Maximum Principles and Singular Elliptic Inequalities

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In this paper, we present a version of the Omori–Yau maximum principle, a Liouville-type result, and a Phragmen–Lindelöf-type theorem for a class of singular elliptic operators on a Riemannian manifold, which include the $p$-Laplacian and the mean curvature operator. Some applications of the results obtained are discussed. © 2002 Elsevier Science (USA)

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0. INTRODUCTION

Let $(M, \langle , \rangle)$ be a smooth, connected, non-compact, complete Riemannian manifold of dimension $m \geq 2$. We fix an origin $o$, and denote by $r(x)$ the distance function from $o$, and by $B_t = \{ x \in M : r(x) < t \}$ and $\partial B_t = \{ x \in M : r(x) = t \}$ the geodesic ball and sphere of radius $t > 0$ centered at $o$.

Let $\varphi$ be a real-valued function in $C^1(0, +\infty) \cap C^0([0, +\infty))$ satisfying the following structural conditions:

(i) $\varphi(0) = 0$;  \hspace{1cm} (ii) $\varphi'(t) > 0 \hspace{1cm} \forall t > 0$. \hspace{1cm} (0.1)

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Further conditions on \( \varphi \) will be imposed when needed. We will focus our attention on the differential operator defined, for \( u \in C^1(M) \), by

\[
div(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u),
\]

and which could be referred to as the \( \varphi \)-Laplacian. Of course, if the vector field in brackets is not \( C^1 \), then the divergence in (0.2) must be understood in distributional sense.

We note that the conditions satisfied by \( \varphi \) could be interpreted as ellipticity conditions for the operator. We also remark that the \( \varphi \)-Laplacian arises naturally when considering the Euler–Lagrange equation associated to the energy functional

\[
\Lambda(u) = \int \Phi(|\nabla u|),
\]

where \( \Phi(t) = \int_0^t \varphi(s) \, ds \).

As important natural examples we mention

1. the Laplace–Beltrami operator, \( \Delta u \), corresponding to \( \varphi(t) = t \);
2. or, more generally, the \( p \)-Laplacian, \( \text{div}(|\nabla u|^{p-2} \nabla u) \), \( p > 1 \), corresponding to \( \varphi(t) = t^{p-1} \);
3. the generalized mean curvature operator, \( \text{div}(\frac{\nabla u}{(1+|\nabla u|^2)^\alpha}) \), \( \alpha > 0 \), corresponding to \( \varphi(t) = t/(1+t^2)\alpha \).

Starting from the classical work of Redheffer [R1] and Vazquez [V], which analyzed the case of the \( p \)-Laplacian on domains of \( \mathbb{R}^m \), a number of authors have considered the problem of establishing the validity of the strong maximum principle for solutions of the differential inequality

\[
div(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u) \leq f(u)
\]

on a domain \( \Omega \subset \mathbb{R}^m \). Thus, denoting by \( H^{-1} \) the inverse function of

\[
H(t) = t\varphi(t) - \int_0^t \varphi(s) \, ds
\]

(the Legendre transform of the integrand \( \Phi \) above), in a recent paper, Pucci and Serrin [PS] (see also [PSZ]) proved that if \( u \) is a non-negative solution of (0.3) such that \( u(x_0) = 0 \) for some \( x_0 \in \Omega \), then \( u \equiv 0 \) on \( \Omega \) provided that either \( f(s) \equiv 0 \) in a right neighborhood of 0 or

\[
\frac{1}{H^{-1}(\int_0^t f(s) \, ds)} \notin L^1(0^+).
\]
In the present paper, we shall make use of this result for \( f \equiv 0 \) and we shall refer to it as the maximum principle for the \( \varphi \)-Laplacian.

It is apparent that the validity of some form of the maximum principle is extremely useful when studying the qualitative behavior of solutions of differential equations and inequalities. Indeed, the natural setting of many genuine problems in Riemannian geometry is non-compact, and the Omori–Yau maximum principle, a kind of maximum principle “at infinity”, has revealed itself a powerful tool towards their solution. One of the aims of this paper is to give a generalized form of the latter valid for the \( \varphi \)-Laplacian. This will be a consequence of the next more sophisticated

**Theorem A.** Let \( \varphi \in C^1((0, +\infty)) \cap C^0([0, +\infty)) \) satisfy (0.1)(i), (ii) and

(iii) \( \varphi(t) \leq A t^\delta \) on \([0, \varepsilon]\);

(iv) \( \varphi(t) \to +\infty \) as \( t \to +\infty \),

for some \( A, \delta, \varepsilon > 0 \). Let \( b \in C^0(M) \) be such that

(i) \( b(x) \geq 0 \) on \( M \);

(ii) \( b(x) \geq q(r(x)) \) for \( r(x) \geq 1 \)

with \( q \in C^0([R_0, +\infty)) \), some \( R_0 > 0 \), and \( q > 0 \). Assume that the radial Ricci curvature of \((M, \langle , \rangle)\) satisfies

\[
\text{Ricc}_{(M, \langle , \rangle)}(\nabla r, \nabla r) > -(m - 1)B^2G(r)
\]

for some constant \( B > 0 \) and some positive \( G \in C^1([0, +\infty)) \) with the following properties:

\[
\begin{cases}
(i) \quad \inf_{[0, +\infty)} \frac{G'}{G^2} > -\infty, \\
(ii) \quad \frac{\sqrt{G}}{q}(t) \leq z(t)^\delta, \quad t \geq 1,
\end{cases}
\]

where \( z(t) \) is a positive, non-decreasing, \( C^1 \) function defined for \( t \geq 1 \) such that \( \frac{1}{\sqrt{\pi}} \notin L^1(+\infty) \). Let \( f \in C^0(\mathbb{R}) \), and \( u \in C^1(M) \) be such that \( u^* = \sup_M u < +\infty \). If \( u \) is a solution of

\[
\text{div}(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u) \geq b(x)f(u) \quad \text{on}
\]

\( A_\eta = \{ x \in M : u(x) > u^* - \eta \} \)

for some \( \eta > 0 \), then \( f(u^*) \leq 0 \). If \( q \) is bounded above, the same conclusion is reached requiring that (0.8) is valid on the smaller set

\[
\hat{A}_\eta = \{ x \in M : u(x) > u^* - \eta \quad \text{and} \quad |\nabla u|(x) < \eta \}.
\]
In accordance to what we have said above, if the vector field

$$|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$$

is not $C^1$, the inequality in (0.8) must be understood in weak sense; explicitly

$$- \int \langle \nabla \psi, |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u \rangle \geq \int \psi \xi f(u)$$

for all $0 \leq \psi \in C_0^\infty(M)$ supported in the appropriate sets.

The next corollary generalizes, even in case of the Laplace–Beltrami operator, the Omori–Yau result [RRS].

**Corollary A1.** Let $\varphi \in C^1((0, +\infty)) \cap C^0([0, +\infty))$ satisfy (0.1)(i), (ii), (iii), and assume that (0.6) holds, and that

\[
\begin{align*}
(i') & \quad G'(t) \geq 0 \quad \text{for } t \gg 1; \\
(ii') & \quad \sqrt{G(t)} \leq z(t)^{\delta} \quad \text{for } t \gg 1
\end{align*}
\]

with $z(t)$ and $\delta$ as in Theorem A. Let $u \in C^2(M)$ and $u^* = \sup_M u < +\infty$, and assume that the vector field $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class at least $C^1$. Then, there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset M$ such that

\[
\begin{align*}
(i) & \quad u(x_k) > u^* - \frac{1}{k}, \\
(ii) & \quad |\nabla u(x_k)| < \frac{1}{k}, \\
(iii) & \quad \text{div}(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u)(x_k) < \frac{1}{k}
\end{align*}
\]

for each $k \in \mathbb{N}$. If $u^* = +\infty$ but

$$u(x) = o\left(\int_1^{r(x)} \frac{dt}{z(t)}\right) \quad \text{as } r(x) \to +\infty,$$

there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ verifying (0.11)(iii).

We observe that the regularity assumption in the statement of Corollary A1 is certainly satisfied for the Laplacian, the $p$-Laplacian with $p \geq 2$, or the generalized mean curvature operators. We stress that the original method of proof of Cheng and Yau [CY] cannot be implemented in the present situation.

As a simple example of application of Corollary A1 let us consider a graph $\Gamma_u : M \to M \times \mathbb{R}$ over the complete manifold $(M, \langle , \rangle)$ determined by
the smooth function $u : M \to \mathbb{R}$. The graph $\Gamma_u$ has constant mean curvature if
\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = c
\]
for some $c \in \mathbb{R}$. In the curvature assumption of Corollary A1 with $\delta = 1$, if
\[
|u(x)| = o\left( \int_1^{r(x)} \frac{dt}{z(t)} \right) \quad \text{as} \quad r(x) \to +\infty, \quad (0.13)
\]
then $\Gamma_u$ is a minimal graph.

Note that if the Cheeger constant $h(M)$ of the manifold is zero, then the above result holds without requiring (0.13). However, if $h(M) > 0$, then some growth condition on $u$ must be imposed as shown by the following example. Let $\mathbb{H}^m$ be the hyperbolic space with canonical metric $\langle , \rangle$ of constant negative curvature $-1$, which we realize, in polar coordinates $(r, \theta) \in (0, +\infty) \times S^{m-1}$ as $\langle , \rangle = dr^2 + (\sinh r)^2 d\theta^2$, $d\theta^2$ being the standard metric on $S^{m-1}$. Then, for any $a \in (0, m - 1]$ the smooth function
\[
u(x) = \int_0^{r(x)} \frac{(\sinh t)^{1-m} \int_0^t a(\sinh s)^{m-1} ds}{\left( 1 - (\sinh t)^{2(1-m)} \int_0^t a(\sinh s)^{m-1} ds \right)^{1/2}} dt
\]
realizes a graph $\Gamma_u : x \mapsto (x, u(x))$ with constant mean curvature $\frac{a}{m}$. Here
\[
h(\mathbb{H}^m) = m - 1 \quad \text{and} \quad u(x) \asymp r(x) \quad \text{as} \quad r(x) \to +\infty.\]
Note that in this example, choosing $z(t) \equiv 2$, the curvature assumptions of Corollary A1 with $\delta = 1$ are satisfied but (0.13) becomes $|u(x)| = o(r(x))$, as $r(x) \to +\infty$.

Further results related to Theorem A, such as Theorem 1.2, shall be presented in Section 1.

We now come to a Liouville-type theorem. Results of this type were first obtained in the seminal paper [S].

**Theorem B.** Let $\varphi \in C^1((0, +\infty)) \cap C^0([0, +\infty))$ satisfy (0.1)(i), (ii), (iv) and
\[
(iii') \quad \varphi'(t) \leq A t^\delta \quad \text{on} \quad [0, +\infty) \quad (0.1)
\]
for some $A, \delta > 0$. Let $h \in C^0([0, +\infty))$ be positive on $(0, +\infty)$ and non-decreasing. Assume that there exist $D, \tilde{c}, \beta > 0, \zeta > 0$ such that on $(0, \tilde{c})$
\[
(i) \quad \varphi(t)^{-\beta} h(t) \quad \text{is non-increasing;} \quad (ii) \quad h(t) \geq D t^\zeta. \quad (0.14)
\]
Suppose that the radial Ricci curvature of \((M, \langle \cdot, \cdot \rangle)\) satisfy (0.6) for some \(B > 0\) and some positive \(G \in C^0([0, +\infty))\) such that

\[ \log t G(t) \in L^1(+\infty). \quad (0.15) \]

Let \(b \in C^0(M)\) with \(b(x) \geq 0\) on \(M\) and

\[ \liminf_{r(x) \to +\infty} \frac{b(x)}{r(x)^p} > 0 \quad (0.16) \]

for some \(\rho > -1 + (m - 1)\zeta/\delta\). Fix

\[ 0 < \gamma \leq \min \left\{ \frac{1}{\delta} \frac{\rho + 1}{m - 1} - \frac{\zeta}{\delta} \right\} \quad (0.17) \]

and let \(u \in C^2(M)\) be such that \(u^* = \sup_M u < +\infty\). If \(u\) is a solution of

\[ \text{div}(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u) \geq b(x) \varphi(|\nabla u|)^{1 + \gamma} h(|\nabla u|) \quad \text{on} \quad (0.18) \]

\[ A_\eta = \{ x \in M : u(x) > u^* - \eta \} \quad (0.9) \]

for some \(\eta > 0\), then \(u\) is constant. If \(\rho \leq 0\), then the same conclusion is reached requiring the validity of (0.18) on the smaller set

\[ \hat{A}_\eta = \{ x \in M : u(x) > u^* - \eta \text{ and } |\nabla u|(x) < \eta \}. \quad (0.10) \]

We observe that if \(u\) is not assumed to be bounded above, but (0.18) holds on all of \(M\), then we can still conclude that \(u\) is constant (and therefore automatically bounded above), see the proof of the theorem.

