Equivariant differential topology in an o-minimal expansion of the field of real numbers

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Abstract

We establish basic properties of differential topology for definable \(C^r\) \(G\) manifolds in an o-minimal expansion \(\mathcal{M}\) of \(\mathbb{R} = (\mathbb{R}, +, \cdot, <)\), where \(0 \leq r \leq \omega\) and \(G\) is a definable \(C^r\) group. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let \(\mathcal{M}\) be an expansion (in the sense of [4]) of the field of real numbers \(\mathcal{R} = (\mathbb{R}, <, +, \cdot)\). By an open interval we mean a subset of \(\mathbb{R}\) of the form \((a, b)\), \(a, b \in \mathbb{R} \cup \{-\infty, +\infty\}\), \(a < b\). We say that \(\mathcal{M}\) is o-minimal if every subset of \(\mathbb{R}\) that is definable with parameters in \(\mathcal{M}\) is a finite union of open intervals and points.

Many results in semialgebraic geometry over \(\mathcal{R}\) hold true in the more general setting of o-minimal expansions of \(\mathcal{R}\). This theory of o-minimal structures has presented a strong interest since Wilkie [27] proved that the expansion

\[ R_{\exp} = (\mathbb{R}, <, +, \cdot, \exp) \]

is o-minimal. See also [5,7,15,26] for other examples and constructions of o-minimal expansions of the field of reals. General references on o-minimal structures are [4,6], see also [22].

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The purpose of this paper is to establish basic properties of equivariant differential topology in an o-minimal expansion of the field of real numbers. For this includes the situation of Nash $G$ manifolds and Nash $G$ vector bundles treated in [10,8].

The term “definable” is used throughout in the sense of “definable with parameters in a fixed but arbitrary o-minimal expansion $M$ of $\mathbb{R}$”, unless otherwise stated. Such expansions come in just two types, see [16]: the polynomially bounded type and the exponential type. (In exponential $M$, the exponential function $x \mapsto e^x: \mathbb{R} \to \mathbb{R}$ is definable.) Some results below will be stated in a stronger form for exponential $M$.

Approximations of maps between $C^r$ manifolds are with respect to the strong $C^r$ Whitney topology, unless otherwise specified. This topology is abbreviated to the $C^r$ topology.

We now list the main results of this paper. (See Section 2 for explanations of terminology.) Let $G$ be a definable $C^r$ group, $1 \leq r \leq \omega$ (including the case $r = \infty$). Let $f$ be a $G$ invariant surjective submersive definable $C^r$ map from a definable $C^r G$ manifold $S$ to a definable $C^r$ manifold $A$. We say that $f$ is piecewise definably $C^r G$ trivial if there exists a finite partition of $A$ into definable $C^r$ submanifolds $C_i$ such that for each $C_i$ there exist a definable $C^r G$ diffeomorphism $k_i: f^{-1}(C_i) \to C_i \times f^{-1}(a_i)$ with $f| f^{-1}(C_i) = p_i \circ k_i$, where $a_i \in C_i$ and $p_i$ denotes the projection $C_i \times f^{-1}(a_i) \to C_i$. One can also consider piecewise definable $C^r G$ triviality of surjective submersive definable $C^r G$ maps between definable $C^r G$ manifolds. But this kind of triviality does not hold in general. For example, it fails when $f$ is the projection onto $S^n$ of the tangent bundle of the standard $n$-dimensional sphere $S^n$ with the standard $O(n+1)$ action for $n \geq 8$ because the action is transitive and this bundle is not trivial.

It is known that $M$ admits the $C^r$ cell decomposition for any non-negative integer $r$ (see [4, 7.3.3.2]). We say that $M$ admits the $C^\omega$ (respectively $C^\infty$) cell decomposition if we can take $r = \omega$ (respectively $r = \infty$).

\textbf{Theorem 1.1.} Let $G$ be a compact definable $C^r$ group and $1 \leq r < \infty$. Let $S$ be a definable $C^r G$ submanifold of a representation of $G$ and let $A$ be a definable $C^r$ submanifold of $\mathbb{R}^n$. Then every $G$ invariant surjective submersive definable $C^r$ map $f: S \to A$ is piecewise definably $C^r G$ trivial. Moreover we can take $r = \omega$ (respectively $r = \infty$) if $M$ admits the $C^\omega$ (respectively $C^\infty$) cell decomposition.

Let $G$ be a definable $C^r$ group and $0 \leq r \leq \omega$. We say that a noncompact definable $C^r G$ manifold is compactifiable as a definable $C^r G$ manifold if it is definably $C^r G$ diffeomorphic (definably $G$ homeomorphic if $r = 0$) to the interior of some compact definable $C^r G$ manifold with boundary.

As an application of Theorem 1.1, we have the following result.

\textbf{Theorem 1.2.} Let $G$ be a compact definable $C^r$ group and $0 \leq r < \infty$. Then every affine definable $C^r G$ manifold $X$ is either compact or compactifiable as a definable $C^r G$ manifold.
The following is a definable version of a well-known fact on Lie groups. Our proof is quite different from that of the fact.

**Theorem 1.3.** Let $H$ be a definable $C^r$ subgroup of a definable $C^r$ group $G$ and $0 \leq r < \infty$. Then $G/H$ admits a definable $C^r$ manifold structure such that:

1. The projection $\pi : G \to G/H$ is a definable $C^r$ map.
2. For any map $\phi$ from $G/H$ to a definable $C^r$ manifold $Y$, $\phi$ is a definable $C^r$ map if and only if so is $\phi \circ \pi$.

Moreover if $H$ is normal, then $G/H$ is a definable $C^r$ group with the same property.

Here the $G$ action on $G/H$ is $G \times G/H \to G/H$, $(g, g'H) \mapsto gg'H$.

If $M$ admits the $C^\omega$ (respectively $C^\infty$) cell decomposition, then we can take $r = \omega$ (respectively $r = \infty$) in Theorem 1.3.

The following is a definable version of the structure theorem on $C^\infty G$ manifolds (see [12]). One of the good references on orbit types is Section 1.1.8 [12].

**Theorem 1.4.** Suppose that $G$ is a compact affine definable $C^\infty$ group and $X$ is a compact definable $C^\infty G$ manifold with only one orbit type $G/H$. Then $X/G$ admits a unique structure of definable $C^\infty$ manifold such that:

1. The projection $\pi : X \to X/G$ is a definable $C^\infty$ map.
2. For any definable $C^\infty$ manifold $Y$ and a map $h : X/G \to Y$, $h$ is a definable $C^\infty$ map if and only if so is $h \circ \pi$.
3. $(X, \pi, X/G, G/H, N(H)/H)$ is a definable $C^\infty$ fiber bundle.

Replacing compactness of $X$ by affineness of $X$, Theorem 1.4 except (3) remains valid because a similar proof as for [9, 4.4] works. Remark that a compact definable $C^\infty G$ manifold is not always affine even if $M = \mathbb{R}$ [10], and that if $M$ is exponential, then every compact definable $C^\infty G$ manifold is affine [9].

The following is a definable version of Whitney’s imbedding theorem.

**Theorem 1.5.** Let $X$ be a definable $C^r$ manifold of dimension $n$ and $2 \leq r < \infty$. If $M$ is polynomially bounded, then $X$ is definably $C^r$ imbeddable into $\mathbb{R}^{2n+1}$. If $M$ is exponential and $X$ is compact, then we can take $r = \infty$.

In Theorem 1.5, we cannot take $r = \infty$ even if $M = \mathbb{R}$ because there exist uncountably many compact nonaffine Nash manifolds [20].

Let $G$ be a compact definable $C^\infty$ group. Let $\Omega$ be a representation of $G$ and $0 \leq r \leq \infty$. A definable $C^r G$ manifold $X$ is said to be subordinate to $\Omega$ if for each $x \in X$, there exist an open $G$ invariant definable neighborhood $U_x$ of $x$ and a definable $C^r G$ imbedding from $U_x$ into $\Omega^t = \{0\}$, where $\Omega^t$ denotes the $t$-fold direct sum of $\Omega$ for some $t$.

The next theorem is an equivariant version of Theorem 1.5 when $X$ is compact.

**Theorem 1.6.** Let $G$ be a compact definable $C^\infty$ group and $2 \leq r < \infty$. If $X$ is an $n$-dimensional compact definable $C^r G$ manifold subordinate to a representation $\Omega$ of $G$,
then every definable \( C^r \) map \( f : X \to \Omega^t \) can be approximated in the \( C^r \) topology by definable \( C^r \) immersions (respectively definable \( C^r \) imbeddings) if \( t \geq 2n \) (respectively if \( t \geq 2n + 1 \)). Moreover if \( M \) is exponential, then for any \( k \in \mathbb{N} \), every definable \( \mathcal{C}^\infty \) map \( f : X \to \Omega^t \) is approximated in the \( \mathcal{C}^k \) topology by definable \( \mathcal{C}^\infty \) immersions (respectively definable \( \mathcal{C}^\infty \) imbeddings) if \( t \geq 2n \) (respectively if \( t \geq 2n + 1 \)).

Let \( G \) be a compact definable \( C^r \) group and \( 0 \leq r \leq \omega \). Let \( \eta \) be a definable \( C^r \) vector bundle over an affine definable \( C^r \) manifold \( X \). We say that \( \eta \) is strongly definable if \( \eta \) admits its classifying map in the definable \( C^r \) category (see Definition 4.4). There exists a non-strongly definable \( \mathcal{C}^\infty \) vector bundle over \( \mathbb{R} \) even if \( M = \mathbb{R} \) (see [2, 12.7.9]). We have the following results on definable \( \mathcal{C}^\infty \) vector bundles over a compact base space.

**Theorem 1.7.** Let \( G \) be a compact definable \( \mathcal{C}^\infty \) group and let \( X \) be a compact affine definable \( \mathcal{C}^\infty \) manifold. Let \( Y \) be an affine definable \( \mathcal{C}^\infty \) manifold.

1. If \( 0 \leq r < \infty \), then every \( \mathcal{C}^r \) map \( f : X \to Y \) is approximated in the \( \mathcal{C}^r \) topology by definable \( \mathcal{C}^\infty \) maps. In particular, for any two compact affine definable \( \mathcal{C}^\infty \) manifolds, they are definably \( \mathcal{C}^1 \) diffeomorphic if and only if they are \( \mathcal{C}^1 \) diffeomorphic.

2. Let \( \eta_1 \) and \( \eta_2 \) be strongly definable \( \mathcal{C}^\infty \) vector bundles over \( X \). If \( \eta_1 \) and \( \eta_2 \) are \( G \) vector bundle isomorphic, then they are definably \( \mathcal{C}^\infty \) vector bundle isomorphic.

3. Let \( \xi \) be a strongly definable \( \mathcal{C}^\infty \) vector bundle over \( Y \). Then for any two definable \( \mathcal{C}^\infty \) maps \( f_1, f_2 : X \to Y \), if they are \( G \) homotopic, then \( f_1^* (\xi) \) and \( f_2^* (\xi) \) are definably \( \mathcal{C}^\infty \) vector bundle isomorphic.

In Theorem 1.7 (2), (3), we cannot drop the condition “strongly definable” even if \( M = \mathbb{R} \) [8].

**Theorem 1.8.** Let \( G \) be a finite group. Let \( X \) be an affine definable \( \mathcal{C}^r \) manifold and \( 0 \leq r < \infty \).

