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Nonlinear conditions for weighted composition operators between Lipschitz algebras ☆

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ABSTRACT

Let $T:\mathrm{Lip}_0(X)\to\mathrm{Lip}_0(Y)$ be a surjective map between pointed Lipschitz *-algebras, where X and Y are compact metric spaces. On the one hand, we prove that if T satisfies the non-symmetric norm *-multiplicativity condition:

$$\left\|T(f)\overline{T(g)}-\mathbf{1}\right\|_\infty=\left\|f\bar{g}-\mathbf{1}\right\|_\infty\quad\left(f,g\in\mathrm{Lip}_0(X)\right),$$

then T is of the form

$$T(f)=\tau\cdot\left(\eta\cdot(f\circ\varphi)+(\mathbf{1}-\eta)\cdot\overline{(f\circ\varphi)}\right)\quad\left(f\in\mathrm{Lip}_0(X)\right),$$

where η and τ are functions on Y such that $\eta(Y)\subseteq\{0,1\}$ and $\tau(Y)\subseteq\{\alpha\in\mathbb{K}\colon\ |\alpha|=1\}$, and $\varphi:Y\to X$ is a base point preserving Lipschitz homeomorphism. On the other hand, if T satisfies the weakly peripherally *-multiplicativity condition:

$$\mathrm{Ran}_\pi(f\bar{g})\cap\mathrm{Ran}_\pi\left(T(f)\overline{T(g)}\right)\neq\emptyset\quad\left(f,g\in\mathrm{Lip}_0(X)\right),$$

where $\mathrm{Ran}_\pi(f)$ denotes the peripheral range of f, then T can be expressed as

$$T(f)=\tau\cdot(f\circ\varphi)\quad\left(f\in\mathrm{Lip}_0(X)\right),$$

with τ and φ as above. As a consequence, we obtain similar descriptions for surjective maps between Lipschitz *-algebras $\mathrm{Lip}(X)$.

1. Introduction

The study of surjective maps between commutative Banach *-algebras that preserve *-multiplicatively the spectrum has attracted the attention of several mathematicians in recent years (see [2,3,6,13]). The first results on the matter concern the C^*-algebra $\mathcal{C}(X)$ of all complex-valued continuous functions on a compact Hausdorff space X with the supremum norm and the complex conjugation involution. Under the additional condition that X satisfies the first countability axiom, Molnár proved in [13, Theorem 6] that every surjective map $T:\mathcal{C}(X)\to\mathcal{C}(X)$ such that

$$\sigma\left(T(f)\overline{T(g)}\right)=\sigma(f\bar{g})\quad\left(f,g\in\mathcal{C}(X)\right),$$

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is of the form

$$T(f)(x)=\tau(x)f\big(\varphi(x)\big)\quad \big(f\in\mathcal{C}(X),x\in X\big),$$

where τ is a continuous function from X into the unit circle $S_{\mathbb{C}}$ of the complex plane $\mathbb{C}$, and φ is a homeomorphism from X onto itself. Hatori, Miura and Takagi showed that Molnár's theorem holds without first countability for X [2, Theorem 3.6]. Notice that for each $f\in\mathcal{C}(X)$, the spectrum of f coincides with its range $f(X)$, but this is not true in general.

For the case of unital *-algebras, the most general result known so far is due to Hatori, Miura and Takagi [3]. With a method of proof that cannot be translated to the non-unital setting, they proved that if $\mathcal{A},\mathcal{B}$ are unital semisimple commutative Banach *-algebras, then every surjective map $T:\mathcal{A}\to\mathcal{B}$ that satisfies the *spectrum *-multiplicativity condition*:

$$\sigma\big(T(f)T(g)^*\big)=\sigma\big(fg^*\big)\quad (f,g\in\mathcal{A}),$$

where $\sigma(f)$ denotes the spectrum of f, is a *-isomorphism multiplied by a unimodular function [3, Theorem 6.2].

In [12], Luttman and Tonev introduced the concept of *peripheral range* of a function $f\in\mathcal{C}(X)$ as the set

$$\operatorname{Ran}_\pi(f)=\big\{f(x)\colon x\in X,\ \big|f(x)\big|=\|f\|_\infty\big\},$$

and they characterized surjective maps between uniform algebras $T:\mathcal{A}\to\mathcal{B}$ satisfying the *peripheral range multiplicativity condition*:

$$\operatorname{Ran}_\pi\big(T(f)T(g)\big)=\operatorname{Ran}_\pi(fg)\quad (f,g\in\mathcal{A}).$$

Later, peripherally multiplicative surjective maps on uniformly closed algebras of complex-valued continuous functions vanishing at infinity, and on Banach algebras of scalar-valued Lipschitz functions have been considered in [4,8], respectively.

In the case of non-unital *-algebras, as far as we know, the unique result on surjective maps fulfilling a spectrum *-multiplicativity condition appears in the paper by Honma [6, Theorem 1.1]. He showed that if a surjective map $T:\mathcal{C}_0(X)\to\mathcal{C}_0(Y)$ satisfies the *peripheral range *-multiplicativity condition*:

$$\operatorname{Ran}_\pi\big(T(f)\overline{T(g)}\big)=\operatorname{Ran}_\pi(f\overline{g})\quad \big(f,g\in\mathcal{C}_0(X)\big),$$

then there exist a continuous function $\tau:Y\to S_{\mathbb{C}}$ and a homeomorphism $\varphi:Y\to X$ such that

$$T(f)(y)=\tau(y)f\big(\varphi(y)\big)\quad \big(f\in\mathcal{C}_0(X),\ y\in Y\big).$$

As usual, $\mathcal{C}_0(X)$ denotes the C^*-algebra of all complex-valued continuous functions vanishing at infinity on a locally compact Hausdorff space X, equipped with the supremum norm and the complex conjugation involution.

Lambert, Luttman and Tonev opened in [10] a new line of research by studying surjective maps between uniform algebras $T:\mathcal{A}\to\mathcal{B}$ satisfying the *weakly peripherally multiplicativity condition*:

$$\operatorname{Ran}_\pi\big(T(f)T(g)\big)\cap\operatorname{Ran}_\pi(fg)\neq\emptyset\quad (f,g\in\mathcal{A}).$$

Jiménez, Luttman and Villegas characterized in [9] those weakly peripherally multiplicative surjections between pointed Lipschitz algebras $\operatorname{Lip}_0(X)$.

Related to a conjecture by O. Hatori, the authors of [10] also proved that every surjective unital map $T:\mathcal{A}\to\mathcal{B}$ with the property that

$$\big\|T(f)T(h)+\alpha\mathbf{1}\big\|_\infty=\|fh+\alpha\mathbf{1}\|_\infty\quad \big(f\in\mathcal{A},\ h\in\mathcal{F}(\mathcal{A}),\ \alpha\in S_{\mathbb{C}}\big),$$

is an isometric algebra isomorphism, where $\mathcal{F}(\mathcal{A})$ denotes the set of all peaking functions in $\mathcal{A}$.

With regard to this property, Honma [7] proved that every surjective map $T:\mathcal{C}(X)\to\mathcal{C}(Y)$ such that $T(\lambda\mathbf{1})=\lambda\mathbf{1}$ for $\lambda\in\{\pm1,\pm i\}$ satisfying the *non-symmetric norm *-multiplicativity condition*:

$$\big\|T(f)\overline{T(g)}-\mathbf{1}\big\|_\infty=\|f\overline{g}-\mathbf{1}\|_\infty\quad \big(f,g\in\mathcal{C}(X)\big),$$

is an isometric algebra isomorphism.

Recently, Hatori, Miura and Takagi [5] and Lambert and Luttman [11] have obtained some nice descriptions of surjective maps between uniform algebras $T:\mathcal{A}\to\mathcal{B}$ fulfilling the *non-symmetric norm multiplicativity condition*:

$$\big\|T(f)T(g)-\lambda\mathbf{1}\big\|_\infty=\|fg-\lambda\mathbf{1}\|_\infty\quad (f,g\in\mathcal{A}),$$

for some $\lambda\in\mathbb{C}\setminus\{0\}$.

Our goal in this paper is to state the main results in [6,7,11], for surjective maps between pointed Lipschitz *-algebras, $\operatorname{Lip}_0(X)$, and Lipschitz *-algebras, $\operatorname{Lip}(X)$, on compact metric spaces X. It is well known (see [16]) that every algebra isomorphism between pointed Lipschitz algebras $T:\operatorname{Lip}_0(X)\to\operatorname{Lip}_0(Y)$ is a composition operator

$$T(f)=f\circ\varphi\quad \big(f\in\operatorname{Lip}_0(X)\big),$$

for some base point preserving Lipschitz homeomorphism $\varphi : Y \to X$. A similar assertion holds for isomorphisms between Lipschitz algebras. In particular, every isomorphism between Lipschitz algebras $\mathrm{Lip}_0(X)$ or $\mathrm{Lip}(X)$ is a *-isomorphism.