In Section 2, we shall comment on the role of the integrability condition (0.15).

To show the usefulness of differential inequalities of the type of (0.18), in Section 2 we shall prove Corollary B1. Recalling that a complete manifold \((M, \langle \cdot, \cdot \rangle)\) is called \(p\)-parabolic, \(p > 1\), if non-negative \(p\)-superharmonic functions on \(M\) are constant, the next result can be considered as a rather fine version of \(p\)-parabolicity.

**Corollary B1.** Let \((M, \langle \cdot, \cdot \rangle)\) be a complete Riemannian manifold of dimension \(m\) satisfying (0.6), (0.15). Let \(p \geq m, \eta > 0\). Let \(u \in C^2(M)\) be a non-negative solution of

\[ \text{div}(|\nabla u|^{p-2} \nabla u) \leq a \frac{|\nabla u|^p}{u + b}, \quad 0 \leq a < 2(p - 1), \quad b > 0 \quad \text{on} \quad (0.19) \]
\[ \hat{B}_\eta = \{ x \in M : u(x) < \eta \text{ and } |\nabla u(x)| < \eta \}. \]  

Then \( u \) is constant.

Note that, in the above curvature assumptions, \((M, \langle , \rangle)\) is not necessarily \(p\)-parabolic for \( p < m \). The above corollary says that, for \( p \geq m \), \((M, \langle , \rangle)\) is in some “strong sense” \(p\)-parabolic.

Our last result is the following Phragmen–Lindelöf-type theorem.

**Theorem C.** Let \( \varphi \in C^1((0, +\infty)) \cap C^0([0, +\infty)) \) satisfy the structural conditions (0.1)(i), (ii), (iii), (iv). Let \( f \in C^0(\mathbb{R}) \) and let \( H \in C^1((0, +\infty)) \cap C^0([0, +\infty)) \) be the function defined in (0.4) and assume that \( \lim_{t \to +\infty} H(t) = +\infty \), and

(i) \( f(t) > 0 \) on \((0, +\infty)\);  
(ii) \( f \) is non-decreasing on \((0, +\infty)\);  
(iii) \[ \frac{1}{H^{-1}(\int_0^t f(s)\,ds)} \not\in L^1(+\infty). \]  

Given an unbounded domain \( \Omega \) in \( M \) with (possibly empty) boundary \( \partial \Omega \), let \( b \in C^0(\Omega) \) be such that

(i) \( b(x) \geq 0 \) on \( \Omega \);  
(ii) \( b(x) \geq q(r(x)) \) \( \text{for } r(x) \geq 1 \)  

with \( q \) a positive, continuous function defined for large \( t \), and assume that the Ricci curvature of \((M, \langle , \rangle)\) satisfies the inequality

\[ \text{Ricc}_{(M, \langle , \rangle)} \geq - (m - 1)\mathcal{B}^2 G(r) \]  

in the sense of quadratic forms, where the functions \( G \) and \( q \) satisfy (0.6), (0.7) in Theorem A. Moreover, assume that for every fixed \( a \) there exists a constant \( C_a > 0 \) such that

\[ \frac{q(t_1)}{q(t_2)} \geq C_a^{-1} \quad \text{and} \quad \frac{G(t_1)}{G(t_2)} \leq C_a \quad \forall |t_1 - t_2| \leq a. \]

Let \( u \in C^0(\bar{\Omega}) \cap C^1(\Omega) \) be a bounded solution of the differential inequality

\[ \text{div}(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u) \geq b(x) f(u) \quad \text{on } \Omega. \]

If \( u \leq v \), \( v \geq 0 \), on \( \partial \Omega \), then \( u \leq v \) on \( \Omega \).

**Remark.** If \( \Omega \) is a bounded domain the result simplifies as follows (for a proof, see [PSZ, Lemma 3]; cf. also [RS, Proposition 2.5]):
Let \( u \in C^0(\Omega) \cap C^1(\Omega) \) be a (weak) solution of the problem

\[
\begin{cases}
\text{div}(|\nabla u|^{-1}\phi(|\nabla u|)\nabla u) \geq 0 & \text{on } \Omega, \\
u \leq v & \text{on } \partial \Omega.
\end{cases}
\]

Then \( u \leq v \) on \( \Omega \).

1. PROOF OF THEOREM A AND RELATED RESULTS

We keep the notation of the Introduction.

Proof of Theorem A. We reason by contradiction and suppose that \( f(u^*) = 2\sigma_0 > 0 \). The idea of the proof is to show that one can construct a suitable function \( v \) such that (a) \( u - v \) attains a positive maximum \( m \) on a bounded open set \( \Omega \), (b) \( u - v < m \) on \( \partial \Omega \), and (c) the inequality

\[
\text{div}(|\nabla u|^{-1}\phi(|\nabla u|)\nabla u) \geq \text{div}(|\nabla v|^{-1}\phi(|\nabla v|)\nabla v)
\]

holds on \( \Omega \), thereby contradicting the weak comparison principle (see, [RS, Proposition 2.5] or [GT, Theorem 10.7]).

First we observe that, by the strong maximum principle for the \( \varphi \)-Laplacian, \( u^* \) cannot be attained at any point of \( M \). Indeed, assume that \( u(z) = u^* \) for some \( z \in M \), so that \( f(u(z)) > 0 \) and \( |\nabla u|(z) = 0 \). Then, the continuity of \( f, u, \) and \( \nabla u \), the non-negativity of \( b(x) \), and (0.8) and (0.10) imply that there exists a ball \( B_T(z) \subset A_\eta \), with \( T > 0 \) sufficiently small, such that

\[
\text{div}(|\nabla u|^{-1}\phi(|\nabla u|)\nabla u) \geq 0 \quad \text{(weakly) on } B_T(z).
\]

By the maximum principle for the \( \varphi \)-Laplacian, \( u = u^* \) on \( B_T(z) \), showing that the set \( \{ z \in M : u(z) = u^* \} \) is non-empty, closed and open in \( M \), and therefore equal to \( M \). Thus \( u = u^* \) on \( M \), which contradicts (0.8), since \( f(u^*) > 0 \) and \( b(x) \) is non-negative and non-identically null.

We choose \( R_1 > 0 \) large enough that (0.5) (ii) holds on the set \( M \setminus \overline{B_{R_1}(o)} \).

We set

\[
w(r) = \sup_{\partial B_r(o)} u, \quad u^*_r = \sup_{B_r(o)} u.
\]

We observe that, since \( u^* \) is not attained on \( M \), there exists a divergent sequence \( \{r_j\}_{j \in \mathbb{N}} \) such that

\[
w(r_j) \to u^* \quad \text{as } j \to +\infty.
\]
We choose $R_2 > R_1$ in such a way that $u_{R_2}^*$ satisfies
\[ u_{R_2}^* > u^* - \eta \] (1.2)
and is sufficiently close to $u^*$ that, whenever
\[ u_{R_2}^* < u(x) < u^* \] (1.3)
we have
\[ f(u(x)) \geq \sigma_0. \] (1.4)

Next, we observe that, without loss of generality, we can suppose that $G \in C^\infty([0, +\infty))$, $G(t)$ even at the origin and $G(0) = 1$. Thus, using the (odd) solution of
\[
\begin{cases}
h'' - B^2 G(r)h = 0, \\
h(0) = 0, \ h'(0) = 1,
\end{cases}
\] (1.5)
we can construct a smooth model, see [GW,KW], with radial Ricci curvature equal to $(m - 1)B^2 G(r)$. By the Laplacian Comparison Theorem [GW,Y]
\[
\Delta r \leq (m - 1) \frac{h'}{h}(r)
\] (1.6)
within the cut locus of $o$. We define, for $D > 0$,
\[
g(r) = D^{-1} \left\{ e^D \int_0^r \sqrt{G(s)} ds - 1 \right\},
\]
then, $g(0) = 0$, $g'(0) = 1$, and assumption (0.7) (i) guarantees that if $D > 0$ is sufficiently large, then $g$ is a subsolution of (1.5). It follows from the Sturm comparison theorem and (1.6) that
\[
\Delta r \leq (m - 1) \frac{g'}{g}(r)
\] (1.7)
within the cut locus of $o$.

Now, fix $x_{R_2} \in (u_{R_2}^*, u^*)$ (this is possible since $u^*$ is not attained on $M$), and, for every $\sigma \in (0, \sigma_0)$, define a function $x_\sigma(t)$ by the formula
\[
x_\sigma(t) = x_{R_2} + \int_{R_2}^t \varphi^{-1} \left( g(s)^{1-m} \int_{R_2}^s \sigma g(y)^{m-1} q(y) dy \right) ds.
\] (1.8)
We collect in the following lemma the key properties of $x_\sigma$. 
Lemma 1.1. The function $\varphi_\sigma$ is defined on $[R_2, +\infty)$, non-decreasing, and satisfies

$$
\begin{align*}
(g^{m-1} \varphi'_\sigma)' &= g^{m-1} \sigma q, \\
\varphi'_\sigma(R_2) &= 0, \quad \varphi_\sigma(R_2) = \varphi_{R_2},
\end{align*}
$$

and

$$
\lim_{t \to +\infty} \varphi_\sigma(t) = +\infty.
$$

Moreover, the following hold:

(a) given $R_3 > R_2$ and $\tilde{\eta} > 0$, there exists $\sigma_1$ such that, for every $\sigma < \sigma_1$,

$$
\varphi_{R_2} \leq \varphi_\sigma(t) \leq \varphi_{R_3} + \tilde{\eta} \quad \text{on } [R_2, R_3];
$$

(b) if $q$ is bounded above, given $\nu > 0$, let $C(\nu) = \varphi^{-1}(\varphi(\nu)/2)$. Then, there exists $\sigma_2 > 0$ such that, for every $\sigma < \sigma_2$ and $r > R_2$

$$
\text{if } \varphi_\sigma(r) < \varphi_{R_2} + C(\nu), \quad \text{then } 0 < \varphi'_\sigma(r) < \nu.
$$

Postponing the proof of the lemma, we continue with the proof of the theorem. To better illustrate the main idea, we first assume that $o$ is a pole of $M$, so that the distance function $r(x)$ is smooth on $M \setminus \{o\}$.

We define a function $v_\sigma$ on $M \setminus B_{R_3}(o)$ by the formula

$$
v_\sigma(x) = \varphi_\sigma(r(x)).
$$

Then, according to (1.10)

$$
v_\sigma(x) \to +\infty \quad \text{as } r(x) \to +\infty
$$

and a computation that uses $\varphi'_\sigma > 0$, (1.7) and (1.9), together with (0.5) (ii), gives

$$
div(|\nabla v_\sigma|^{-1} \varphi(|\nabla v_\sigma|) \nabla v_\sigma) \leq \sigma q < \sigma_0 q < \sigma_0 b \quad \text{on } M \setminus \overline{B_{R_3}(o)}.
$$

We claim that, if $\sigma$ is sufficiently small, then $u - v_\sigma$ attains a positive maximum $m_\sigma$ on $M \setminus \overline{B_{R_3}(o)}$. Indeed, by (1.1) we may choose $\tilde{j}$ sufficiently large that, having set $R_3 = r_j$, we have

$$
R_3 > R_2 \quad \text{and} \quad w(R_3) > \varphi_{R_2}.
$$

We select $\tilde{\eta} > 0$ small enough that $\varphi_{R_2} + \tilde{\eta} < w(R_3)$. Finally, we choose $\sigma = \sigma(R_3, \tilde{\eta}) \in (0, \sigma_0)$ so small that (1.11) holds on $[R_2, R_3]$. For every such $\sigma$
we have
\[ v_\sigma(x) = \alpha_\sigma(R_2) = \alpha_{R_2} > u_{R_2}^* \geq w(R_2) \geq u(x) \quad \forall x \in \partial B_{R_2}(o) \]
so that \( u - v_\sigma < 0 \) on \( \partial B_{R_2}(o) \). Furthermore, if \( \tilde{x} \in \partial B_{R_3}(o) \) is such that \( w(R_3) = u(\tilde{x}) \), we have
\[ u(\tilde{x}) - v_\sigma(\tilde{x}) = w(R_3) - \alpha_{R_3} \geq w(R_3) - \alpha_{R_2} - \tilde{\eta} > 0. \]
Finally, (1.13) and the fact that \( u \) is bounded above imply that \( u(x) - v_\sigma(x) < 0 \) for \( r(x) \) sufficiently large. Thus, \( u - v_\sigma \) attains its absolute positive maximum \( m_\sigma \) on \( M \setminus B_{R_2}(o) \), and the set \( \Gamma_\sigma \) of points where \( m_\sigma \) is attained is compact and contained in \( M \setminus B_{R_2}(o) \).