1. Every definable \( \mathcal{C}^r \) vector bundle over \( X \) is strongly definable.

2. If \( 0 \leq s \leq r \) and \( X \) is compact, then each definable \( \mathcal{C}^s \) vector bundle over \( X \) admits a definable \( \mathcal{C}^r \) vector bundle structure over \( X \). For any two definable \( \mathcal{C}^r \) vector bundles over \( X \), if they are definably \( \mathcal{C}^r \) vector bundle isomorphic, then they are definably \( \mathcal{C}^r \) vector bundle isomorphic.

3. If \( M \) is exponential and \( X \) is compact, then we can take \( r = \infty \) in (1) and (2).

There is a new topology of the set of definable \( \mathcal{C}^k \) maps between affine definable \( \mathcal{C}^k \) manifolds (see Section 4). We call it the definable \( \mathcal{C}^k \) topology, and it is useful to consider approximations of maps between noncompact affine definable \( \mathcal{C}^k \) manifolds.

**Proposition 1.9.** Let \( G \) be a compact definable \( \mathcal{C}^\infty \) group. Let \( X \) and \( Y \) be definable \( \mathcal{C}^\infty \) submanifolds of representations \( \Omega \) and \( \Xi \) of \( G \), respectively, and \( 0 \leq k < r < \infty \). If
If $G$ is finite, or $X$ has only one orbit type, then every definable $C^k_G$ map $f : X \to Y$ can be approximated in the definable $C^k$ topology by definable $C^r_G$ maps.

The following two corollaries are applications of Proposition 1.9, and in Proposition 1.9 and their corollaries, we can take $r = \infty$ if $M = \mathbb{R}$.

**Corollary 1.10.** Let $G$ be a compact definable $C^\infty$ group and let $X$ be a definable $C^\infty_G$ submanifold in a representation $\Omega$ of $G$ with only one orbit type. Let $Y$ be an affine definable $C^r_G$ manifold and $1 \leq r < \infty$. Then $X$ and $Y$ are definably diffeomorphic if and only if they are definably $C^1_G$ diffeomorphic.

In Corollary 1.10, even if $M = \mathbb{R}$ and $G = 1$, we cannot replace “definably $C^1_G$ diffeomorphic” by “$C^1_G$ diffeomorphic” because there exist two affine Nash manifolds which are $C^\infty$ diffeomorphic but not Nash diffeomorphic [21] and for any two affine Nash manifolds, they are Nash diffeomorphic if they are $C^1$ Nash diffeomorphic (see [23, II.4.4]).

**Corollary 1.11.** Let $G, X$ be as in Corollary 1.10 and $0 \leq r < \infty$. Let $\eta_1$ and $\eta_2$ be strongly definable $C^r_G$ vector bundles over $X$. If they are definably $G$ vector bundle isomorphic, then they are definably $C^r_G$ vector bundle isomorphic.

This paper is organized as follows.

In Section 2, we recall definable $C^r_G$ manifolds [9] and consider properties of definable $C^r(0 \leq r \leq \omega)$ groups, definable $C^r_G(0 \leq r \leq \omega)$ manifolds, and their orbit spaces.

Imbeddings of definable $C^r_G$ manifolds generalizing Whitney’s imbedding theorem are considered in Section 3.

In Section 4, we consider approximations of definable $C^s_G$ maps by definable $C^r_G$ maps when $0 \leq s < r \leq \infty$, and homotopy properties of definable $C^r_G$ vector bundles.

### 2. Definable $C^r$ groups, definable $C^r_G$ manifolds, and their orbit spaces

Let $K \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^m$ be definable sets. We say that a map $f : K \to L$ is definable if the graph of $(K \times L \subset \mathbb{R}^n \times \mathbb{R}^m)$ is definable. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open definable sets and $0 \leq r \leq \omega$. A $C^r$ map $f : U \to V$ is called a definable $C^r$ map if it is definable. A definable $C^r$ map $h : U \to V$ is called a definable $C^r$ diffeomorphism (a definable homeomorphism if $r = 0$) if there exists a definable $C^r$ map $k : V \to U$ such that $h \circ k = \text{id}$ and $k \circ h = \text{id}$.

Recall definable $C^r$ groups and definable $C^r_G$ manifolds [9].

**Definition 2.1.** Suppose that $0 \leq r \leq \omega$.

(1) A definable subset $X$ of $\mathbb{R}^n$ is called a *d-dimensional definable $C^r$ submanifold of $\mathbb{R}^n$* if for any $x \in X$ there exists a definable $C^r$ diffeomorphism (a definable
Definition 2.2. (b) By o-minimality, a definable \( \mathbb{C}^r \) group

(a) The definition of definable subsets of a definable \( \mathbb{C}^r \) manifold with a finite system of charts \( \{ \phi_i : U_i \rightarrow \mathbb{R}^n \} \) such that for each \( i \) and \( j \), \( \phi_i(U_i \cap U_j) \) is an open definable subset of \( \mathbb{R}^d \) and the map \( \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_j \cap U_j) \) is a definable \( \mathbb{C}^r \) diffeomorphism (a definable homeomorphism if \( r = 0 \)). We call this atlas definable \( \mathbb{C}^r \). Definable \( \mathbb{C}^r \) manifolds with compatible atlases are identified.

A subset \( Y \) of \( X \) is said to be definable if each \( \phi_i(U_i \cap Y) \) is a definable subset of \( \mathbb{R}^d \). A definable subset \( Z \) of \( X \) is called a \( k \)-dimensional definable \( \mathbb{C}^r \) submanifold of \( X \) if each point \( x \in Z \) there exist an open definable neighborhood \( U_x \) of \( x \) in \( X \) and a definable \( \mathbb{C}^r \) diffeomorphism \( \phi_x \) from \( U_x \) to some open definable subset \( V_x \) of \( \mathbb{R}^d \) such that \( \phi_x(x) = 0 \) and \( U_x \cap Y = \phi_x^{-1}(\mathbb{R}^k \cap V_x) \), where \( \mathbb{R}^k \subset \mathbb{R}^d \) is the vectors whose last \( (d-k) \) components are zero.

(3) Let \( X \) (respectively \( Y \)) be a definable \( \mathbb{C}^r \) manifold with definable \( \mathbb{C}^r \) charts \( \{ \phi_i : U_i \rightarrow \mathbb{R}^n \} \) (respectively \( \{ \psi_j : V_j \rightarrow \mathbb{R}^n \} \)). A \( \mathbb{C}^r \) map \( f : X \rightarrow Y \) is said to be a definable \( \mathbb{C}^r \) map if for any \( i \) and \( j \), \( \phi_i(f^{-1}(V_j) \cap U_i) \) is open and definable in \( \mathbb{R}^n \) and the map \( \psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \rightarrow \mathbb{R}^n \) is a definable \( \mathbb{C}^r \) map.

(4) Let \( X \) and \( Y \) be definable \( \mathbb{C}^r \) manifolds. We say that \( X \) is definably \( \mathbb{C}^r \) diffeomorphic to \( Y \) (definably homeomorphic to \( Y \) if \( r = 0 \)) if one can find definable \( \mathbb{C}^r \) maps \( f : X \rightarrow Y \) and \( h : Y \rightarrow X \) such that \( f \circ h = \text{id} \) and \( h \circ f = \text{id} \).

(5) A definable \( \mathbb{C}^r \) manifold is said to be affine if it is definably \( \mathbb{C}^r \) diffeomorphic (definably homeomorphic if \( r = 0 \)) to a definable \( \mathbb{C}^r \) submanifold of some \( \mathbb{R}^l \).

(6) A definable \( \mathbb{C}^r \) submanifold of \( \mathbb{R}^n \) with boundary and a definable \( \mathbb{C}^r \) manifold with boundary are defined similarly.

Remark.

(a) The definition of definable subsets of a definable \( \mathbb{C}^r \) manifold \( X \) is not depends on the choice of definable \( \mathbb{C}^r \) charts of \( X \).

(b) By o-minimality, a definable \( \mathbb{C}^r \) submanifold of \( \mathbb{R}^n \) admits a finite family of definable \( \mathbb{C}^r \) charts, thus it is of course a definable \( \mathbb{C}^r \) manifold. In Definition 2.1 (2), by o-minimality, \( Z \) is covered by finitely many such neighborhoods. Hence \( Z \) is also a definable \( \mathbb{C}^r \) manifold.

(c) We can consider a definable \( \mathbb{C}^r \) manifold \( X \) with possibly different dimensions on different connected components of \( X \). In this paper, we assume that every connected component of a definable \( \mathbb{C}^r \) manifold has the same dimension.

Definition 2.2. Let \( 0 \leq r \leq \omega \).

(1) A group \( G \) is called a definable \( \mathbb{C}^r \) group (respectively an affine definable \( \mathbb{C}^r \) group) if \( G \) is a definable \( \mathbb{C}^r \) manifold (respectively an affine definable \( \mathbb{C}^r \) manifold) and that the multiplication \( G \times G \rightarrow G \) and the inversion \( G \rightarrow G \) are definable \( \mathbb{C}^r \) maps.
Let $G$ be a definable $C^r$ group.
(2) A subgroup $H$ of $G$ is called definable if it is a definable subset of $G$.
(3) A subgroup $K$ of $G$ is said to be a definable $C^r$ subgroup of $G$ if $K$ is a definable $C^r$ submanifold of $G$.
(4) A group homomorphism (respectively an group isomorphism) between two definable $C^r$ groups is a definable $C^r$ group homomorphism (respectively a definable $C^r$ group isomorphism) if it is a definable $C^r$ map (respectively a definable $C^r$ diffeomorphism (a definable homeomorphism if $r = 0$)).
(5) A representation map of $G$ is a group homomorphism from $G$ to some $O_n(\mathbb{R})$ which is a definable $C^r$ map. A representation of $G$ means some $\mathbb{R}^n$ with the linear action induced by a representation map $G \to O_n(\mathbb{R})$. In this paper, we assume that every representation of $G$ is orthogonal.
(6) A definable $C^r$ submanifold of a representation $\Omega$ of $G$ is called a definable $C^rG$ submanifold if it is $G$ invariant.
(7) A definable $C^rG$ manifold is a pair $(X, \theta)$ consisting of a definable $C^r$ manifold $X$ and a group action $\theta$ of $G$ on $X$ such that $\theta: G \times X \to X$ is a definable $C^r$ map. For simplicity of notation, we write $X$ instead of $(X, \theta)$.
(8) A definable $C^r$ submanifold of a definable $C^rG$ manifold $X$ is called a definable $C^rG$ submanifold of $X$ if it is $G$ invariant.

One can consider definable $C^rG$ maps, definable $C^rG$ diffeomorphisms, definable $G$ homeomorphisms, and affine definable $C^rG$ manifolds in a similar way.

If $M = \mathcal{R} = (\mathbb{R}, +, \cdot, <)$, then a definable subset of $\mathbb{R}^n$ is a semialgebraic subset of it [24]. A definable $C^\omega$ manifold (respectively definable $C^\omega$ group, definable $C^\omega G$ manifold) in $\mathcal{R}$ is called a Nash manifold (respectively a Nash group, a Nash $G$ manifold). We say that an affine definable $C^\omega$ group (respectively an affine definable $C^\omega G$ manifold) in $\mathcal{R}$ is an affine Nash group (respectively an affine Nash $G$ manifold). Nash $G$ manifold structures of $C^\omega G$ manifolds are studied in [10] and one-dimensional connected Nash groups are classified by [14].

A definable subset $Y$ of a definable set $X \subset \mathbb{R}^n$ is called large if $\dim(X - Y) < \dim X$. By the $C^r$ cell decomposition theorem for $0 \leq r < \infty$ (see [4, 7.3.3.2]) and [17, 1.1], we have the next lemma.

