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# Risk theory in a stochastic economic environment

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We introduce a general model to describe the risk process of an insurance company. This model allows for stochastic rate of return on investments as well as stochastic level of inflation, thus in theory enabling a decision maker to choose between insurance and investment risk. In the first part of the paper we discuss the model in itself and in the second part the problem of finding the probability of eventual ruin is posed. We obtain some integro-differential equations that in some cases lead us to the exact probability of eventual ruin and in other cases to inequalities. Examples are given showing that stochastic economic factors may have a serious impact on this probability.

risk process * semimartingale * stochastic differential equation * process with stationary independent increments * ruin probability * characteristic function * Markov process * integro-differential equation

## 1. Introduction

Since the appearance of Gerber's (1973) paper, the effect on an insurance portfolio of rate of return on investments and level of inflation has been subject to much study. In this paper we will consider a model for the risk process that takes into account stochastic rates of return and inflation, thus departing from former models which assume these quantities to be deterministic (see Segerdahl, 1942, 1959; Gerber, 1973, 1979; Harrison, 1977; Taylor, 1979; Moriconi, 1985, 1986; Delbaen and Haezendonck, 1987; and Dassios and Embrechts, 1989). Recently Dufresne (1990) studied the case with stochastic interest rates, but with a different motivation.

We will first introduce a rather general model and then go on to analyze in some detail a more restricted version. Then following the ideas of Harrison (1977), we will develop some integro-differential equations that may be useful in the calculation of the probability of eventual ruin. Some effort will be taken to find conditions that allow us to use these equations. Just as in Harrison (1977), we will be able to find exact values of the probability of eventual ruin in the special cases when the uninflated risk process follows a Brownian motion or a compound Poisson process with exponentially distributed claims. Otherwise only inequalities are obtained.

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To motivate our model we will write the model in e.g. Delbaen and Haezendonck (1987) in a manner that makes it suitable for generalizations. If we denote the risk process measured in real units by $Y$, then $Y$ is obtained through the following steps.

*Step 1.* We start with the surplus generating process $P$ given by

$$P_t = y + pt - \sum_{i=1}^{N_t} S_i,$$

where $N$ is a Poissonprocess and the $S_i$'s are i.i.d. random variables independent of $N$. The process $P$ is measured in real units unaffected by inflation.

*Step 2.* There is an inflation generating process $I_t = \bar{i}t$ so that the level of inflation $\bar{I}$ is given as the solution of

$$\mathrm{d}\bar{I}_t = \bar{I}_{t-}\,\mathrm{d}I_t \quad \text{where } \bar{I}_0 = 1. \tag{1.1}$$

*Step 3.* Claims and premiums in the surplus generating process are then subject to inflation and we obtain the inflated surplus process $\bar{P}$ as the integral

$$\bar{P}_t = y + \int_0^t \bar{I}_{s-}\,\mathrm{d}P_s. \tag{1.2}$$

*Step 4.* There is a return on investment generating process $R_t = rt$ so that the risk process in terms of nominal units is given as the solution of

$$\mathrm{d}\bar{Y}_t = \mathrm{d}\bar{P}_t + \bar{Y}_{t-}\,\mathrm{d}R_t \quad \text{where } Y_0 = y. \tag{1.3}$$

*Step 5.* The risk process in terms of real units at time $t$ is then given as

$$Y_t = \bar{I}_t^{-1}\bar{Y}_t. \tag{1.4}$$

It is easy to see that the solution of (1.4) is

$$Y_t = \mathrm{e}^{(r-\bar{i})t}\left(y + \int_0^t \mathrm{e}^{(\bar{i}-r)s}\,\mathrm{d}P_s\right) = \mathrm{e}^{(r-\bar{i})t}y + \int_0^t \mathrm{e}^{(r-\bar{i})(t-s)}\,\mathrm{d}P_s. \tag{1.5}$$

By letting $\bar{i} = 0$ we obtain the expression on p. 67 of Harrison (1977).

**Remark 1.1.** If we consider the real interest generating process $R - I$ and make the calculations in terms of real units we can define

$$\mathrm{d}\tilde{Y}_t = \mathrm{d}P_t + \tilde{Y}_{t-}\,\mathrm{d}(R-I)_t.$$

It is easy to see that in this case $\tilde{Y} = Y$. The reason why we distinguish between $\tilde{Y}$ and $Y$ is that they are normally not equal when $R$ and $I$ are general semimartingales, as will be explained in Remark 2.1.

A major drawback of this model is that the only source of uncertainty allowed for is in the number and severity of claims. Rate of return on investments and level of inflation are assumed known. But the reason that insurance companies run into financial trouble is just as often due to low or even negative return on investments, and this is of course unforeseeable. Unexpected levels of inflation may also have an impact on the solidity of an insurance company.

In this paper we will therefore allow for uncertainty in Steps 2 and 4 above. We start with a very general model where the surplus generating process $P$, the inflation generating process $I$ and the return on investment generating process $R$, all are semimartingales. This level of generality allows us to obtain the solution $Y$ of (1.4), but not very much more. So we are forced to put some restrictions on these processes. It turns out that assuming the vector process $(P, I, R)$ to be a process with stationary independent increments (and hence a semimartingale, see Jacod and Shiryaev, 1987, Corollary 4.19, p. 107) with a finite number of jumps on each finite interval, and in addition that $P$ is independent of $(I, R)$, the process $Y$ becomes fairly manageable.

## 2. The model

We will let all processes and random variables be defined on a filtered probability space $(\Omega, \mathscr{F}, F, P)$ satisfying the usual conditions (i.e. $\mathscr{F}_t$ is right continuous and $P$-complete). This is just the notation used by Jacod and Shiryaev (1987, p. 2).

We will now repeat Steps 1 to 5 in the introduction for our more general model. It is assumed that each semimartingale will be $\mathscr{F}_t$ adapted.

*Step 1.* The surplus generating process $P$ is a semimartingale with $P_0 = y$.

*Step 2.* The inflation generating process $I$ is a semimartingale with $I_0 = 0$. Then (1.1) is just the stochastic differential equation for the exponential formula, hence $\bar{I}$ is given as (see e.g. Jacod and Shiryaev, 1987, formula 4.64, p. 59)

$$\bar{I}_t = \mathscr{E}(I)_t = e^{I_t - 1/2\langle I^c, I^c\rangle_t} \prod_{0 \le s \le t} (1 + \Delta I_s)\, e^{-\Delta I_s} \tag{2.1}$$

where $\langle I^c, I^c\rangle$ is the predictable quadratic variation of the continuous martingale part $I^c$ of the semimartingale $I$.

*Step 3.* The inflated surplus process is as in (1.2). We will frequently use the standard notation

$$\bar{P} = y + \bar{I}_- \cdot P \quad \text{where } \bar{I}_{0-} = 0. \tag{2.2}$$

*Step 4.* The return on investment generating process $R$ is a semimartingale with $R_0 = 0$. Then with the above notation (1.3) becomes

$$\bar{Y} = \bar{P} + \bar{Y}_- \cdot R \quad \text{where } \bar{Y}_{0-} = 0. \tag{2.3}$$

*Step 5.* The risk process in terms of real units at time $t$ is then given as

$$Y_t = \bar{I}_t^{-1} \bar{Y}_t \quad \text{where } Y_{0-} = 0. \tag{2.4}$$

Here we have assumed that $\bar{I}_t > 0\, \forall t$, see Remark 2.2 for a justification of this assumption.

The unique solution of (2.4) is given in Jacod (1979, p. 194) as

$$\begin{aligned} Y &= \sum_{n\geq 0} Y^{(n)} 1_{[\![T_n, T_{n+1}[\![}, \\ Y^{(n)} &= \bar{I}^{-1}\bar{R}_{(n)}(\Delta\bar{P}_{T_n} + (1/\bar{R}^{(n)}_{-})\cdot(\bar{P}^{T_{n+1}} - \bar{P}^{T_n}) \\ &\qquad - ((1/\bar{R}^{(n)})1_{[\![0,T_{n+1}[\![})\cdot[\bar{P}, R^{T_{n+1}} - R^{T_n}]), \\ \bar{R}^{(n)} &= \mathscr{E}(R^{T_{n+1}} - R^{T_n}). \end{aligned} \tag{2.5}$$

Here $T_0 = 0$, $T_{n+1} = \inf\{t > T_n: \Delta R_t = -1\}$ and $[\bar{P}, R^{T_{n+1}} - R^{T_n}]$ is the optional quadratic covariation between $\bar{P}$ and $R^{T_{n+1}} - R^{T_n}$. By expressions like $X_t^{T_n}$ is meant $X_{t\wedge T_n}$.

Now from (2.2) and general results in stochastic calculus $\Delta\bar{P}_0 = y$, $\Delta\bar{P}_{T_n} = \bar{I}_{T_n-}\Delta P_{T_n}$ when $n \geq 1$, $\bar{P}^{T_{n+1}} - \bar{P}^{T_n} = \bar{I}_- \cdot (P^{T_{n+1}} - P^{T_n})$ and $[\bar{P}, R^{T_{n+1}} - R^{T_n}] = \bar{I}_- \cdot [P, R^{T_{n+1}} - R^{T_n}]$. Since $(\bar{I}/\bar{R}^{(n)})_-$ is locally bounded, associativity of the stochastic integral gives

$$\begin{aligned} Y^{(n)} = \bar{I}^{-1}\bar{R}^{(n)}(\tilde{I}_{T_n-}\Delta P_{T_n} &+ ((\bar{I}/\bar{R}^{(n)})_- \cdot (P^{T_{n+1}} - P^{T_n}) \\ &- ((\bar{I}_-/\bar{R}_{(n)})1_{[\![0,T_{n+1}[\![})\cdot[P, R^{T_{n+1}} - R^{T_n}]) \end{aligned} \tag{2.6}$$

where $\tilde{I}_{0-} = 1$ and $\tilde{I}_{T_n-} = I_{T_n-}$, $n \geq 1$.