The contents of this manuscript are organized as follows. Section 2 presents some preliminary information on Lipschitz algebras and peaking functions. Section 3 focuses on surjective maps $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ that satisfy the *norm multiplicativity condition*:

$$\left\|T(f)T(g)\right\|_\infty = \|fg\|_\infty \quad \bigl(f, g \in \mathrm{Lip}_0(X)\bigr).$$

We prove that such a map gives rise to a base point preserving bijective map $\psi : X \to Y$ in such a way that

$$\left|T(f)\bigl(\psi(x)\bigr)\right| = \left|f(x)\right| \quad \bigl(f \in \mathrm{Lip}_0(X),\ x \in X\bigr).$$

Notice that the corresponding result for uniform algebras was proved in [10].

Section 4 is devoted to surjective maps $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ with the non-symmetric norm *-multiplicativity condition. We see that such a map is norm-multiplicative, and can be expressed as

$$T(f) = \tau \cdot \bigl(\eta \cdot (f \circ \varphi) + (\mathbf{1} - \eta) \cdot \overline{(f \circ \varphi)}\bigr) \quad \bigl(f \in \mathrm{Lip}_0(X)\bigr),$$

where η and τ are functions on Y such that $\eta(Y) \subseteq \{0, 1\}$ and $\tau(Y) \subseteq S_{\mathbb{K}}$, and $\varphi : Y \to X$ is a Lipschitz homeomorphism.

These results are applied in Section 5 in order to prove that every surjective map $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ satisfying the *weakly peripherally *-multiplicativity condition*:

$$\mathrm{Ran}_\pi\bigl(T(f)\overline{T(g)}\bigr) \cap \mathrm{Ran}_\pi(f\bar{g}) \neq \emptyset \quad \bigl(f, g \in \mathrm{Lip}_0(X)\bigr)$$

is a weighted composition operator. Moreover, T is an algebra isomorphism provided T preserves an approximate identity for the supremum norm.

Similar results are stated for surjective maps between algebras $\mathrm{Lip}(X)$.

2. Preliminaries

Throughout the paper, we will denote by d the distance on a metric space. A map between metric spaces $f : X \to Y$ is said to be *Lipschitz* if there exists a constant $a \geqslant 0$ such that $\mathrm{d}(f(x), f(z)) \leqslant a\mathrm{d}(x, z)$ for every $x, z \in X$. If f is bijective and both f and f^{-1} are Lipschitz, then f is a *Lipschitz homeomorphism*. A *pointed metric space* is a metric space X with a distinguished element $e_X \in X$ called *base point*. A map between pointed metric spaces $f : X \to Y$ *preserves base point* if $f(e_X) = e_Y$.

As usual, $\mathbb{K}$ stands for the field of real or complex numbers, and $S_{\mathbb{K}}$ denotes the set of all unimodular elements of $\mathbb{K}$. Given a metric space X, we represent the function constantly equal 1 on X by $\mathbf{1}$ and the diameter of X by $\mathrm{diam}(X)$.

Let X be a pointed compact metric space. We denote by $\mathrm{Lip}_0(X)$ the *-algebra of all Lipschitz functions $f : X \to \mathbb{K}$ vanishing at e_X, equipped with the complex conjugation involution and the Lipschitz norm:

$$L(f) = \sup\left\{\frac{|f(x) - f(z)|}{\mathrm{d}(x, z)} \colon x, z \in X,\ x \neq z\right\}.$$

If we need to specify the base field, we will write $\mathrm{Lip}_0(X, \mathbb{R})$ or $\mathrm{Lip}_0(X, \mathbb{C})$.

Given a compact metric space X, $\mathrm{Lip}(X)$ stands for the *-algebra of all Lipschitz functions $f : X \to \mathbb{K}$ with the complex conjugation involution and the norm

$$\|f\| = \max\bigl\{\|f\|_\infty, L(f)\bigr\}.$$

Since $L(fg) \leqslant 2\,\mathrm{diam}(X)L(f)L(g)$ for all $f, g \in \mathrm{Lip}_0(X)$ and $\|fg\| \leqslant 2\|f\|\|g\|$ for all $f, g \in \mathrm{Lip}(X)$, $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}(X)$ are commutative Banach algebras after renorming.

Both algebras are closely related. According to Weaver [16], if a metric space X is *spherical*, that is, it has a base point e_X such that $\mathrm{d}(x, e_X) = 1$ for all $x \neq e_X$, then $\mathrm{Lip}_0(X)$ is isometrically isomorphic to $\mathrm{Lip}(X \setminus \{e_X\})$. Conversely, given a metric space X, if X_0 denotes the metric space obtained by remetrizing X with $\mathrm{d}_0(x, y) = \min\{2, \mathrm{d}(x, y)\}$ and adding a base point e_X such that $\mathrm{d}_0(x, e_X) = 1$ for all $x \in X$, then X_0 is spherical, and $\mathrm{Lip}(X)$ is isometrically isomorphic to $\mathrm{Lip}_0(X_0)$. This isometric algebra isomorphism is given by

$$\Psi_X(f)(x) = f(x) \quad (x \in X), \qquad \Psi_X(f)(e_X) = 0. \tag{2.1}$$

We refer the reader to the book by Weaver [16], for details and more background on the algebras of Lipschitz functions.

For our purposes, we next present two families of functions in $\mathrm{Lip}_0(X)$. The first one is formed by the called peaking functions. These functions have played an important role in uniform algebra theory (see [1]).

For any $x \in X \setminus \{e_X\}$, define the set of functions peaking at x as

$$P_x(X) = \bigl\{h \in \mathrm{Lip}_0(X)\colon \mathrm{Ran}_\pi(h) = \{1\},\ h(x) = 1\bigr\},$$

where

$$\mathrm{Ran}_\pi(h) = \{h(x)\colon x \in X,\ |h(x)| = \|h\|_\infty\}$$

is the peripheral range of h, and let $P(X) = \bigcup_{x \in X \setminus \{e_X\}} P_x(X)$ be the set of all peaking functions of $\mathrm{Lip}_0(X)$. Notice that $P(X) = \{h \in \mathrm{Lip}_0(X)\colon \mathrm{Ran}_\pi(h) = \{1\}\}$.

Given $x \in X \setminus \{e_X\}$, set

$$F_x(X) = \{f \in \mathrm{Lip}_0(X)\colon |f(x)| = \|f\|_\infty = 1\}.$$

We will also write $F(X) = \bigcup_{x \in X \setminus \{e_X\}} F_x(X)$. Clearly, $P_x(X) \subseteq F_x(X)$ for all $x \in X \setminus \{e_X\}$, and $P(X) \subseteq F(X)$.

There is no shortage of elements of $P_x(X)$. In fact, for every $x \in X \setminus \{e_X\}$ and $0 < \delta \leqslant \mathrm{d}(x, e_X)$, the function $h_{x,\delta} : X \to [0, 1]$ given by

$$h_{x,\delta}(z) = \max\left\{0, 1 - \frac{\mathrm{d}(z,x)}{\delta}\right\} \quad (z \in X) \tag{2.2}$$

lies in $P_x(X)$, $h_{x,\delta}(z) < 1$ if $z \neq x$, and $h_{x,\delta}(z) = 0$ whenever $\mathrm{d}(z,x) \geqslant \delta$. In particular, $h_{x,\delta} \in F_x(X) \setminus F_z(X)$ for $z \neq x$, and the next lemma becomes an easy observation.

Lemma 2.1. *Let $x, z \in X \setminus \{e_X\}$. If $F_x(X) \subseteq F_z(X)$, then $x = z$.*

In the subsequent sections, we will use the following two lemmas. The first one, which is just [9, Lemma 2.1(iii)], is a version for $\mathrm{Lip}_0(X)$ of a Bishop's theorem for uniform algebras (see, for example, [1, Theorem 2.4.1]). The second one provides us with a method to identify the modulus of two functions of $\mathrm{Lip}_0(X)$ by using peaking functions.

Lemma 2.2. *Let X be a pointed compact metric space. Given $f \in \mathrm{Lip}_0(X)$ and $x \in X$ with $f(x) \neq 0$, there exists a nonnegative real function $h_{f,x} \in P_x(X)$ such that $h_{f,x}(z) < 1$ and $|f(z)h_{f,x}(z)| < |f(x)|$ for all $z \neq x$. In particular, $\mathrm{Ran}_\pi(fh) = \{f(x)\}$.*

The following result was stated for uniform algebras in [10] and [12].

Lemma 2.3. *Let X be a pointed compact metric space.*

(1) *For all $f \in \mathrm{Lip}_0(X)$ and $x \in X \setminus \{e_X\}$,*

$$|f(x)| = \inf\{\|fh\|_\infty\colon h \in P_x(X)\} = \inf\{\|fh\|_\infty\colon h \in F_x(X)\}.$$

(2) *Let $f, g \in \mathrm{Lip}_0(X)$. Then $|f| \leqslant |g|$ if and only if $\|fh\|_\infty \leqslant \|gh\|_\infty$ for all $h \in P(X)$.*

Proof. Let $f \in \mathrm{Lip}_0(X)$ and $x \in X \setminus \{e_X\}$. It is clear that $|f(x)| = |f(x)h(x)| \leqslant \|fh\|_\infty$ for all $h \in P_x(X)$. Moreover, for every $\varepsilon > 0$, since f is continuous at x, there exists $\delta \in]0, \mathrm{d}(x, e_X)] > 0$ such that $|f(z)| < |f(x)| + \varepsilon/2$ if $\mathrm{d}(z,x) < \delta$. Take $h_{x,\delta} \in P_x(X)$ as defined in (2.2). Then

$$\|fh_{x,\delta}\|_\infty = \sup\{|f(z)h_{x,\delta}(z)|\colon \mathrm{d}(z,x) < \delta\} < |f(x)| + \varepsilon,$$

and this proves the first equality of (1). The second one follows easily.