**Lemma 2.3.** Let $X$ and $Y$ be definable $C^r$ submanifolds of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $1 \leq r < \infty$. For any continuous definable map $f : X \to Y$, there exists an open definable large subset $Z$ of $X$ such that $f|Z : Z \to Y$ is a definable $C^r$ map.

In Lemma 2.3, we can take $r = \omega$ (respectively $r = \infty$) if $\mathcal{M}$ admits the $C^\omega$ (respectively $C^\infty$) cell decomposition.

**Theorem 2.4** (see [4, 9.1.2] and [4, 6.2.5]). Let $S \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$ be definable sets and let $f : S \to A$ be a continuous definable map. Then there exists a finite partition $\{A_i\}$ of $A$ into definable sets such that each $f|f^{-1}(A_i) : f^{-1}(A_i) \to A_i$ is definably trivial.
In a more general setting, the above piecewise definable triviality remains valid (see [4, 9.1.2] and [4, 6.2.5]).

A definable subset \( G \) of \( \mathbb{R}^n \) is called a **definable topological group** if \( G \) is a topological group and its group operations \( G \times G \to G \) and \( G \to G \) are definable. A subgroup of a definable topological group \( G \) is a **definable subgroup** if it is a definable subset of \( G \). A topological group homomorphism (respectively a topological group isomorphism) between two definable topological groups is a **definable topological group homomorphism** (respectively a **definable topological group isomorphism**) if it is definable. Similarly, we can consider definable topological homomorphisms and definable topological isomorphisms between definable \( C^r \) groups and definable topological groups.

Let \( G \) be a definable topological group. A **definable \( G \) set** is a pair \((X, \theta)\) consisting a definable set \( X \) and a group action \( \theta : G \times X \to X \) which is a continuous definable map.

Let \( X \) and \( Y \) be two definable \( G \) sets. A **definable \( G \) map** \( f : X \to Y \) is called a **definable \( G \) map** if it is definable. We say that \( X \) and \( Y \) are **definably \( G \)-homeomorphic** if there exist continuous definable \( G \) maps \( h : X \to Y \) and \( k : Y \to X \) such that \( h \circ k = \text{id} \) and \( k \circ h = \text{id} \).

One can define piecewise definable \( G \) triviality of continuous definable \( G \) maps between definable \( G \) sets. We have a piecewise definable \( G \) trivial property of \( G \)-invariant continuous definable maps from a definable \( G \) set to a definable set, which is an equivariant version of Theorem 2.4.

**Theorem 2.5.** Let \( G \) be a compact definable topological group and let \( S \) be a definable \( G \) set in some representation \( \Omega \) of \( G \). Let \( A \) be a definable set in some \( \mathbb{R}^n \) and let \( f : S \to A \) be a \( G \)-invariant continuous definable map. Then there exists a finite partition \( \{A_i\} \) of \( A \) into definable sets such that each \( f \mid f^{-1}(A_i) : f^{-1}(A_i) \to A_i \) is definably \( G \) trivial.

To prove Theorem 2.5, we need some preparations.

**Theorem 2.6** (see [4, 10.2.18]). Let \( G \) be a compact definable topological group and let \( X \) be a definable \( G \) set. Then the orbit space \( X/G \) exists as a definable set and the orbit map \( \pi : X \to X/G \) is definable, continuous and proper.

In a more general setting, Theorem 2.6 is true (see [4, 10.2.8]).

By a way similar to [3, Theorem II.3.1], we have the following lemma.

**Lemma 2.7.** Let \( G \) be a compact definable topological group and let \( K, H \) be definable subgroups of \( G \) with \( K \subset H \). If \( X \) is a definable \( K \) set, then the map \( G \times_K X \to G \times_H (H \times_K X), [g, x] \mapsto [g, [e, x]] \) is a definable \( G \) homeomorphism.

**Proof of Theorem 2.5.** By Theorem 2.6, the orbit space \( S/G \) is a definable set and the orbit map \( \pi : S \to S/G \) is definable, continuous and proper. Thus we can find a continuous definable map \( \overline{f} : S/G \to A \) induced from \( f \) with \( f = \overline{f} \circ \pi \). By Theorem 2.4, \( \overline{f} \) is piecewise definably trivial. Thus it suffices to prove that \( \pi \) is piecewise definably \( G \) trivial.

Since \( S \subset \Omega \) and \( \Omega \) is a representation of \( G \), \( S \) has only finitely many orbit types, say \((H_1), \ldots, (H_l)\). Then for each \((H_i)\), \( S(H_i) \) is a definable \( G \) subset of \( S \) because \( S(H_i) = \)}
\[ \{ x \in S \mid (G_x) = (H_i) \} = \{ x \in S \mid \exists g \in G \ gG_xg^{-1} = H_i \}. \] Moreover \( \pi(S(H_i)) = S(H_i)/G \) is also a definable subset of \( \pi(S) = S/G \). Considering the restriction \( \pi|S(H_i) : S(H_i) \to S(H_i)/G \) of \( \pi \), we may assume that \( S \) has only one orbit type, say \( (H) \).

Then the \( H \) fixed point set \( S^H = \{ s \in S \mid hs = s \ \forall h \in H \} \) is a closed \( N \) subset of \( S \), where \( N \) denotes the normalizer of \( H \) which is a closed definable subgroup of \( G \). The map \( \alpha : G \times N S^H \to S, \alpha([g, x]) = gx \) is a definable \( G \) homeomorphism because the graph of \( \alpha \) is the image of that of the restriction on \( G \times S^H \) of the action map \( G \times S \to S \) by the projection \( \pi_1 : G \times S^H \to G \times N S^H \). Moreover the inclusion \( j : S^H \to S \) induces a definable homeomorphism \( \beta : S^H/N \to S/G \). By Theorem 2.4, there exists a finite partition \( \{ A_i \}_{i=1}^k \) of \( S^H/N \) into definable sets such that for each \( i \), there exists a definable homeomorphism \( \phi_i : N/H \times A_i \to \pi^{-1}_H(A_i) \) with \( \pi_H \circ \phi_i = p_1 \), where \( \pi_H : S^H \to S^H/N \) denotes the orbit map and \( p_1 \) stands for the projection \( N/H \times A_i \to A_i \).

Let \( B_i = \beta(A_i) \). Since \( H \) acts trivially on \( A_i \), \( N \times_H A_i \cong N/H \times A_i \) and \( G \times_H A_i \cong G/H \times A_i \). Therefore \( \psi_i : G/H \times B_i \to \pi^{-1}_H(B_i) \), \( \psi_i(gH, x) = g \circ \phi(eH, \beta^{-1}(x)) \) is a definable \( G \) homeomorphism because Lemma 2.7 and

\[
\begin{align*}
G/H \times B_i &\cong G/H \times A_i \cong G \times_H A_i \cong G \times_N (N \times_H A_i) \\
&\cong G \times_N (N/H \times A_i) \cong G \times_N \pi^{-1}_H(A_i) \\
&\cong G \times_N (\pi^{-1}(B_i)H) \cong \pi^{-1}(B_i). \quad \Box
\end{align*}
\]

By the classical inverse function theorem, we have the inverse function theorem for definable \( C^r \) maps.

**Theorem 2.8.** Let \( f : U \to \mathbb{R}^n \) be a definable \( C^r \) map on an open definable set \( U \subset \mathbb{R}^n \) and \( 1 \leq r \leq \infty \). If \( a \in U \) is a point such that the Jacobian \( df : \mathbb{R}^n \to \mathbb{R}^n \) of \( f \) at \( a \) is invertible, then there exist open definable neighborhoods \( U' \subset U \) of \( a \) and \( V' \) of \( f(a) \) such that \( f : U' \to V' \) is a definable \( C^r \) diffeomorphism.

Let \( G \) be a definable \( C^r \) group, \( X \) and \( Y \) definable \( C^r \) manifolds, and \( r > 0 \). Suppose that \( f : X \to Y \) is a definable \( C^r \) imbedding. Then by Theorem 2.8, the remark after Definition 2.1, and an argument similar to \( C^\infty \) manifold cases, \( f(X) \) is a definable \( C^r \) submanifold of \( Y \) and \( f : X \to f(X) \) is a definable \( C^r \) diffeomorphism.

We consider definable \( C^r \) triviality and piecewise definable \( C^r \) triviality of definable \( C^r \) maps. Let \( G \) be a definable \( C^r \) group and \( 0 \leq r \leq \infty \). Let \( S \) and \( A \) be definable \( C^r \) manifolds and let \( f : S \to A \) be a surjective definable \( C^r \) map. We say that \( f \) is definably \( C^r \) trivial, if there exist a definable \( C^r \) manifold \( F \) and a definable \( C^r \) map \( h : S \to F \) such that \( f \circ h : S \to A \times F \), \( (f \circ h)(s) = (f(s), h(s)) \) is a definable \( C^r \) diffeomorphism (a definable \( G \) homeomorphism if \( r = 0 \)).

Assume that \( f \) is submersive and \( G \) invariant. If \( f : S \to A \) is definably \( C^r \) trivial and \( r \geq 1 \), then for every \( a \in A \) the fiber \( f^{-1}(a) \) of \( a \) is a definable \( C^r \) submanifold of \( S \) which is definably \( C^r \) diffeomorphic to \( F \). Hence one can find a definable \( C^r \) diffeomorphism \( \phi : S \to A \times f^{-1}(a) \) such that \( f = p \circ \phi \), where \( p \) denotes the projection \( A \times f^{-1}(a) \to A \).
We call $f$ piecewise definably $C^rG$ trivial if there exists a finite partition $\{C_i\}$ of $A$ into definable $C^rG$ submanifolds of $A$ such that each $f^{-1}(C_i)$ is a definable $C^rG$ submanifold of $S$ and every $f|f^{-1}(C_i): f^{-1}(C_i) \rightarrow C_i$ is definably $C^rG$ trivial.

**Proof of Theorem 1.1.** We proceed by induction on dim $S$. If dim $S = 0$, the result is clear. Assume dim $S = k > 0$. Using [4, 10.1.8], $G$ is definably topologically group isomorphic to a definable topological group. By Theorem 2.5, there exists a finite partition $\{D_j\}$ of $A$ into definable sets such that each $f|f^{-1}(D_j): f^{-1}(D_j) \rightarrow D_j$ is definably $G$ trivial. Using a $C^r$ cell decomposition of $A$ compatible with $\{D_j\}$, we may assume that each member of $\{D_j\}$ is a definable $C^r$ submanifold of $A$.

Let $S_j = f^{-1}(D_j)$. Then $S_j$ is a definable $C^rG$ submanifold of $S$. We have only to consider $f|S_j: S_j \rightarrow D_j$ with dim $S = k$. If dim $S_j < k$, then $f|S_j: S_j \rightarrow D_j$ is piecewise definably $C^rG$ trivial by the inductive hypothesis. Note that $f|S_j: S_j \rightarrow D_j$ is a submersion.

