the bottom row. Finally note that $\bar{\xi}$ and ξ define the same class in $K(BG) = \hat{K}_G(*) = (RG)\hat{}$. The completed representation ring satisfies induction with respect to the Sylow subgroups of G. Hence the bundles are determined by the completed representations $(\mathrm{Res}_P T_y U)\hat{}$, where P ranges over the Sylow subgroups and $y \in U^P$. They do not depend on y, D^P is connected, and they are the same for the spaces U and D.

Proof of Theorem 2. We want to show that every disk which satisfies the assumptions in the theorem can be constructed in an easy way; namely by starting out with a neighborhood U of the singular set M and then adding and subtracting handles. These handles are in at most half the dimension of the disk and a small neighborhood U chosen as regular neighborhood of U^s stays unchanged. Secondly we build up such a disk D_1 within an arbitrary disk D_2 to prove the theorem.

Let D be some disk, and U a neighborhood of the singular set. Pick bundle data $(\xi, \bar{b})$ for D, and let (ξ, b) be bundle data for U obtained by restricting those for D to U. By a sequence of handle attachings and subtractions U can be made into a disk D_1 (Lemma 4). The claim is that this can be done inside of D. Let us show this.

Suppose the handle operations give us a sequence of spaces $U = U_0, \ldots, U_k = D_1$. Let U_{j+1} be obtained from U_j by subtracting a handle. So we have to imbed a handle (together with its translates under the G action) $(G \times D^k \times D^{n-k}, G \times S^{k-1} \times D^{n-k}) \to (U_j, \partial U_j)$ which we remove. Hirsch's Lemma (the relative version) gives us some information about the appropriate relative homotopy class which we have to choose such that the boundary of U_{j+1} will be again a smooth manifold with bundle data. The choice is just such that we could glue the handle back in and obtain the bundle data back we had to begin with. That implies that $U_{j+1} \subset U_j$ and the bundle data for U_{j+1} are obtained by restriction from $(\xi, \bar{b})$. Now suppose U_{j+1} is obtained from U_j by attaching a handle along a class $\mu \in \pi_k(\partial U_j)$. By duality μ gives us a class $\tilde{\mu} \in \pi_{k+1}(D - \mathring{U}_j, \partial U_j)$. We assume that we killed already the lower dimensional homotopy groups, so the Hurewicz homomorphism can be used to compare homology and homotopy. So to attach a handle to U_j we subtract it from $D-\mathring{U}_j$. For $D-\mathring{U}_j$, called also $\tilde{U}$, we have completed bundle data $(\xi, \tilde{b})$ obtained by restricting $(\xi, \tilde{b})$, the data for D. Use $\tilde{\mu}$ to do surgery on $\tilde{U}$ with bundle data $(\xi, \tilde{b})$, and call the result $\tilde{U}_1$. Then $U_{j+1} = D - \mathrm{int}(\tilde{U}_1)$, and these spaces have the same bundle data. This can be arranged using the relative version of Hirsch's Lemma. Namely we choose first the immersion for $\partial\tilde{\mu} = \mu \in \pi_k(\partial U_j)$, and over ∂U_j the bundle data (ξ, b) and $(\xi, \tilde{b})$ agree. Relative to this choice we choose the imbedding representing $\tilde{\mu}$ to do the handle subtraction. Note that we had no problems applying arguments to cancel intersection points for an immersion representing $\tilde{\mu}$ as we never attach handles in the middle dimension to ∂U_j, so dim $\tilde{\mu}$ is in at most half the dimension.

Until now we constructed some disk D_1 in D. Obviously $D-\mathring{D}_1$ is an h-cobordism (which is a product on the singular set). So (D, D_1) defines an element τ in Wh(G), see [6, 13]. By attaching handles in two consecutive dimension τ can be killed, and attaching these handles can be done by surgery just like in the first part of the proof. Then $\tau(D, D_1) = 0$ and it follows from the equivariant s-cobordism theorem [6, 13] that D and D_1 are diffeomorphic.

We did use the following slightly *in precise notation.* $D-\mathring{D}_1$ (or $D-\mathring{U}_j$) should mean that we remove from D all points in the interior of D_1 and furthermore we remove these points of ∂D_1 which are in the interior of $\partial D_1 \cap \partial D$.