We pick a point \( \zeta \in \Gamma_\sigma \) and \( 0 < \mu < m_\sigma \), and let \( \Omega_{\mu,\zeta} \) be the connected component containing \( \zeta \) of the set
\[ \{ x \in M \setminus B_{R_2}(o) : (u - v_\sigma)(x) > \mu \} . \]
Clearly, \( \Omega_{\mu,\zeta} \) is bounded, contains \( \zeta \), and since \( u - v_\sigma < 0 \) on \( \partial B_{R_2}(o) \), \( \tilde{\Omega}_{\mu,\zeta} \) is contained in \( M \setminus B_{R_2}(o) \). Furthermore, \( u = v_\sigma + \mu \) on \( \partial \Omega_{\mu,\zeta} \) and
\[ u(x) > v_\sigma(x) + \mu \geq \alpha_{R_2} > u_{R_2}^* > u^* - \eta \quad \text{on} \quad \Omega_{\mu,\zeta} \quad (1.15) \]
so that \( \Omega_{\mu,\zeta} \subseteq A_\eta \).

According to (0.8), (1.14), (1.4), and (1.15) on \( \Omega_{\mu,\zeta} \) we have
\[ \text{div}(\nabla u^{-1}\phi(\nabla u)\nabla u)(x) \geq b(x)f(u(x)) \]
\[ > \sigma_0 b(x) > \text{div}(\nabla v^{-1}\phi(\nabla v)\nabla v)(x). \]
Applying Theorem 10.7 of [GT] we deduce that
\[ u(x) \leq v_\sigma(x) + \mu \quad \text{on} \quad \Omega_{\mu,\zeta} \]
contradicting \( \zeta \in \Omega_{\mu,\zeta} \).

Next, we assume that \( q \) is bounded above.

By what seen above, for every \( \sigma \) sufficiently small, the function \( u - v_\sigma \) attains its positive absolute maximum \( m_\sigma \) on a compact set \( \Gamma_\sigma \) contained in \( M \setminus B_{R_2}(o) \). We may conclude as above provided we show that if \( \sigma \) is sufficiently small, and \( \mu \) is sufficiently close to \( m_\sigma \), then the set \( \Omega_{\mu,\sigma} \) is contained in the smaller set \( \hat{A}_\eta \).

To this end, we first claim that, for every \( c > 0 \), there exists \( \sigma_1 > 0 \) such that if \( \sigma < \sigma_1 \), then
\[ v_\sigma(\zeta) < c + \alpha_{R_2} \quad \forall \zeta \in \Gamma_\sigma. \quad (1.16) \]
Suppose by contradiction that this is not true. Then, there exists \( \tilde{c} > 0 \) a sequence \( \sigma_n \downarrow 0 \), and for every \( n \) a point \( \zeta_n \in \Gamma_{\sigma_n} \) such that

\[
v_{\sigma_n}(\zeta_n) \geq \tilde{c} + \alpha_{R_2} \quad \forall n.
\]

Thus, for every \( n \)

\[
(u - v_{\sigma_n})(\zeta_n) < u^* - \alpha_{R_2} - \tilde{c}.
\]  
(1.17)

We choose \( j \) large enough that \( w(r_j) > u^* - \tilde{c}/3 \). Since \( \sigma_n \) tends to zero, by (a) in Lemma 1.1, we may choose \( n \) large enough that

\[
\alpha_{\sigma_n}(r_j) \leq \alpha_{R_2} + \tilde{c}/3.
\]  
(1.18)

Thus, if \( \hat{x} \) is such that \( r(\hat{x}) = r_j \) and \( u(\hat{x}) = w(r_j) \), then

\[
u(\hat{x}) - v_{\sigma_n}(\hat{x}) \geq u^* - \alpha_{R_2} - \frac{2}{3} \tilde{c}
\]

and therefore

\[
(u - v_{\sigma_n})(\zeta_n) \geq u^* - \alpha_{R_2} - \frac{2}{3} \tilde{c} > u^* - \alpha_{R_2} - \tilde{c},
\]  
(1.19)

which contradicts (1.17). Thus (1.16) holds.

We now claim that for every \( \nu > 0 \), there exists \( \sigma_2 > 0 \) such that, if \( 0 < \sigma < \sigma_2 \)

\[
|\nabla v_{\sigma}(\zeta)| < \nu \quad \text{for each} \quad \zeta \in \Gamma_{\sigma}.
\]  
(1.20)

Indeed, by the previous claim there exists \( \sigma_1 \) such that, for every \( \sigma < \sigma_1 \), (1.16) holds with the constant \( c = C(\nu) \) defined in Lemma 1.1. Set \( r_\zeta = r(\zeta) \) for every \( \zeta \in \Gamma_{\sigma} \). It follows from (b) in the lemma that there exist \( 0 < \sigma_2 < \sigma_1 \) such that for every \( 0 < \sigma < \sigma_2 \) and \( \zeta \in \Gamma_{\sigma} \), \( \alpha_{\sigma}(r_\zeta) < \nu \), and (1.20) follows.

We may therefore choose \( \sigma \) sufficiently small that

\[
u > u^* - \eta/2 \quad \text{and} \quad |\nabla v_{\sigma}| < \eta/2 \quad \text{on} \Gamma_{\sigma},
\]

and since \( |\nabla u(\zeta)| = |\nabla v_{\sigma}(\zeta)| \) on \( \Gamma_{\sigma} \), we conclude that there exists a sufficiently small neighborhood of \( \Gamma_{\sigma} \) contained in \( \hat{A}_{\eta} \). Finally, we may choose \( 0 < \mu < m_{\sigma} \) sufficiently close to \( m_{\sigma} \) such that the closure of the set

\[
\{ x \in M \setminus \overline{B_{R_2}(o)} : u(x) > v_{\sigma}(x) + \mu \}
\]

is contained in \( \hat{A}_{\eta} \). In particular, if \( \zeta \) is in \( \Gamma_{\sigma} \), the set \( \Omega_{\mu,\zeta} \) is contained in \( \hat{A}_{\eta} \), as required.

We now drop the assumption that \( o \) is a pole of \( M \), and describe how to adapt the argument to deal with the lack of smoothness of the distance function \( r(x) \) on the cut locus \( \text{cut}(o) \).
We only consider the case where \( q \) is bounded. The other case is easier. The argument given above may be carried out without changes to deduce that there exists a sufficiently small \( s \) such that \( u/C_0 v_s \) attains its maximum on a compact set \( \Gamma_s \). Moreover, \( u(\xi) > u^- - \eta/2 \) and \( 0 < x'_\sigma(r_\xi) < \eta/2 \) for every \( \zeta \) in \( \Gamma_s \).

The problem comes from the fact that we can no longer deduce that \( v_s \) is smooth and, in particular that \( |\nabla v_s| < \eta/2 \) on \( \Gamma_s \). To circumvent the problem, we adapt an argument of Calabi as follows. Let \( z \) be a point on \( \Gamma_s \) at maximum distance from \( o \), and choose a minimizing geodesic \( g \), parametrized by arc length, joining \( o \) to \( z \). For \( \varepsilon \) suitably small, denote by \( o_\varepsilon \) the point \( g(\varepsilon) \) and by \( r_\varepsilon \) the distance function from the point \( o_\varepsilon \). Clearly, the entire geodesic \( g \) from \( o_\varepsilon \) to \( z \) is contained in the complement \( D_\varepsilon \) of the cut locus \( \text{cut}(o_\varepsilon) \) of \( o_\varepsilon \). We note that, since the cut locus is closed, \( D_\varepsilon \) is an open set in \( M \). The idea is to replace in the whole reasoning the function \( v_s \) with the function \( v_\varepsilon \) defined by

\[
v_\varepsilon(x) = \varepsilon_\sigma(\varepsilon + r_\varepsilon(x)).
\]

We begin by observing that \( v_\varepsilon = v_s \) on \( \gamma([\varepsilon, +\infty)) \cap D_\varepsilon \) (\( D_\varepsilon \) is the complement of the cut locus of \( o \)), while, by the triangle inequality, and the strict monotonicity of \( \varepsilon_\sigma \), \( v_\varepsilon > v_s \) elsewhere. Thus, the maximum of the function \( u - v_\varepsilon \) is \( m_s \) and is attained on the set \( \Gamma = \gamma([\varepsilon, r_\varepsilon]) \cap \Gamma_s \). By what said above \( \Gamma \) is contained in the open set \( D_\varepsilon \), and it follows that \( v_\varepsilon \) is smooth in a neighborhood of \( \Gamma \), and \( |\nabla u| = |\nabla v_s| < \eta/2 \) on \( \Gamma \). In particular, \( \Gamma \) is contained in \( \hat{A}_\eta \).

Now, recalling that \( \varepsilon_\sigma \) satisfies (1.9), it is easy to check that \( v_\varepsilon \) satisfies

\[
div(\nabla v_s)^{-1} \varphi(\nabla v_s) \nabla v_\varepsilon(x)
\]

\[
= [\varphi(\varepsilon_\sigma')]'(\varepsilon + r_\varepsilon(x)) + \varphi(\varepsilon_\sigma')(\varepsilon + r_\varepsilon(x)) \Delta r_\varepsilon(x)
\]

\[
\leq \sigma q(\varepsilon + r_\varepsilon(x)) + \varphi(\varepsilon_\sigma')(\varepsilon + r_\varepsilon(x)) \left[ \Delta r_\varepsilon(x) - (m - 1) \frac{g'}{g} (\varepsilon + r_\varepsilon(x)) \right].
\]

Since \( r = \varepsilon + r_\varepsilon \) on the part of \( \gamma \) between \( o_\varepsilon \) and \( \zeta \), (0.6) shows that there \( \text{Ricc}(M,\zeta,\gamma)(\nabla r_\varepsilon, \nabla r_\varepsilon) > -(m - 1) B^2 G(\varepsilon + r_\varepsilon(x)) \). By continuity, the inequality holds replacing \( G(\varepsilon + r_\varepsilon) \) with \( G(r_\varepsilon) \) on the right-hand side, provided \( \varepsilon \) is sufficiently small. Then, the proof of the Laplacian Comparison Theorem (see [GW, pp. 22–28]) shows that

\[
\Delta r_\varepsilon(x) \leq (m - 1) \frac{g'}{g} (r_\varepsilon(x))
\]
at every point of $\gamma$ between $o_e$ and $\zeta$. Since $q$ is continuous and strictly positive, and $\sigma < \sigma_0$, we may choose $\varepsilon$ small enough that

$$\text{div}(|\nabla v_e|^{-1} \varphi(|\nabla v_e|) \nabla v_e) (x) < \sigma_0 q(\varepsilon + r_e(x)) \leq \sigma_0 b(x)$$

on $\Gamma$. Thus, there is a sufficiently small neighborhood of $\Gamma$ contained in $D_\varepsilon$ where

$$\text{div}(|\nabla v_e|^{-1} \varphi(|\nabla v_e|) \nabla v_e) < \text{div}(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u).$$

The rest of the proof proceeds as before.

It remains to prove Lemma 1.1.

*Proof of Lemma 1.1.* Since $\varphi(t) \to +\infty$ as $t \to +\infty$, it is clear from the definition (1.8) that $z_\sigma$ is defined on $[R_2, +\infty)$, is non-decreasing, satisfies (1.9), and tends to zero uniformly on compact intervals as $\sigma \to 0$.