Since $f|S_j: S_j \rightarrow D_j$ is definably $G$ trivial, there exists a continuous definable $G$ map $h_j: S_j \rightarrow F_j$ such that $(f|S_j, h_j): S_j \rightarrow D_j \times F_j$ is a definable $G$ homeomorphism, where $F_j = f^{-1}(a_j), a_j \in D_j$. Applying Lemma 2.3 to $h_j$, we have a closed $G$ invariant definable subset $S_j'$ of $S_j$ such that $\dim S_j' < k$ and $h_j|S_j - S_j' : S_j - S_j' \rightarrow h_j(S_j - S_j') \subset F_j$ is a definable $C^rG$ map. Since $S_j - S_j'$ is open and $G$ invariant in $S_j$, $f(S_j - S_j')$ is an open $G$ invariant definable subset of $f(S_j)$. Thus $(f, h_j)|S_j - S_j' : S_j - S_j' \rightarrow f(S_j - S_j')$ is a definable $C^rG$ map. Applying the same argument to the inverse of $(f, h_j)|S_j - S_j'$, we have a closed $G$ invariant definable subset $W_j$ of $S_j - S_j'$ and a closed $G$ invariant definable subset $W'_j$ of $f(S_j - S_j') \times h_j(S_j - S_j')$ such that $\dim W_j, \dim W'_j < k$ and $(f, h_j)(S_j - S_j' - W_j): S_j - S_j' - W_j \rightarrow (f(S_j - S_j') \times h_j(S_j - S_j')) - W'_j$ is a definable $C^rG$ diffeomorphism. Take a $C^r$ cell decomposition $\{U_j^i\}$ of $S_j - S_j'$ - $W_j$. Thus each $(f, h_j)(U_j^i): U_j^i \rightarrow f(U_j^i) \times h_j(U_j^i)$ is a definable $C^rG$ diffeomorphism because $(f, h_j)(W_j) = W'_j$.

Take a $C^r$ cell decomposition $\{T_j\}$ of $f(S_j' \cup W_j)$. Then each $f^{-1}(T_j)$ is a definable $C^rG$ submanifold of $S$ and $f|f^{-1}(T_j): f^{-1}(T_j) \rightarrow T_j$ satisfies the inductive hypothesis. Thus it is piecewise definably $C^rG$ trivial. Therefore the proof of the first half is complete.

The latter half follows similarly. \[\square\]

Even in the non-equivariant category, $f: S \rightarrow A$ is not always definably trivial. For example, let $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 \mid y = 2\} \subset \mathbb{R}^2, A = \mathbb{R}, f: S \rightarrow A, f(x, y) = x$. Then $f$ is a surjective submersive definable $C^0$ map, and it is piecewise definably $C^0$ trivial but not definably trivial.

**Theorem 2.9** [6]. Let $A \subset \mathbb{R}^r$ be a closed definable set and $0 \leq r < \infty$. Then there exists a definable $C^r$ function $f$ on $\mathbb{R}^n$ with $A = f^{-1}(0)$.

**Proposition 2.10.** Let $G$ be a compact definable $C^r$ group and let $X$ be a definable $C^rG$ manifold in a representation $\Omega$ of $G$ and $0 \leq r < \infty$. Then $X$ is definably $C^rG$ imbeddable.
into $\Omega \times \mathbb{R}^2$ such that $X$ is bounded and $\overline{X} - X$ consists of at most one point, where $\overline{X}$ denotes the closure of $X$.

**Proof.** We may assume that $X$ is noncompact. Then $\overline{X} - X$ is a $G$ invariant closed definable subset of $\Omega$. Let $\pi : \Omega \rightarrow \Omega / G \subset \mathbb{R}^2$ denote the orbit map. Then $i \circ \pi : \Omega \rightarrow \mathbb{R}^2$ is a proper polynomial map (see [9, Section 4]), where $i : \Omega / G \rightarrow \mathbb{R}^2$ denotes the inclusion. Hence $i \circ \pi(\overline{X} - X : \overline{X} - X \rightarrow \mathbb{R}^2$ is proper because $\overline{X} - X$ is closed in $\Omega$. Thus $i \circ \pi(\overline{X} - X) (= \pi(\overline{X} - X))$ is a closed definable subset of $\mathbb{R}^2$. Applying Theorem 2.9, there exists a definable $C^r$ function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $\pi(\overline{X} - X) = f^{-1}(0)$. Hence $F := f \circ \pi : \Omega \rightarrow \mathbb{R}$ is a $G$ invariant definable $C^r$ function with $\overline{X} - X = F^{-1}(0)$. Therefore replacing $X$ by the graph of $1/F$, we may assume that $X$ is closed in $\Omega \times \mathbb{R}$. Applying the stereographic projection $s : \Omega \times \mathbb{R} \rightarrow S(\Omega \times \mathbb{R}^2)$, $s(X)$ satisfies our requirements, where $S(\Omega \times \mathbb{R}^2)$ denotes the standard sphere of $\Omega \times \mathbb{R}^2$. $\square$

**Proof of Theorem 1.2.** Let $X$ be a noncompact affine definable $C^r G$ manifold. By Proposition 2.10, one can find a representation $\Omega$ of $G$ and a definable $C^r G$ imbedding $i : X \rightarrow \Omega$ such that $i(X)$ is bounded and $\overline{i(X)} - i(X) = \{0\}$, where $\overline{i(X)}$ denotes the closure of $i(X)$ in $\Omega$.

Let $f : i(X) \rightarrow \mathbb{R}$, $f(x) = \|x\|^{-1}$, where $\|x\|$ denotes the standard norm of $x$ in $\Omega$. Let $r > 0$. Applying Theorem 1.1, one can find a positive integer $k$ and a definable $C^r G$ diffeomorphism $h := (f, h_1) : f^{-1}((k, \infty)) \rightarrow (k, \infty) \times f^{-1}(k)$. If $k$ is sufficiently large, then $f^{-1}([0, k + 1])$ is a compact definable $C^r G$ manifold with boundary. Hence using $h$, $i(X)$ is definably $C^r G$ diffeomorphic to $f^{-1}([0, k + 1])$ which is the interior of $f^{-1}([0, k + 1])$.

The case $r = 0$ is proved in a similar way using Theorem 2.5. $\square$

Petrie [18] proved that if $G$ is a compact Lie group, then any nonsingular algebraic $G$ set is either compact or compactifiable as a $C^\infty G$ manifold, namely it is either compact or $C^\infty G$ diffeomorphic to the interior of some compact $C^\infty G$ manifold with boundary. An argument similar to his proof works for the next proposition because the number of connected components of the zeros of a definable $C^\infty$ map is finite.

**Proposition 2.11.** Let $G$ be a compact definable $C^\infty$ group and let $X$ be a non-compact affine definable $C^\infty G$ manifold. Then $X$ is compactifiable as a $C^\infty G$ manifold.

**Theorem 2.12.** Let $G$ be a definable topological group and $0 \leq r < \infty$. Then there exists a unique definable $C^r$ group $G'$ such that $G$ is definably topologically isomorphic to $G'$ up to definable $C^r$ group isomorphism. Moreover any definable subgroup of $G$ is closed.

Theorem 2.12 is obtained, using [17, 1.10], in a similar way as [19, 2.5] and [19, 2.8] since the group operations of $G$ are continuous.

If $\mathcal{M}$ admits the $C^\omega$ (respectively $C^\infty$) cell decomposition, then we can take $r = \omega$ (respectively $r = \infty$) in Theorem 2.12 and Corollary 2.13.
If we do not assume that the group operations of $G$ are continuous, then the topology of a definable $C^r$ group structure $G'$ of $G$ does not necessarily coincide with the original topology of $G$. Let $\mathcal{M} = \mathbb{R}$, and let $G = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, \ y = 0 \text{ or } x < 0, \ y = 1\} \subset \mathbb{R}^2$. Then $G$ becomes a group with semialgebraic group operations. The Nash group structure $G'$ of $G$ obtained in [19] is $\mathbb{R} = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. Hence there exists a semialgebraic bijection between $G$ and $G'$, but $G$ is not homeomorphic to $G'$.

**Corollary 2.13.** If $0 \leq r < \infty$, then the definable $C^r$ group isomorphism classes of definable $C^r$ groups corresponds bijectively to the definable topological group isomorphism classes of definable topological groups.

**Remark 2.14.** If $\mathcal{M}$ admits the $C^\omega$ (respectively $C^\infty$) cell decomposition, then the underlying space of a definable $C^\omega$ group (respectively a definable $C^\infty$ group) is a definable subset of some $\mathbb{R}^n$.

**Proof.** Let $G$ be a definable $C^\omega$ (respectively $C^\infty$) group. By [4, 10.1.8], $G$ is definably imbeddable into some $\mathbb{R}^n$. Thus its image $G'$ becomes a definable topological group such that the imbedding is a definable topological isomorphism. Since $\mathcal{M}$ admits the $C^\omega$ (respectively $C^\infty$) cell decomposition and by Theorem 2.12, $G'$ admits a definable $C^\omega$ (respectively $C^\infty$) group structure. Using [17, 1.12], $G$ and $G'$ are definable $C^\omega$ (respectively $C^\infty$) group isomorphic. $\square$

Note that every definable $C^r$ group is definably topologically group isomorphic to a definable topological group. Hence we have the following by [17, 2.17], which is a definable version of a well-known fact that a closed subgroup $H$ of a Lie group $G$ is a Lie subgroup.

**Theorem 2.15.** Let $G$ be a definable $C^r$ group and $0 \leq r < \infty$. Then every definable subgroup of $G$ is a definable $G'$ subgroup of $G$.

If $\mathcal{M}$ admits the $C^\omega$ (respectively $C^\infty$) cell decomposition, then in Theorem 2.15 we can take $r = \omega$ (respectively $r = \infty$).

In particular, in the Nash category, we have the following corollary because $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ admits the $C^\omega$ cell decomposition.

**Corollary 2.16.** A semialgebraic subgroup of a Nash group is a Nash subgroup.

In general, a closed subgroup of a definable group is not necessarily a definable subgroup. For example, if $G = \mathbb{R}$ and $H = \mathbb{Z}$, then $H$ is a closed Lie subgroup but not a definable one.

**Proposition 2.17** (see [17, 2.12]). Let $X$ be a definable set with definable transitive action of a definable group and let $Y$ be a large definable subset of $X$. Then finitely many translates of $Y$ cover $X$. 

Proof of Theorem 1.3. By the proof of Remark 2.14, every definable $C^r$ group is definably topologically group isomorphic to a definable topological group. Note that $\mathcal{M}$ has elimination of imaginaries. Thus if $H$ is a definable subgroup of a definable topological group $G$, then we can consider $G/H$ as a definable object (see [17]). By Theorem 2.19 below, $G/H$ admits a definable $C^r G$ manifold structure such that $\pi$ is a definable map.

Using [17, the proof of 1.11], $\pi$ is a definable $C^r$ map. Applying (the non-equivariant version of) Theorem 1.1, there exists a finite partition $\{C_1, \ldots, C_k\}$ of $G/H$ into definable $C^r$ submanifolds of $G/H$ such that each $\pi|\pi^{-1}(C_i) : \pi^{-1}(C_i) \to C_i$ is definably $C^r$ trivial. Thus, each $\pi|\pi^{-1}(C_i)$ admits a definable $C^r$ section $s_i : C_i \to \pi^{-1}(C_i)$. Let $V$ be the union of Int $C_i$ with $\text{dim} C_i = \text{dim} G/H$.

Then $V$ is open and large in $G/H$, and $\pi|\pi^{-1}(V) : \pi^{-1}(V) \to V$ admits a definable $C^r$ section. Hence by Proposition 2.17, finitely many translates of $V$ cover $G/H$. Thus there exists a finite open definable covering $\{V_l\}$ of $G/H$ such that each $\pi|\pi^{-1}(V_l) : \pi^{-1}(V_l) \to V_l$ admits a definable $C^r$ section. Hence property (2) is proved because $\pi$ admits a local definable $C^r$ section.

If $H$ is normal, then $G/H$ is a group. By the above argument, $G/H$ is a definable $C^r$ manifold. Therefore using property (2), $G/H$ becomes a definable group. $\square$

Applying Theorem 1.3 to the Nash case, we obtain the following corollary because $\mathcal{R} = (\mathbb{R}, +, \cdot, \prec)$ admits the $C^\infty$ cell decomposition.