This expression is rather complicated so let us look for reasonable assumptions that make it easier to handle.

First we will assume that the surplus generating process $P$ and the return on investment generating process $R$ are independent. Since these processes model different aspects of economic activity, this assumption is quite reasonable. It implies that $[P, R^{T_{n+1}} - R^{T_n}]$ is indistinguishable from the zero process, so (2.6) takes the simplified form:

$$Y^{(n)} = \bar{I}^{-1}\bar{R}^{(n)}(\tilde{I}_{T_n-}\Delta P_{T_n} + (\bar{I}/\bar{R}^{(n)})_- \cdot (P^{T_{n+1}} - P^{T_n})). \tag{2.7}$$

Next we will assume that it is impossible that all the assets of the insurance company become worthless in one stroke due to negative return on investment. This is perhaps a stronger assumption, but see Remark 2.2 below for a discussion in connection with ruin theory. To state it mathematically, we assume that $P(T_1 < \infty) = 0$. Then using (2.7), $Y$ in (2.5) takes the following form:

$$Y = U^{-1}(y + U_- \cdot P), \quad U = \bar{I}\bar{R}^{-1}, \quad \bar{R} = \mathscr{E}(R). \tag{2.8}$$

Note the similarity between (2.8) and (1.5).

**Remark 2.1.** The process $U^{-1} = \bar{R}\bar{I}^{-1} = \mathscr{E}(R)/\mathscr{E}(I)$ is a measure of real return on investment. In Remark 1.1 we considered the process $\tilde{Y}$ given by

$$\tilde{Y} = P + \tilde{Y}_- \cdot (R - I).$$

Under the same assumptions as above it follows that the unique solution is given by

$$\tilde{Y} = \tilde{U}^{-1}(y + \tilde{U}_- \cdot P)$$

where $\tilde{U} = (\mathscr{E}(R-I))^{-1}$ is also a measure of real return on investment, but is generally different from $U$. Indeed it follows from Protter (1990, Corollary, p. 79) that

$$\mathscr{E}(R-I)\mathscr{E}(I) = \mathscr{E}(R+[R-I, I])$$

and therefore

$$\tilde{U} = U\mathscr{E}(R)(\mathscr{E}(R+[R-I, I]))^{-1}.$$

This implies that $\tilde{U} = U$ and hence $\tilde{Y} = Y$ if and only if $[R-I, I] = 0$. A sufficient condition for this is that either $R-I$ or $I$ is a continuous deterministic process.

Finally we will assume that the vector process $\bar{X} = (P, I, R)$ is a process with stationary independent increments with a finite number of jumps on each finite interval. Then $\bar{X}$ has representation (see Gihman and Skorohod, 1969, Chapter VI, for the necessary theory of processes with independent increments)

$$\bar{X}_t = \bar{X}_0 + \bar{a}t + \bar{C}\bar{W}_t + \bar{V}_t \quad \text{where } X_0 = (y, 0, 0)^{\mathrm{T}}. \tag{2.9}$$

Here $\bar{a}$ is a constant vector, $\bar{C}$ is a $3\times 3$ matrix with the property

$$\bar{C}\bar{C}^{\mathrm{T}} = \begin{bmatrix} \sigma_P^2 & 0 & 0 \\ 0 & \sigma_I^2 & \rho\sigma_I\sigma_R \\ 0 & \rho\sigma_I\sigma_R & \sigma_R^2 \end{bmatrix} \tag{2.10}$$

where $|\rho| \leqslant 1$, $\bar{W}$ is a three-dimensional Brownian motion and $\bar{V}$ is a three dimensional compound Poisson process, independent of $\bar{W}$. We will assume that the first component of $\bar{V}$ is independent of the other two, and so (2.10) implies that $P$ and $(I, R)$ are independent. That $P$ and $I$ are independent is not necessary to obtain (2.8), but it will become so in our further work. It can also be justified by the same arguments as why $P$ and $R$ may be assumed independent. In terms of the components of $\bar{X}$ we have

$$P_t = y + pt + W_{P,t} - \sum_{i=1}^{N_{P,t}} S_{P,i}, \tag{2.11}$$

$$I_t = \bar{i}t + W_{I,t} + \sum_{i=1}^{N_{I,t}} \tilde{S}_{I,i}, \tag{2.12}$$

$$R_t = rt + W_{R,t} + \sum_{i=1}^{N_{R,t}} \tilde{S}_{R,i}, \tag{2.13}$$

where $(W_P, W_I, W_R)^{\mathrm{T}} = \bar{C}\bar{W}$, $N_P$, $N_I$ and $N_R$ are three Poisson processes with intensities $\lambda_P$, $\lambda_I$ and $\lambda_R$ respectively, and $N_P$ is independent of $(N_I, N_R)$. Also the summands in each sum are i.i.d. and $S_{P,i}$ and $(\tilde{S}_{I,j}, \tilde{S}_{R,j})$ are independent $\forall i, j$.

For future reference we will write

$$F_P(s) = P(S_P \leqslant s), \quad F_I(s) = P(1+\tilde{S}_I \leqslant s), \quad F_R(s) = P(1+\tilde{S}_R \leqslant s). \tag{2.14}$$

We will assume that $F_I(0) = F_R(0) = 0$.

**Remark 2.2.** The assumption $F_I(0)=0$ excludes the possibility that inflation is $-100\%$ or more, i.e. it is impossible that all assets in the economy become worthless or of negative value. So from a practical point of view this assumption is no restriction at all.

Similarly the assumption that $F_R(0)=0$ excludes the possibility that all assets of the insurance company become worthless or of negative value due to negative return on investments. As financial institutions often commit themselves to financial responsibilities far larger than their own assets, this is a much stricter assumption. However, the following argument justifies at least why we in the context of ruin theory in an infinite time interval may assume $F_R(0-)=0$. Let $T=\inf\{t\colon \Delta R_t<-1\}$. Since $P$ and $R$ are assumed independent, we see from (2.3) that $\Delta\bar{Y}_T=\bar{Y}_{T-}\Delta R_T$, hence $\bar{Y}_T<0$. Therefore ruin occurs at time $T$ (if not before). But $\Delta R_T=\tilde{S}_{R,N_{R,T}}$, so if we define $M_t=\sum_{i=0}^{N_{R,t}}\tilde{S}_{R,i}1_{\{\tilde{S}_{R,i}<-1\}}$, then $M$ is a compound Poisson process with intensity $\lambda_R F_R(0-)$ and $T=\inf\{t\colon M_t\neq 0\}$. Therefore $F_R(0-)>0$ implies that $P(T<\infty)=1$, i.e. ruin occurs with probability one. This argument also shows that $F_R(0)=0$ implies that $P(T_1<\infty)=0$, hence leading from (2.7) to (2.8).

**Remark 2.3.** Although going from (2.5) to our present model implies lots of assumptions, we are still at a level of generality that includes many models in the theory of finance, including the much celebrated Black and Scholes (1973) option pricing formula (see also Merton, 1973). There the underlying asset $S$ is assumed to follow a geometric Brownian motion, i.e. $S$ is the solution of $\mathrm{d}S_t=S_{t-}\,\mathrm{d}R_t$ where $R_t$ is as in (2.13) with $\lambda_R=0$. Thus it is a special case of our model (see (2.3)) with $\bar{P}_t=S_0$ a constant. Also in the more general jump-diffusion option valuation formula of Merton (1976), $S$ is the solution of the same equation, but now $\lambda_R>0$ and $1+\tilde{S}_R$ is assumed to be lognormally distributed, hence $F_R(0)=0$. On the other hand, in the constant elasticity of variance option pricing formula by Cox and Ross (1976), the underlying asset is the solution of $\mathrm{d}S_t=rS_{t-}\,\mathrm{d}t+S_{t-}^{1/2}\,\mathrm{d}W_{R,t}$, hence it is not a special case of (2.3) and (2.13).

We will now proceed to compute $Y$. From (2.12) we see that $I^{\mathrm{c}}=W_I$, hence $\langle I^{\mathrm{c}},I^{\mathrm{c}}\rangle_t=EW_{I,t}^2=\sigma_I^2 t$. Also since

$$\prod_{0\le s\le t}(1+\Delta I_s)\,\mathrm{e}^{-\Delta I_s}=\left(\prod_{i=1}^{N_{I,t}}(1+\tilde{S}_{I,i})\right)\exp\left\{-\sum_{i=1}^{N_{I,t}}\tilde{S}_{I,i}\right\},$$

we obtain from (2.1) that

$$\bar{I}_t=\exp\{(\bar{\imath}-\tfrac{1}{2}\sigma_I^2)t+W_{I,t}\}\prod_{i=1}^{N_{I,t}}(1+\tilde{S}_{I,i}). \tag{2.15}$$

Similarly

$$\bar{R}_t=\exp\{(r-\tfrac{1}{2}\sigma_R^2)t+W_{R,t}\}\prod_{i=1}^{N_{R,t}}(1+\tilde{S}_{R,i}). \tag{2.16}$$

Since $U = \bar{R}^{-1}\bar{I}$, we obtain

$$U_t = \exp\{-\alpha_U t + \sigma_U W_{U,t}\} \prod_{i=1}^{N_{I,t}} (1+\tilde{S}_{I,i}) \prod_{i=1}^{N_{R,t}} \frac{1}{1+\tilde{S}_{R,i}} \tag{2.17}$$

where $W_U$ is a Brownian motion so that $\sigma_U W_U = W_I - W_R$, hence

$$\alpha_U = r - \bar{i} + \tfrac{1}{2}(\sigma_I^2 - \sigma_R^2), \qquad \sigma_U^2 = \sigma_I^2 - 2\rho\sigma_I\sigma_R + \sigma_R^2. \tag{2.18}$$

The problem with (2.17) is that $\prod_{i=1}^{N_{I,t}} (1+\tilde{S}_{I,i})$ and $\prod_{i=1}^{N_{R,t}} (1+\tilde{S}_{R,i})^{-1}$ will normally not be independent. But from (2.9) we see that $I$ and $R$ can alternatively be represented as

$$I_t = \bar{i}t + W_{I,t} + \sum_{i=1}^{N_{U,t}} S_{I,i}, \qquad R_t = rt + W_{R,t} + \sum_{i=1}^{N_{U,t}} S_{R,i},$$

where $N_U$ is a Poisson process with intensity $\lambda_U$, independent of $N_P$, and the vectors $(S_{I,i}, S_{R,i})$ are i.i.d., independent of the $S_{P,i}$'s.