The 'only if' part of (2) is trivial. The 'if' part is [9, Lemma 2.2], but now it follows immediately from (1). □

3. Norm multiplicativity condition

Our purpose in this section is to show that every norm-multiplicative surjective map $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ brings a bijective map $\psi : X \to Y$ in such a way that

$$|T(f)(\psi(x))| = |f(x)| \quad (f \in \mathrm{Lip}_0(X),\ x \in X).$$

Lemma 3.1. *Let $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a map satisfying the conditions*:

(1) $\|T(f)\|_\infty = \|f\|_\infty$ *for all $f \in \mathrm{Lip}_0(X)$.*
(2) *For any $f, g \in \mathrm{Lip}_0(X)$, $|f| \leqslant |g|$ if and only if $|T(f)| \leqslant |T(g)|$.*

Then the following assertions hold

(i) *For every $x \in X \setminus \{e_X\}$, there exists $y \in Y \setminus \{e_Y\}$ such that $T(F_x(X)) \subseteq F_y(Y)$.*

(ii) *If* $x, z \in X \setminus \{e_X\}$ *and* $T(F_x(X)) \subseteq T(F_z(X))$, *then* $x = z$.

If, in addition, T is surjective, then:

(iii) *For every* $y \in Y \setminus \{e_Y\}$, *there exists* $z \in X \setminus \{e_X\}$ *such that* $F_y(Y) \subseteq T(F_z(X))$.
(iv) *For every* $x \in X \setminus \{e_X\}$, *there exists a unique* $y \in Y \setminus \{e_Y\}$ *such that* $T(F_x(X)) = F_y(Y)$.

Proof. (i) We follow here the method of proof used in [14] for uniform algebras. Let $x \in X \setminus \{e_X\}$. For each $f \in F_x(X)$, define

$$F(f) = \{y \in Y \setminus \{e_Y\}\colon\ T(f) \in F_y(Y)\}.$$

Since Y is compact, we deduce from (1) that $F(f)$ is nonempty. Statement (i) follows if we show that $\bigcap_{f \in F_x(X)} F(f)$ is nonempty. To this end, it suffices to prove that $\{F(f)\colon\ f \in F_x(X)\}$ has the finite intersection property, because $F(f)$ is a closed subset of the compact Hausdorff space Y for each $f \in F_x(X)$. Pick $f_1, \ldots, f_n \in F_x(X)$. By (1), we have $\|T(f_k)\|_\infty = \|f_k\|_\infty = 1$ for all $k \in \{1, \ldots, n\}$. Let $g = f_1 \cdots f_n \in F_x(X)$ and let $y \in Y \setminus \{e_Y\}$ be such that $T(g) \in F_y(Y)$. We deduce from (2) that for each $k \in \{1, \ldots, n\}$, $|T(g)| \leqslant |T(f_k)|$ and therefore $1 = |T(g)(y)| \leqslant |T(f_k)(y)| \leqslant 1$. Hence $|T(f_k)(y)| = \|T(f_k)\|_\infty = 1$ for all $k \in \{1, \ldots, n\}$ and thus $y \in \bigcap_{k=1}^n F(f_k)$ as desired.

(ii) Let $x, z \in X \setminus \{e_X\}$, and assume that $T(F_x(X)) \subseteq T(F_z(X))$. Take $h_{x,\delta} \in \mathrm{Lip}_0(X)$ as in (2.2). Notice that $h_{x,\delta}(x) = 1$ and $|h_{x,\delta}(w)| < 1$ if $w \neq x$. Because $h_{x,\delta} \in F_x(X)$, there exists $g \in F_z(X)$ such that $T(h_{x,\delta}) = T(g)$. Then (2) gives $|h_{x,\delta}| = |g|$, which yields $z = x$.

(iii) Since T is surjective, there is a map $S : \mathrm{Lip}_0(Y) \to \mathrm{Lip}_0(X)$ such that $T \circ S$ is the identity map on $\mathrm{Lip}_0(Y)$ and, obviously, S also fulfills conditions (1) and (2). By applying (i) to S instead of T we obtain that for every $y \in Y \setminus \{e_Y\}$, there exists $z \in X \setminus \{e_X\}$ such that $S(F_y(Y)) \subseteq F_z(X)$. This implies that $F_y(Y) \subseteq T(F_z(X))$.

(iv) Let $x \in X \setminus \{e_X\}$. By (i) and (iii), there exist $y \in Y \setminus \{e_Y\}$ and $z \in X \setminus \{e_X\}$ such that $T(F_x(X)) \subseteq F_y(Y) \subseteq T(F_z(X))$. By (ii), it follows that $x = z$ and thus $T(F_x(X)) = F_y(Y)$. The uniqueness of y is deduced from Lemma 2.1. □

The next theorem has been proved in [10] in the context of uniform algebras. The proof provided here is an adaptation for Lipschitz algebras of the original proof from [10].

Theorem 3.2. *Let X and Y be pointed compact metric spaces and let T be a surjective map from* $\mathrm{Lip}_0(X)$ *to* $\mathrm{Lip}_0(Y)$ *satisfying*

$$\|T(f)T(g)\|_\infty = \|fg\|_\infty \quad (f, g \in \mathrm{Lip}_0(X)). \tag{3.3}$$

Then there exists a unique bijective map $\psi : X \to Y$ *such that* $\psi(e_X) = e_Y$ *and*

$$|T(f)(\psi(x))| = |f(x)| \quad (f \in \mathrm{Lip}_0(X),\ x \in X). \tag{3.4}$$

The map ψ *will be referred to as the map associated to* T.

Proof. By taking $g = f$ in (3.3) we see that T satisfies condition (1) of Lemma 3.1. We next show that T also fulfills condition (2). To see this, let $f, g \in \mathrm{Lip}_0(X)$. If $|f| \leqslant |g|$, then $\|fh\|_\infty \leqslant \|gh\|_\infty$ for all $h \in \mathrm{Lip}_0(X)$. Since T is surjective, for each $k \in P(Y)$, there is $h \in \mathrm{Lip}_0(X)$ such that $k = T(h)$. By using (3.3), we have

$$\|T(f)k\|_\infty = \|T(f)T(h)\|_\infty = \|fh\|_\infty \leqslant \|gh\|_\infty = \|T(g)T(h)\|_\infty = \|T(g)k\|_\infty.$$

Since k is arbitrary in $P(Y)$, we infer from Lemma 2.3(2) that $|T(f)| \leqslant |T(g)|$. The other implication is proved likewise.

Then, by Lemma 3.1(iv), for every $x \in X \setminus \{e_X\}$, there exists a unique point $\psi(x) \in Y \setminus \{e_Y\}$ such that $T(F_x(X)) = F_{\psi(x)}(Y)$. Put $\psi(e_X) = e_Y$. We have thus defined a map $\psi : X \to Y$. The injectivity of ψ follows from Lemma 3.1(ii) and its surjectivity from Lemma 3.1(iii), and Lemma 2.1. To prove the equality (3.4), take $f \in \mathrm{Lip}_0(X)$. It is clear that

$$|T(f)(\psi(e_X))| = |T(f)(e_Y)| = 0 = |f(e_X)|,$$

and if $x \in X \setminus \{e_X\}$, we have

$$\begin{aligned} |f(x)| &= \inf\{\|fg\|_\infty\colon\ g \in F_x(X)\} \\ &= \inf\{\|T(f)T(g)\|_\infty\colon\ g \in F_x(X)\} \\ &= \inf\{\|T(f)h\|_\infty\colon\ h \in F_{\psi(x)}(Y)\} \\ &= |T(f)(\psi(x))|, \end{aligned}$$

by using Lemma 2.3(1).

For the uniqueness of ψ, let $\psi' : X \to Y$ be another bijection satisfying $\psi'(e_X) = e_Y$ and (3.4). It is easy to see that $T(F_x(X)) = F_{\psi'(x)}(Y)$ for all $x \in X \setminus \{e_X\}$. Then, Lemma 2.1 implies that $\psi'(x) = \psi(x)$ for all $x \in X \setminus \{e_X\}$, and thus $\psi' = \psi$. □

4. Non-symmetric norm *-multiplicativity condition

In this section, we show that every surjective map $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ satisfying the non-symmetric norm *-multiplicativity condition

$$\big\| T(f)\overline{T(g)} - \mathbf{1}\big\|_\infty = \| f\bar{g} - \mathbf{1}\|_\infty \quad \big(f, g \in \mathrm{Lip}_0(X)\big) \tag{4.5}$$

can be expressed as sum of a weighted isomorphism and a weighted conjugate-isomorphism from $\mathrm{Lip}_0(X)$ onto $\mathrm{Lip}_0(Y)$ in the form given in Theorem 4.8.

First we need to see that T is norm-multiplicative.