Corollary 2.18. Let $G$ be a Nash group and let $H$ be a normal Nash subgroup of $G$. Then $G/H$ is a unique Nash group structure such that:

1. The projection $\pi : G \to G/H$ is a Nash map.
2. For any map $\phi$ from $G/H$ to a Nash manifold $Y$, $\phi$ is a Nash map if and only if so is $\phi \circ \pi$.

We are concerned with definable $C^r G$ manifold structures on definable sets with transitive continuous definable $G$ actions, where $G$ is a definable $C^r$ group and $0 \leq r < \infty$. It is a refinement of [17, 2.11].

Theorem 2.19. Let $G$ be a definable $C^r$ group, and let $X$ be a definable $G$ set contained in $\mathbb{R}^n$ with transitive continuous definable $G$ action and $0 \leq r < \infty$. Then $X$ admits a definable $C^r$ manifold structure such that it makes $X$ a definable $C^r G$ manifold. Moreover the definable $G$ homeomorphism classes of transitive continuous definable $G$ action on $X$ corresponds bijectively to the definable $C^r G$ diffeomorphism classes of transitive definable $C^r G$ manifold structures of $X$.

If $\mathcal{M}$ admits the $C^\omega$ (respectively $C^\infty$) cell decomposition, then in Theorem 2.19 we can take $r = \omega$ (respectively $r = \infty$).

Proof. By the proof of Remark 2.14, $G$ is definably topologically group isomorphic to a definable topological group. Take a $C^r$ cell decomposition of $G$. Let $U_1, \ldots, U_k$ be the
Corollary 2.20. Let $G$ be a Nash group and let $X$ be a semialgebraic subset of $\mathbb{R}^n$ with transitive continuous semialgebraic action. Then $X$ admits a unique Nash $G$ manifold structure up to Nash $G$ diffeomorphism.

An argument similar to the proof of Theorem 1(1) [10] shows that if $G$ is a compact Lie group and $X$ is a compact $C^\infty$ manifold, then $X$ is $C^\infty$ diffeomorphic to an affine Nash $G$ manifold $Y$. Using this fact and Theorem 2.19, we can drop the affineness condition on $G$ in Theorem 1(2) [10].

Corollary 2.21. Let $G$ be a compact Nash group and let $X$ be a compact $C^\infty$ manifold with transitive action. Then $X$ is $C^\infty$ diffeomorphic to an affine Nash $G$ manifold $Y$ such that for any Nash $G$ manifold $Z$ which is $C^\infty$ diffeomorphic to $Y$, $Z$ is Nash $G$ diffeomorphic to $Y$. 

In the Nash category, we have the following corollary because $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ admits the $C^\omega$ cell decomposition.
Proposition 2.22. Let $G$ be a compact definable $C^r$ group and $1 \leq r < \infty$. Let $X$ be a definable $C^r G$ manifold and $x \in X$. Then the orbit $G(x)$ is a definable $C^r G$ submanifold of $X$.

If $\mathcal{M}$ admits the $C^\omega$ (respectively $C^\infty$) cell decomposition, then in Proposition 2.22 we can take $r = \omega$ (respectively $r = \infty$).

Proof. By Theorem 1.3, $G/G_x$ is a definable $C^r G$ manifold. Since $G$ is compact, $f: G/G_x \to X, f(gG_x) = gx$ is a $C^r G$ imbedding. Let $\overline{f}: G \to X, \overline{f}(g) = gx$. Then $\overline{f}$ is a definable $C^r G$ map. Hence by Theorem 1.3, $f$ is a definable $C^r G$ map. Thus $f$ is a definable $C^r G$ imbedding. Therefore $G(x)$ is a definable $C^r G$ submanifold of $X$ and $f$ is a definable $C^r G$ diffeomorphism onto $G(x)$. 

Let $G$ be a compact affine definable $C^\infty$ group. Let $X$ be a definable $C^\infty G$ manifold and let $x \in X$. A $G_x$ invariant definable $C^\infty$ submanifold $S$ of $X$ is called a linear definable $C^\infty$ slice at $x$ in $X$ if there exist a representation $\Omega$ of $G_x$ and a definable $C^\infty G_x$ imbedding $j: \Omega \to X$ such that $j(\Omega) = S$, $j(0) = x$, $GS$ is open in $X$, $S$ is affine as a definable $C^\infty G_x$ manifold, and

$$\mu: G \times G_x S \to GS \subset X, \quad [g, x] \mapsto gx$$

is a definable $C^\infty G$ diffeomorphism.

Theorem 2.23 [9]. Suppose that $G$ is a compact affine definable $C^\infty$ group, $X$ is a definable $C^\infty G$ manifold, and that $x \in X$. Then there exists a linear definable $C^\infty$ slice at $x$ in $X$.

The following proposition is obtained in a way similar to the usual equivariant Nash cases [10].

Proposition 2.24. Let $r = \infty$ or $\omega$ and let $G$ be a compact definable $C^r$ group.

1. Every definable $C^r G$ submanifold $X$ of a representation $\Omega$ of $G$ has a definable $C^r G$ tubular neighborhood $(U, p)$ of $X$ in $\Omega$.

2. Any compact affine definable $C^r G$ manifold $X$ with boundary $\partial X$ admits a definable $C^r G$ collar, namely there exists a definable $C^r G$ imbedding $\phi: \partial X \times [0, 1] \to X$ such that $\phi(\partial X \times 0) = \text{id}_{\partial X}$, where the action the closed unit interval $[0, 1]$ is trivial.

Proposition 2.25. Let $G$ be a compact definable $C^\infty$ group and let $X$ be a definable $C^\infty G$ manifold. Then the fixed point set $X^G$ is a definable $C^\infty$ submanifold closed in $X$.

Proof. Since $X^G = \{ x \in X \mid gx = x, \forall g \in G \}$ and $G$ acts on $X$ continuously and definably, $X^G$ is a closed definable subset of $X$.

Take $x \in X^G$. Then by Theorem 2.23, $x$ has an open $G$ invariant definable neighborhood $U$ in $X$ such that $U$ is affine definable $C^\infty G$ manifold. Hence we may assume that $X$ is
affine. Since \( G_x = G \), a definable \( C^\infty \)-tubular neighborhood of \( \{ x \} \) is \( f : T_x(X) \to X \). Then \( f \) is a definable \( C^\infty \)-embedding and \( T_x(X)^G \) is a linear subspace of \( T_x(X) \) and
\[
f(T_x(X)) \cap X^G = f(T_x(X)^G). \quad \square
\]

**Definition 2.26.** Let \( 0 \leq r \leq \omega \).

1. A topological fiber bundle \( \eta = (E, \pi, X, F, K) \) is called a definable \( C^r \) fiber bundle over \( X \) with fiber \( F \) and structure group \( K \) if the following two conditions are satisfied:
   - (a) The total space \( E \) and the base space \( X \) are definable \( C^r \) manifolds, the structure group \( K \) is a definable \( C^r \) group, the fiber \( F \) is a definable \( C^r \) \( K \)-manifold with effective action, and the projection \( \pi : E \to X \) is a definable \( C^r \) map.
   - (b) There exists a finite family of local trivializations \((U_i, \phi_i : \pi^{-1}(U_i) \to U_i \times F)\) of \( \eta \) such that each \( U_i \) is an open definable subset of \( X \), \([U_i]\), is a finite covering of \( X \). For any \( x \in U_i \), let \( \phi_{i,x} : \pi^{-1}(x) \to F, \phi_{i,x}(z) = p_i^* \circ \phi_i(z) \), where \( p_i^* \) stands for the projection \( U_i \times F \to F \). For any \( i \) and \( j \) with \( U_i \cap U_j \not= \emptyset \), the transition function \( \theta_{ji} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K \) is a definable \( C^r \) map.
   We call these trivializations definable \( C^r \). Definable \( C^r \) fiber bundles with compatible local trivializations are identified.

2. Let \( \eta = (E, \pi, X, F, K) \) and \( \xi = (E', \pi', X', F, K) \) be definable \( C^r \) fiber bundles whose local trivializations are \((U_i, \phi_i), \) and \((V_j, \psi_j), \) respectively. A definable \( C^r \) map \( \overline{f} : E \to E' \) is said to be a definable \( C^r \) bundle morphism if the following two conditions are satisfied:
   - (a) There exists a definable \( C^r \) map \( f : X \to X' \) such that \( f \circ \pi = \pi' \circ f \).
   - (b) For any \( i, j \) such that \( U_i \cap U_j \not= \emptyset \) and for any \( x \in U_i \cap \pi^{-1}(V_j) \), the map \( f_{ji}(x) := \phi_{j,x} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F \) lies in \( K \), and \( f_{ji} : U_i \cap \pi^{-1}(V_j) \to K \) is a definable \( C^r \) map.
   We say that \( \overline{f} : E \to E' \) is a definable \( C^r \) fiber bundle isomorphism if \( X = X' \), \( f = \text{id}_X \) and there exists a definable \( C^r \) fiber bundle morphism \( \overline{f} : E \to E' \) such that \( f' = \text{id}_X, \overline{f} \circ \overline{f}' = \text{id}, \) and \( \overline{f}' \circ \overline{f} = \text{id} \).

3. A continuous section \( s : X \to E \) of a definable fiber bundle \( \eta = (E, \pi, X, F, K) \) is a definable \( C^r \) section if for any \( i \), the map \( \phi \circ s : [U_i : U_i \to U_i \times F] \) is a definable \( C^r \) map.

4. We say that a definable \( C^r \) fiber bundle is a principal definable \( C^r \) fiber bundle if \( F = K \) and the \( K \) action on \( F \) is defined by the multiplication of \( K \).

5. A definable \( C^r \) vector bundle is a definable fiber bundle with fiber \( \mathbb{R}^n \) and structure group \( GL_n(\mathbb{R}) \).

6. Let \( \eta_1 = (E, p, X) \) and \( \eta_2 = (E', p', X) \) be definable \( C^r \) vector bundle over \( X \). A definable \( C^r \) vector bundle morphism \( \eta_1 \to \eta_2 \) is a definable \( C^r \) map \( f : E \to E' \) such that \( p = p' \circ f \) and \( f \) is linear on each fiber. A definable \( C^r \) vector bundle morphism \( h : \eta \to \xi \) is said to be a definable \( C^r \) vector bundle isomorphism if there exists a definable \( C^r \) vector bundle morphism \( \overline{h} : \xi \to \eta \) such that \( h \circ \overline{h} = \text{id} \) and \( h \circ \overline{h} = \text{id} \).
Theorem 2.27. Let $G$ be a compact affine definable $C^\infty$ group and let $X$ be a compact definable $C^\infty$ manifold with free $G$ action. Then $X/G$ admits a unique structure of definable $C^\infty$ manifold such that:

1. The projection $\pi : X \to X/G$ is a definable $C^\infty$ map.
2. For any definable $C^\infty$ manifold $Y$ and a map $h : X/G \to Y$, $h$ is a definable $C^\infty$ map if and only if so is $h \circ \pi$.
3. $(X, \pi, X/G, G)$ is a principal definable $C^\infty$ fiber bundle whose fiber is $G$.

Proof. By Theorem 2.23 and since $G$ acts on $X$ freely, for any $x \in X$, there exist a vector space $V_x$ and a definable $C^\infty$ imbedding $f_x : G \times V_x \to X$ onto some open $G$ invariant definable subset $U_x$ of $X$ such that $f_x(e, 0) = x$. Then the induced map $\overline{f}_x : (G \times V_x)/G \to X/G$ obtained from $f_x$ is an imbedding.