As in the calculations leading to (2.17), we find that $U$ can be written as

$$U_t = \exp\{-\alpha_U t + \sigma_U W_{U,t}\} \prod_{i=1}^{N_{U,t}} S_{U,i}, \tag{2.19}$$

where the two products are independent and $S_{U,i} = (1+S_{I,i})/(1+S_{R,i})$. If we let

$$F_U(s) = P(S_U \leqslant s), \tag{2.20}$$

then $F_I(0) = F_R(0) = 0$ implies $F_U(0) = 0$ and $F_U(\infty) = 1$. In case $\prod_{i=1}^{N_{I,t}} (1+\tilde{S}_{I,i})$ and $\prod_{i=1}^{N_{R,t}} (1+\tilde{S}_{R,i})$ are independent, we have the following relationship between (2.17) and (2.19).

**Lemma 2.1.** *Assume $\sum_{i=1}^{N_{I,t}} \tilde{S}_{I,t}$ in (2.12) and $\sum_{i=1}^{N_{R,t}} \tilde{S}_{R,t}$ in (2.13) are independent. Let*

$$V_t = \prod_{i=1}^{N_{I,t}} (1+\tilde{S}_{I,t}) \prod_{i=1}^{N_{R,t}} \frac{1}{1+\tilde{S}_{R,i}}.$$

*Then $V$ can be written as*

$$V = \prod_{i=1}^{N_{V,t}} S_{V,i}$$

*where $N_V$ is a Poisson process with intensity $\lambda_V = \lambda_I + \lambda_R$, the $S_{V,i}$'s are i.i.d. independent of $N_V$ and $S_V$ has the distribution*

$$F_V(s) = \frac{\lambda_I}{\lambda_V} F_I(s) + \frac{\lambda_R}{\lambda_V}\left(1 - F_R\left(\frac{1}{s-}\right)\right).$$

**Proof.** We have that

$$\log V_t = \sum_{i=1}^{N_{I,t}} \log(1+\tilde{S}_{I,i}) - \sum_{i=1}^{N_{R,t}} \log(1+\tilde{S}_{R,i})$$

so if we define

$$\psi_I(u)=E[\exp\{\mathrm{i}u\log(1+\tilde{S}_I)\}] \quad \text{and} \quad \psi_R(u)=E[\exp\{-\mathrm{i}u\log(1+\tilde{S}_R)\}],$$

we get by independence and the formula for the characteristic function of a compound Poisson process (Feller, 1971, formula 2.4, p. 504),

$$\begin{aligned} E[\exp\{\mathrm{i}u\log V_t\}] &= \exp\{\lambda_I t(\psi_I(u)-1)+\lambda_R t(\psi_R(u)-1)\} \\ &= \exp\{\lambda_V t(\psi_V(u)-1)\} \end{aligned}$$

where $\psi_V(u)=(\lambda_I/\lambda_V)\psi_I(u)+(\lambda_R/\lambda_V)\psi_R(u)$ is the characteristic function of the mixture $G_V(s)=(\lambda_I/\lambda_V)G_I(s)+(\lambda_R/\lambda_V)G_R(s)$, $G_I(s)=P(\log(1+\tilde{S}_I)\leq s)$ and $G_R(s)=P(-\log(1+\tilde{S}_R)\leq s)$. This means that $\log V_t$ is a compound Poisson process with intensity $\lambda_V$, i.e.

$$\log V_t=\sum_{i=1}^{N_{V,t}}\log S_{V,i}$$

where $\log S_V$ has the distribution $G_V$, hence $S_V$ has the distribution

$$F_V(s)=\frac{\lambda_I}{\lambda_V}P((1+\tilde{S}_I)\leq s)+\frac{\lambda_R}{\lambda_V}P\left(\frac{1}{1+\tilde{S}_R}\leq s\right). \qquad \square$$

## 3. Ruin theory

In this section we will retain the assumptions of Section 2, i.e. $Y$ is given as $Y=U^{-1}(y+U_-\cdot P)$ where $U$ and $P$ are given in (2.19) and (2.11). If in addition we define

$$m_{P,k}=E[S_P^k], \qquad m_{U,k}=E[S_U^k],$$

then we will assume that $m_{P,2}$ and $m_{U,2}$ both exist and are finite.

Throughout the section we will let $T_R=\inf\{t\colon Y_t<0\}=\inf\{t\colon \bar{Y}_t<0\}$ and $T_R=\infty$ if $Y_t\geq 0\ \forall t$. Then $T_R$ is the time of ruin, and we will let $R(y)=P(T_R<\infty)$ be the probability of eventual ruin. If we define the semimartingale $Z$ by $Z=U_-\cdot P$, then since $U_t>0\ \forall t$ (remember $F_I(0)=F_R(0)=0$),

$$T_R=\inf\{t\colon Z_t<-y\}. \tag{3.1}$$

This fact is made full use of in Harrison (1977), and we shall follow his steps with our more general model. We start by computing some expectations. By independence

$$E[U_t^k]=\exp\{-k\alpha_U t\}E[\exp\{k\sigma_U W_{U,t}\}]\cdot E\left[\prod_{i=1}^{N_{U,t}}S_{U,i}^k\right].$$

But $k\sigma_U W_{P,t}$ is normally distributed with zero mean and variance equal to $k^2\sigma_U^2 t$, hence $E[\exp\{k\sigma_U W_{U,t}\}]=\exp\{\frac{1}{2}k^2\sigma_U^2 t\}$. Furthermore, by conditioning on $N_{U,t}$,

$$E\left[\prod_{i=1}^{N_{U,t}}S_{U,i}^k\right]=E[m_{U,k}^{N_{U,t}}]=\sum_{n=0}^{\infty}m_{U,k}^n\frac{(\lambda_U t)^n}{n!}\mathrm{e}^{-\lambda_U t}=\exp\{(m_{U,k}-1)\lambda_U t\}.$$

We thus end up with

$$E[U_t^k] = \exp\{-(k\alpha_U - \tfrac{1}{2}k^2\sigma_U^2 - \lambda_U(m_{U,k} - 1))t\} \overset{\text{def}}{=} e^{-\mu_k t}. \tag{3.2}$$

Here we have tacitly assumed that $E[S_U^k]$ exists. It follows from Jensen's inequality that if $\mu_k > 0$, then $\mu_l > 0$ for $l \leq k$. By using (2.18) we find

$$\begin{aligned} \mu_1 &= r - \bar{i} + \rho\sigma_I\sigma_R - \sigma_R^2 + \lambda_U(1 - m_{U,1}), \\ \mu_2 &= 2(r - \bar{i}) + 4\rho\sigma_I\sigma_R - \sigma_I^2 - 3\sigma_R^2 + \lambda_U(1 - m_{U,2}). \end{aligned} \tag{3.3}$$

By Fubini's theorem and the fact that $l(s\colon U_{s-} \neq U_s) = 0$ where $l$ denotes Lebesgue measure, we have

$$m_1(t) \overset{\text{def}}{=} E\left[\int_0^t U_{s-}\,\mathrm{d}s\right] = \frac{1}{\mu_1}(1 - e^{-\mu_1 t}). \tag{3.4}$$

It is easy to see that for $s \geq u$, $U_u$ and $U_s/U_u$ are independent and that $U_s/U_u$ has the same distribution as $U_{s-u}$. (Consider $\log U_u$ and $\log(U_s/U_u) = \log U_s - \log U_u$.) Therefore

$$E[U_s U_u] = E[U_u^2]E[U_{s-u}] = e^{-\mu_1 s}\, e^{-(\mu_2 - \mu_1)u}.$$

So by the same arguments leading to (3.4),

$$\begin{aligned} m_2(t) \overset{\text{def}}{=} E\left[\left(\int_0^t U_{s-}\,\mathrm{d}s\right)^2\right] &= 2\int_0^t \int_0^s E[U_s U_u]\,\mathrm{d}u\,\mathrm{d}s \\ &= 2\left(\frac{1}{\mu_1\mu_2} + \frac{1}{\mu_2(\mu_2 - \mu_1)} e^{-\mu_2 t} - \frac{1}{\mu_1(\mu_2 - \mu_1)} e^{-\mu_1 t}\right). \end{aligned} \tag{3.5}$$

And from what was said after (3.2), it follows that

$$m_2(t) \to \frac{2}{\mu_1\mu_2} \quad \text{when } t \to \infty \quad \text{iff} \quad \mu_2 > 0. \tag{3.6}$$

We can now state the following theorem. The notation and assumptions are the same as above.