Lemma 4.1. *Let $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a surjective map satisfying condition (4.5). Then $\|T(f)T(g)\|_\infty = \|fg\|_\infty$ for every $f, g \in \mathrm{Lip}_0(X)$.*

Proof. We claim that

$$\big|T(rf)\big| = r\big|T(f)\big| \quad \big(r \in \mathbb{R}^+,\ f \in \mathrm{Lip}_0(X)\big).$$

Let $r \in \mathbb{R}^+$ and $f \in \mathrm{Lip}_0(X)$. It is obvious that $|T(rf)(e_Y)| = 0 = r|T(f)(e_Y)|$. Given $y \in Y \setminus \{e_Y\}$, we distinguish two cases.

First, if $T(f)(y) \neq 0$, take

$$h_n = -n\big(T(f)(y)/\big|T(f)(y)\big|\big)h_{T(f),y} \in \mathrm{Lip}_0(Y) \quad (n \in \mathbb{N}),$$

where $h_{T(f),y}$ is the function given in Lemma 2.2. Let $g_n \in \mathrm{Lip}_0(X)$ be such that $T(g_n) = h_n$. An easy calculation gives

$$\begin{aligned} n\big|T(rf)(y)\big| - 1 &= \big|T(rf)(y)\overline{h_n(y)}\big| - 1 \leqslant \big\|T(rf)\overline{h_n} - \mathbf{1}\big\|_\infty \\ &= \|rf\overline{g_n} - \mathbf{1}\|_\infty \leqslant r\|f\overline{g_n} - \mathbf{1} + \mathbf{1}\|_\infty + 1 \leqslant r\|f\overline{g_n} - \mathbf{1}\|_\infty + r + 1 \\ &= r\big\|T(f)\overline{h_n} - \mathbf{1}\big\|_\infty + r + 1 = r\big(n\big|T(f)(y)\big| + 1\big) + r + 1. \end{aligned}$$

Therefore $|T(rf)(y)| \leqslant r|T(f)(y)| + (2r+2)/n$. As n is arbitrary, it follows that $|T(rf)(y)| \leqslant r|T(f)(y)|$.

Now assume that $T(f)(y) = 0$. Since $T(f)$ is continuous at y, given $n \in \mathbb{N}$, there exists $\delta_n \in \,]0, \mathrm{d}(y, e_Y)]$ such that $|T(f)(z)| < 1/n$, whenever $\mathrm{d}(z, y) < \delta_n$. Take $h_n = nh_{y,\delta_n}$ and let $g_n \in \mathrm{Lip}_0(X)$ be such that $T(g_n) = h_n$. A trivial verification yields $\|T(f)\overline{h_n}\|_\infty \leqslant 1$ and, arguing as above, we can see that

$$n\big|T(rf)(y)\big| - 1 \leqslant r\big\|T(f)\overline{h_n} - \mathbf{1}\big\|_\infty + r + 1 \leqslant r\big\|T(f)\overline{h_n}\big\|_\infty + 2r + 1 \leqslant 3r + 1.$$

Hence $|T(rf)(y)| \leqslant (3r+2)/n$ and thus $|T(rf)(y)| = 0 = r|T(f)(y)|$.

This proves that $|T(rf)| \leqslant r|T(f)|$. Since r and f are arbitrary, the previous inequality also holds for $1/r$ and rf instead of r and f. Then

$$\big|T(f)\big| = \left|T\left(\frac{1}{r}rf\right)\right| \leqslant \frac{1}{r}\big|T(rf)\big| \leqslant \big|T(f)\big|,$$

and from this we conclude that $|T(rf)| = r|T(f)|$, as claimed.

In order to prove that T satisfies condition (3.3), take $f, g \in \mathrm{Lip}_0(X)$ and $n \in \mathbb{N}$. From the claim proved above, it follows easily that $\|T(nf)T(g)\|_\infty = n\|T(f)T(g)\|_\infty$. Then

$$n\|fg\|_\infty - 1 \leqslant \|nf\bar{g} - \mathbf{1}\|_\infty = \big\|T(nf)\overline{T(g)} - \mathbf{1}\big\|_\infty \leqslant n\big\|T(f)T(g)\big\|_\infty + 1,$$

which shows that $\|fg\|_\infty \leqslant \|T(f)T(g)\|_\infty + 2/n$. Making $n \to \infty$ we get $\|fg\|_\infty \leqslant \|T(f)T(g)\|_\infty$. The contrary inequality is deduced similarly by taking into account that

$$n\big\|T(f)T(g)\big\|_\infty - 1 \leqslant \big\|T(nf)\overline{T(g)} - \mathbf{1}\big\|_\infty = \|nf\bar{g} - \mathbf{1}\|_\infty \leqslant n\|fg\|_\infty + 1$$

for all $n \in \mathbb{N}$. □

Since T satisfies condition (3.3), we can consider its associated map ψ and then equality (3.4) holds. Roughly speaking, our next aim is to eliminate the modulus in (3.4). To get this, we study the homogeneity of T on products of scalars in $S_{\mathbb{K}}$ by functions in $P_x(X)$ $(x \in X \setminus \{e_X\})$.

We begin with a straightforward lemma that will make easier the reading of the proofs.

Lemma 4.2. *Let $\alpha, \beta \in \mathbb{C}$.*

(i) *If $|\alpha - 1| = |\beta| + 1$ and $|\alpha| = |\beta|$, then $\alpha = -|\beta|$.*

(ii) *If* $|\alpha - 1| \leqslant |\alpha| - 1$, *then* $\alpha \in \mathbb{R}^+$.
(iii) *If* $|\alpha| = 1$ *and* $0 \leqslant r \leqslant 1$, *then* $|2\alpha r - 1| \leqslant |2\alpha - 1|$.
(iv) *If* $|\beta| = |\alpha|$, $|\beta - 1| \leqslant |\alpha - 1|$ *and* $|\beta + 1| \leqslant |\alpha + 1|$, *then* $\beta = \alpha$ *or* $\beta = \overline{\alpha}$.

Lemma 4.3. *Let* $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ *be a surjective map satisfying condition* (4.5) *and let* $\psi : X \to Y$ *be its associated map. For every* $x \in X \setminus \{e_X\}$, $\alpha \in S_{\mathbb{K}}$ *and* $h \in P_x(X)$,

(i) $T(\alpha h)(\psi(x)) = T(\alpha h_{x,\delta})(\psi(x))$ *where* $h_{x,\delta} \in P_x(X)$ *is given in* (2.2),
(ii) $T(-\alpha h)(\psi(x)) = -T(\alpha h)(\psi(x))$.

Proof. Let $x \in X \setminus \{e_X\}$, $\alpha \in S_{\mathbb{K}}$ and $h \in P_x(X)$. Since

$$\big\| T(-\alpha h_{x,\delta})\overline{T(\alpha h)} - \mathbf{1}\big\|_\infty = \|-h_{x,\delta}\bar{h} - \mathbf{1}\|_\infty = 2,$$

ψ is surjective and Y is compact, we can find $z \in X$ such that

$$\big|T(-\alpha h_{x,\delta})\big(\psi(z)\big)\overline{T(\alpha h)\big(\psi(z)\big)} - 1\big| = 2.$$

Equality (3.4) allows us to obtain that

$$2 \leqslant \big|T(-\alpha h_{x,\delta})\big(\psi(z)\big)\overline{T(\alpha h)\big(\psi(z)\big)}\big| + 1 = \big|h_{x,\delta}(z)\big|\big|h(z)\big| + 1 \leqslant \big|h_{x,\delta}(z)\big| + 1.$$

This clearly forces $z = x$. Consequently, we have

$$\big|T(-\alpha h_{x,\delta})\big(\psi(x)\big)\overline{T(\alpha h)\big(\psi(x)\big)} - 1\big| = 2, \qquad \big|T(-\alpha h_{x,\delta})\big(\psi(x)\big)\overline{T(\alpha h)\big(\psi(x)\big)}\big| = 1.$$

Moreover, according to Lemma 4.2(i), $T(-\alpha h_{x,\delta})(\psi(x))\overline{T(\alpha h)(\psi(x))} = -1$, and since $|T(\alpha h)(\psi(x))| = 1$, it follows that

$$T(-\alpha h_{x,\delta})\big(\psi(x)\big) = -T(\alpha h)\big(\psi(x)\big).$$

Since h is arbitrary, taking $h = h_{x,\delta}$ above we get

$$T(-\alpha h_{x,\delta})\big(\psi(x)\big) = -T(\alpha h_{x,\delta})\big(\psi(x)\big).$$

Hence $T(\alpha h)(\psi(x)) = T(\alpha h_{x,\delta})(\psi(x))$, which proves (i). By replacing α with $-\alpha$ above,

$$T(-\alpha h)\big(\psi(x)\big) = T(-\alpha h_{x,\delta})\big(\psi(x)\big) = -T(\alpha h_{x,\delta})\big(\psi(x)\big) = -T(\alpha h)\big(\psi(x)\big),$$

and the proof is complete. □

The first part of Lemma 4.3 motivates the following definition.

Definition 4.4. Let $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a surjective map satisfying condition (4.5). Let $\psi : X \to Y$ be the associated map to T, and $\varphi = \psi^{-1}$. Let $\tau : Y \to S_{\mathbb{K}}$ be the function defined by

$$\tau(e_Y) = 1, \qquad \tau(y) = T(h)(y) \quad \big(y \in Y \setminus \{e_Y\}\big),$$

where h is any function in $P_{\varphi(y)}(X)$.