Let $j_x : V_x \to G \times V_x, j_x(v) = (e, v)$ and let $\pi_x : G \times V_x \to (G \times V_x)/G$ be the projection. Then $k_x := \pi \circ f_x : V_x \to (G \times V_x)/G$ is a homeomorphism. Let $[V_x] = \overline{f}_x \circ k_x^{-1} : [V_x] \to V_x$ is a chart around $\pi(x)$. Since $X$ is compact, $X/G$ is compact. Thus one can take a finite family of charts $[[V_{x_i}], \psi_{x_i} : [V_{x_i}] \to V_{x_i}]$ of $X/G$.

An argument similar to [12, 4.11] proves that $X/G$ is a definable $C^\infty$ manifold with definable charts $[[V_{x_i}], \psi_{x_i} : [V_{x_i}] \to V_{x_i}]$ satisfying (1), (2) and (3). Uniqueness of definable $C^\infty$ manifold structure of $X/G$ follows from (2). □

In the above proof, we only use compactness of $X$ to prove finiteness of its charts. In Theorem 2.27 if $X$ is affine, then $X/G$ admits a unique definable $C^\infty$ manifold structure satisfying (1) and (2) [9].

By a way similar to [12, the proof of 4.18], we have Theorem 1.4 which is a slightly general version of Theorem 2.27.

3. Generalization of Whitney’s imbedding theorem

Proof of Theorem 1.5. If $\mathcal{M}$ is polynomially bounded, then we may assume that $X$ is affine. Thus it is either compact or compactifiable by Theorem 1.2. Hence we may suppose that $X$ is affine and compact at the beginning. Since $r \geq 2$, applying an argument similar to [25, the proof of 1.4], any definable $C^r$ map $f : X \to \mathbb{R}^{2n+1}$ can be approximated in the $C^r$ topology by injective definable $C^r$ immersions. Hence there exists an injective definable $C^r$ immersion $h : X \to \mathbb{R}^{2n+1}$. Since $X$ is compact, $h$ is a definable $C^r$ imbedding.

If $\mathcal{M}$ is exponential, $r = \infty$ and $X$ is compact, then $X$ is definably $C^\infty$ imbeddable into some $\mathbb{R}^l$ by [9]. Hence $X$ is definably $C^\infty$ imbeddable into $\mathbb{R}^{2n+1}$ by a way similar to the above argument. □

To prove Theorem 1.6, we prepare some results.

By a way similar to [25, the proofs of 1.1–1.4], we have the following proposition.

Proposition 3.1. Let $G$ be a compact definable $C^\infty$ group and let $2 \leq r \leq \infty$. Suppose that $X$ is a compact definable $C^rG$ manifold of dimension $n$ admitting a definable $C^r$
immersion (respectively a definable $C^r$ imbedding) into the $t$-fold direct sum $\Omega^t$ of a representation $\Omega$ of $G$ for some $t$. Then for any $k$ with $1 \leq k \leq r$ and $k < \infty$, any definable $C^r$ map $f : X \to \Omega^{2n}$ (respectively $f : X \to \Omega^{2n+1}$) can be $C^k$ approximated by definable $C^r$ immersions (respectively definable $C^r$ imbeddings).

**Lemma 3.2.** Let $G$ be a compact definable $C^\infty$ group. Suppose that $D_1$ and $D_2$ are open balls in a representation $\Omega$ of $G$ of radius $a$ and $b$ with same center the origin and $a < b$. If $A, B \in \mathbb{R}$, $A \neq B$ and $0 \leq r < \infty$, then there exists a $G$ invariant definable $C^r$ function $f$ on $\Omega$ such that $f = A$ on $D_1$ and $f = B$ on $\Omega - D_2$. Moreover if $\mathcal{M}$ is exponential, then we can take $r = \infty$.

**Proof.** We may assume that $A = 1$ and $B = 0$. First we construct the required function when $\Omega = \mathbb{R}$. Let $D_1 = (-a, a), D_2 = (-b, b)$. Take a definable $C^r$ function

$$F : \mathbb{R} \to \mathbb{R}, \quad F(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x^{r+1} & \text{if } x > 0. \end{cases}$$

Then the function

$$\psi : \mathbb{R} \to \mathbb{R}, \quad \psi(x) = F(b - x)F(b + x)/(F(b - x)F(b + x) + F(x^2 - a^2))$$

is the required function.

Thus for a general representation $\Omega$ of $G$, $f : \Omega \to \mathbb{R}$, $f(x) = \psi(\|x\|)$ is the required function, where $\|x\|$ denotes the standard norm of $x$ in $\Omega$.

If $\mathcal{M}$ is exponential, then in the above argument, replacing $F(x)$ by

$$H(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x} & \text{if } x > 0, \end{cases}$$

we have the lemma. □

**Proof of Theorem 1.6.** Since $X$ is compact and subordinate to $\Omega$, one can find a finite set $\{x_i\}_{i=1}^{k}$ of points in $X$ and two finite families $\{U_i\}_{i=1}^{k}$ and $\{V_i\}_{i=1}^{k}$ of open definable $G$ invariant subsets of $X$ such that $x_i \in V_i \subset U_i, X = \bigcup_{i=1}^{k} V_i$ and that each $U_i$ admits a definable $C^r$ imbedding $f_i : V_i \to \Omega^{(i)} - \{0\}$ such that $f_i(U_j)$ (respectively $f_j(U_i)$) lies in the $2\varepsilon$-neighborhood (respectively the $\varepsilon$-neighborhood) of $f_j(x_i)$ in $\Omega^{(i)} - \{0\}$, for some $i \in \mathbb{N}$ and some $\varepsilon > 0$. We may replace $i(i)$ by $i$, where $i = \max_{1 \leq i \leq k} i(i)$. Using Lemma 3.2, we can find $G$ invariant definable $C^r$ functions $\lambda_i : X \to \mathbb{R}, 1 \leq i \leq k$ such that $0 \leq \lambda_i \leq 1, \lambda_i(V_i) = 1$ and $\lambda_i(X - \overline{U}) = 0$. Hence a definable $C^r$ $G$ map

$$F : X \to \Omega^{k+1}$$

defined by

$$F(x) = (\lambda_1(x)f_1(x), \ldots, \lambda_k(x)f_k(x))$$

is an immersion, where each $\lambda_i f_i$ is extended to 0 outside of $U_i$. Assume $F(x) = F(y)$. If $x, y \in V_i$, then $\lambda_i(x) = \lambda_i(y) = 1$. Thus $f_i(x) = f_i(y)$ and $x = y$. If $x \in V_i, y \notin U_i$, then $\lambda_i(x) = 1, \lambda_i(y) = 0$. Thus this case cannot occur because $f_i(U_i) \subset \Omega^i - \{0\}$. Similarly, the case where $x \in V_i, y \in U_i - V_i$ does not occur. Hence $F$ is injective. Since $X$ is
compact, \(F\) is a definable \(C^rG\) imbedding. By Proposition 3.1, we complete the proof of the first case.

If \(M\) is exponential, then using Lemma 3.2, one can prove in a way similar to the first case. \(\square\)

4. Approximations of definable \(G\) maps, and definable \(C^rG\) vector bundles

**Definition 4.1.** Let \(G\) be a definable \(C^r\) group and \(0 \leq r \leq \omega\).

(1) A **definable \(C^rG\) vector bundle** is a definable \(C^r\) vector bundle \(\eta = (E, p, X)\) satisfying the following three conditions.

(a) The total space \(E\) and the base space \(X\) are definable \(C^rG\) manifolds.

(b) The projection \(p : E \rightarrow X\) is a definable \(C^rG\) map.

(c) For any \(x \in X\) and \(g \in G\), the map \(p^{-1}(x) \rightarrow p^{-1}(gx)\) is linear.

(2) Let \(\eta\) and \(\zeta\) be definable \(C^rG\) vector bundles over \(X\). A **definable \(C^rG\) vector bundle morphism** \(\eta \rightarrow \zeta\) is called a **definable \(C^rG\) vector bundle isomorphism** if there exists a definable \(C^rG\) vector bundle morphism \(h : \zeta \rightarrow \eta\) such that \(f \circ h = \text{id}\) and \(h \circ f = \text{id}\). If \(r = 0\), then a definable \(C^0G\) vector bundle (respectively a definable \(C^0G\) vector bundle morphism, a definable \(C^0G\) vector bundle isomorphism) is simply called a **definable \(G\) vector bundle** (respectively a **definable \(G\) vector bundle morphism**, a **definable \(G\) vector bundle isomorphism**).

(3) A definable \(C^r\) section of a definable \(C^rG\) vector bundle is a **definable \(C^rG\) section** if it is a \(G\) map.

Typical examples of definable \(C^rG\) vector bundles are algebraic \(G\) vector bundles over a nonsingular algebraic \(G\) set and Nash \(G\) vector bundles.

Recall universal \(G\) vector bundles and the averaging process of \(C^r\) maps from a \(C^rG\) manifold to a representation of \(G\).

**Definition 4.2.** Let \(G\) be a compact definable \(C^r\) group and \(0 \leq r \leq \omega\). Let \(\Omega\) be an \(n\)-dimensional representation of \(G\) and let \(B\) be the representation map \(G \rightarrow O_n(\mathbb{R})\) of \(\Omega\). Suppose that \(M(\Omega)\) denotes the vector space of \(n \times n\)-matrices with the action \((g, A) \in G \times M(\Omega) \rightarrow B(g)AB(g)^{-1} \in M(\Omega)\). For any positive integer \(k\), we define the vector bundle \(\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))\) as follows:

\[
G(\Omega, k) = \{ A \in M(\Omega) \mid A^2 = A, A = A', \text{Tr} A = k \},
\]

\[
E(\Omega, k) = \{ (A, v) \in G(\Omega, k) \times \Omega \mid Av = v \},
\]

\[
u : E(\Omega, k) \rightarrow G(\Omega, k) : u((A, v)) = A,
\]

where \(A'\) denotes the transposed matrix of \(A\) and \(\text{Tr} A\) stands for the trace of \(A\). Then \(\gamma(\Omega, k)\) is an algebraic vector bundle. Since the action on \(\gamma(\Omega, k)\) is algebraic, it is an algebraic \(G\) vector bundle. We call it the **universal \(G\) vector bundle associated with \(\Omega\) and
Remark that $G(\Omega, k) \subset M(\Omega)$ and $E(\Omega, k) \subset M(\Omega) \times \Omega$ are nonsingular algebraic $G$ sets.

Let $G$ be a compact Lie group and $0 \leq r \leq \omega$. Let $f$ be a map from a $C^r G$ manifold $X$ to a representation $\Omega$ of $G$. Denote the Haar measure of $G$ by $dg$ and let $C^r(X, \Omega)$ denote the set of $C^r$ maps from $X$ to $\Omega$. Define

$$A : C^r(X, \Omega) \to C^r(X, \Omega), \quad A(f)(x) = \int_G g^{-1} f(gx) \, dg.$$ 

In particular, if $G = \{g_1, \ldots, g_n\}$ then $A(f)(x) = (1/n) \sum_{i=1}^n g_i^{-1} f(g_i x)$.

By an observation similar to [11, 2.6] shows the following.

**Proposition 4.3.** Let $G$ be a compact definable $C^\infty$ group.

1. $A(f)$ is equivariant, and $A(f) = f$ if $f$ is equivariant.
2. If $0 \leq r < \infty$ and $f \in C^r(X, \Omega)$, then $A(f) \in C^r(X, \Omega)$.
3. If $f$ is a polynomial map, then so is $A(f)$.
4. If $X$ is compact and $0 \leq r < \infty$, then $A : C^r(X, \Omega) \to C^r(X, \Omega)$ is continuous in the $C^r$ topology.
5. If $G$ is a finite group and $0 \leq r \leq \omega$, $X$ is a definable $C^r G$ manifold and $f$ is a definable $C^r$ map, then $A(f)$ is a definable $C^r G$ map.