**Theorem 3.1.** *Let* $\beta_P = p - \lambda_P m_{P,1}$. *Then* $Z_t = \int_0^t U_{s-}\,\mathrm{d}P_s$ *is a*

| | |
|---|---|
| *supermartingale* | *if* $\beta_P < 0$, |
| *martingale* | *if* $\beta_P = 0$, |
| *submartingale* | *if* $\beta_P > 0$. |

*Assume* $\mu_1 > 0$. *Then* $\lim_{t\to\infty} Z_t = Z_\infty$ *exists and convergence takes place both almost surely and in* $L^1$. *The expectation of* $Z_\infty$ *is*

$$E[Z_\infty] = \beta_P/\mu_1.$$

*Finally if* $\mu_2 > 0$ *then* $E[Z_\infty^2] < \infty$.

**Proof.** Decompose the semimartingale $P$ into $P_t = M_t + \beta_P t$ where $M_t = W_{P,t} + \sum_{i=1}^{N_{P,t}} S_{P,i} - \lambda_P m_{P,1} t$ is a martingale. Then

$$Z_t = \int_0^t U_{s-}\,\mathrm{d}M_s + \beta_P \int_0^t U_{s-}\,\mathrm{d}s. \tag{3.7}$$

The increasing process $V_t = \int_0^t U_{s-}\,\mathrm{d}s$ is square integrable by (3.5), and by (3.4), $E[V_\infty] < \infty$ iff $\mu_1 > 0$. By (3.6) $E[V_\infty^2] < \infty$ iff $\mu_2 > 0$.

We now consider the local martingale $N_t = \int_0^t U_{s-}\,\mathrm{d}M_s$. Let $N_t^* = \sup_{0\le s\le t}|N_s|$ and $N_\infty^* = \sup_t |N_t|$. By the Burkholder-Davis-Gundy inequality (Dellacherie and Meyer, 1980, Chapter VII, Theorem 92),

$$E[(N_t^*)^p] \le c_p E[[N, N]_t^{p/2}], \quad 0 \le t \le \infty, \tag{3.8}$$

for $p \ge 1$ and some constant $c_p > 0$. Since $N = U_- \cdot M$, it is well known that $[N, N] = U_-^2 \cdot [M, M]$. By definition of the optional quadratic variation process we see that $[M, M]_t = \sigma_P^2 t + \sum_{i=1}^{N_{P,t}} S_{P,i}^2$. Let $T_1 < T_2 < \cdots$ be the times of jumps of $N_P$. Since $N_P$ and $N_U$ are independent, we have a.s. $U_{T_i -} = U_{T_i}$. Therefore we have a.s.

$$[N, N]_t = \sigma_P^2 \int_0^t U_s^2\,\mathrm{d}s + \sum_{i=1}^{N_{P,t}} U_{T_i}^2 S_{P,i}^2$$

and

$$[N, N]_\infty = \sigma_P^2 \int_0^\infty U_s^2\,\mathrm{d}s + \sum_{i=1}^{\infty} U_{T_i}^2 S_{P,i}^2 \tag{3.9}$$

and

$$[N, N]_\infty^{1/2} \le \sigma_P \left( \int_0^\infty U_s^2\,\mathrm{d}s \right)^{1/2} + \sum_{i=1}^{\infty} U_{T_i} |S_{P,i}|. \tag{3.10}$$

By conditioning on $N_{P,t}$, using (3.2) and the fact that given $N_{P,t} = m$, $T_1, \ldots, T_m$ have the same distribution as $m$ ordered uniformly distributed random variables on $[0, t]$, some calculations give

$$E[N, N]_t = \frac{1}{\mu_1} (\sigma_P^2 + \lambda_P E[S_P^2])(1 - \mathrm{e}^{-\mu_2 t}).$$

Therefore by Protter (1990, Theorem 47, p. 35), $N_t$ is a square integrable martingale. This finishes the first part of the theorem.

Using (3.2) and the fact that $T_i$ is gamma distributed with parameters $\lambda_P$ and $i$, we obtain for $k = 1, 2$,

$$\begin{aligned} E\left[ \sum_{i=1}^{\infty} U_{T_i}^k |S_{P,i}|^k \right] &= E[|S_P|^k] \sum_{i=1}^{\infty} E[\mathrm{e}^{-\mu_k T_i}] \\ &= E[|S_P|^k] \sum_{i=1}^{\infty} \left( \frac{\lambda_P}{\lambda_P + \mu_k} \right)^i < \infty \quad \text{iff} \quad \mu_k > 0. \end{aligned} \tag{3.11}$$

Furthermore

$$E\left[\left(\int_0^\infty U_s^2\,\mathrm{d}s\right)^{1/2}\right] \le E\left[\left(\sum_{n=0}^\infty \sup_{n\le s<n+1} U_s^2\right)^{1/2}\right] \le E\left[\sum_{n=0}^\infty \sup_{n\le s<n+1} U_s\right].$$

Now for $s>u$, $U_s$ and $U_s/U_u$ are independent with $U_s/U_u$ having the same distribution as $U_{s-u}$. Therefore

$$E\left[\sup_{n\le s<n+1} U_s\right] = E[U_n]E\left[\sup_{0\le s<1} U_s\right].$$

But $\sup_{0\le s<1} U_s \le \mathrm{e}^{\sigma_U B_1}\prod_{i=1}^{N_{U,1}}(S_{U,i}\vee 1)$ where $B_t = \max_{0\le s\le t} W_{U,s}$. By Karatzas and Shreve (1988, formula (8.3), p. 96), $E[\mathrm{e}^{\sigma_U B_1}]<\infty$, hence by independence,

$$E\left[\sup_{0\le s<1} U_s\right] = a < \infty. \tag{3.12}$$

Therefore

$$E\left[\left(\int_0^\infty U_s^2\,\mathrm{d}s\right)^{1/2}\right] \le a\sum_{n=0}^\infty \mathrm{e}^{-\mu_1 n} < \infty \quad \text{iff} \quad \mu_1 > 0. \tag{3.13}$$

Combining (3.8), (3.10), (3.11) and (3.13) gives that $\mu_1>0$ implies $E[N_\infty^*]<\infty$, and again by Protter (1990, Theorem 4.7, p. 35), $N_t$ is a uniformly integrable martingale. From (3.2), (3.8), (3.9) and (3.11) we see that $\mu_2>0$ implies $E[(N_\infty^*)^2]<\infty$. This finishes the proof of the theorem. □

**Remark 3.1.** With the exception of the final statement, Theorem 3.1 holds under the weaker conditions $E[|S_P|]<\infty$ and $E[|S_U|]<\infty$.

**Remark 3.2.** From (3.2) we see that if $\sigma_U^2>0$, then for $k$ sufficiently large the term $k^2\sigma_U^2$ will be dominant in $\mu_k$, hence $\mu_k<0$ for all $k\ge K$ say. By (3.7) and calculations similar to (3.5), this implies that $E[|Z_t|^k]\to\infty$ as $t\to\infty$, hence $Z_\infty$ in Theorem 3.1 can only have a finite number of finite moments. The same argument applies if $\lambda_U>0$ provided $S_U$ has positive probability of assuming values larger than one.

In the rest of this paper we will always assume $\mu_1>0$, without explicitly stating so all the time. We will also assume that the model is not totally degenerate, i.e. we will assume that either

1. $\sigma_U^2>0$ or $\lambda_U>0$ (and of course if $\lambda_U>0$, then $F_U(\{1\})=F_U(1)-F_U(1-)<1$),

or

2. $\sigma_P^2>0$ or $\lambda_P>0$ with $F_P(\{0\})<1$.

If neither of the above conditions are satisfied, then $Z_\infty = p/(r-\bar{i})$. In papers dealing with the classical ruin problem as well as those cited in the introduction, assumption 2 is satisfied, while assumption 1 is not.

The following result is an extension of Proposition 2.2 and Theorem 2.3 of Harrison (1977). The proof follows closely those of Harrison, but because our model is more complicated, we will give it here.

**Theorem 3.2.** *Let $H$ be the distribution function of $Z_\infty$, where $Z_\infty$ is given in Theorem 3.1. Then $H$ is continuous and the probability of eventual ruin is given by*

$$R(y)=\frac{H(-y)}{E[H(-Y_T)\,|\,T<\infty]}.$$

**Proof.** By assumptions of nondegeneracy, it is easy to see that $H$ is not concentrated at one point. Let

$$V_t=U_{t-}^{-1}\int_t^\infty U_{s-}\,\mathrm{d}P_s=\int_t^\infty\left(\frac{U_s}{U_t}\right)_-\mathrm{d}P_s=\int_0^\infty \tilde{U}_{s-}\,\mathrm{d}\tilde{P}_s \tag{3.14}$$

where $\tilde{U}_s=U_{s+t}/U_t\sim U_s$ and $\tilde{P}_s=P_{t+s}-P_t\sim P_s$. (By $X\sim Y$ we will mean that $X$ and $Y$ have the same distribution, see the argument after (3.4).) Since $(P, I, R)$ is a process of stationary, independent increments it follows that both $\tilde{U}_s$ and $\tilde{P}_s$ are independent of $\mathscr{F}_t$, hence $V_t$ is independent of $\mathscr{F}_t$ and $V_T$ is independent of $\mathscr{F}_T$ where $T$ is any $\mathscr{F}_t$ stopping time. It also follows from (3.14) that $V_t\sim Z_\infty$, and hence that $V_T\sim Z_\infty$ for any $\mathscr{F}_t$ stopping time $T$.