Moreover, in the complex-valued case, define $\gamma : Y \to S_{\mathbb{C}}$ by

$$\gamma(e_Y) = i, \qquad \gamma(y) = T(ih)(y) \quad \big(y \in Y \setminus \{e_Y\}\big),$$

where h is any function in $P_{\varphi(y)}(X)$.

Notice that Lemma 4.3(i) guarantees that the definitions of τ and γ do not depend on the choise of h.

We will need the following fact about the real homogeneity of T.

Lemma 4.5. *Let* $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ *be a surjective map satisfying condition* (4.5), *and* $\psi : X \to Y$ *be its associated map. Let* $x \in X \setminus \{e_X\}$ *and* $f \in F_x(X)$. *Then* $T(rf)(\psi(x)) = rT(f)(\psi(x))$ *for all* $r \geqslant 2$.

Proof. Let $r \in \mathbb{R}$, $r \geqslant 2$. An easy verification gives

$$\big|T(rf)\big(\psi(x)\big)\overline{T(f)\big(\psi(x)\big)} - 1\big| \leqslant \big\|T(rf)\overline{T(f)} - \mathbf{1}\big\|_\infty = \big\|r|f|^2 - \mathbf{1}\big\|_\infty = r - 1,$$

and in view of equality (3.4), we have $|T(rf)(\psi(x))\overline{T(f)(\psi(x))}| = r$. Hence

$$T(rf)\big(\psi(x)\big)\overline{T(f)\big(\psi(x)\big)} = r$$

by Lemma 4.2(ii), and as $|T(f)(\psi(x))| = 1$, we conclude that $T(rf)(\psi(x)) = rT(f)(\psi(x))$. □

If $\mathbb{K} = \mathbb{R}$, for any $x \in X \setminus \{e_X\}$, Lemma 4.3(ii) shows that $T(\alpha h)(\psi(x)) = \alpha T(h)(\psi(x))$ for all $\alpha \in S_{\mathbb{R}}$ and $h \in P_x(X)$. In the complex-valued case, a little more effort is needed.

Lemma 4.6. *Let $T : \mathrm{Lip}_0(X, \mathbb{C}) \to \mathrm{Lip}_0(Y, \mathbb{C})$ be a surjective map fulfilling* (4.5)*, $\psi : X \to Y$ its associated map, and τ and γ the functions introduced in Definition* 4.4*. Given $x \in X \setminus \{e_X\}$, one has*

(i) *Either $\gamma(\psi(x)) = i\tau(\psi(x))$ or $\gamma(\psi(x)) = -i\tau(\psi(x))$.*
(ii) *If $\gamma(\psi(x)) = i\tau(\psi(x))$, then*

$$T(\alpha h)\big(\psi(x)\big) = \alpha T(h)\big(\psi(x)\big) \quad \big(\alpha \in S_{\mathbb{C}},\ h \in P_x(X)\big).$$

(iii) *If $\gamma(\psi(x)) = -i\tau(\psi(x))$, then*

$$T(\alpha h)\big(\psi(x)\big) = \overline{\alpha} T(h)\big(\psi(x)\big) \quad \big(\alpha \in S_{\mathbb{C}},\ h \in P_x(X)\big).$$

Proof. Let $\alpha \in S_{\mathbb{C}}$ and $h \in P_x(X)$. From equality (3.4) we can deduce that

$$\big|T(\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)}\big| = 2.$$

By using Lemma 4.2(iii) and taking into account that $h(x) = 1$, it follows that

$$\big|T(\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)} - 1\big| \leqslant \big\|T(\alpha h)\overline{T(2h)} - \mathbf{1}\big\|_\infty = \big\|2\alpha|h|^2 - \mathbf{1}\big\|_\infty = |2\alpha - 1|,$$

and from Lemmas 4.3(ii), 4.2(iii) and condition (4.5), we obtain

$$\begin{aligned}\big|T(\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)} + 1\big| &= \big|T(-\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)} - 1\big| \\ &\leqslant \big\|T(-\alpha h)\overline{T(2h)} - \mathbf{1}\big\|_\infty = \big\|-2\alpha|h|^2 - \mathbf{1}\big\|_\infty \\ &= |-2\alpha - 1| = |2\alpha + 1|.\end{aligned}$$

Now Lemma 4.2(iv) gives

$$T(\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)} = 2\alpha \quad \text{or} \quad T(\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)} = 2\overline{\alpha}. \tag{4.6}$$

Since $|T(h)(\psi(x))| = 1$, from (4.6) and Lemma 4.5 it follows that

$$T(\alpha h)\big(\psi(x)\big) = \alpha T(h)\big(\psi(x)\big) \quad \text{or} \quad T(\alpha h)\big(\psi(x)\big) = \overline{\alpha} T(h)\big(\psi(x)\big).$$

For $\alpha = i$ we have $T(ih)(\psi(x)) = iT(h)(\psi(x))$ or $T(ih)(\psi(x)) = -iT(h)(\psi(x))$. According to Definition 4.4, this means that either $\gamma(\psi(x)) = i\tau(\psi(x))$ or $\gamma(\psi(x)) = -i\tau(\psi(x))$, which proves (i).

We next show (ii), and (iii) follows analogously. So, assume that $\gamma(\psi(x)) = i\tau(\psi(x))$. Then $T(ih)(\psi(x)) = iT(h)(\psi(x))$ and, Lemmas 4.3(ii) and 4.5, give

$$\begin{aligned}\big|iT(\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)} - 1\big| &= \big|T(-\alpha h)\big(\psi(x)\big)\overline{T(2ih)\big(\psi(x)\big)} - 1\big| \\ &\leqslant \big\|T(-\alpha h)\overline{T(2ih)} - \mathbf{1}\big\|_\infty = \big\|2i\alpha|h|^2 - \mathbf{1}\big\|_\infty \\ &= |2i\alpha - 1|,\end{aligned}$$

and

$$\begin{aligned}\big|iT(\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)} + 1\big| &= \big|T(\alpha h)\big(\psi(x)\big)\overline{T(2ih)\big(\psi(x)\big)} - 1\big| \\ &\leqslant \big\|T(\alpha h)\overline{T(2ih)} - \mathbf{1}\big\|_\infty = \big\|-2i\alpha|h|^2 - \mathbf{1}\big\|_\infty \\ &= |-2i\alpha - 1| = |2i\alpha + 1|.\end{aligned}$$

By (3.4), it is clear that

$$\big|iT(\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)}\big| = |2i\alpha|.$$

Thus, taking into account Lemma 4.2(iv),

$$\mathrm{Re}\big(iT(\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)}\big) = 2\mathrm{Re}(i\alpha),$$

or equivalently

$$\operatorname{Im}\big(T(\alpha h)\big(\psi(x)\big)\overline{T(2h)\big(\psi(x)\big)}\big) = 2\operatorname{Im}(\alpha).$$

From (4.6) we deduce that $T(\alpha h)(\psi(x))\overline{T(2h)(\psi(x))} = 2\alpha$, which together with equality (3.4) and Lemma 4.5 give

$$T(\alpha h)\big(\psi(x)\big) = \alpha T(h)\big(\psi(x)\big). \qquad \square$$

We will need the following lemma in order to prove the automatic continuity of a surjective map between pointed Lipschitz algebras satisfying the non-symmetric norm *-multiplicativity condition. Recall that if E and F are real or complex normed spaces and $S : E \to F$ is an $\mathbb{R}$-linear map, then S is continuous if and only if there exists $\beta \in \mathbb{R}^+$ such that $\|S(e)\| \leqslant \beta\|e\|$ for every $e \in E$.

Lemma 4.7. *Let $S : \operatorname{Lip}_0(X) \to \operatorname{Lip}_0(Y)$ be an $\mathbb{R}$-linear and continuous map with respect to the supremum norm. Then S is continuous with respect to the Lipschitz norm.*

Proof. If $\mathbb{K} = \mathbb{C}$, let us define $S_1, S_2 : \operatorname{Lip}_0(X) \to \operatorname{Lip}_0(Y)$ by

$$S_1(f) = S(f) - iS(if), \qquad S_2(f) = \overline{S(f) + iS(if)} \quad \big(f \in \operatorname{Lip}_0(X)\big).$$

It is easily seen that both S_1 and S_2 are $\mathbb{C}$-linear. Moreover, by the continuity of S, there exists $\beta \in \mathbb{R}^+$ such that

$$\big\|S_j(f)\big\|_\infty \leqslant \big\|S(f)\big\|_\infty + \big\|S(if)\big\|_\infty \leqslant 2\beta\|f\|_\infty \quad \big(j \in \{1,2\}\big).$$

Hence S_1 and S_2 are $\|\cdot\|_\infty$-continuous.

If $\mathbb{K} = \mathbb{R}$, we can take $S_1 = S_2 = S$ and thus S_1 and S_2 are linear and $\|\cdot\|_\infty$-continuous as well.