**Definition 4.4.** Let $G$ be a compact definable $C^r$ group and $0 \leq r \leq \omega$. Let $\eta$ be a definable $C^r G$ vector bundle over an affine definable $C^r G$ manifold $X$. We say that $\eta$ is strongly definable if there exist a representation $\Omega$ of $G$ and a definable $C^r G$ map $f : X \to G(\Omega, k)$ such that $\eta$ is definably $C^r G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where $k$ denotes the rank of $\eta$.

By a way similar to [8, the proof of 3.1], we have the following.

**Proposition 4.5.** Let $G$ be a compact definable $C^r$ group and $0 \leq r \leq \omega$. Let $\eta$ and $\xi$ be strongly definable $C^r G$ vector bundles over an affine definable $C^r G$ manifold. Then $\text{Hom}(\eta, \xi)$ is also a strongly definable $C^r G$ vector bundle.

To prove Theorem 1.7, we need the following.

**Theorem 4.6** [1,13]. Let $G$ be a compact Lie group. Let $X$ be a paracompact $G$ space and let $Y$ be a $G$ space. If $f, h : X \to Y$ are $G$ homotopic continuous $G$ maps and $\eta$ is a $G$ vector bundle over $Y$, then $f^*(\eta)$ and $h^*(\eta)$ are $G$ vector bundle isomorphic.

**Proof of Theorem 1.7.** (1) Let $\Omega$ be a representation of $G$ including $Y$ as a definable $C^\infty G$ submanifold. By Proposition 2.24, one can find a definable $C^\infty G$ tubular neighborhood $(U, q)$ of $Y$ in $\Omega$. Let $f' = i \circ f$, where $i : Y \to \Omega$ denotes the inclusion. Applying the polynomial approximation theorem, one can find a polynomial map $p : X \to$...
Let $G$ be a compact definable $C'$ group and $0 \leq r \leq \omega$. Let $\eta = (E, \pi, X)$ be a definable $C' G$ vector bundle over an affine definable $C' G$ manifold $X$. Then the set $\Gamma(E)$ of $C'$ global sections of $E$ has the natural $G$ action, namely $(g \cdot s)(x) = g(s(g^{-1}(x)))$, $s \in \Gamma(E)$, $g \in G$ and $x \in X$.

**Lemma 4.7.** Let $G$ be a finite group and $0 \leq r \leq \omega$. Let $\eta = (E, \pi, X)$ be a definable $C' G$ vector bundle over an affine definable $C' G$ manifold $X$. Then $\eta$ is strongly definable if and only if there exists a finite family of (non-equivariant) definable $C'$ sections $s_1, \ldots, s_n : X \to E$ such that each fiber $\pi^{-1}(x)$ is generated by $s_1(x), \ldots, s_n(x)$ and that $s_1, \ldots, s_n$ generate a finite dimensional $G$ invariant vector subspace of $\Gamma(E)$.

**Proof.** Assume that $\eta$ is strongly definable. Then there exists a representation $\Omega$ of $G$ and a definable $C' G$ map $f : X \to G(\Omega, k)$ such that $\eta$ and $f^*(\gamma(\Omega, k))$ are definably $C' G$ vector bundle isomorphic, where $k$ denotes the rank of $\eta$. Note that a definable $C'$ section $s$ of $\eta$ can be identified with a definable $C'$ map $\tilde{s} : X \to \Omega$ such that $f(x)\tilde{s}(x) = \tilde{s}(x)$ for any $x \in X$. A trivial $G$ vector bundle $X \times \Omega$ clearly has a finite family of definable $C'$ sections $s_1, \ldots, s_m$, where $m = \dim \Omega$. Therefore we have the desired finite family of definable $C'$ sections $f s_1, \ldots, f s_m$.

Conversely, assume that $\eta$ admits such a finite family of sections. Then $s_1, \ldots, s_n$ determine a representation $\Sigma$ of $G$, and for each $x \in X$, $s_1(x), \ldots, s_n(x)$ define a vector subspace $V_x$ of $\Sigma$. Hence the orthogonal projection $F$ from $\Sigma$ onto $V_x$ is a definable $C' G$ map, and $\eta$ and $F^*(\gamma(\Sigma, k))$ are definably $C' G$ vector bundle isomorphic. □
To prove Theorem 1.8, we need a definable partition of unity and an approximation theorem for definable $C^r G$ maps.

**Proposition 4.8.** Let $X$ be a definable $C^r$ submanifold of $\mathbb{R}^n$ and let $\{U_i\}_{i=1}^r$ be a finite open definable covering of $X$ and $0 \leq r < \infty$. Then there exist definable $C^r$ functions $\lambda_1, \ldots, \lambda_i : X \to \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, $\text{supp} \lambda_i \subset U_i$ and $\sum_{i=1}^r \lambda_i(x) = 1$ for any $x \in X$. If $\mathcal{M}$ is exponential and $X$ is compact, then we can take $r = \infty$.

We call $\{\lambda_i\}$ in Proposition 4.8 a definable $C^r$ partition of unity subordinate to $\{U_i\}$.

**Proof.** By [9, 3.2], we may assume that $X$ is closed in $\mathbb{R}^n$. Thus every $\mathbb{R}^n - U_i$ is a closed definable subset of $\mathbb{R}^n$. Hence by [6], for each $i$, there exists a definable $C^r$ function $h_i : \mathbb{R}^n \to \mathbb{R}$ with $h_i^{-1}(0) = \mathbb{R}^n - U_i$. For any $i$, define $V_i = \{x \in X \mid h_i(x) > \frac{1}{2}\max_{1 \leq j \leq n} h_j(x)\}$. Then $\{V_i\}_{i=1}^r$ forms an open definable covering of $X$ and the closure $\overline{V_i}$ of $V_i$ in $X$ lies in $U_i$. By the above argument, for each $i$ we can find a definable $C^r$ function $h_i^0 : \mathbb{R}^n \to \mathbb{R}$ with $h_i^0(0) = \mathbb{R}^n - V_i$. Therefore $\lambda_i = h_i^0 / \sum_{j=1}^r h_j^0, 1 \leq i \leq l$, are the required definable $C^r$ functions.

Assume that $\mathcal{M}$ is exponential and $X$ is compact. It follows from the definition of definable $C^r$ manifolds and Lemma 3.2 that for any $x \in X$, there exists an open definable neighborhood $U_x$ of $x$ in $X$ and a definable $C^\infty$ function $f_x : U_x \to \mathbb{R}$ such that $U_x$ lies in some $U_i, 0 \leq f_x \leq 1$, $\text{supp} f_x \subset U_i$, and $X = \bigcup_{x \in X} \{y \in U_x \mid f_x(y) > 0\}$. Since $X$ is compact, there exist $x_1, \ldots, x_m$ such that $X = \bigcup_{i=1}^m \{y \in U_x \mid f_x(y) > 0\}$. Let $I_l = \{j \in \{1, \ldots, m\} \mid U_{x_j} \subset U_i\}, 1 \leq i \leq l$ and let $\lambda_{ij} = \sum_{j \in I_l} f_{x_j}$. Then $\lambda_i = \lambda_{ij} / \sum_{j=1}^r \lambda_j, 1 \leq i \leq l$ are the required functions. □

Recall the argument defining $C^r X$ topology in [22, Section II.5].

Let $X$ and $Y$ be definable $C^r$ submanifolds of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $0 \leq r \leq \infty$. Let $C^r_{\text{def}}(X, Y)$ denote the set of definable $C^r$ maps from $X$ to $Y$.

For $f \in C^r_{\text{def}}(X, Y)$ and $x \in X$, the differential $df_x$ of $f$ at $x$ means a linear map from the tangent space $T_x X$ of $X$ at $x$ to $\mathbb{R}^m$. Composing it with the orthogonal projection $\mathbb{R}^n \to T_x X$, one can extend $df_x$ to a linear map $\mathbb{R}^n \to \mathbb{R}^m$. Then $Df : X \to M(m, n; \mathbb{R}) = \mathbb{R}^{mn}$ is defined as the matrix representation of $df$.

For each $1 \leq k \leq r$, we inductively define a $C^{r-k}$ map

$$D^k f : X \to \mathbb{R}^{n^k}, \quad D^k f = D(D^{k-1} f).$$

Let $\|f\|_r$ denote the definable function on $X$ defined by

$$\|f\|_r(x) = |f(x)| + |Df(x)| + \cdots + |D^r f(x)|.$$

For a positive continuous definable function $\varepsilon : X \to \mathbb{R}$, let

$$U_\varepsilon = \{h \in C^r_{\text{def}}(X, Y) \mid \|h\|_r < \varepsilon\}.$$

We call the **definable $C^r$ topology** on $C^r_{\text{def}}(X, Y)$ the topology defined by choosing $\{h + U_\varepsilon\}_\varepsilon$ as a fundamental neighborhood system of $h$ in $C^r_{\text{def}}(X, Y)$. 

If $X$ is compact, then this topology coincides with the $C^r$ topology [22, p. 156].

The following is a fundamental property of the definable $C^r$ topology, which is obtained by (i), (ii), (iii) in [22, p. 159] and [22, II.5.11] because the image of a definable set by a definable map is definable.

**Proposition 4.9** [22]. Let $X$ and $Y$ be definable $C^r$ submanifolds $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $Z$ be another definable $C^r$ submanifold of $\mathbb{R}^l$ and let $f \in C^r_{\text{def}}(X,Y)$ and $h \in C^r_{\text{def}}(Y,Z)$.

1. The map $h^* : C^r_{\text{def}}(X,Y) \to C^r_{\text{def}}(X,Z)$, $h^* (k) = h \circ k$ is continuous.
2. The map $f^* : C^r_{\text{def}}(Y,Z) \to C^r_{\text{def}}(X,Z)$, $f^* (k) = k \circ f$ is continuous if and only if $f$ is proper, namely $f^{-1}(C)$ is compact if $C \subset Y$ is compact.

The following proposition is obtained by [22, II.5.3] and [22, II.5.11].

**Proposition 4.10** [22]. Let $X$ and $Y$ be definable $C^r$ submanifolds of $\mathbb{R}^n$ and $0 < r < \infty$. Let $f : X \to Y$ be a definable $C^r$ map. If $f$ is an immersion (respectively a diffeomorphism, a diffeomorphism onto its image), then an approximation of $f$ in the definable $C^r$ topology is an immersion (respectively a diffeomorphism, a diffeomorphism onto its image). Moreover if $f$ is a diffeomorphism, then $h^{-1} \to f^{-1}$ as $h \to f$.

By [22, II.5.2] and [22, II.5.11], we have the following.

**Theorem 4.11** [22]. Let $X$ and $Y$ be affine definable $C^r$ manifolds and $0 \leq k < r < \infty$. Then every definable $C^k$ map $f : X \to Y$ is approximated in the definable $C^k$ topology by definable $C^r$ maps.

In Theorem 4.11 we can take $r = \infty$ if $\mathcal{M} = \mathbb{R}$ (see [23, II.4.1]).