In order to prove that $z_\sigma(t) \to +\infty$ as $t \to +\infty$, it suffices to show that

$$\varphi^{-1}\left(g(t)^{1-m} \int_{R_2}^t \sigma g(y)^{m-1} q(y) \, dy\right) \geq \frac{C}{z(t)}, \quad t \geq 1 \tag{1.21}$$

for some constant $C > 0$. This follows if there exists $C > 0$ such that

$$\tilde{h}(t) = \frac{\int_{R_2}^t g(y)^{m-1} q(y) \, dy}{g(t)^{m-1} \varphi\left(\frac{z}{z(t)}\right)} \geq \frac{1}{\sigma}, \quad t \geq 1. \tag{1.22}$$

We shall show that inequality (1.22) holds provided

$$0 < C < \left[\frac{\sigma}{DA(m-1)}\right]^{1/\delta}. \tag{1.23}$$

Without loss of generality, we may assume that $z(t) \to +\infty$ as $t \to +\infty$; thus (0.1) (iii) gives

$$\tilde{h}(t) \geq \frac{A(t)}{B(t)}$$

with

$$A(t) = z(t)^{\delta} \int_{R_2}^t g(y)^{m-1} q(y) \, dy, \quad B(t) = AC^\delta g(t)^{m-1}.$$
Now, since \( A(t) \to +\infty \) as \( t \to +\infty \), (1.22) is trivially satisfied if \( B(t) \) is bounded. Otherwise,

\[
\liminf_{t \to +\infty} \frac{A(t)}{B(t)} \geq \liminf_{t \to +\infty} \frac{A'(t)}{B'(t)}.
\]

A computation that uses \( z' \geq 0 \), \( q > 0 \) and (0.7)(ii), shows that

\[
\frac{A'(t)}{B'(t)} \geq \frac{1}{AC^3(m-1)} \frac{\sqrt{G(t)}}{g(t)}, \quad t \geq 1,
\]

and since

\[
\frac{g'}{g}(t) \geq D \sqrt{G(t)},
\]

the condition imposed on \( C \) implies that

\[
\liminf_{t \to +\infty} \frac{A'(t)}{B'(t)} > \frac{1}{\sigma} \tag{1.24}
\]

so that (1.22) holds even in this case. This proves (1.10).

Finally, we prove the implication (b) in the statement. Since \( a' \to 0 \) uniformly on compact sets, it is enough to consider the case where \( r > R_2 + 1 \). Assume by contradiction that (b) does not hold. Thus, there exist \( \tilde{v} > 0 \), a sequence \( \sigma_n \downarrow 0 \), and, for every \( n \), a point \( r_n \geq R_2 + 1 \)

\[
\varphi_{\sigma_n}(r_n) < \varphi_{R_2} + C(\tilde{v}) \quad \text{and} \quad \varphi_{\sigma_n}'(r_n) \geq \tilde{v}.
\]

It follows from (1.9) that, for every \( n \in \mathbb{N} \)

\[
[\varphi(\varphi_{\sigma_n}')]' + (m-1) \frac{g'}{g} \varphi(\varphi_{\sigma_n}') = \sigma_n q \quad \text{on } [R_2, +\infty).
\]

Since \( g' \geq 0 \), \( \varphi > 0 \), and \( q \leq \sup q = q^* \), we have

\[
[\varphi(\varphi_{\sigma_n}')]' \leq \sigma_n q^*,
\]

whence, integrating between \( t \) and \( r_n \), \( R_2 \leq t \leq r_n \), and recalling that \( \varphi(\varphi_{\sigma_n}(r_n)) \geq \varphi(\tilde{v}) \), we obtain

\[
[\varphi(\varphi_{\sigma_n}'(t))]' \geq \sigma_n q^*(t - r_n) + \varphi(\varphi_{\sigma_n}'(r_n))
\geq \sigma_n q^*(t - r_n) + \varphi(\tilde{v}) \quad \text{on } [R_2, +\infty).
\]
We deduce that, for $n$ sufficiently large, and every $t$ in $[r_n - 1, r_n]$

$$[\varphi(x_\sigma(t))]' \geq -\sigma_n q^* + \varphi(\tilde{v}) \geq \varphi(\tilde{v})/2$$

and therefore

$$x_\sigma'(t) \geq \varphi^{-1}(\varphi(\tilde{v})/2) = C(\tilde{v}).$$

Integrating over $[r_n, r_n]$, and using the monotonicity of $x_\sigma$, we conclude that

$$x_\sigma(r_n) \geq x_\sigma(r_n - 1) + C(\tilde{v})$$

$$\geq x_\sigma(R_2) + C(\tilde{v}) = x_{R_2} + C(\tilde{v})$$

for every sufficiently large $n$. This yields a contradiction, and (1.12) follows.

In order to prove Corollary A1 we shall make use of the following version of Theorem A, which is of independent interest.

**Theorem 1.2.** Let $\varphi \in C^1((0, +\infty)) \cap C^0([0, +\infty))$ satisfy (0.1)(i), (ii), (iii). Let $b(x)$ satisfy (0.5)(i), (ii) with

$$\sup_{[R_0, +\infty)} q < +\infty.$$ 

Assume that the radial Ricci curvature of $(M, \langle , \rangle)$ satisfies (0.6) with a function $G$ verifying assumptions (0.7)(i), (ii) ($z$ being as in Theorem A). Let $f \in C^0(\mathbb{R})$. Let $u \in C^1(M)$ be such that $u^* = \sup_M u < +\infty$ and let $\eta > 0$. If $u$ is a solution of (0.8) on $\hat{A}_\eta$ as in (0.10) then $f(u^*) \leq 0$.

**Proof.** If $\varphi(t) \to +\infty$ as $t \to +\infty$, Theorem 1.2 is contained in Theorem A. It remains to analyze the case $\varphi(t) \to \lambda > 0$ as $t \to +\infty$. We note that, assuming without loss of generality $q \leq 1$,

$$g(s)^{1-m} \int_{R_2}^s g(y)^{m-1} q(y) \, dy \leq g(s)^{1-m} \int_{R_2}^s g(y)^{m-1} \, dy = h(s),$$

$g$ being defined as in the proof of Theorem A. Furthermore, since $G$ is non-decreasing, $g(r) \to +\infty$ as $r \to +\infty$ and condition (0.7)(i) is satisfied. A simple checking shows that $\lim_{s \to +\infty} h(s) < +\infty$. It follows that there exists $\sigma_1 > 0$ such that, for each $0 < \sigma < \sigma_1$

$$\max_{s \in [R_2, +\infty)} \left( g(s)^{1-m} \int_{R_2}^s \sigma g(y)^{m-1} q(y) \, dy \right) \in [0, \lambda)$$

the domain of $\varphi^{-1}$. 


Hence, (1.8) yields a genuine solution to (1.9) as before. The rest of the proof is the same as that of Theorem A in the case where \( q \) is bounded above. 

Corollary A1 follows immediately from the next

Corollary 1.3. Let \( \varphi \in C^1((0, +\infty)) \cap C^0([0, +\infty)) \) satisfy (0.1)(i), (ii), (iii), and let (0.6) and (0.7) (i), (ii) hold. Let \( u \in C^2(M) \) be such that \( u^* = \sup_M u < +\infty \), and assume that the vector field \( |\nabla u|^{-1}\varphi(|\nabla u|)\nabla u \) is of class at least \( C^1 \). For each \( k \in \mathbb{N} \), define the sets

\[
A_k = \{ x \in M : u(x) > u^* - 1/k, \ |\nabla u|(x) < 1/k \}
\]

and

\[
B_k = \{ x \in M : \text{div}(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u) < 1/k \}.
\]

Then, \( A_k \cap B_k \neq \emptyset \), \( k \in \mathbb{N} \). If \( u^* = +\infty \), but (0.12) holds, then we conclude that \( B_k \neq \emptyset \) for each \( k \in \mathbb{N} \).

Proof. First, we suppose that \( u \) is bounded above. If \( u^* \) is attained at some point \( \tilde{x} \in M \), then, at \( \tilde{x} \), \( u(\tilde{x}) = u^* \), \( |\nabla u(\tilde{x})| = 0 \) and \( \text{div}(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u)(\tilde{x}) \leq 0 \) by the maximum principle for the \( \varphi \)-Laplacian. We are therefore left with the case where \( u^* \) is not attained on \( M \). It is easy to see that \( A_k \) is open and non-empty. This is a consequence of [HU], but can also be seen arguing as in the proof of Theorem A. Namely, given \( k \in \mathbb{N} \), construct a function \( v_\sigma \) as in Theorems A and 1.2 with \( q \equiv 1 \). If \( \sigma \) is sufficiently small, then the points \( \zeta \in M \backslash \overline{B_{R_1}(0)} \) where \( u - v_\sigma \) attains its positive absolute maximum \( m_\sigma \) are in \( A_k \).

By contradiction, suppose now that \( A_k \cap B_k = \emptyset \). Then, on \( A_k \)

\[
\text{div}(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u) \geq 1/k.
\]

Applying Theorem 1.2 with \( f(t) = 1/k, \ q \equiv 1, \) and \( b \equiv 1 \), we obtain a contradiction.

We now suppose that \( u \) is not bounded above, and that (0.12) holds. Again, we argue by contradiction, and assume that \( B_k = \emptyset \) for some fixed \( k \). Thus, (1.25) holds on \( M \). We construct the function \( v_\sigma \) as above, with \( q \equiv 1 \), and \( 0 < \sigma < 1/k \) and \( \varepsilon_{R_1} \) chosen in such a way that the set \( \Gamma_\sigma \subset M \backslash \overline{B_{R_1}(0)} \) of points where \( u - v_\sigma \) attains its absolute positive maximum \( m_\sigma \) is non-empty. Note that this is possible, since, according to the estimate (1.21) in Theorem A

\[
\varepsilon_{\sigma}(t) \geq \varepsilon_{R_2} + C \int_{R_2}^{t} \frac{ds}{z(s)},
\]
$u^* = +\infty$ and $u$ satisfies (0.12). Reasoning as in the proof of Theorem A, we may assume that $\Gamma_\sigma$ is contained in the complement $D_o$ of the cut locus of $o$. Since $v_\sigma(x)$ satisfies

$$\text{div}(|\nabla v_\sigma|^{-1} \varphi(|\nabla v_\sigma|) \nabla v_\sigma) \leq \sigma$$

in $D \setminus \overline{B_R(o)}$, there we have

$$\text{div}(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u) \geq 1/k > \sigma \geq \text{div}(|\nabla v_\sigma|^{-1} \varphi(|\nabla v_\sigma|) \nabla v_\sigma).$$

Now fix $\zeta \in \Gamma_\sigma$ and choose $0 < \mu < m_\sigma$, so that, if $\Omega_{\mu,\zeta}$ is the connected component containing $\zeta$ of the set $\{x \in M: u(x) > v(x) + \mu\}$, then $\Omega_{\mu,\zeta}$ lies in a neighborhood of $\Gamma_\sigma$ whose closure is contained in $D \setminus \overline{B_R(o)}$. Since on $u(x) = v(x) + \mu$ on $\partial \Omega_{\mu,\zeta}$, applying the weak comparison principle we deduce that $u(x) \leq v(x) + \mu$ on $\Omega_{\mu,\zeta}$, contradicting the fact that $\zeta \in \Omega_{\mu,\zeta}$.

The argument in the last part of the proof of Corollary 1.3 can be used to establish the validity of the following.

**Proposition 1.4.** Let $\varphi \in C^1((0, +\infty)) \cap C^0((0, +\infty))$ satisfy (0.1)(i), (ii), (iii), (iv) and let $b \in C^0(M)$ satisfy (0.5)(i), (ii). Assume that (0.6) and (0.7)(i), (ii) hold and suppose that

$$u(x) = o\left(\int_1^{r(x)} \frac{ds}{z(s)}\right) \quad \text{as } r(x) \to +\infty,$$

and that $u$ is a solution on $M \setminus B_R(o)$ of

$$\text{div}(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u) \geq b(x),$$

for some $R > 0$. Then $u$ is bounded above and attains its maximum over $M \setminus B_R(o)$ on $\partial B_R(o)$. If $q$ is bounded above and we assume $G' \geq 0$ instead of (0.7)(i), then the same conclusion is reached without having to assume (0.1)(iv).

**Proof.** To prove that $u$ is bounded above, one argues as in the proof of the previous corollary. Thus $u^* = \sup_{M \setminus B_R(o)} < +\infty$. If $u^*$ were not attained, one could argue as in the proof of Theorem A, with $f \equiv 1$, and reach a contradiction. By the maximum principle for the $\varphi$-Laplacian, $u^*$ is attained on $\partial B_R(o)$.