Pick $j \in \{1,2\}$. We next see that S_j is $L(\cdot)$-continuous. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\operatorname{Lip}_0(X)$ such that $L(f_n)$ converges to 0, and assume that $L(S_j(f_n) - g)$ converges to 0 for some $g \in \operatorname{Lip}_0(Y)$. For every $n \in \mathbb{N}$, we have

$$\|g\|_\infty \leqslant \big\|S_j(f_n) - g\big\|_\infty + \big\|S_j(f_n)\big\|_\infty \leqslant \operatorname{diam}(Y) L\big(S_j(f_n) - g\big) + 2\beta \operatorname{diam}(X) L(f_n).$$

Hence $g = 0$. Then S_j is $L(\cdot)$-continuous by the Closed Graph Theorem. Since $S(f) = (1/2)(S_1(f) + \overline{S_2(f)})$ for all $f \in \operatorname{Lip}_0(X)$, the lemma follows. $\square$

We are now ready to prove the main result of this section.

Theorem 4.8. *Let X, Y be pointed compact metric spaces, and let $T : \operatorname{Lip}_0(X) \to \operatorname{Lip}_0(Y)$ be a surjective map satisfying*

$$\big\|T(f)\overline{T(g)} - \mathbf{1}\big\|_\infty = \|f\bar{g} - \mathbf{1}\|_\infty \quad \big(f, g \in \operatorname{Lip}_0(X)\big).$$

Then there exist a function $\eta : Y \to \{0,1\}$ with $\eta(e_Y) = 1$, a function $\tau : Y \to S_{\mathbb{K}}$ with $\tau(e_Y) = 1$, and a base point preserving Lipschitz homeomorphism $\varphi : Y \to X$ such that

$$T(f) = \tau \cdot \big(\eta \cdot (f \circ \varphi) + (\mathbf{1} - \eta) \cdot \overline{(f \circ \varphi)}\big) \quad \big(f \in \operatorname{Lip}_0(X)\big).$$

Proof. Let ψ be the associated map to T and τ be the function given in Definition 4.4. Set $\varphi = \psi^{-1}$.

Assume for a moment that $\mathbb{K} = \mathbb{C}$ and consider as well the function γ presented in Definition 4.4. Define now $\eta = (1/2)(\mathbf{1} - i\gamma\bar{\tau})$. Using the surjectivity of ψ and Lemma 4.6(i), it is easy to see that $(\gamma\bar{\tau})(Y) \subseteq \{-i, i\}$. Then $\eta(Y) \subseteq \{0,1\}$ and it is obvious that $\eta(e_Y) = 1$. If $\mathbb{K} = \mathbb{R}$, set $\eta = \mathbf{1}$.

Let $f \in \operatorname{Lip}_0(X)$, $y \in Y$ and $x = \varphi(y)$. If $f(x) = 0$, from (3.4) we infer that

$$T(f)(y) = 0 = \tau(y)\big(\eta(y) f\big(\varphi(y)\big) + \big(1 - \eta(y)\big)\overline{f\big(\varphi(y)\big)}\big).$$

Suppose $f(x) \neq 0$ and take the peaking function $h_{f,x}$ given in Lemma 2.2. Set

$$\alpha = -T(h_{f,x})\big(\psi(x)\big)\overline{T(f)\big(\psi(x)\big)}/\big|f(x)\big|,$$
$$\lambda = \operatorname{Re}(\alpha) + \big(1 - 2\eta\big(\psi(x)\big)\big)\operatorname{Im}(\alpha)i.$$

Notice that if $\mathbb{K} = \mathbb{R}$, then $\lambda = \alpha$; and for $\mathbb{K} = \mathbb{C}$,

$$\lambda = \begin{cases} \bar{\alpha} & \text{if } \gamma(\psi(x)) = i\tau(\psi(x)), \\ \alpha & \text{if } \gamma(\psi(x)) = -i\tau(\psi(x)). \end{cases}$$

By applying Lemma 4.3(ii) in the real-valued case, Lemma 4.6 in the complex-valued case, and (3.4), we obtain

$$\big|T(f)\big(\psi(x)\big)\overline{T(\lambda h_{f,x})\big(\psi(x)\big)}-1\big|=\big|T(f)\big(\psi(x)\big)\alpha\overline{T(h_{f,x})\big(\psi(x)\big)}-1\big|=\big|f(x)\big|+1,$$

and since $|T(f)(\psi(z))\overline{T(\lambda h_{f,x})(\psi(z))}-1|\leqslant|f(z)h_{f,x}(z)|+1\leqslant|f(x)|+1$ for all $z\in X$, we deduce that

$$\big|f(x)\big|+1=\big\|T(f)\overline{T(\lambda h_{f,x})}-\mathbf{1}\big\|_\infty=\|\bar\lambda f h_{f,x}-\mathbf{1}\|_\infty.$$

Besides, as $|\bar\lambda f(z)h_{f,x}(z)-1|\leqslant|f(z)h_{f,x}(z)|+1<|f(x)|+1$ for all $z\in X$ with $z\neq x$, the compactness of X grants

$$\big|\bar\lambda f(x)-1\big|=\|\bar\lambda f h_{f,x}-\mathbf{1}\|_\infty=\big|f(x)\big|+1.$$

In view of Lemma 4.2(i), this shows that $\bar\lambda f(x)=-|f(x)|$. As a consequence,

$$f(x)=\begin{cases}\overline{T(h_{f,x})(\psi(x))}T(f)(\psi(x)) & \text{if } \eta(\psi(x))=1,\\ T(h_{f,x})(\psi(x))\overline{T(f)(\psi(x))} & \text{if } \eta(\psi(x))=0.\end{cases}$$

According to the definition of τ, we conclude that

$$T(f)\big(\psi(x)\big)=\begin{cases}\tau(\psi(x))f(x) & \text{if } \eta(\psi(x))=1,\\ \tau(\psi(x))\overline{f(x)} & \text{if } \eta(\psi(x))=0,\end{cases}$$

or, equivalently,

$$T(f)(y)=\tau(y)\big(\eta(y)f\big(\varphi(y)\big)+\big(1-\eta(y)\big)\overline{f\big(\varphi(y)\big)}\big).$$

Finally, we prove that φ is a Lipschitz homeomorphism. From the expressions obtained above for T, we deduce immediately that T is $\mathbb{R}$-linear and T preserves the supremum norm. Hence T is continuous with respect to the Lipschitz norm by Lemma 4.7. Then there exists $\beta\in\mathbb{R}^+$ such that $L(T(f))\leqslant\beta L(f)$ for all $f\in\mathrm{Lip}_0(X)$. Let $y,z\in Y$ be with $y\neq z$. We can assume without lost of generality that $\mathrm{d}(\varphi(y),e_X)\leqslant\mathrm{d}(\varphi(z),e_X)$, and then $\mathrm{d}(\varphi(y),\varphi(z))\leqslant2\mathrm{d}(\varphi(z),e_X)$. Take $\delta=\min\{\mathrm{d}(\varphi(z),e_X),\mathrm{d}(\varphi(y),\varphi(z))\}$ and $f=\mathrm{d}(\varphi(y),\varphi(z))h_{\varphi(z),\delta}$. It is clear that $f\in\mathrm{Lip}_0(X)$ with $L(f)\leqslant2$, $f(\varphi(z))=\mathrm{d}(\varphi(y),\varphi(z))$ and $f(\varphi(y))=0$. An easy calculation yields

$$\begin{aligned}\mathrm{d}\big(\varphi(y),\varphi(z)\big)&=\big|\tau(z)f\big(\varphi(z)\big)-\tau(y)f\big(\varphi(y)\big)\big|=\big|T(f)(z)-T(f)(y)\big|\\&\leqslant L\big(T(f)\big)\mathrm{d}(y,z)\leqslant2\beta\mathrm{d}(y,z),\end{aligned}$$

and thus φ is Lipschitz.

Clearly, T is bijective and T^{-1} has the same properties as T. Therefore we can apply the same arguments as above to obtain a Lipschitz bijection $\varphi':X\to Y$ with $\varphi'(e_X)=e_Y$ such that

$$\big|T^{-1}(g)(x)\big|=\big|g\big(\varphi'(x)\big)\big|\quad\big(g\in\mathrm{Lip}_0(Y),\ x\in X\big).$$

From this formula we deduce that $|T(f)(\varphi'(x))|=|f(x)|$ for all $f\in\mathrm{Lip}_0(X)$ and $x\in X$. Since ψ is unique by Theorem 3.2, it follows that $\varphi'=\psi$, and thus ψ is Lipschitz. □

Remark 4.9. The functions which appear in the representation of T given in the previous theorem are unique as we see next. Let η, τ and φ as in Theorem 4.8. If $\eta':Y\to\{0,1\}$ and $\tau':Y\to S_{\mathbb{K}}$ are functions for which $\eta'(e_Y)=1=\tau'(e_Y)$ and $\varphi':Y\to X$ is a bijection with $\varphi'(e_Y)=e_X$ such that

$$T(f)=\tau'\cdot\big(\eta\cdot(f\circ\varphi')+(\mathbf{1}-\eta')\cdot\overline{(f\circ\varphi')}\big)\quad\big(f\in\mathrm{Lip}_0(X)\big),$$

one has

(1) If $\mathbb{K}=\mathbb{R}$, then $\varphi'=\varphi$ and $\tau'=\tau$.
(2) If $\mathbb{K}=\mathbb{C}$, then $\varphi'=\varphi$, $\tau'=\tau$ and $\eta'=\eta$.