Let $0 < r < \infty$ and let $G$ be a compact definable $C^r$ group. Let $X$ be a definable $C^r G$ submanifold in a representation $\Omega$ of $G$. A definable $C^r G$ bent tubular neighborhood $(U, \chi, \psi, X)$ of $X$ in $\Omega$ consists of a $G$ invariant open definable neighborhood $U$ of $X$ in $\Omega$, a definable $C^r G$ submersion $\chi : U \to X$, a definable $C^r G$ map $\psi : X \to G(\Omega, \dim \Omega - \dim X)$ and a positive continuous definable function $\varepsilon : X \to \mathbb{R}$ such that $\chi^{-1}(x) = x + \{y \in (x) | \|y\| < \varepsilon\}, x \in X$ and the map $(x, y) \in X \times \Omega \mid y \in \chi^{-1}(x)) \to \Omega, (x, y) \mapsto x + y \in \Omega$ is a definable $C^r G$ diffeomorphism onto $U$, where $\|y\|$ denotes the standard norm of $y$ in $\Omega$.

**Lemma 4.12.** If $G$ is a finite group and $0 < r < \infty$, then every definable $C^r G$ submanifold $X$ in a representation $\Omega$ admits a definable $C^r G$ bent tubular neighborhood in $\Omega$.

**Proof.** Let $\psi_1 : X \to G(\Omega, \dim X)$ and $\psi_2 : X \to G(\Omega, \dim \Omega - \dim X)$ be the characteristic maps of the tangent bundle of $X$ and the normal bundle of $X$ in $\Omega$, respectively. As in the proof of Proposition 4.14 below, $\psi_1$ and $\psi_2$ are definable $C^{r-1} G$ maps. By
Theorem 4.11 and Propositions 4.3 and 2.24, \( \psi_2 \) is approximated by a definable \( C^r G \) map \( \tilde{\psi}_2 : X \to G(\Omega, \dim \Omega - \dim X) \) such that \( \psi_1(x) + \tilde{\psi}_2(x) = \Omega \) for any \( x \in X \). Let \( (L, q, X) = \tilde{\psi}_2^*(\gamma(\Omega, \dim \Omega - \dim X)) \). Then \( \tilde{\psi}_2^*(\gamma(\Omega, \dim \Omega - \dim X)) \) is a definable \( C^r G \) vector bundle over \( X, L = \{(x, y) \in X \times \Omega \mid y \in \tilde{\psi}_2(x)\} \) and \( q \) is the restriction of the projection from \( X \times \Omega \) onto \( X \). Let \( \theta : L \to \Omega, \theta(x, y) = x + y \). Then there exists a positive continuous definable function \( \varepsilon \) such that the restriction of \( \theta \) to \( L_\varepsilon := \{(x, y) \in L \mid \|y\| < \varepsilon(x)\} \) is a definable \( C^r G \) imbedding. Therefore \( U = \theta(L_\varepsilon), x = q \circ \theta^{-1}, \psi = \psi_2 \) fulfill the requirements. \( \square \)

By Theorem 4.11, Proposition 4.3, Lemma 4.12, and Proposition 4.9, we have the following equivariant version of Theorem 4.11.

**Theorem 4.13.** Let \( G \) be a finite group. Let \( X \) and \( Y \) be affine definable \( C^r G \) manifolds and \( 0 \leq k < r < \infty \). Then every definable \( C^k G \) map \( f : X \to Y \) is approximated in the definable \( C^k \) topology by definable \( C^r G \) maps.

**Proof of Theorem 1.8.** (1) It suffices to construct a finite family of definable \( C^r \) sections of \( \eta \) as in Lemma 4.7. By the definition of definable \( C^r \) vector bundles, \( \eta \) has a finite family of definable local trivializations \( (U_i, \phi_i : U_i \times \mathbb{R}^k \to \pi^{-1}(U_i))_{i=1}^\ell \), where \( k \) denotes the rank of \( \eta \). By Proposition 4.8, we can find a definable \( C^r \) partition of unity \( \{\lambda_i\} \) subordinate to \( \{U_i\} \). Thus we have global definable \( C^r \) sections \( s_{ij}(x) = \phi_i(x, \lambda_i(x) e_j) \), \( 1 \leq i \leq \ell, 1 \leq j \leq k \), where \( e_j \) denotes the \( j \)th fundamental vector of \( \mathbb{R}^k \). Since \( G \) is finite, we obtain the required finite family of sections \( \{g \cdot s_{ij} \mid 1 \leq i \leq \ell, 1 \leq j \leq k, g \in G\} \).

(2) Let \( \zeta \) be a definable \( C^r G \) vector bundle over \( X \) of rank \( k' \). Then by (1), we can find a representation \( \Omega \) of \( G \) and a definable \( C^r G \) map \( h : X \to G(\Omega, k') \) such that \( \zeta \) and \( f^* (\gamma(\Omega, k')) \) are definably \( C^r G \) vector bundle isomorphic. By Theorem 4.13, there exists a definable \( C^r G \) map \( h : X \to G(\Omega, k') \) as an approximation of \( f \). It suffices to show that \( \zeta_1 := f^* (\gamma(\Omega, k')) \) and \( \zeta_2 := h^* (\gamma(\Omega, k')) \) are definably \( C^r G \) vector bundle isomorphic because \( \zeta_2 \) is a definable \( C^r G \) vector bundle. By Proposition 4.5, \( \zeta_1 \) is a strongly definable \( C^r G \) vector bundle. By Theorem 4.6 and since \( f \) and \( h \) are \( G \) homotopic, \( \zeta_1 \) is \( G \) vector bundle isomorphic to \( \zeta_2 \). Since \( X \) is compact, every continuous \( G \) map is approximated by polynomial \( G \) maps. Thus by a way similar to the proof of Theorem 1.7(2), \( \zeta_1 \) and \( \zeta_2 \) are definably \( C^r G \) vector bundle isomorphic. Using Theorem 4.13, the latter half of (2) follows from a way similar to the above argument.

(3) Assume that \( \mathcal{M} \) is exponential and \( X \) is compact. Let \( \xi = (E, p, X) \) be a definable \( C^\infty \) vector bundle over \( X \). By the definition of definable \( C^\infty \) vector bundles, there exist a finite open definable covering \( \{U_i\}_{i=1}^m \) of \( X \) and local definable \( C^\infty \) sections \( h_{ij} : U_i \to E, 1 \leq i \leq m, 1 \leq j \leq m' \), where \( m' \) denotes the rank of \( \xi \). By Proposition 4.8, we can find a definable \( C^\infty \) partition of unity \( \{\lambda_i\}_{i=1}^m \) subordinate to \( U_i \). Thus \( \sum_{i=1}^m \lambda_i(x) h_{ij}(x), 1 \leq i \leq m, 1 \leq j \leq m' \). By a way similar to (1), we have the required definable \( C^\infty \) sections. Therefore we can take \( r = \infty \) in (1).

Using the polynomial approximation theorem and Proposition 4.3, we have the latter half of (3). \( \square \)
Note that the total space of a definable $C^r G$ vector bundle is affine if $G$ is finite and $r < \infty$, or $G$ is finite, $M$ is exponential, $X$ is compact and $r = \infty$. If $r = \infty$, then the following proposition holds, but which is not used to prove Proposition 1.9, Corollaries 1.10 and 1.11.

**Proposition 4.14.** Let $G$ be a compact definable $C^\infty$ group and let $\eta = (E, p, X)$ be a definable $C^\infty G$ vector bundle of rank $k$ over an affine definable $C^\infty G$ manifold $X$. Then $\eta$ is strongly definable if and only if $E$ is affine.

**Proof.** If $\eta$ is strongly definable, then there exist a definable $C^\infty G$ map $f$ from $X$ to some $G(\Omega, k)$ such that $\eta$ is definably $C^\infty G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$. Since the total space of $f^*(\gamma(\Omega, k))$ is affine, $E$ is affine.

Conversely, we assume that $E$ is a definable $C^\infty G$ submanifold of a representation $\Xi$ of $G$. Let

$\begin{align*}
F_1 : X &\to M(\Xi), \quad F_1(x) = \text{the matrix projecting } T_x \Xi \text{ onto } T_x E, \\
F_2 : X &\to M(\Xi), \quad F_2(x) = \text{the matrix projecting } T_x \Xi \text{ onto } T_x X.
\end{align*}$

Then $F_1$ and $F_2$ are definable maps by a way similar to [23, proof of I.3.3], thus they are definable $C^\infty$ maps. By the definition of $G$ action, they are $G$ maps. Hence they are definable $C^\infty G$ maps. Let

$F : X \to G(\Xi, k), \quad F = (\text{id} - F_2)F_1.$

Then $\eta$ is definably $C^\infty G$ vector bundle isomorphic to $F^*(\gamma(\Xi, k))$. Therefore $\eta$ is strongly definable. $\square$

**Proof of Proposition 1.9.** If $G$ is a finite group, then the result follows from Theorem 4.13. Assume that $X$ has only one orbit type. Then by the paragraph after Theorem 1.4, and by Theorem 2.6, $X/G$ is an affine definable $C^\infty G$ manifold and the projection $\pi : X \to X/G$ is a proper definable $C^\infty$ map. Since $f$ is equivariant, $f$ induces a definable $C^k$ map $\overline{f} : X/G \to Y$ such that $f = \overline{f} \circ \pi$. Applying Theorem 4.11, we have a definable $C^r$ map $\overline{\pi} : X/G \to Y$ as an approximation of $\overline{f}$ in the definable $C^k$ topology. Thus $\overline{\pi} \circ \pi$ is the desired approximation of $f$ because $\pi$ is proper and Proposition 4.9.

If $M = \mathbb{R}$, then we can take $r = \infty$ by the above argument and Theorem 4.11. $\square$

Corollary 1.10 follows from Propositions 1.9 and 4.10.

**Proof of Corollary 1.11.** (1) By Proposition 4.5, $\text{Hom}(\eta_1, \eta_2)$ is a strongly definable $C^r G$ vector bundle over $X$. Take a representation $\Xi$ of $G$ and a definable $C^r G$ map $f : X \to \Xi$ such that $\text{Hom}(\eta_1, \eta_2)$ and $f^*(\gamma(\Xi, k))$ are definably $C^r G$ vector bundle isomorphic, where $k$ denotes the rank of $\text{Hom}(\eta_1, \eta_2)$. Thus a continuous definable $G$ section (respectively a definable $C^r G$ section) $s$ of $\text{Hom}(\eta_1, \eta_2)$ corresponds a continuous definable $G$ map (respectively a definable $C^r G$ map) $\tilde{s} : X \to \Xi$ such that $f(x)\tilde{s}(x) = \tilde{s}(x)$ for any $x \in X$. Since $\eta_1$ and $\eta_2$ are definable $G$ vector bundle isomorphic, there exists
a continuous definable $G$ section $s$ of $\text{Iso}(\eta_1, \eta_2) \subset \text{Hom}(\eta_1, \eta_2)$. Thus it defines a continuous definable $G$ map $s': X \to \Xi$ such that $f(x)s'(x) = s'(x)$ for any $x \in X$. By Proposition 1.9, we can find a definable $C^r G$ map $s'': X \to \Xi$ as an approximation of $s'$ in the definable $C^0$ topology. Hence $\tilde{s}(x) := f(x)s''(x)$ is an approximation of $s$ by Proposition 4.9 and since $s'(x) = f(x)s'(x)$ for any $x \in X$. Thus $\tilde{s}$ defines a definable $C^r G$ section $\tilde{s}$ of $\text{Hom}(\eta_1, \eta_2)$. If this approximation is sufficiently close, then $\tilde{s} \in \text{Iso}(\eta_1, \eta_2) \subset \text{Hom}(\eta_1, \eta_2)$. Therefore $\tilde{s}$ give a definable $C^r G$ vector bundle isomorphism between $\eta_1$ and $\eta_2$.

By the above argument, we can take $r = \infty$ if $\mathcal{M} = \mathcal{R}$. □

References