**Remark.** When $f(t) > 0$ on $\mathbb{R}$ and $\liminf_{t \to +\infty} f(t) > 0$, using Proposition 1.4 and Theorems A and 1.2, we immediately obtain a non-existence result for solutions of (0.8) on $M$ which satisfy (0.12). We leave the precise statement to the interested reader.
To further illustrate the scope of Theorem A we describe another application. We consider the stationary Schrödinger equation
\[\Delta u - b(x)u = 0,\]  \hspace{1cm} (1.26)
where $b$ is a continuous function on $M$, $b(x) \geq 0$, $b(x) \neq 0$. A function $u$ is said to be $b$-harmonic if it satisfies (1.26). Following Grigor’yan [G, Sect. 13.2], we say that $M$ has the $b$-Liouville property if the only bounded $b$-harmonic function on $M$ is $u \equiv 0$. We note that

(i) if $b \in C^\infty_o(M)$, then the $b$-Liouville property is equivalent to the parabolicity of $M$;

(ii) if $b \equiv c > 0$ is a positive constant, then the $b$-Liouville property is equivalent to the stochastic completeness of $M$.

**Proposition 1.5.** Assume that $b$ satisfies condition (0.5), and that (0.6) and (0.7) in the statement of Theorem A hold with $\delta = 1$. Then $M$ has the $b$-Liouville property.

**Remark.** Since $b$ is strictly positive outside a compact set (see assumption (0.5)), Proposition 1.5 cannot be used to establish the parabolicity of $M$. In this connection we note that Theorem A is unlikely to hold if we only assume that $q \geq 0$, and correspondingly replace (0.7)(ii) with $\sqrt{G} \leq z(t)^\delta q(t)$. Indeed, consider $M = \mathbb{R}^m$, $m \geq 3$, and choose any smooth, compactly supported function $b \geq 0$, $b \neq 0$ on $M$. Since $M$ is not parabolic, it follows from [G, Theorem 5.1] that (1.26) has a bounded positive solution $u$. In this case, $f(t) = t$, so that $f(u^*) > 0$, and the conclusion of Theorem A fails.

**Proof of Proposition 1.5.** Let $u$ be a bounded solution of (1.26), and set $u^* = \sup_M u$, $u_* = \inf_M u$. According to Theorem A with $f(t) = t$, $u^* \leq 0$. On the other hand,
\[\Delta(-u) = b(x)(-u)\] on $M$,
so that, we also have $u_* \geq 0$ and $u \equiv 0$.

It seems worth to state the following “dual” version of Theorem A. The argument follows the lines of that of Theorem A.

**Theorem 1.6.** In the assumptions of Theorem A, let $v \in C^1(M)$ be such that $v_* = \inf_M v > -\infty$ and let $\eta > 0$. If $v$ is a solution of
\[\text{div}(|\nabla v|^{-1} \varphi(|\nabla v|)\nabla v) \leq -b(x)f(v)\]  \hspace{1cm} (1.27)
on the set

\[ A_\eta = \{ x \in M : v(x) < v_* + \eta \}, \]

then \( f(v_*) \leq 0 \). If we assume that \( q \) is bounded above, then the same conclusion holds requiring that (1.27) is valid on the smaller set

\[ \hat{A}_\eta = \{ x \in M : v(x) < v_* + \eta \text{ and } |\nabla v(x)| < \eta \}. \]

**Remark.** Theorem 1.2 also admits a “dual” version, and Proposition 1.5 can be extended to the \( \varphi \)-Laplacian. It follows, for instance, that if \( M \) satisfies the assumptions of Theorem A with \( A = \delta = 1 \), and \( q \) is bounded above, then the only bounded solutions of the capillarity equation

\[ \text{div}(\nabla u/\sqrt{1 + |\nabla u|^2}) = b(x)u \]

is \( u \equiv 0 \).

We observe that Proposition 1.5 compares with Theorem 13.9 of [G]. However, it is not implied by the latter, as shown by the following example:

Let \( g \in C^\infty([0, +\infty)) \) be positive on \((0, +\infty)\) and such that

\[ g(r) = \begin{cases} r & \text{on } [0, 1], \\ \exp(\int_1^r \sqrt{\tilde{G}(s)} \, ds) & \text{on } [2, +\infty), \end{cases} \tag{1.28} \]

where \( \tilde{G} \) is a function defined on \((1, +\infty)\) and satisfying

\[ \tilde{G}(t) = t^{2(\gamma+1)}(\log t)^\beta \quad \text{with } \gamma > -2, \quad 0 < \beta \leq 2. \tag{1.29} \]

Then \( g \) defines a model metric \( \langle \cdot , \cdot \rangle = dt^2 + g(r)^2 \, d\theta^2 \) \((d\theta^2 \text{ being the standard metric on } S^{m-1})\) on \((0, +\infty) \times S^{m-1}\), which extends smoothly to all of \( \mathbb{R}^m \). Further,

\[ \text{Ricc}_{(\mathbb{R}^m, \langle \cdot , \cdot \rangle)}(\nabla r, \nabla r) = -(m - 1) \begin{cases} -1 & \text{on } B_1(o), \\ \tilde{G} + \frac{1}{2} \tilde{G}^{-1/2} \tilde{G}' & \text{on } \mathbb{R}^m \setminus B_2(o). \end{cases} \]

It follows that there exists a positive smooth function \( G \) on \([0, +\infty)\) with \( G(r) = \tilde{G}(r) \) for \( r \geq 2 \) and \( B > 0 \) such that

\[ \text{Ricc}_{(\mathbb{R}^m, \langle \cdot , \cdot \rangle)}(\nabla r, \nabla r) \geq -(m - 1)B^2 G(r) \quad \text{on } M \setminus B_2(o). \]

By our choice of \( \gamma \) \( G \) satisfies (0.7)(i). Next, we choose a positive function \( q \in C^\omega([0, +\infty)) \), with the property that \( q(t) = t^\beta \) for \( t \geq 2 \). It follows from
Finally, we choose \( z(t) \) non-decreasing and such that \( z(t) = (\sqrt{G}/q)(t) \) for \( t \geq 1 \). All the assumptions of Proposition 1.5 are satisfied and we conclude that the only bounded solution of \( \Delta u - q(r)u = 0 \) on \((M, \langle , \rangle)\) is \( u \equiv 0 \).

However, Theorem 13.9 of [G] does not apply. Indeed, using the definition of the quantities involved, it is easy to check that, for every \( C > 0 \),

\[
\frac{\text{vol } B_r}{r^2 \exp \{C(\int_0^t \sqrt{g(s)} \, ds)^2\}} \to +\infty \quad \text{as } t \to +\infty
\]

so that assumption (13.9) of [G] is not satisfied.

We now show that the condition that \( 1/z(t) \notin L^1(\mathbb{R}_+) \) in Theorem A cannot be dropped. To this end, we proceed as in the previous example, and consider the model metric \( \langle , \rangle \) on \( \mathbb{R}^m \) obtained starting with a function \( g \) defined by (1.28), and where

\[
\tilde{G}(t) = t^\alpha (\log t)^\beta \quad \text{on } [1, +\infty), \quad \beta \geq 0, \quad \alpha > -2.
\]

Thus, (0.6) holds with a function \( G \) which is equal to \( \tilde{G} \) if \( t \geq 2 \) and satisfies (0.7)(i). Moreover, \( g(r) \) diverges as \( r \to +\infty \).

We also choose a positive function \( q \in C^0([0, +\infty)) \) such that, on \([0, 1]\)

\[
q(t) = \begin{cases} 
1 & \text{if } 1 < p \leq 2, \\
\frac{p}{p-2} & \text{if } p > 2
\end{cases}
\]

and

\[
q(t) = t^\gamma
\]

on \([2, +\infty)\), with \( \gamma \in \mathbb{R} \). We consider the case of the \( p \)-Laplacian, for any fixed \( p > 1 \), so that \( q(t) = t^{p-1} \), and \( \delta = p - 1 \).

We define the function

\[
u(x) = \int_0^{r(x)} g(t)^{-\delta/(p-1)} \left\{ \int_0^t q(s) g(s)^{\delta-1} \, ds \right\}^{1/(p-1)} \, dt.
\]
Then $u$ is a $C^2$, radial, non-negative solution of
\[
div(|\nabla u|^{p-2}\nabla u)(x) = q(r(x)) \quad \text{on } (\mathbb{R}^m, \langle \cdot, \cdot \rangle).
\]

Further, applying de L'Hospital's rule, it is easily verified that $u$ is bounded if and only if $(q/\sqrt{G})^{1/(p-1)} \in L^1(+\infty)$.

To conclude, we choose
\[
z = 2(p - 1) + 2\gamma, \quad \beta > 2(p - 1), \quad \gamma \geq -p.
\]

Then, $z > -2$, $(\sqrt{G}/q)(t) = t^{p-1}(\log t)^{\beta/2}$ for $t \geq 2$, and assumption (0.7)(ii) holds with $z(t) = (\sqrt{G}/q)(t)^{1/(p-1)} = t(\log t)^{\beta/2(p-1)}$. The condition on $\beta$ implies that $u$ is bounded, and that $1/z$ is integrable at infinity. This yields the required counterexample to Theorem A, with $f(t) \equiv 1$ and $1/z \in L^1(+\infty)$.

We note that this example, with $\gamma = 0$ and $q \equiv 1$, also shows that the assumption $1/z \notin L^1(+\infty)$ is essential for the conclusion of Corollary A1 to hold. Indeed, the function $u$ defined above is bounded, radial, strictly increasing, so that $u^* = \sup u = \lim_{r(x) \to +\infty} u(x)$, and satisfies
\[
div(|\nabla u|^{p-2}\nabla u)(x) = 1
\]
on $\mathbb{R}^m$.

Corollary A1 compares with some recent work of the second and third named authors, [RS, Theorem D], where the conclusion (0.11)(i), (iii) was established assuming that $\varphi$ satisfies the same structural condition as in the statement of Corollary A1, but replacing the curvature assumption (0.6), (0.7)(i)' and (ii)' with the volume growth condition
\[
\liminf_{r \to +\infty} \frac{\log \text{vol } B_r}{r^{1+\delta}} < +\infty.
\] (1.30)

This, however, is not implied by the curvature condition assumed in the statement of Corollary A1. We consider the metric on $\mathbb{R}^m$ constructed in the last example with $z = 2(p - 1)$, $\gamma = 0$ and $0 < \beta \leq 2(p - 1)$. Then (0.6) holds with $G(t) = t^{2(p-1)}(\log t)^\beta$ for $t \geq 2$. Thus, (0.7) (i)' and (ii)'' hold with $\delta = p - 1$ and $z(t) = t(\log t)^{\beta/2(p-1)}$. By our choice of $\beta$, $1/z$ is not integrable at infinity, so Corollary A1 applies. On the other hand, an easy estimate shows that
\[
\log \text{vol } B_r \geq C r^p (\log r)^{\beta/2}
\]
for some constant $C > 0$, so that the volume growth condition (1.30) with $\delta = p - 1$ fails.
2. PROOF OF THEOREM B

To prove Theorem B we shall make use of the following.

**Lemma 2.1** Let \( g \in C^1((0, +\infty)) \) be positive and such that
\[
g(t) \to +\infty \quad \text{as} \quad t \to +\infty. \tag{2.1}\]
Let \( \varphi \in C^0([0, +\infty)) \cap C^1((0, +\infty)) \) satisfy \((0.1)(i), (ii), (iv)\). Let also \( h : [0, +\infty) \to [0, +\infty) \) be continuous, non-decreasing and positive in \((R, +\infty)\) for some \( R > 0 \), and assume that \((0.14)(i), (ii)\) hold for some constants \( D, \tilde{\varepsilon}, \beta > 0 \), and \( \zeta \geq 0 \). Fix \( E, \gamma > 0, \rho \in \mathbb{R} \) and \( m \geq 2 \). Then, there exists \( \tilde{r} > R \) such that, for every \( a > r_0 \geq \tilde{r} \), the differential inequality
\[
g(t)^{1-m} \varphi(|x'|) < E\rho^\gamma \varphi(|x'|)^{1+\gamma} h(|x'|) \tag{2.2}\]
has a \( C^2 \) solution \( x_a \) defined on \([r_0, a)\) satisfying \( x_a(r_0) = 0 \) and \( x'_a > 0 \) on \([r_0, a)\).