Indeed, notice that for every $f\in\mathrm{Lip}_0(X)$ and all $x\in X$, we have

$$T(f)\big(\varphi'^{-1}(x)\big)=\tau'\big(\varphi'^{-1}(x)\big)f(x)\quad\text{if } \eta'\big(\varphi'^{-1}(x)\big)=1,$$
$$T(f)\big(\varphi'^{-1}(x)\big)=\tau'\big(\varphi'^{-1}(x)\big)\overline{f(x)}\quad\text{if } \eta'\big(\varphi'^{-1}(x)\big)=0.$$

Therefore $|T(f)(\varphi'^{-1}(x))|=|f(x)|$ for all $f\in\mathrm{Lip}_0(X)$ and $x\in X$. By Theorem 3.2, it follows that $\varphi'^{-1}=\psi$ and thus $\varphi'=\varphi$. Now, let $y\in Y\setminus\{e_Y\}$ and $h\in P_{\varphi(y)}(X)$. Then $\tau'(y)=T(h)(y)=\tau(y)$ and, if $\mathbb{K}=\mathbb{C}$, we have $\eta'(y)=\eta(y)$ since

$$\tau(y)\big(2i\eta'(y)-i\big)=\tau'(y)\big(\eta'(y)i+\big(1-\eta'(y)\big)\bar i\big)=T(ih)(y)=\tau(y)\big(2i\eta(y)-i\big).$$

In consequence, $\tau'=\tau$ and, in the complex-valued case, $\eta'=\eta$.

The following corollary characterizes surjective maps $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ that satisfy the non-symmetric norm *-multiplicativity condition.

Recall that every algebra isomorphism T from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ is a composition operator $T(f) = f \circ \varphi$ for all $f \in \mathrm{Lip}(X)$, where $\varphi : Y \to X$ is a Lipschitz homeomorphism [15, Theorem 5.1]. As a consequence, every isomorphism from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ is also a *-isomorphism.

Corollary 4.10. *Let X and Y be compact metric spaces and let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective map with the property that*

$$\big\|T(f)\overline{T(g)} - \mathbf{1}\big\|_\infty = \|f\bar{g} - \mathbf{1}\|_\infty \quad \big(f, g \in \mathrm{Lip}(X)\big).$$

(1) *If $\mathbb{K} = \mathbb{C}$, there exist two unique Lipschitz functions $\eta : Y \to \{0, 1\}$ and $\tau : Y \to S_{\mathbb{C}}$ and a unique Lipschitz homeomorphism $\varphi : Y \to X$ such that*

$$T(f) = \tau \cdot \big(\eta \cdot (f \circ \varphi) + (\mathbf{1} - \eta) \cdot \overline{(f \circ \varphi)}\big) \quad \big(f \in \mathrm{Lip}(X, \mathbb{C})\big).$$

If, in addition, $T(\mathbf{1}) = \mathbf{1}$ and $T(i\mathbf{1}) = i\mathbf{1}$ ($T(i\mathbf{1}) = -i\mathbf{1}$), then T is an algebra isomorphism (respectively, a conjugate-isomorphism).

(2) *If $\mathbb{K} = \mathbb{R}$, there exist a unique Lipschitz function $\tau : Y \to S_{\mathbb{R}}$, and a unique Lipschitz homeomorphism $\varphi : Y \to X$ such that*

$$T(f) = \tau \cdot (f \circ \varphi) \quad \big(f \in \mathrm{Lip}(X, \mathbb{R})\big).$$

Moreover, if $T(\mathbf{1}) = \mathbf{1}$, then T is an algebra isomorphism.

Proof. Let $\Psi_X : \mathrm{Lip}(X) \to \mathrm{Lip}_0(X_0)$ and $\Psi_Y : \mathrm{Lip}(Y) \to \mathrm{Lip}_0(Y_0)$ be the isometric isomorphisms given by (2.1), where $X_0 = X \cup \{e_X\}$ and $Y_0 = Y \cup \{e_Y\}$. Then $T_0 = \Psi_Y \circ T \circ \Psi_X^{-1}$ is a map from $\mathrm{Lip}_0(X_0)$ onto $\mathrm{Lip}_0(Y_0)$. We first prove that T_0 satisfies condition (4.5). Pick $f, g \in \mathrm{Lip}_0(X_0)$. We have

$$\begin{aligned}\big|f(x)\overline{g(x)} - 1\big| &= \big|\Psi_X^{-1}(f)(x)\overline{\Psi_X^{-1}(g)(x)} - 1\big| \leqslant \big\|\Psi_X^{-1}(f)\overline{\Psi_X^{-1}(g)} - \mathbf{1}\big\|_\infty \\ &= \big\|T\big(\Psi_X^{-1}(f)\big)\overline{T\big(\Psi_X^{-1}(g)\big)} - \mathbf{1}\big\|_\infty \leqslant \big\|T_0(f)\overline{T_0(g)} - \mathbf{1}\big\|_\infty \quad (x \in X),\end{aligned}$$

and

$$\big|f(e_X)\overline{g(e_X)} - 1\big| = 1 = \big|T_0(f)(e_Y)\overline{T_0(g)(e_Y)} - 1\big| \leqslant \big\|T_0(f)\overline{T_0(g)} - \mathbf{1}\big\|_\infty.$$

Therefore $\|f\bar{g} - \mathbf{1}\|_\infty \leqslant \|T_0(f)\overline{T_0(g)} - \mathbf{1}\|_\infty$. The converse inequality is obtained similarly.

Then, by Theorem 4.8, there exist $\eta_0 : Y_0 \to \{0, 1\}$ with $\eta_0(e_Y) = 1$, $\tau_0 : Y_0 \to S_{\mathbb{K}}$ with $\tau_0(e_Y) = 1$, and a base point preserving Lipschitz homeomorphism $\varphi_0 : Y_0 \to X_0$ such that

$$T_0(g)(y) = \tau_0(y)\big(\eta_0(y)g\big(\varphi_0(y)\big) + \big(1 - \eta_0(y)\big)\overline{g\big(\varphi_0(y)\big)}\big) \quad \big(g \in \mathrm{Lip}_0(X_0),\ y \in Y_0\big).$$

Define $\tau = \tau_0|_Y$, $\eta = \eta_0|_Y$ and $\varphi = \varphi_0|_Y$. Then φ is a Lipschitz homeomorphism from Y onto X and, for every $f \in \mathrm{Lip}(X)$ and $y \in Y$, it is easy to see that

$$\tau(y)\big(\eta(y)f\big(\varphi(y)\big) + \big(1 - \eta(y)\big)\overline{f\big(\varphi(y)\big)}\big) = T(f)(y).$$

Moreover, $\tau = T(\mathbf{1}) \in \mathrm{Lip}(Y)$ and, if $\mathbb{K} = \mathbb{C}$, $\eta = (1/2)(\mathbf{1} - i\bar{\tau}T(i\mathbf{1})) \in \mathrm{Lip}(Y)$. Finally, the uniqueness of φ, τ and, in the complex-valued case, of η follows from Remark 4.9. $\square$

5. Weakly peripherally *-multiplicativity condition

This last section is concerned with surjective maps between pointed Lipschitz algebras satisfying the weakly peripherally *-multiplicativity. We prove that such a map is an algebra isomorphism multiplied by a unimodular function.

Theorem 5.1. *Let X, Y be pointed compact metric spaces, and let $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a surjective map that satisfies*

$$\mathrm{Ran}_\pi\big(T(f)\overline{T(g)}\big) \cap \mathrm{Ran}_\pi(f\bar{g}) \neq \emptyset \quad \big(f, g \in \mathrm{Lip}_0(X)\big). \tag{5.7}$$

There exist a unique function $\tau : Y \to S_{\mathbb{K}}$ with $\tau(e_Y) = 1$, and a unique base point preserving Lipschitz homeomorphism $\varphi : Y \to X$ such that

$$T(f) = \tau \cdot (f \circ \varphi) \quad \big(f \in \mathrm{Lip}_0(X)\big).$$

Proof. From (5.7), it is clear that

$$\left\|T(f)T(g)\right\|_\infty = \|fg\|_\infty \quad \left(f, g \in \mathrm{Lip}_0(X)\right).$$

Hence, by Theorem 3.2, there exists a unique bijective map $\psi : X \to Y$ such that $|T(f)(\psi(x))| = |f(x)|$ for all $f \in \mathrm{Lip}_0(X)$ and $x \in X$.