Now let $p$ be the largest probability of any point mass of $Z_\infty$. Assume $P(Z_\infty=c_i)=p$, $i=1,\dots,K$, and let $G_t$ be the distribution of $U_{t-}^{-1}(c_1-Z_t)$. Then since $Z_\infty=Z_t+U_{t-}V_t$,

$$p=P(Z_\infty=c_1)=P(V_t=U_{t-}^{-1}(c_1-Z_t))=\int_{-\infty}^{\infty}H(\{z\})\,\mathrm{d}G_t(z),$$

which implies that $G_t(\{c_1,\dots,c_K\})=1$ $\forall t$. But $Z_t\to Z_\infty$ a.s. and $U_t\to 0$ a.s. as $t\to\infty$. Hence

$$\begin{aligned}H(\{c_1\})=P(Z_\infty=c_1)&\geqslant P\left(\limsup_n\{Z_n=c_1-U_{n-}\{c_1,\dots,c_K\}\}\right)\\&\geqslant \limsup_n P(Z_n=c_1-U_{n-}\{c_1,\dots,c_K\})=1,\end{aligned}$$

a contradiction, hence $p=0$ and $H$ is continuous.

For notational simplicity we replace $T_R$ by $T$. On $\{T<\infty\}$ we have a.s.

$$y+Z_\infty=y+Z_T+U_{T-}V_T=U_{T-}[U_{T-}^{-1}(y+Z_T)+V_T]=U_T[Y_T+V_T]$$

since a.s. $\bar{I}_T=\bar{I}_{T-}$ and $\bar{R}_T=\bar{R}_{T-}$. This is because $F_U(0)=0$, hence ruin will occur as a result of the behaviour of $P$ at time $T$, and we have assumed that $P$ and $(I, R)$ are independent. Therefore by continuity of $H$ (see (3.1)),

$$\begin{aligned}H(-y)&=P(y+Z_\infty<0)=P(T<\infty,\,y+Z_\infty<0)\\&=P(T<\infty,\,V_T<-Y_T)=\int_{\{T<\infty\}}P(V_T<-Y_T\,|\,\mathscr{F}_T)\,\mathrm{d}P\\&=\int_{\{T<\infty\}}H(-Y_T)\,\mathrm{d}P=E[H(-Y_T)\,|\,T<\infty]P(T<\infty).\end{aligned}$$

Here the third equality follows since $U_t > 0\ \forall t$, and the fourth is just the definition of conditional probability. Note that $T$ is an $\mathscr{F}_t$ stopping time. The fifth equality follows since $V_T$ is independent of $\mathscr{F}_T$ and has the same distribution as $Z_\infty$ (see above), and $Y_T$ is $\mathscr{F}_T$ measurable. □

**Remark 3.3.** In Propositions 3.4 and 3.5 we will give sufficient conditions to ensure that $H$ is twice continuously differentiable, thus strengthening the first part of Theorem 3.2.

Now assume ruin is caused by a claim $S_{P,N_{P,T}}$, and not by drift in the term $W_{P,t}$. For simplicity we again replace $T_R$ by $T$. Then a.s. (see (2.3))

$$\Delta Y_T = \bar{I}_{T-}^{-1}\Delta \bar{P}_T = \Delta P_T = -S_{P,N_{P,T}}.$$

Assume $S_P$ exponentially distributed. By definition of $T_R$, $Y_{T-} \geq 0$ and $Y_T < 0$, so we know that $S_{P,N_{P,T}} > Y_{T-}$. But then the memoryless property of the exponential distribution implies that $-Y_T$ has the same distribution as $S_P$.

More generally if $S_P$ has an increasing failure rate, i.e. $P(S_P > t+s \mid S_P > t) \leq P(S_P > s)\ \forall t, s$, then

$$E[H(-Y_T) \mid T < \infty] \leq E[H(S_P)],$$

hence

$$R(y) \geq H(-y)/E[H(S_P)].$$

Similarly if $S_P$ has a decreasing failure rate, we reverse the above inequality. Note that a mixture of decreasing failure rates is again a decreasing failure rate (Ross, 1983, Theorem 8.1.5, p. 254).

If ruin is caused by drift in $W_P$, then $-Y_T = 0$, so in this case

$$R(y) = H(-y)/H(0).$$

To summarize we have proved:

**Corollary 3.1.** *We always have*

$$R(y) \leq H(-y)/H(0)$$

*with equality if* $\lambda_P = 0$.

*If* $S_P$ *has increasing failure rate, then*

$$R(y) \geq H(-y)/E[H(S_P)].$$

*If* $\sigma_P^2 = 0$ *and* $S_P$ *has decreasing failure rate, then*

$$R(y) \leq H(-y)/E[H(S_P)]$$

*with equality if* $S_P$ *is exponentially distributed.* □

Motivated by this result we will now set forth to find expressions for $H(z)$. Since $H$ is the distribution of $Z_\infty$, which is just a randomly discounted infinite time income process so that $E[Z_\infty]=\beta_P/\mu_1$ may be regarded as net present value, our results may have applications other than those proposed in this article. See Dufresne (1990) who considers a special case.

We define

$$\nu(u)=\mathrm{i}up-\tfrac{1}{2}u^2\sigma_P^2-\lambda_P(1-\phi(-u)) \tag{3.15}$$

where

$$\phi(u)=E[\mathrm{e}^{\mathrm{i}uS_P}]. \tag{3.16}$$

This implies that

$$E[\mathrm{e}^{\mathrm{i}uP_t}]=\mathrm{e}^{\nu(u)t}. \tag{3.17}$$

Finally define

$$\psi(u)=E[\mathrm{e}^{\mathrm{i}uZ_\infty}]. \tag{3.18}$$

The following proposition is an extension of a result in Proposition 2.2 in Harrison (1977).

**Proposition 3.1.** *With the above definitions we have*

$$\psi(u)=E\left[\exp\left\{\int_0^\infty \nu(uU_s)\,\mathrm{d}s\right\}\right]=E^u\left[\exp\left\{\int_0^\infty \nu(U_s)\,\mathrm{d}s\right\}\right]$$

*where in the first expectation $U_0=1$ while in the second $U_0=u$.*

**Proof.** The equality of the two expectations follows from the fact that $U_s=U_0(U_s/U_0)=U_0\tilde{U}_s$ where $\tilde{U}_0=1$ and $\tilde{U}_s$ is independent of $U_0$.

To prove the first equality, let $\mathscr{G}=\sigma\{U_s\colon s\geq 0\}$. Note that the $\sigma$-algebras $\mathscr{G}$ and $\sigma\{P_s\colon s\geq 0\}$ are independent. Let $\delta_k^{(n)}=k2^{-n}$, $k=0,1,\ldots,2^n-1$. Also define $t_k=t\delta_k^{(n)}$, $U_k=U_{t_k}$ and $P_k=P_{t_k}$. Then if

$$Z_t^{(n)}=\sum_{k=0}^{2^n-1} U_k(P_{k+1}-P_k)$$

it follows from Dellacherie and Meyer (1980, Theorem VIII-15) that $Z_t^{(n)}\xrightarrow{\mathrm{P}} Z_t$ as $n\to\infty$. Therefore

$$\lim_{n\to\infty} E[\mathrm{e}^{\mathrm{i}uZ_t^{(n)}}]=E[\mathrm{e}^{\mathrm{i}uZ_t}].$$

And since $Z_t\to Z_\infty$ a.s. as $n\to\infty$,

$$\lim_{t\to\infty}\lim_{n\to\infty} E[\mathrm{e}^{\mathrm{i}uZ_t^{(n)}}]=\psi(u).$$

Now by independence and (3.17),

$$\begin{aligned} E[\mathrm{e}^{\mathrm{i}uZ_t^{(n)}}] &= E\left[E\left[\exp\left\{\mathrm{i}u \sum_{k=0}^{2^n-1} U_k(P_{k+1}-P_k)\right\} \Big| \mathscr{G}\right]\right] \\ &= E\left[\prod_{k=0}^{2^n-1} E[\exp\{\mathrm{i}uU_k(P_{k+1}-P_k)\}\,|\,\mathscr{G}]\right] \\ &= E\left[\prod_{k=0}^{2^n-1} \exp\{\nu(uU_k)(t_{k+1}-t_k)\}\right] \\ &= E\left[\exp\left\{\sum_{k=0}^{2^n-1} \nu(uU_k)(t_{k+1}-t_k)\right\}\right]. \end{aligned}$$

By (3.15) and (3.16), $\mathrm{Re}(\nu(uU_k)) \leq 0$, and since $\nu$ is continuous, dominated convergence gives

$$\lim_{n\to\infty} E[\mathrm{e}^{\mathrm{i}uZ_t^{(n)}}] = E\left[\exp\left\{\int_0^t \nu(uU_s)\,\mathrm{d}s\right\}\right].$$

Letting $t\to\infty$, dominated convergence yields the desired result. □

From now on we will always assume that $\mu_2 > 0$. One problem with $\nu(u)$ is that it is unbounded. We therefore define

$$u_n^+ = \min\{u \geq 0: |\nu(u)| = n\}, \qquad u_n^- = \max\{u \leq 0: |\nu(u)| = n\}$$

and

$$\nu_n(u) = \nu((u \wedge u_n^+) \vee u_n^-). \tag{3.19}$$

Then $|\nu_n(u)| \leq n$ and $\nu_n(u) \to \nu(u)$ as $n\to\infty$. We also set

$$\psi_n(u) = E\left[\exp\left\{\int_0^\infty \nu_n(uU_s)\,\mathrm{d}s\right\}\right]. \tag{3.20}$$

By $\psi^{(k)}(u)$ we will mean the $k$th derivative of $\psi(u)$, $\psi^{(0)} = \psi(u)$. Similarly with $\psi_n^{(k)}(u)$.