If we assume that \((0.1)(iii)'\) holds and that \( \gamma \leq 1/\delta \), then
\[
x_a(t) \to +\infty \quad \text{as} \quad t \to a -. \tag{2.3}\]

If we assume that \((0.1)(iii)'\) holds and
\[
\frac{t^\rho}{g(t)^{(m-1)\gamma+\zeta/\delta}} \notin L^1(+\infty), \tag{2.4}\]
then, for every \( r_1 > r_0 \) and \( \eta > 0 \), there exists \( a_0 > r_0 \) such that, for every \( a \geq a_0 \)
\[
x_a(t) < \eta \quad \text{on} \quad [r_0, r_1]. \tag{2.5}\]

**Remark.** As the proof below shows, if \( \zeta = 0 \), \((0.1)(iii)'\) is not necessary to obtain \((2.5)\).

**Proof.** By absorbing the constant \( E \) into \( h \), we may assume that \( E = 1 \).

Since \( \varphi \) is equal to 0 at \( t = 0 \), is strictly increasing, and tends to infinity as \( t \) tends to infinity, the inverse function \( \varphi^{-1} : [0, +\infty) \to [0, +\infty) \) is defined, strictly increasing, \( \varphi^{-1}(0) = 0 \), and \( \varphi^{-1}(t) \to +\infty \) as \( t \to +\infty \). We fix \( \tilde{\varepsilon} > 0 \) such that \( \varphi^{-1}(\tilde{\varepsilon}) \leq \tilde{\varepsilon} \). By \((2.1)\), there exists \( \tilde{r} > R \) such that
\[
g(t)^{1-m} < \tilde{\varepsilon} \quad \text{on} \quad [\tilde{r}, +\infty). \tag{2.6}\]
We fix $a > r_0 \geq \tilde{r}$, $\lambda > 0$, and define a function $z = z_a$ on $[r_0, a)$ by means of the formula

$$z_a(r) = z(r) = \int_{r_0}^r \varphi^{-1}(g(s)^{1-m}z(s)^{-1/\gamma}) \, ds,$$

where, for ease of notation, we have set

$$z(s) = \lambda \log \left[1 + \gamma \int_s^a \frac{t^\rho h(\varphi^{-1}(g(t)^{1-m}))}{g(t)^{m-1/\gamma}} \, dt\right] \quad r_0 \geq s < a.$$

Then,

$$z'(r) = \varphi^{-1}(g^{1-m}z^{-1/\gamma}) > 0$$

and a simple computation shows that, on $[r_0, a)$

$$g^{1-m}(g^{-1} \varphi(z'))' = \lambda t^\rho e^{-z/\lambda} \varphi(z')^{1+\gamma} h(\varphi^{-1}(g^{1-m})).$$

We now show that (2.2) is satisfied for an appropriate choice of $\lambda$. We need to consider two cases. If $0 < z(t) \leq 1$, the monotonicity of $\varphi^{-1}$ and $h$ gives

$$h(z') = h(\varphi^{-1}(g^{1-m}z^{-1/\gamma})) \leq h(\varphi^{-1}(g^{1-m})).$$

Thus, the right-hand side of (2.7) is bounded above by $\lambda t^\rho \varphi(z')^{1+\gamma} h(\varphi^{-1}(g^{1-m}))$, and (2.2) holds for every $\lambda \in (0, 1)$.

We consider next the case $z(t) \geq 1$. By (2.6)

$$g(t)^{1-m}z(t)^{-1/\gamma} \leq g(t)^{1-m} < \tilde{\varepsilon}$$

and by the monotonicity of $\varphi^{-1}$

$$z'(t) = \varphi^{-1}(g^{1-m}z^{-1/\gamma}) \leq \varphi^{-1}(g^{1-m}) \leq \varphi^{-1}(\tilde{\varepsilon}) < \tilde{\varepsilon}.$$

Since, by (0.14)(i), $s^{-\beta} h(\varphi^{-1}(s))$ is non-increasing, it follows that

$$(g^{1-m}z^{-1/\gamma})^{-\beta} h(\varphi^{-1}(g^{1-m}z^{-1/\gamma})) \geq g^{-\beta(1-m)} h(\varphi^{-1}(g^{1-m})).$$

Thus, the right-hand side of (2.7) is bounded above by

$$\lambda t^\rho \varphi(z')^{1+\gamma} h(\varphi^{-1}(g^{1-m})),$$

It is easily verified that there exists $\lambda \in (0, 1)$, which depends only on $\beta$ and $\gamma$, such that, for every $z \geq 1$,

$$\lambda z^{\beta/\gamma} e^{-z/\lambda} < 1.$$
With this choice of $\lambda$, (2.2) follows from (2.7) even in this case, and we conclude that $x_a$ satisfies (2.2) on $[r_0, a]$. Clearly, $x_a(r_0) = 0$.

To prove (2.3), note that $z(s) \simeq (a - s)$ as $s \to a^-$. Together with (0.1)(iii)', and the definition of $x_a$, this implies that

$$x_a(r) \geq C \int_{r_0}^{r} (a - t)^{-1/\delta y} \, dt$$

showing that, if $\gamma \leq 1/\delta$, then $x_a(r) \to +\infty$ as $r \to +\infty$.

Finally, assume that (0.1)(iii)' and (2.4) hold. To prove (2.5), it suffices to show that as $a \to +\infty$, $z(s) \to +\infty$ uniformly on $[r_0, r_1]$. This is equivalent to the fact that

$$t^\gamma h((\varphi^{-1}(g(t)^{1-m})) g(t)^{(m-1)/\gamma}) \notin L^1(+#),$$

which follows easily from our assumptions. 

**Proof of Theorem B.** To avoid the repetition of cumbersome technical details, we will give the proof under the further assumption that $o$ is a pole of $M$. In the general case, points on the cut locus of $o$ are dealt with adapting the ideas used in the proof of Theorem A.

As in the proof of Theorem A, without loss of generality we may assume that $G$ is smooth, even at the origin, and such that $G(0) = 1$. Applying the Laplacian and the Sturm comparison theorems we deduce that

$$\Delta r \leq (m - 1) \frac{g'}{g}, \quad (2.8)$$

where

$$g(r) = \int_{0}^{r} e^\int_{0}^{s} \frac{G(s)}{g(s)} ds dt.$$

Using (0.15), it is easy to check that $g(r) \simeq r$ as $r \to +\infty$. In particular, $g(r) \to +\infty$ as $r \to +\infty$, and since $\gamma \leq (\frac{p+1}{m-1} - \frac{\delta}{2})$, (2.4) holds.

It follows from (0.16) that, if $R > 0$ is sufficiently large, then

$$b(x) \geq Er(x)\rho \quad \text{on} \quad M \setminus B_R(o). \quad (2.9)$$

Since $\gamma \leq 1/\delta$, we may apply Lemma 2.1 to deduce that there exists $r_0 > R$, such that, for every $a > r_0$ there exists a function $z_a$ defined on $[r_0, a]$ with $z(r_0) = 0$, $z' > 0$, $z(r) \to +\infty$ as $r \to a^-$ and satisfying

$$g^{1-m}(g^{m-1} \varphi(z'))' < Er^\rho \varphi(z')^{1+\gamma} h(z') \quad \text{on} \quad [r_0, a]. \quad (2.10)$$
We define
\[ v(x) = v_a(x) = \alpha_a(\rho(x)) \quad \text{on } B_a(o) \setminus B_{r_0}(o). \]

It is clear that \( v \equiv 0 \) on \( \partial B_{r_0} \), \( v(x) \to +\infty \) as \( r(x) \to a^- \), and \( |\nabla v| > 0 \) on \( B_a(o) \setminus B_{r_0}(o) \). It follows from (2.8) and (2.10) that \( v \) satisfies
\[
\text{div}(|\nabla v|^{-1} \phi(|\nabla v|) \nabla v) < E r^\rho \phi(|\nabla v|)^{1+\gamma} h(|\nabla v|). \tag{2.11}
\]

By adding a constant to \( u \) we may suppose that \( u^* > 0 \). We want to show that \( u = u^* \) on \( M \). Assume by contradiction \( u \) is not identically equal to \( u^* \).

First of all, \( u < u^* \) on \( M \), by the strong maximum principle for the \( \varphi \)-Laplacian. We let \( u_{r_0}^* = \sup_{B_{r_0}} u < u^* \), and assume, without loss of generality, that \( \eta < u^* - u_{r_0}^* \). Next, we choose \( \tilde{x} \in M \setminus \overline{B_{r_0}(o)} \) such that \( u(\tilde{x}) > u^* - \eta / 2 \), and take a sufficiently large that \( \alpha_a(r) < \eta / 2 \) for every \( r \) in \( [r_0, r(\tilde{x})] \). It follows that \( u(\tilde{x}) - v(\tilde{x}) > u^* - \eta \).

Since, \( u(x) - v(x) \leq u_{r_0}^* < u^* - \eta \) if \( x \in \partial B_{r_0}(o) \), and \( u(x) - v(x) \to -\infty \) as \( r(x) \to a^- \), we deduce that the function \( u - v \) attains a positive absolute maximum \( m_a \) on \( B_a(o) \setminus \overline{B_{r_0}(o)} \). Let \( \Gamma_a \) be a connected component of the set of points where the \( m_a \) is attained. Clearly, \( \Gamma_a \) is a compact subset of \( B_a(o) \setminus \overline{B_{r_0}(o)} \). Also, \( u(\zeta) > u(\xi) - v(\xi) > u^* - \eta \) and \( |\nabla u(\xi)| = |\nabla v(\xi)| > 0 \) for every \( \xi \in \Gamma_a \). In particular, \( \Gamma_a \) is contained in \( A_{m_a} \), and therefore (0.18), (2.9) and (2.11) imply that, for every \( \zeta \in \Gamma_a \),
\[
\text{div}(|\nabla u|^{-1} \phi(|\nabla u|) \nabla u) \geq b(\zeta) \phi(|\nabla u|)^{1+\gamma} h(|\nabla u|) = b(\zeta) \phi(|\nabla v|)^{1+\gamma} h(|\nabla v|) > \text{div}(|\nabla v|^{-1} \phi(|\nabla v|) \nabla v).
\]

We conclude that a sufficiently small neighborhood \( O \) of \( \Gamma_a \) is contained in \( A_{m_a} \cap (B_a(o) \setminus \overline{B_{r_0}(o)}) \) and there
\[
\text{div}(|\nabla u|^{-1} \phi(|\nabla u|) \nabla u) > \text{div}(|\nabla v|^{-1} \phi(|\nabla v|) \nabla v).
\]

Finally, fix a point \( \zeta \in \Gamma_a \), and, for every \( 0 < \mu < m_a \), denote by \( \Omega_{\zeta, \mu} \) the connected component containing \( \zeta \) of the set \( \{ x \colon u(x) > v(x) + \mu \} \). If we choose \( \mu \) close enough to \( m_a \), then \( \Omega_{\zeta, \mu} \) is contained in \( O \). Since \( u(x) = v(x) + \mu \) on \( \partial \Omega_{\zeta, \mu} \), by the weak comparison principle \( u(x) \leq v(x) + \mu \) on \( \Omega_{\zeta, \mu} \), contradicting \( \zeta \in \Omega_{\zeta, \mu} \).

Now assume that \( \rho \equiv 0 \). The idea of the proof is similar to that of the last part of Theorem A (and was inspired by the proof of Theorem VI in [R2]).
Since the conditions imposed on $h$ concern only its behavior for $t$ small, we may assume, without loss of generality, that $h$ is bounded above, and in fact that $h \leq 1$. As in the proof of Theorem A, to conclude it suffices to show that, as $a \to +\infty$, $|\nabla v_a| \to 0$ uniformly on $\Gamma_a$. We begin by observing that $v_a(\zeta) \to 0$ as $a \to +\infty$, uniformly on $\Gamma_a$. Indeed, since $\sup_{B_{r_1}(0)B_{a_0}(0)} u \to u^*$ as $r_1 \to +\infty$, given $\epsilon > 0$ there exists $r_\epsilon$ such that $\sup_{B_{r_1}(0)B_{a_0}(0)} u > u^* - \epsilon/2$. On the other hand, since $x_a(r) \to 0$ as $a \to \infty$, uniformly on any interval $[r_a, r_1]$, there exists $a_\epsilon > 0$ such that, for every $a \geq a_\epsilon$, $0 \leq x_a < \epsilon/2$ in $[r_a, r_\epsilon]$. It follows that, for every $a > a_\epsilon$, max $B_{r_1}(0)B_{a_0}(0) (u - v_a) > u^* - \epsilon$. Thus, for every $a > a_\epsilon$ and $\zeta \in \Gamma_a$, $u^* > u(\zeta) > u^* - \epsilon$ and $0 < v_a(\zeta) < \epsilon$.