Let $f, g \in \mathrm{Lip}_0(X)$ and $x \in X$. Assume first $f(x) \neq 0 \neq g(x)$. Let $h_{f,x}, h_{g,x} \in P_x(X)$ be the functions given in Lemma 2.2, and $h = h_{f,x}h_{g,x} \in P_x(X)$. Evidently, $\mathrm{Ran}_\pi(fh) = \{f(x)\}$, and so by assumption, there exists $z \in X$ such that $f(x) = T(f)(\psi(z))\overline{T(h)(\psi(z))}$. Then

$$\left|f(x)\right| = \left|T(f)\left(\psi(z)\right)\right|\left|T(h)\left(\psi(z)\right)\right| = \left|f(z)\right|\left|h(z)\right|,$$

and since $|f(w)h(w)| < |f(x)|$ for all $w \neq x$, we have $z = x$. Since $|T(h)(\psi(x))| = |h(x)| = 1$, it follows that $T(f)(\psi(x)) = T(h)(\psi(x))f(x)$. Similarly, it is proved that $T(g)(\psi(x)) = T(h)(\psi(x))g(x)$. Thus

$$T(f)\left(\psi(x)\right)\overline{T(g)\left(\psi(x)\right)} = T(h)\left(\psi(x)\right)f(x)\overline{T(h)\left(\psi(x)\right)g(x)} = f(x)\overline{g(x)}.$$

If now $f(x) = 0$ or $g(x) = 0$, obviously

$$T(f)\left(\psi(x)\right)\overline{T(g)\left(\psi(x)\right)} = 0 = f(x)\overline{g(x)}.$$

So, we have proved that

$$T(f)\left(\psi(x)\right)\overline{T(g)\left(\psi(x)\right)} = f(x)\overline{g(x)} \quad \left(f, g \in \mathrm{Lip}_0(X),\ x \in X\right). \tag{5.8}$$

As a consequence, we have $\|T(f)\overline{T(g)} - \mathbf{1}\|_\infty = \|f\overline{g} - \mathbf{1}\|_\infty$ for all $f, g \in \mathrm{Lip}_0(X)$.

Then, in the real-valued case, Theorem 4.8 and Remark 4.9 yield a unique function $\tau : Y \to S_{\mathbb{R}}$ with $\tau(e_Y) = 1$ and a unique base point preserving Lipschitz homeomorphism $\varphi : Y \to X$ such that

$$T(f)(y) = \tau(y)f\left(\varphi(y)\right) \quad \left(f \in \mathrm{Lip}_0(X, \mathbb{R}),\ y \in Y\right),$$

which proves the theorem in this case.

In the complex-valued case, by the aforementioned results, we have a unique function $\eta : Y \to \{0, 1\}$ with $\eta(e_Y) = 1$, a unique function $\tau : Y \to S_{\mathbb{C}}$ with $\tau(e_Y) = 1$ and a unique base point preserving Lipschitz homeomorphism $\varphi : Y \to X$ such that

$$T(f)(y) = \tau(y)\left(\eta(y)f\left(\varphi(y)\right) + \left(1 - \eta(y)\right)\overline{f\left(\varphi(y)\right)}\right) \quad \left(f \in \mathrm{Lip}_0(X, \mathbb{C}),\ y \in Y\right). \tag{5.9}$$

Let $y \in Y \setminus \{e_Y\}$, $x = \in X \setminus \{e_X\}$ for which $\psi(x) = y$ and $h_{x,\delta} \in P_x(X)$. Applying (5.8) gives

$$T(ih_{x,\delta})(y)\overline{T\left(h_{x,\delta}(y)\right)} = ih_{x,\delta}(x)\overline{h_{x,\delta}(x)} = i,$$

but, by using (5.9), we also have

$$T(ih_{x,\delta})(y)\overline{T(h_{x,\delta})(y)} = \left(2\eta(y) - 1\right)ih_{x,\delta}\left(\varphi(y)\right)^2.$$

It follows that $\eta(y) = 1$ and this completes the proof. □

Let us recall that a net $\{a_j\}_{j\in I}$ in a commutative Banach algebra $\mathcal{A}$ is an *approximate identity* if $\lim_{j\in I} \|a_j x - x\| = 0$ for each $x \in \mathcal{A}$. Notice that $\mathrm{Lip}_0(X)$ may do not have an approximate identity. In fact, for a pointed compact metric space X, the following conditions are equivalent.

(i) $\mathrm{Lip}_0(X)$ has a unity.
(ii) $\mathrm{Lip}_0(X)$ has an approximate identity.
(iii) e_X is an isolated point.

Only (ii) ⇒ (iii) deserves some comments. Assume that $\{h_j\}_{j\in I}$ is an approximate identity for $\mathrm{Lip}_0(X)$ and e_X is not an isolated point. Let $f \in \mathrm{Lip}_0(X)$ be the function defined as $f(x) = \mathrm{d}(x, e_X)$. For every $j \in I$, since h_j is continuous at e_X, there exists $\delta_j > 0$ such that $|h_j(x)| = |h_j(x) - h_j(e_X)| < 1/2$ if $0 < \mathrm{d}(x, e_X) < \delta_j$. Therefore,

$$L(fh_j - f) \geqslant \frac{|f(x)h_j(x) - f(x)|}{\mathrm{d}(x, e_X)} = \left|h_j(x) - 1\right| \geqslant 1 - \left|h_j(x)\right| > \frac{1}{2},$$

whenever $0 < \mathrm{d}(x, e_X) < \delta_j$. Since $\lim_{j\in I} L(fh_j - f) = 0$, we arrive at a contradiction and therefore e_X is isolated.

However, for any pointed compact metric space X, the sequence $\{h_n\}_{n\in\mathbb{N}}$ defined by

$$h_n(x) = \min\{1, nd(x, e_X)\} \quad (x \in X,\ n \in \mathbb{N}),$$

is an approximate identity for the supremum norm in $\mathrm{Lip}_0(X)$.

Following an idea by Honma in [6], we next provide a sufficient condition for a weakly peripherally *-multiplicative surjection between *-algebras $\mathrm{Lip}_0(X)$ to be an algebra isomorphism.

Corollary 5.2. *Let X, Y be pointed compact metric spaces and let $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a surjective map satisfying condition* (5.7)*. Then T is an algebra isomorphism if T preserves an approximate identity for the supremum norm.*

Proof. We know that $T(f) = \tau \cdot (f \circ \varphi)$ for all $f \in \mathrm{Lip}_0(X)$, with τ and φ as in Theorem 5.1. To prove the corollary, it suffices to show that $\tau = \mathbf{1}$. Suppose that T preserves an approximate identity for the supremum norm $\{h_j\}_{j\in I}$ in $\mathrm{Lip}_0(X)$. For each $x \in X \setminus \{e_X\}$, we can take a $f \in \mathrm{Lip}_0(X)$ for which $f(x) = 1$. Then, for all $j \in I$, we have

$$\left|h_j(x) - 1\right| = \left|h_j(x)f(x) - f(x)\right| \leqslant \|h_j f - f\|_\infty.$$

Since $\lim_{j\in I} \|h_j f - f\|_\infty = 0$, it follows that $\lim_{j\in I} h_j(x) = 1$. In the same way, as $\{T(h_j)\}_{j\in I}$ is an approximate identity for the supremum norm in $\mathrm{Lip}_0(Y)$, we get $\lim_{j\in I} T(h_j)(y) = 1$ for each $y \in Y \setminus \{e_Y\}$. Then

$$\tau(y) = \lim_{j\in I} \tau(y)h_j\big(\varphi(y)\big) = \lim_{j\in I} T(h_j)(y) = 1$$

for all $y \in Y \setminus \{e_Y\}$, which is the desired conclusion. □

Taking into account that elements with equal ranges have equal peripheral ranges, from Theorem 5.1 we deduce the following version for algebras $\mathrm{Lip}_0(X)$ of the result obtained by Hatori, Miura and Takagi [2, Theorem 3.6].

Corollary 5.3. *Let X and Y be pointed compact metric spaces. Every surjective map $T : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ fulfilling*

$$\big(T(f)\overline{T(g)}\big)(Y) = (f\bar{g})(X) \quad \big(f, g \in \mathrm{Lip}_0(X)\big)$$

is a weighted composition operator

$$T(f)(y) = \tau(y) f\big(\varphi(y)\big) \quad \big(f \in \mathrm{Lip}_0(X),\ y \in Y\big),$$

where τ is a unimodular function on Y, and φ is a base point preserving Lipschitz homeomorphism from Y onto X.

From Theorem 5.1 we deduce the next result that characterizes surjective maps between *-algebras $\mathrm{Lip}(X)$ satisfying the weakly peripherally *-multiplicativity condition. Its proof follows by the same method used in Corollary 4.10.

Corollary 5.4. *Let X and Y be compact metric spaces and let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective map such that*

$$\mathrm{Ran}_\pi\big(T(f)\overline{T(g)}\big) \cap \mathrm{Ran}_\pi(f\bar{g}) \neq \emptyset \quad \big(f, g \in \mathrm{Lip}(X)\big).$$

Then there exist a unique Lipschitz function $\tau : Y \to S_{\mathbb{K}}$ and a unique Lipschitz homeomorphism $\varphi : Y \to X$ such that

$$T(f)(y) = \tau(y) f\big(\varphi(y)\big) \quad \big(f \in \mathrm{Lip}(X),\ y \in Y\big).$$

In particular, if T preserves the unity, then T is an algebra isomorphism.

Finally, we deduce from Corollary 5.4 the version for algebras $\mathrm{Lip}(X)$ of the aforementioned result by Hatori–Miura–Takagi.

Corollary 5.5. *Let X and Y be compact metric spaces. Every surjective map $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ such that*

$$\big(T(f)\overline{T(g)}\big)(Y) = (f\bar{g})(X) \quad \big(f, g \in \mathrm{Lip}(X)\big)$$

is of the form

$$T(f)(y) = \tau(y) f\big(\varphi(y)\big) \quad \big(f \in \mathrm{Lip}(X),\ y \in Y\big),$$

where $\tau : Y \to S_{\mathbb{K}}$ is Lipschitz and $\varphi : Y \to X$ is a Lipschitz homeomorphism.

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