**Lemma 3.1.** *$\psi$ and $\psi_n$ are both twice continuously differentiable, and there exists constants $M_k$, $k=0,1,2$ so that $\forall u, n$,*

$$|\psi^{(k)}(u)| \vee |\psi_n^{(k)}(u)| \leq M_k, \quad k=0,1,2.$$

*Also* $\lim_{n\to\infty} \psi_n^{(k)}(u) = \psi^{(k)}(u)$, $k=0,1,2$.

**Remark 3.4.** According to Theorem 3.1, $\mu_2 > 0$ implies that $E[Z_\infty^2] < \infty$. Therefore, since $\psi$ is the characteristic function of $Z_\infty$, the fact that $\psi$ has the above properties follows from standard results on characteristic functions. But this does not apply to $\psi_n$, so the tedious proof given below seems necessary. Some of the results obtained during the proof will also be needed later.

**Proof of Lemma 3.1.** Using $|e^{-ix}-1| \leqslant |x|$ we have by independence of $S_P$ and $U_s$,

$$|\phi(-uU_s)-1| = |E[(\exp\{-iuU_sS_P\}-1) \mid U_s]| \leqslant |u| U_s E[|S_P|].$$

And similarly

$$\left|\frac{d}{du}(\phi(-uU_s)-1)\right| \leqslant U_s E[|S_P|] \quad \text{and} \quad \left|\frac{d^2}{du^2}(\phi(-uU_s)-1)\right| \leqslant U_s^2 E[|S_P^2|].$$

So by the above there exists a constant $K>0$ such that (see (3.15))

$$|\nu(uU_s)| \leqslant K|u|U_s + \tfrac{1}{2}\sigma_P^2 u^2 U_s^2, \tag{3.21}$$

$$\left|\frac{d}{du}\nu(uU_s)\right| \leqslant KU_s + \sigma_P^2|u|U_s^2, \tag{3.22}$$

$$\left|\frac{d^2}{du^2}\nu(uU_s)\right| \leqslant KU_s^2. \tag{3.23}$$

Since by assumption $E[\int_0^\infty U_s^k \, ds] < \infty$, hence $\int_0^\infty U_s^k \, ds < \infty$ a.s. for $k=1,2$, we obtain by Billingsley (1986, Theorem 16.8, p. 215),

$$\frac{d^k}{du^k}\int_0^\infty \nu(uU_s)\, ds = \int_0^\infty \frac{d^k}{du^k}\nu(uU_s)\, ds, \quad k=1,2. \tag{3.24}$$

Now let $N_u = \{s: \nu(uU_s) = \{u_n^-, u_n^+\}\}$. If $l$ denotes Lebesgue measure, then by Fubini's theorem,

$$E[l(N_u)] = E\left[\int_0^\infty 1_{\{uU_s=\{u_n^-,u_n^+\}\}}\, ds\right] = \int_0^\infty P(uU_s = \{u_n^-, u_n^+\})\, ds = 0,$$

hence $l(N_u)=0$ a.s. Furthermore if $S$ is a time of jump of $\prod_{i=1}^{N_{U,t}} S_{U,i}$, then $P(uU_S = \{u_n^-, u_n^+\}) = 0$, hence $uU_s$ will attain $\{u_n^-, u_n^+\}$ at a point of continuity of $U_s$. Since $\nu$ is continuous, this implies that $N_u$ is closed, hence $N_u^c$ is open. When $s \in N_u^c$, $\nu_n(uU_s) \neq \{u_n^-, u_n^+\}$, and since $\nu_n$ is continuous, there is a neighbourhood $O_s$ around $u$ so that $\nu_n(vU_s) \neq \{u_n^-, u_n^+\}$ when $v \in O_s$. Therefore $\nu_n(uU_s)$ is twice continuously differentiable when $s \in N_u^c$. So if we define $(d/du)\nu_n(uU_s) = (d^2/du^2)\nu_n(uU_s) = 0$ when $s \in N_u$, we have

$$|\nu_n(uU_s)| \leqslant (K|u|U_s + \tfrac{1}{2}\sigma_P^2 u^2 U_s^2) \wedge n, \tag{3.21'}$$

$$\left|\frac{d}{du}\nu_n(uU_s)\right| \leqslant (KU_s + \sigma_P^2|u|U_s^2)1_{[u_n^-,u_n^+]}(uU_s), \tag{3.22'}$$

$$\left|\frac{d^2}{du^2}\nu_n(uU_s)\right| \leqslant KU_s^2 1_{[u_n^-,u_n^+]}(uU_s). \tag{3.23'}$$

Let $g_n(u) = \int_0^\infty \nu_n(uU_s)\, ds$. Then

$$\frac{g_n(u+h)-g_n(u)}{h} = \int_0^\infty \frac{\nu_n((u+h)U_s) - \nu_n(uU_s))}{h}\, ds \tag{3.25}$$

and since $|\nu_n((u+h)U_s)-\nu_n(uU_s)|\leq|\nu((u+h)U_s)-\nu(uU_s)|$, we have from dominated convergence (as in Billingsley, 1986, Theorem 16.8), the fact that $\nu_n(uU_s)$ is differentiable on $N_u^c$ and that $l(N_u)=0$, that the limit as $h\to 0$ on the right side of (3.25) exists and is the same whether $h$ approaches zero from below or above. Therefore $g_n'(u)$ exists, and (3.24) applies for $\nu_n$ when $k=1$. Similarly we can prove that (3.24) applies for $\nu_n$ when $k=2$.

Let

$$X(u)=\exp\left\{\int_0^\infty \nu(uU_s)\,\mathrm{d}s\right\},\qquad X_n(u)=\exp\left\{\int_0^\infty \nu_n(uU_s)\,\mathrm{d}s\right\}. \tag{3.26}$$

Since $\mathrm{Re}(1-\phi(-uU_s))\geq 0$, we have

$$|X(u)|\leq\exp\left\{-\tfrac{1}{2}\sigma_P^2u^2\int_0^\infty U_s^2\,\mathrm{d}s\right\}. \tag{3.27}$$

And since $\nu_n(uU_s)=\nu((uU_s)\wedge u_n^+)\vee u_n^-)$,

$$|X_n(u)|\leq\exp\left\{-\tfrac{1}{2}\sigma_P^2u^2\int_0^\infty U_s^2 1_{[u_n^-,u_n^+]}(uU_s)\,\mathrm{d}s\right\}. \tag{3.27'}$$

By (3.24),

$$X'(u)=X(u)\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}u}\nu(uU_s)\,\mathrm{d}s. \tag{3.28}$$

So by (3.22) and (3.27),

$$\begin{aligned}|X'(u)|&\leq\int_0^\infty (KU_s+\sigma_P^2|u|U_s^2)\,\mathrm{d}s\cdot\exp\left\{-\tfrac{1}{2}\sigma_P^2u^2\int_0^\infty U_s^2\,\mathrm{d}s\right\}\\&\leq K\int_0^\infty U_s\,\mathrm{d}s+\sigma_P^2|u|\int_0^\infty U_s^2\,\mathrm{d}s\cdot\exp\left\{-\tfrac{1}{2}\sigma_P^2u^2\int_0^\infty U_s^2\,\mathrm{d}s\right\}\\&\leq K\int_0^\infty U_s\,\mathrm{d}s+\mathrm{e}^{-1/2}\sigma_P\left(\int_0^\infty U_s^2\,\mathrm{d}s\right)^{1/2}.\end{aligned} \tag{3.29}$$

The last inequality follows from the fact that for $a>0$,

$$a|u|\,\mathrm{e}^{-au^2}\leq\mathrm{e}^{-1/2}\sqrt{\tfrac{1}{2}a}. \tag{3.30}$$

By using (3.22′) and (3.27′), we find that (3.29) is valid for $|X_n'(u)|$ as well. (We may replace $U_s$ by $U_s1_{[u_n^-,u_n^+]}(uU_s)$, but this will not be needed in the sequel.) Since $\psi(u)=E[X(u)]$ and $\psi_n(u)=E[X_n(u)]$, we have from (3.29) and dominated convergence,

$$\begin{aligned}&|\psi'(u)|=\left|\frac{\mathrm{d}}{\mathrm{d}u}E[X(u)]\right|=|E[X'(u)]|''\leq E[|X'(u)|]\leq M_1,\\&|\psi_n'(u)|\leq M_1,\end{aligned} \tag{3.31}$$

where $M_1$ is some constant.

For $n$ sufficiently large, $\nu_n(uU_s)=\nu(uU_s)$, therefore $(\mathrm{d}/\mathrm{d}u)\nu_n(uU_s)\to(\mathrm{d}/\mathrm{d}u)\nu(uU_s)$ as $n\to\infty$. Hence by (3.22′), (3.23′), (3.28) (which is valid for $X_n$ as well), and dominated convergence,

$$X_n(u) \to X(u) \text{ and } X'_n(u) \to X'(u) \text{ a.s. as } n \to \infty.$$

This implies

$$\lim_{n\to\infty} \psi'_n(u) = \lim_{n\to\infty} E[X'_n(u)] = E\left[\lim_{n\to\infty} X'_n(u)\right]$$
$$= E[X'(u)] = \frac{\mathrm{d}}{\mathrm{d}u} E[X(u)] = \psi'(u) \tag{3.32}$$

by (3.29) and dominated convergence.