The fact that U can be chosen such that it is not changed in constructing D is obvious. We pick U as regular neighborhood of $D^s = U^s$, the singular set. A diffeomorphic smaller copy of U can then be chosen such that every surgery step misses it as it misses U^s.

Now we can say that each disk D comes with a precise instruction how it is built up from (U, ξ, b) by attaching and subtracting handles, each in at most half the dimension of D. Finally we have to see that we can construct D also in another disk D', where D and D' have the same U as neighborhood of the singular set. By choosing a smaller neighborhood we can assume that it is a regular neighborhood, so it stays unchanged in our surgery steps.

By Lemma 5 we assume bundle data $(\xi, \bar{b})$ for D and $(\xi, \bar{b}')$ for D'. The bundles over BG for these bundle data are the same by the remark after that lemma. Now it is obvious that each step used to construct D can be carried out inside of D'. The procedure is just as above where we built up some disk D_1 in D; the only assumption which was used extensively is that the bundle data for U are obtained by restriction from the bundle data of D (now in this application those of D'). This proves the theorem.

Proof of Theorem 3. Suppose D_1 and D_2 are two disks as assumed in the theorem which have equivalent neighborhoods of their respective singular sets. Then D_1 can be imbedded in D_2 such that $D_2-\mathring{D}_1$ is an h-cobordism which is a product on the singular set. So (D_2, D_1) defines an element in Wh(G) which vanishes if and only if D_2 and D_1 are diffeomorphic. By attaching an h-cobordism along the boundary of any disk we can realize each element in Wh(G).

References

[1] A. Assadi, Finite group actions on simply connected manifolds and CW complexes, Thesis, Princeton University 1979 (to appear with an appendix as Memoir AMS No. 257).

[2] A. Assadi, Extensions of finite group actions from submanifolds of disks, Preprint 1979, see also Proc. Topology Conference in London, Ontario (1981), Canad. Math. Soc.

[3] W. Browder and T. Petrie, Diffeomorphisms of manifolds and semifree actions on homotopy spheres, Bull. AMS 77 (1971) 160–163.

[4] K.H. Dovermann and T. Petrie, G surgery II, Memoir AMS, No. 260 (1982).

[5] K.H. Dovermann and M. Rothenberg, An equivariant surgery sequence and equivariant diffeomorphism and homeomorphism classification (A survery), Springer Lecture Notes in Mathematics, Vol. 788 (Springer-Verlag, Berlin, 1980) 257–278.

[6] S. Illman, Whitehead torsion and group actions, Ann. Acad. Sci. Fenn. 588 (1974).

[7] L. Jones, The converse to the fixed point theorem of P.A. Smith I, Ann. of Math. 94 (1971) 52–68.

[8] J. Milnor, Lectures on the h-cobordism theorem, Notes by L. Siebenmann and J. Sondow (Princeton University Press, 1965).

[9] R. Oliver, Fixed point sets of group actions on finite acyclic complexes, Comm. Math. Helv. 50 (1975) 155–177.

[10] R. Oliver and T. Petrie, G CW surgery and $\tilde{K}_0(Z[G])$, Math. Zeitschrift 179 (1982) 11–42.

[11] T. Petrie, G Surgery II, preprint 1977.

[12] T. Petrie, G maps and the projective class group, Comm. Math. Helv. 51 (1976) 611–626.

[13] M. Rothenberg, Torsion invariants and finite transformation groups, Proc. of Sym. in Pure Math., Vol. 32 (I) (1978) 267–311.

[14] M. Rothenberg, Differentiable group actions on spheres, Proc. Adv. Study Inst. Alg. Top. Aarhus (1970).

[15] M. Rothenberg and J. Sondow, Nonlinear smooth representations of compact Lie groups, Pacific J. Math. 84 (1979) 427–444.

[16] R. Schultz, Differentiable group actions on homotopy spheres, Inv. Math. 31 (1975) 105–128.

[17] D. Smith, Invariants for group actions on spheres, Thesis, University of Chicago, 1980.

[18] C.T.C. Wall, Surgery on compact manifolds (Academic Press, New York, 1970).