Next, assume by contradiction that, as $a \to +\infty$, $|\nabla v_a|$ does not converge to zero uniformly on $\Gamma_a$. It follows that there exists a sequence $a_n \to +\infty$, and for every $n$ a point $\zeta_n \in \Gamma_{a_n}$ such that $|\nabla v_{a_n}(\zeta_n)| > \nu$, for some $\nu > 0$. Having set $r_{a_n} = r(\zeta_n)$, this amounts to saying that $x_{a_n}(r_{a_n}) > \nu$. Since $x_{a_n} \to 0$ uniformly on compacts, as $a \to 0$, we may assume that $r_{a_n} > r_0 + 1$ for every $n$. By what proved above,

$$x_{a_n}(r_{a_n}) \to 0 \quad \text{as} \quad n \to +\infty. \quad (2.12)$$

By construction $x_{a_n}$ satisfies

$$(\varphi(x_{a_n}))' + (m - 1)\frac{g'}{g} \varphi(x_{a_n}) < E\rho \varphi(x_{a'_n})^{1+\gamma} h(x_{a'_n}) \quad \text{on} \quad [r_0, a_n).$$

Using $h \leq 1$, $g' > 0$, $\varphi > 0$, and setting for notational convenience $y_n = \varphi(x_{a_n})$, we deduce that $y_n$ satisfies

$$y_n' < S_1 y_n^{1+\gamma} \quad \text{on} \quad [r_0, a_n)$$

with $S = E\rho$. Moreover, $y_n(r_{a_n}) \geq \varphi(\nu)$ for every $n$, and, since $x_{a_n} > 0$, then $y_n > 0$ on $[r_0, a_n)$. We integrate the above differential inequality over $[r, r_{a_n}]$ and obtain

$$y_n^{-\gamma}(r) \leq y_n^{-\gamma}(r_{a_n}) + \gamma S(r_{a_n} - r) \leq \varphi(\nu)^{-\gamma} + \gamma S(r_{a_n} - r).$$

Rearranging, this yields

$$y_n(r) \geq [\varphi(\nu)^{-\gamma} + \gamma S(r_{a_n} - r)]^{-1/\gamma}.$$

According to (0.1)(iii), $x_{a_n} = \varphi^{-1}(y_n) \geq A^{-1/\delta} y_n^{1/\delta}$, and we obtain the chain of inequalities

$$A^{1/\delta} \int_{r_0}^{r_{a_n}} x_{a_n}' \geq \int_{r_0}^{r_{a_n}} y_n^{1/\delta} \geq [\varphi(\nu)^{-\gamma} + \gamma S]^{-1/(\gamma\delta)},$$

where $A$ and $\gamma$ are fixed.
that is

$$Z_{\alpha\alpha}(r_{\alpha\alpha}) \geq A^{-1/\delta}[\varphi(v)^{-\gamma} + \gamma S]^{-1/(\gamma\delta)}$$

for every $n$. This contradicts (2.12), and completes the proof. □

**Remark 1.** The proof of Theorem B shows that if $u$ is a $C^2$ solution of (0.18) on $M$, then $u$ is necessarily constant.

**Remark 2.** Assumption (0.15) holds in particular if $\liminf_{t \to +\infty} t^\sigma G(t) \geq 0$ for some $\sigma > 2$. The case where $\sigma = 2$ is delicate. To illustrate the situation, we consider $G(r) = (1 + r^2)^{-1}$, which is positive, smooth, even at the origin, and such that $G(0) = 1$. As in the proof of Theorem B, applying the Laplacian and Sturm comparison theorems we deduce that $\Delta r \leq (m - 1)g'/g$ with

$$g(r) = \frac{1}{B^r(r + \sqrt{1 + r^2})^B - 1},$$

where we have set $B^r = [1 + \sqrt{1 + 4B^2}]/2$. Since $g(r) \approx r^B$ as $r \to +\infty$, we see that, in this case, assumption (2.4) in the statement of Lemma 2.1 is satisfied provided

$$0 < \gamma \leq \min \left\{ \frac{1}{\delta}, \frac{1 + \rho}{(m - 1)B^r} - \frac{\zeta}{\delta} \right\}. \quad (2.13)$$

The conclusion of Theorem B continues to hold when $G(r) = (1 + r^2)^{-1}$, if we replace (0.17) with (2.13), and leave the remaining assumptions unchanged. We stress however that, contrary to what happens when $tG(t) \in L^1(+\infty)$, the range of admissible $\gamma$’s depends on the coefficient in the curvature bound

$$\text{Ricc}_{M,\langle,\rangle}((\nabla r, \nabla r)) \geq -(m - 1)B^2(1 + r^2)^{-1}. \quad (2.14)$$

The case of a generic function $G(r)$ satisfying $r^\sigma G(r) \leq C$ with $\sigma = 2$ may be treated in general, but the technical details are cumbersome, and we leave it to the interested reader.

**Proof of Corollary B1.** Let $u$ be a non-negative $C^2$ function satisfying (0.19) on $\bar{B}_n$. Without loss of generality, we may assume that $\inf_M u = 0$. We fix $\mu$ in $(0, b)$, and set $v = (u + \mu)^{-1}$. Then $\sup_M v = 1/\mu$, and a straightforward computation shows that

$$\text{div}(|\nabla v|^{p - 2}\nabla v) \geq \mu[2(p - 1) - a]|\nabla v|^p$$
on the set
\[
\tilde{\mathcal{B}} = \left\{ x \in M : v(x) \geq \frac{1}{\mu} - \frac{\eta}{\mu(\mu + \eta)} \quad \text{and} \quad |\nabla v(x)| \leq \frac{\eta}{\mu(\mu + \eta)} \right\}.
\]

We apply Theorem B to \( v \) with \( h \equiv 1, \phi(t) = t^{p-1}, b \equiv \mu(2(p-1) - a), \delta = p - 1, \gamma = 1/(p-1), \rho = 0 \) and \( \zeta = 0 \). Note that the hypothesis \( p \geq m \) ensures that condition (0.17) is satisfied.

We remark that if we assume the curvature bound (2.14), then the conclusion of Corollary B1 holds provided
\[
p \geq \frac{m-1}{2} \left[ 1 + \sqrt{1 + 4B^2} \right] > m.
\]

**Remark 3.** The range \( p \geq m \) in Corollary B1 is sharp. Indeed, let \( f \in C^\infty([0, +\infty)) \) be a non-negative function such that
\[
f(t) = \begin{cases} 
0 & \text{if } t \in [0, 1], \\
t^{-\sigma} & \text{if } t \in [2, +\infty)
\end{cases}
\]
for some \( \sigma > 2 \), and define
\[
g(r) = \int_0^r e^{\int_0^t sf(s) \, ds} \, dt.
\]
Then \( g \) is smooth, strictly increasing, vanishes at \( r = 0 \), and \( g(r) = r \) if \( r \in [0, 1] \). Hence it defines a model metric \( \langle \cdot, \cdot \rangle = dr^2 + g(r)^2 \, d\theta^2 \) on \( \mathbb{R}^m \). Easy computations show that
\[
g''(r) = rf(r)e^{\int_0^t sf(t) \, dt} \geq 0, \quad g(r) \sim Cr \quad \text{as} \quad r \to +\infty.
\]
It follows that the radial Ricci curvature of the manifold satisfies (0.6) with \( B = 1 \) and \( G(r) = g''(r)/g(r) \). Moreover,
\[
\limsup_{t \to +\infty} t^\sigma G(t) < +\infty.
\]
Given \( p > 1 \), let \( a \in C^\infty([0, +\infty)) \) be non-negative, identically zero on \([2, +\infty)\) and such that on \([0, 1]\)
\[
a(t) = \begin{cases} 
1 & \text{if } 1 < p \leq 2, \\
t^{p-2} & \text{if } p > 2
\end{cases}
\]
and define
\[ u(x) = \int_0^{r(x)} g(t)^{-(m-1)/(p-1)} \left\{ \int_0^t a(s)g(s)^{m-1} \, ds \right\}^{1/(p-1)} \, dt. \]

Then \( u \) is radial, \( C^2 \), non-negative and non-constant on \( (\mathbb{R}^m, \langle , \rangle) \), and satisfies
\[ \text{div}(\nabla |u|^{p-2} \nabla u)(x) = a(r(x)) \geq 0. \]

Easy estimates show that, as \( r(x) \to +\infty \),
\[ u(x) \sim C \begin{cases} \frac{r(x)^{(p-m)/(p-1)}}{\log r(x)} & \text{if } p \neq m, \\ r(x)^{(p-m)/(p-1)} & \text{if } p = m. \end{cases} \]

Thus, if \( p < m \), \( u \) is a non-constant, positive, bounded, \( p \)-subharmonic function, showing that \( (\mathbb{R}^m, \langle , \rangle) \) is not \( p \)-parabolic.

**Remark 4.** We now return to the role of the condition \( tG(t) \in L^1(+\infty) \).
We have already discussed the case where \( G(t) \sim t^{-2} \) as \( t \to +\infty \), and seen that the conclusion of Theorem B continues to hold provided the range of admissible \( \gamma \)'s is suitably restricted (see (2.13)). The next example shows that Theorem B fails if we assume that \( G(t) \sim t^{-\sigma} \) with \( 0 < \sigma < 2 \). Indeed, fix \( 0 < \sigma < 2 \), and denote by \( I_n \) the modified Bessel function of order \( n \). Setting \( \nu = 2 - \sigma \), we define a function \( g \in C^\infty([0, +\infty)) \) such that \( g(t) > 0 \) on \( (0, +\infty) \) and
\[ g(r) = \begin{cases} r & \text{if } 0 \leq r \leq 1, \\ r^{1/2}I_{\nu/2}(r^{\nu/2}/\nu) & \text{if } r \geq 10. \end{cases} \]

As usual, we define a model metric on \( \mathbb{R}^m \) by the formula
\[ \langle , \rangle = dr^2 + g(r)^2 \, d\theta^2. \]

Since \( g \) satisfies
\[ g''(r) = \frac{1}{4} r^{-\sigma} g(r) \quad \text{on } [10, +\infty) \]
the radial Ricci curvature satisfies condition (0.6) with \( B = 1 \) and with a function \( G \) such that
\[ G(r) = \frac{1}{4(m-1)} r^{-\sigma} \quad \text{on } [10, +\infty). \]

Thus (0.15) holds with the given \( \sigma \). For future use, we also note that
\[ g(r) = r^{\sigma/4} e^{r^{\nu/2}/\nu} \quad \text{as } r \to +\infty. \]
Again, we consider the case of the \( p \)-Laplacian, \( p > 1 \), and choose \( h \equiv 1 \). Then (0.14) is satisfied with \( \zeta = 0 \) and arbitrary \( \beta, \bar{\epsilon} > 0 \). We also take \( b(x) \equiv 1 \), so that (0.16) holds with \( \rho = 0 \). Next, let \( \gamma > 0 \) to be specified later, and define

\[
z(s) = \log \left[ e + \gamma \int_{s}^{+\infty} \frac{dt}{g(t)^{(m-1)\gamma}} \right]
\]

for \( t \in [1, +\infty) \). By (2.15) \( z \) is well defined, and

\[
z^* = \sup_{[1, +\infty)} z = z(1) < + \infty.
\]

Finally, we define \( u(x) \) by the formula

\[
\int_{1}^{r(x)} [g(s)^{1-m}z(s)^{-1/\gamma}]^{1/(p-1)} ds \quad \text{if} \quad r(x) \geq 1
\]

and extend it to a smooth radial function defined on \( \mathbb{R}^m \) and negative on \( B_1(o) \). Since \( u \) is non-decreasing and positive on \( A = \mathbb{R}^m \setminus B_1(o) \), a computation shows that \( u \) satisfies

\[
\text{div}(|\nabla u|^{p-2}\nabla u) = e^{-z_1}|\nabla u|^{(p-1)(1+\gamma)} \geq e^{-z^*}|\nabla u|^{(p-1)(1+\gamma)}
\]

on \( A = \{x: u(x) > 0\} \). It is clear that \( u \) is not constant, and, using (2.15) it is easily checked that \( u \) is bounded above provided \( \gamma < \sigma/(p-1) \). If we choose \( \gamma \) sufficiently small, both this condition and (0.17) hold, showing that Theorem B fails if \( \sigma < 2 \).