By (3.24) and (3.28),

$$X''(u) = X(u)\left[\int_0^\infty \frac{\mathrm{d}^2}{\mathrm{d}u^2} \nu(uU_s)\,\mathrm{d}s + \left(\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}u} \nu(uU_s)\,\mathrm{d}s\right)^2\right]. \tag{3.33}$$

So by (3.22), (3.23) and (3.27),

$$\begin{aligned} |X''(u)| &\leq K \int_0^\infty U_s^2\,\mathrm{d}s \\ &\quad + \left(K \int_0^\infty U_s\,\mathrm{d}s + \sigma_P^2|u| \int_0^\infty U_s^2\,\mathrm{d}s\right)^2 \exp\left\{-\tfrac{1}{2}\sigma_P^2 u^2 \int_0^\infty U_s^2\,\mathrm{d}s\right\} \\ &\leq (K + 2\sigma_P^2) \int_0^\infty U_s^2\,\mathrm{d}s + K^2\left(\int_0^\infty U_s\,\mathrm{d}s\right)^2 \\ &\quad + 2\mathrm{e}^{-1/2} K\sigma_P \left(\int_0^\infty U_s\,\mathrm{d}s\right)\left(\int_0^\infty U_s^2\,\mathrm{d}s\right)^{1/2} \end{aligned} \tag{3.34}$$

where we have used that for $a > 0$ and $b > 0$ (see (3.30)),

$$(a + 2b|u|)^2\,\mathrm{e}^{-bu^2} \leq a^2 + 2\sqrt{2b}\,\mathrm{e}^{-1/2}a + 4b.$$

Now by (3.2), (3.5) and the Cauchy–Schwarz inequality the expectation of all terms on the right of (3.34) are finite, hence for some constant $M_2$,

$$\psi''(u) = \frac{\mathrm{d}^2}{\mathrm{d}u^2} E[X(u)] = E[X''(u)] \Rightarrow |\psi''(u)| \leq E[|X''(u)|] \leq M_2. \tag{3.35}$$

By using (3.21′)–(3.23′) and (3.27′), we see that (3.34) and thus (3.35) are valid for $X_n$ as well. That $\lim_{n\to\infty} \psi''_n(u) = \psi''(u)$ follows as in (3.32).

Finally it follows from (3.33), (3.34) and dominated convergence that $\psi''(u)$ is continuous. This finishes the proof of the lemma. □

**Lemma 3.2.** *Let $\nu_n$ and $\psi_n$ be as in* (3.19) *and* (3.20), *and let $A$ be the weak generator of $U$. Then $\psi_n \in \mathcal{D}_A$ and is the solution of*

$$A\psi_n = -\nu_n\psi_n.$$

**Proof.** It is easy to see that $U$ is a stochastically continuous, conservative Feller process, so by Dynkin (1965, p. 58), the domain $\mathscr{D}_A$ of the weak generator $A$ consists of all functions $R_\alpha g$ of the form

$$R_\alpha g(u) = E^u\left[\int_0^\infty e^{-\alpha t} g(U_t)\,dt\right]$$

where $\alpha > 0$ and $g$ is bounded and continuous.

Let $\alpha > 0$ and define

$$\begin{aligned} z(u) &= E^u\left[\int_0^\infty \nu_n(U_t)\exp\left\{-\alpha t - \int_0^t -(\alpha + \nu_n(U_s))\,ds\right\}dt\right] \\ &= E^u\left[\int_0^\infty \frac{d}{dt}\exp\left\{\int_0^t \nu_n(U_s)\,ds\right\}dt\right] = \psi_n(u) - 1. \end{aligned}$$

It follows from Lemma 3.1 that $z$ is bounded. By (3.21′),

$$\begin{aligned} E^u\Bigg[\int_0^\infty \int_0^t \Bigg| & e^{-\alpha t}\nu_n(U_t)(-\alpha - \nu_n(U_s)) \\ & \times \exp\left\{-\int_s^t -(\alpha + \nu_n(U_u))\,du\right\}\Bigg|\,ds\,dt\Bigg] \\ &\leqslant (n+\alpha)E^u\left[\int_0^\infty \int_0^t |\nu_n(U_t)\,e^{-\alpha s}|\,ds\,dt\right] \\ &\leqslant (n+\alpha)\int_0^\infty E^u[|\nu_n(U_t)|]\,dt < \infty. \end{aligned}$$

We can therefore repeat verbatim the proof of Karatzas and Shreve (1988, pp. 272–273) to obtain

$$R_\alpha(-(\alpha + \nu_n)z) = R_\alpha \nu_n - z. \tag{3.36}$$

By using the inversion formula (Dynkin, 1965, Theorem 1.7, p. 40),

$$(\alpha - A)R_\alpha g = g$$

with $g = \nu_n$ and $g = -(\alpha + \nu_n)z$, then using (3.36) in the latter case and finally subtracting the two expressions, we obtain

$$A(\psi_n - 1) = (\alpha - (\alpha + \nu_n))(\psi_n - 1) - \nu_n.$$

Using that $A1 = 0$ gives the desired result. □

We will denote by $C_b^2(R)$ the space of all bounded twice continuously differentiable functions with a bounded first and second derivative. For such functions we have:

**Lemma 3.3.** *The integro-differential operator L defined by*

$$Lf(u)=\tfrac{1}{2}\sigma_U^2 u^2 f''(u)-(\alpha_U-\tfrac{1}{2}\sigma_U^2)uf'(u)$$
$$+\lambda_U\int_0^\infty (f(us)-f(u))\,\mathrm{d}F_U(s) \tag{3.37}$$

*equals the weak generator A of U on* $\mathscr{D}_A\cap C_b^2(R)$. *Here* $\alpha_U$ and $\sigma_U^2$ *are given in* (2.18).

**Proof.** As in (1.1) and (2.15) we have that $U$ (see (2.19)) is the solution of

$$\mathrm{d}U_t=U_{t-}\,\mathrm{d}S_t\quad\text{where } u_0=u. \tag{3.38}$$

Here $S_t=a_Ut+\sigma_UW_{U,t}+\sum_{i=1}^{N_{U,t}}(S_{U,i}-1)$ where $a_U=-(\alpha_U-\frac{1}{2}\sigma_U^2)$ and $W_{U,t}$ is a Brownian motion. Writing for simplicity $W_t=W_{U,t}$, Itô's formula (Jacod and Shiryaev, 1987, Theorem 4.57, p. 57) and (3.38) gives

$$\begin{aligned}f(U_t)-f(u)&=\int_0^t f'(U_{s-})\,\mathrm{d}U_s+\tfrac{1}{2}\int_0^t f''(U_{s-})\,\mathrm{d}\langle U^c,U^c\rangle_s\\&\quad+\sum_{0\le s\le t}[f(U_s)-f(U_{s-})-f'(U_{s-})\Delta U_s]\\&=\int_0^t (a_UU_{s-}f'(U_{s-})+\tfrac{1}{2}\sigma_U^2U_{s-}^2f''(U_{s-}))\,\mathrm{d}s\\&\quad+\sigma_U\int_0^t U_{s-}f'(U_{s-})\,\mathrm{d}W_s+\sum_{0\le s\le t}[f(U_s)-f(U_{s-})].\end{aligned} \tag{3.39}$$

Since $f'$ is bounded and $E[\int_0^t U_s^2\,\mathrm{d}s]<\infty$, we have that

$$E\left[\int_0^t U_{s-}f'(U_{s-})\,\mathrm{d}W_s\right]=0. \tag{3.40}$$

Now let $c$ be a constant such that $|a_Uf'(x)|+|\frac{1}{2}\sigma_U^2f''(x)|\le c\ \forall x$. Let $r>0$ be given. Then

$$\sup_{0\le t\le r}\left|\frac{1}{t}\int_0^t (a_UU_{s-}f'(U_{s-})+\tfrac{1}{2}\sigma_U^2U_{s-}^2f''(U_{s-}))\,\mathrm{d}s\right|\le c\sup_{0\le t\le r}(U_t+U_t^2),$$

and similarly as in the proof of (3.12) we find

$$E\left[\sup_{0\le t\le r}U_t^2\right]<\infty.$$

Therefore by dominated convergence, continuity of $f'$ and $f''$, stochastic continuity of $U_t$, (3.39) and (3.40),

$$\begin{aligned}Af(u)&=\lim_{t\to 0}E^u\left[\frac{1}{t}(f(U_t)-f(u))\right]\\&=a_Uuf'(u)+\tfrac{1}{2}\sigma_U^2u^2f''(u)+\lim_{t\to 0}E^u\left[\frac{1}{t}\sum_{0\le s\le t}(f(U_s)-f(U_{s-}))\right].\end{aligned}$$

Finally since $P(N_{U,t} \geqslant 2) = o(t)$ and $f$ is bounded, it follows readily that the last limit equals $\lambda_U E[f(uS_U) - f(u)]$. $\square$

**Remark 3.5.** If $\sum_{i=1}^{N_{I,t}} \tilde{S}_{I,i}$ and $\sum_{i=1}^{N_{R,t}} \tilde{S}_{R,i}$ are independent, then using Lemma 2.1 and a change of variable in the integral in (3.37), it is easily verified that $Lf$ can alternatively be written as

$$\begin{aligned} Lf(u) &= \tfrac{1}{2}\sigma_U^2 u^2 f''(u) - (\alpha_U - \tfrac{1}{2}\sigma_U^2) u f'(u) \\ &\quad + \lambda_I \int_0^\infty (f(us) - f(u))\, \mathrm{d}F_I(s) \\ &\quad + \lambda_R \int_0^\infty (f(u/s) - f(u))\, \mathrm{d}F_R(s). \end{aligned} \tag{3.41}$$

By Lemma 3.1 both $\psi_n$ and $\psi$ belong to $C_b^2(R)$, and since $\psi_n \in \mathcal{D}_A$, $A\psi_n = L\psi_n$ where $L\psi_n$ is given in (3.37) with $f$ replaced by $\psi_n$. But then Lemma 3.1 and dominated convergence implies that $L\psi_n(u) \to L\psi(u)$ and $\nu_n(u)\psi_n(u) \to \nu(u)\psi(u)$ as $n \to \infty$. Therefore we have:

**Theorem 3.3.** *Let $\nu$ and $\psi$ be given by* (3.15) *and* (3.18) *respectively. Then $\psi$ is the solution of*

$$L\psi(u) = -\nu(u)\psi(u) \tag{3.42}$$

*where $L$ is given by* (3.37) (*or* (3.41) *when it applies*). *Initial conditions are*

$$\psi(0) = 1,$$

$$\psi'(0) = iE[Z_\infty] = i\frac{\beta_P}{\mu_1} \quad (\textit{see Theorem } 3.1). \qquad \square$$

Theorem 3.3 gives us an equation for the characteristic function $\psi$ of $Z_\infty$. But as we want to use Theorem 3.2 and Corollary 3.1 we are more interested in $H$, the distribution function of $Z_\infty$. The following theorem gives an equation for $H$.