Our last example shows that assumption (0.17) is optimal. Indeed, let us consider the model metric on \( \mathbb{R}^m \) constructed in Remark 3. Fix \( p > 1, \rho \in \mathbb{R}, \zeta \geq 0, \) and

\[
\gamma > \frac{1 + \rho}{m - 1} - \frac{\zeta}{p - 1} \geq 0.
\]

The asymptotic behavior of \( g(r) \) as \( r \to +\infty \), and (2.18) guarantee that

\[
z_1(s) = \log \left[ e + \gamma \int_{s}^{+\infty} \frac{t^\rho}{g(t)^{(m-1)\gamma+(m-1)\zeta/(p-1)}} dt \right]
\]

is well defined on \( [1, +\infty) \), \( z_1 > 1 \) and \( z_1^* = \sup_{[1, +\infty)} z_1 < + \infty \). We define a smooth function \( u \) on \( \mathbb{R}^m \) as in the previous example, using \( z_1 \) instead of \( z \) in (2.17). One checks that \( u \) satisfies

\[
\text{div}(|\nabla u|^{p-2}\nabla u) = r^\rho e^{-z_1}|\nabla u|^{(p-1)(1+\gamma)} g^{-(m-1)\zeta/(p-1)}
\]
on $A = \{x: u(x) > 0\}$. The properties of $z_1$ and the definition of $u$ imply that $|\nabla u| \leq g^{-m}/(p-1)$, and therefore that

$$\text{div}(|\nabla u|^{p-2} \nabla u) \geq r^\rho e^{-z_1^\gamma} |\nabla u|^{(p-1)(1+\gamma)+\zeta}$$
on $A$.

On the other hand, it is easy to check that $u$ is bounded above if $m > p$. We also note that, if $\rho$ and $\zeta$ satisfy

$$(1 + \zeta)m - 1 > 1 + \rho > \zeta m - 1,$$

then

$$\frac{1}{p - 1} > \frac{1 + \rho}{m - 1} - \frac{\zeta}{p - 1} \geq 0.$$

Thus we may choose $\gamma$

$$\frac{1}{p - 1} > \gamma > \frac{1 + \rho}{m - 1} - \frac{\zeta}{p - 1}$$

in such a way that (0.17) barely fails to be satisfied while the remaining assumptions of Theorem B hold. The function $u$ constructed above shows that the conclusion of Theorem B fails.

3. PROOF OF THEOREM C

Our proof of Theorem C requires some facts about solutions of the radial version of the equation $\text{div}(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u) = b(x)f(u)$, which we collect in the following lemma.

**Lemma 3.1.** Let $\varphi \in C^1([0, +\infty)) \cap C^0([0, +\infty))$ satisfy (0.1)(i), (ii), (iv) and assume that

$$\lim_{t \to +\infty} H(t) = +\infty,$$

where $H$ is the function defined in (0.4). Let $g : [0, +\infty) \to [0, +\infty)$ be a $C^1$ function such that

(i) $g(0) = 0$; \quad (ii) $g(t) > 0$ on $(0, +\infty)$; \quad (iii) $g'(t) \geq 0$ on $(0, +\infty)$.

(3.2)
Let $f \in C^0(\mathbb{R})$ satisfy

\begin{align*}
(i) & \quad f(t) \geq 0 \quad \text{on } [0, +\infty); \quad (ii) \quad f \neq 0; \\
(iii) & \quad f \text{ is non-decreasing on } (0, +\infty);
\end{align*}

Suppose that

\begin{align*}
\frac{1}{H^{-1}(\int_0^t f(s) \, ds)} \notin L^1 (+\infty).
\end{align*}

Finally, let $q : [0, +\infty) \to [0, +\infty)$ be a continuous function. Then, for every $\alpha_0 \in (0, +\infty)$, the problem

\begin{align*}
\begin{cases}
(i) & \quad (g^{m-1} \phi(|\alpha'|) \text{sign}(\alpha'))' = qg^{m-1}f(\alpha), \\
(ii) & \quad \alpha'(0) = 0; \quad \alpha(0) = \alpha_0
\end{cases}
\end{align*}

has a non-decreasing solution $\alpha$ defined in $[0, +\infty)$.

**Proof.** We suppose $q \neq 0$, the other case being trivial. If $f(\alpha_0) = 0$ then $\alpha(t) \equiv \alpha_0$ solves (3.5). Thus we may also assume that $f(\alpha_0) > 0$. By a variant of Picard’s iteration procedure, a solution $\alpha$ of (3.5) exists. Let $[0, R)$, $0 < R \leq +\infty$, be the maximal interval where $\alpha$ is defined. We claim that $\alpha$ is non-decreasing on $[0, R)$. To see this, we integrate (3.5) (i) over $[0, t]$, $0 < t < R$. Using $\alpha'(0) = 0$ and (0.1) (i) we have

\begin{align*}
\phi(|\alpha'(t)|) \text{sign}(\alpha'(t)) = g(t)^{1-m} \int_0^t q(s)g(s)^{m-1}f(\alpha(s)) \, ds.
\end{align*}

We note that (0.1)(i), (ii) imply $\phi(t) > 0$ for $t > 0$. Thus, the non-negativity of $g$ and $q$, (3.6),(3.3)(i) and $\alpha(0) > 0$ show that

\begin{align*}
\alpha'(t) \geq 0, \quad \text{on } [0, R)
\end{align*}

as claimed. In particular, $\alpha$ satisfies

\begin{align*}
(\phi(\alpha'))' + (m-1) \frac{g'}{g} \phi(\alpha') = qf(\alpha) \quad \text{on } (0, R).
\end{align*}

We next show that $R = +\infty$. We reason by contradiction and assume $R < +\infty$. Then (3.7) implies that the limit $\lim_{t \to R^-} \alpha(t)$ exists. We claim that

\begin{align*}
\lim_{t \to R^-} \alpha(t) = +\infty.
\end{align*}
Indeed, if this were not the case, (3.6), (3.7) would yield the existence of the limit

$$\lim_{t \to R^-} \varphi(\alpha(t)) = C \in \mathbb{R}^+$$

and, therefore, by of (0.1)(ii) and (iv), the existence of

$$\lim_{t \to R^-} \alpha(t) = D \in \mathbb{R}^+.$$ 

Therefore, it would be possible to extend the solution beyond $R$ contradicting the maximality of $[0, R]$. This proves the claim.

Now, we set $q_0 = \max_{[0, R]} q + 1$. Since $g'/g \geq 0$, from (3.8) and the non-negativity of $\varphi$ and $f(x)$, we get

$$(\varphi(\alpha'))' \leq q_0 f(x) \quad \text{on } [0, R). \quad (3.10)$$

Multiplying (3.10) times $\alpha' \geq 0$, noting that the identity $H(t) = \int_0^{\varphi(t)} \varphi^{-1}(v) \, dv$ implies that $[H(\alpha')]' = [\varphi(\alpha')]' \alpha'$, integrating over $[0, t]$, $0 < t < R$, and using $\alpha'(0) = 0$, we obtain, after some manipulations,

$$H(\alpha(t)) \leq \int_{z_0}^{\beta(t)} q_0 f(s) \, ds.$$ 

Thus, using (3.3)(ii), (3.9) and the positivity of $H$ on $(0, +\infty)$ we get

$$\frac{\alpha'(t)}{H^{-1}(\int_{z_0}^{\beta(t)} q_0 f(s) \, ds)} \leq 1$$

for $t$ sufficiently near to $R^-$, say $t \geq t_1$. We integrate again over $[t_1, t]$, $t_1 < t < R$, and perform the change of variable $u = \alpha(t)$ to obtain

$$\int_{\alpha(t_1)}^{\alpha(t)} \frac{du}{H^{-1}(\int_{z_0}^{u} q_0 f(s) \, ds)} \leq t - t_1.$$ 

Whence, letting $t \to R^-$ and using (3.9) we deduce

$$\int_{\alpha(t_1)}^{+\infty} \frac{1}{H^{-1}(\int_{z_0}^{u} q_0 f(s) \, ds)} \leq R - t_1.$$ 

This implies that

$$\frac{1}{H^{-1}(\int_{0}^{u} f(s) \, ds)} \in L^1(+\infty),$$
as one can easily verify changing variables, and using the monotonicity of $H$ and $f$. This contradicts (3.4).

**Proof of Theorem C.** The arguments are very similar to those used in the first part of the proof of Theorem A. We reason by contradiction and we assume that there exists $x_0 \in \Omega$ with $u(x_0) > v$. Without loss of generality, we can suppose that $x_0 = o$. Note that assumption (0.24) implies that all the assumptions in the statement are satisfied with respect to the new origin, possibly with a different constant $B$ in (0.6), and scaling $q$ and $z$ in (0.7).

Arguing as in Theorem A, the Laplacian comparison theorem shows that

$$\Delta r \leq (m - 1) \frac{g'(r)}{g(r)}$$

within the cut locus of $o$, where

$$g(r) = D^{-1} \{e^D \int_0^r \sqrt{G(s)} \, ds - 1\}$$

and $D > 0$ is a suitable constant. We observe that $g$ is smooth on $[0, +\infty)$ and satisfies

(i) $g(0) = 0$;        (ii) $g'(0) = 1$;  (iii) $g'(t) \geq 0$ on $(0, +\infty)$;

(iv) $(g'/g)(t) \geq D \sqrt{G(t)}$.

We next consider the function $q$. Because of (0.22)(i), $q$ can be extended to a non-negative, continuous function on $[0, +\infty)$ satisfying (0.22) (ii) on all of $\Omega$. We still denote with $q$ such an extension. Now, we apply Lemma 3.1 to deduce that, for each $z_0 > 0$, there exists a non-decreasing solution $z$ on $[0, +\infty)$ of

$$\left\{ \begin{array}{l}
(g^{m-1} \varphi(|x'|) \text{sign}(x'))' = qg^{m-1}f(z), \\
z'(0) = 0; \quad z(0) = z_0.
\end{array} \right. \quad (3.12)$$

We claim that $\lim_{t \to +\infty} z(t) = +\infty$. Indeed, from (0.21) (ii) we have

$$f(z(t)) \geq f(z_0) \quad \text{on } [0, +\infty).$$

Since $q, g \geq 0$ it follows that

$$\int_0^t q(s)g(s)^{m-1}f(z(s)) \, ds \geq f(z_0) \int_0^t q(s)g(s)^{m-1} \, ds.$$
Whence, using (3.12), we obtain
\[ g^{m-1}(t) \varphi(\alpha'(t)) \geq f(\alpha_0) \int_0^t q(s) g(s)^{m-1} \, ds. \]

Since \( \varphi^{-1} \) is strictly increasing
\[ \alpha'(t) \geq \varphi^{-1} \left( f(\alpha_0) g(t)^{1-m} \int_0^t q(s) g(s)^{m-1} \, ds \right), \]
so that, integrating over \([0, t] \), and using \( \alpha(0) = \alpha_0 \) yield
\[ \alpha(t) \geq \alpha_0 + \int_0^t \varphi^{-1} \left( f(\alpha_0) g(s)^{1-m} \int_0^s q(y) g(y)^{m-1} \, dy \right) \, ds. \]

Thus, the claim follows provided we show that
\[ \varphi^{-1} \left( f(\alpha_0) g(s)^{1-m} \int_0^s q(y) g(y)^{m-1} \, dy \right) \geq \frac{C}{s} \]
for some constant \( C > 0 \); this is done exactly as in Theorem A.

We define
\[ v(x) = \alpha(r(x)), \quad \text{on } M \]
so that
(i) \( v(o) = \alpha_0 \); (ii) \( v(x) > 0 \) on \( M \); (iii) \( \lim_{r(x) \to +\infty} v(x) = +\infty \) (3.13)

and we assume to have chosen \( \alpha_0 \in (v, u(o)) \) in (3.12) so that \( u(o) - v(o) > 0 \).
Since \( u - v \) is negative on \( \partial \Omega \) and tends to \( -\infty \) as \( r(x) \to \infty \), we conclude
that \( u - v \) attains a positive maximum \( m \) on \( \Omega \), and that the set \( \Gamma \) where
the maximum is attained is compact in \( \Omega \). Arguing as in the proof of Theorem A
we may assume that \( \Gamma \) is contained within the cut locus of \( o \).

Then, using (3.11), (3.12) and the properties of \( \alpha \) we have
\[ \text{div}(\nabla v)^{-1} \varphi(|\nabla u|) \nabla v) \leq qf(v) \leq hf(v) \]
within the cut locus of \( o \). Since \( f \) is non-decreasing and \( u > v \) on \( \Gamma \), it follows
from 0.25 that
\[ \text{div}(\nabla u)^{-1} \varphi(|\nabla u|) \nabla u) > \text{div}(\nabla v)^{-1} \varphi(|\nabla v|) \nabla v \]
in a neighborhood of \( \Gamma \). A contradiction is then reached arguing as in
Theorem A. \( \square \)
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