**Theorem 3.4.** *Assume*:

(A1) *If $\sigma_U^2 > 0$ or $\sigma_P^2 > 0$ then*

$$\int_{-\infty}^{\infty} |u\psi(u)|\, \mathrm{d}u < \infty.$$

*Otherwise it is sufficient that*

$$\int_{-\infty}^{\infty} |\psi(u)|\, \mathrm{d}u < \infty.$$

(A2) $\displaystyle\int_{-\infty}^{\infty} |\psi'(u)|\, \mathrm{d}u < \infty.$

(A3) $E[|\log S_U|] < \infty.$

*Then the distribution function $H$ of $Z_\infty$ is twice continuously differentiable and is the solution of*

$$\tfrac{1}{2}(\sigma_U^2 z^2+\sigma_P^2)H''(z)+((\alpha_U+\tfrac{1}{2}\sigma_U^2)z-p)H'(z)-(\lambda_U+\lambda_P)H(z)$$
$$+\lambda_U\int_0^\infty H(z/s)\,\mathrm{d}F_U(s)+\lambda_P\int_{-\infty}^\infty H(z+s)\,\mathrm{d}F_P(s)=0. \tag{3.43}$$

*Boundary conditions are $H(-\infty)=0$ and $H(\infty)=1$. Also*

$$\int_{-\infty}^0 H(z)\,\mathrm{d}z+\int_0^\infty (1-H(z))\,\mathrm{d}z=\frac{\beta_P}{\mu_1}. \tag{3.44}$$

*If $\sigma_U^2=\sigma_P^2=0$ the weaker version of* (A1) *is sufficient, and in this case $H$ is the once continuously differentiable solution of* (3.43).

**Proof.** Assume for the moment that in case $\sigma_U^2>0$, (A1)-(A3) imply

$$\int_{-\infty}^\infty |u\psi''(u)|\,\mathrm{d}u<\infty. \tag{3.45}$$

Since by (A1) and Feller (1971, formula (3.11), p. 511),

$$H(z)=\frac{1}{2\pi}\lim_{a\to-\infty}\int_{-\infty}^\infty \frac{\mathrm{e}^{-\mathrm{i}ua}-\mathrm{e}^{-\mathrm{i}uz}}{\mathrm{i}u}\psi(u)\,\mathrm{d}u, \tag{3.46}$$

we multiply each term in (3.42) with $(\mathrm{e}^{-\mathrm{i}ua}-\mathrm{e}^{-\mathrm{i}uz})/2\mathrm{i}\pi u$, integrate from $-\infty$ to $\infty$ (must check that the integrals exist), and let $a\to-\infty$. The calculations are term by term (see (3.37)),

$$\begin{aligned}
\lim_{a\to-\infty}\int_{-\infty}^\infty \frac{\mathrm{e}^{-\mathrm{i}ua}-\mathrm{e}^{-\mathrm{i}uz}}{\mathrm{i}u}u^2\psi''(u)\,\mathrm{d}u &= \mathrm{i}\int_{-\infty}^\infty u\,\mathrm{e}^{-\mathrm{i}uz}\psi''(u)\,\mathrm{d}u\\
&= -\frac{\mathrm{d}}{\mathrm{d}z}\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}uz}\psi''(u)\,\mathrm{d}u\\
&= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\left(z\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}uz}\psi'(u)\,\mathrm{d}u\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(z^2\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}uz}\psi(u)\,\mathrm{d}u\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(z^2\frac{\mathrm{d}}{\mathrm{d}z}(2\pi H(z))\right)\\
&= 2\pi(z^2H''(z)+2zH'(z)).
\end{aligned} \tag{3.47}$$

Here the first equality follows from the Riemann-Lebesgue Lemma (Feller, 1971, Lemma 3, p. 513) and (3.45). The other equalities are just integration by parts, use of (A1), (A2), (3.45) and Billingsley (1986, Theorem 16.8, p. 215). Similarly we find that

$$\lim_{a\to-\infty}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}ua}-\mathrm{e}^{-\mathrm{i}uz}}{\mathrm{i}u}\,u\psi'(u)\,\mathrm{d}u=-2\pi H'(z). \tag{3.48}$$

We will now prove that

$$I=\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\frac{\mathrm{e}^{-\mathrm{i}ua}-\mathrm{e}^{-\mathrm{i}uz}}{\mathrm{i}u}\,\psi(us)\right|\,\mathrm{d}F_U(s)\,\mathrm{d}u<\infty. \tag{3.49}$$

For some constant $c>0$ we have $|(\mathrm{e}^{-\mathrm{i}ua}-\mathrm{e}^{-\mathrm{i}uz})/\mathrm{i}u|\leqslant c\ \forall u$, and since $|\psi(u)\leqslant 1$, we get

$$\int_{-1}^{1}\int_{0}^{\infty}\left|\frac{\mathrm{e}^{-\mathrm{i}ua}-\mathrm{e}^{-\mathrm{i}uz}}{\mathrm{i}u}\,\psi(us)\right|\,\mathrm{d}F_U(s)\,\mathrm{d}u\leqslant 2c.$$

Also by Fubini and a change of variables,

$$\begin{aligned}
&\int_{1}^{\infty}\int_{0}^{\infty}\left|\frac{\mathrm{e}^{-\mathrm{i}ua}-\mathrm{e}^{-\mathrm{i}uz}}{\mathrm{i}u}\,\psi(us)\right|\,\mathrm{d}F_U(s)\,\mathrm{d}u\\
&\quad\leqslant 2\int_{0}^{\infty}\int_{1}^{\infty}\left|\frac{\psi(us)}{u}\right|\,\mathrm{d}u\,\mathrm{d}F_U(s)\\
&\quad=2\int_{0}^{\infty}\int_{s}^{\infty}\left|\frac{\psi(v)}{v}\right|\,\mathrm{d}v\,\mathrm{d}F_U(s)\\
&\quad=2\int_{0}^{\infty}\int_{0}^{v}\left|\frac{\psi(v)}{v}\right|\,\mathrm{d}F_U(s)\,\mathrm{d}v\\
&\quad\leqslant 2\int_{0}^{1}\int_{0}^{v}\frac{1}{v}\,\mathrm{d}F_U(s)\,\mathrm{d}v+2\int_{1}^{\infty}\left|\frac{\psi(v)}{v}\right|\,F_U(v)\,\mathrm{d}v\\
&\quad\leqslant 2\int_{0}^{1}\int_{s}^{1}\frac{1}{v}\,\mathrm{d}v\,\mathrm{d}F_U(s)+2\int_{1}^{\infty}\left|\frac{\psi(v)}{v}\right|\,\mathrm{d}v<\infty
\end{aligned} \tag{3.50}$$

since the first integral on the right equals $E[|\log(S_U\wedge 1)|]$ which is finite by (A1). The integral from $-\infty$ to $-1$ in (3.49) is similar to (3.50). Therefore

$$\begin{aligned}
&\lim_{a\to-\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}ua}-\mathrm{e}^{-\mathrm{i}uz}}{\mathrm{i}u}\,\psi(us)\,\mathrm{d}F_U(s)\,\mathrm{d}u\\
&\quad=\lim_{a\to-\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}ua}-\mathrm{e}^{-\mathrm{i}uz}}{\mathrm{i}u}\,\psi(us)\,\mathrm{d}u\,\mathrm{d}F_U(s)\\
&\quad=\lim_{a\to-\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}va/s}-\mathrm{e}^{-\mathrm{i}vz/s}}{\mathrm{i}v}\,\psi(v)\,\mathrm{d}v\,\mathrm{d}F_U(s)\\
&\quad=2\pi\lim_{a\to-\infty}\int_{0}^{\infty}(H(z/s)-H(a/s)\,\mathrm{d}F_U(s)\\
&\quad=2\pi\int_{0}^{\infty}H(z/s)\,\mathrm{d}F_U(s),
\end{aligned} \tag{3.51}$$

where the first equality is Fubini and (3.49), the second a change of variables $v = us$ and the last monotone convergence.

This ends the calculations for the expressions on the left of (3.42). We proceed to the expressions on the right. As in (3.47),

$$\lim_{a\to-\infty}\int_{-\infty}^{\infty}\frac{e^{-iua}-e^{-iuz}}{iu}\,u^2\psi(u)\,du = i\int_{-\infty}^{\infty}u\,e^{-iuz}\psi(u)\,du$$
$$= -2\pi H''(z), \tag{3.52}$$

and

$$\lim_{a\to-\infty}\int_{-\infty}^{\infty}\frac{e^{-iua}-e^{-iuz}}{iu}\,u\psi(u)\,du = 2\pi i H'(z). \tag{3.53}$$

It is straightforward to verify that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{e^{-iua}-e^{-iuz}}{iu}\,e^{-ius}\psi(u)\right|\,dF_P(s)\,du<\infty.$$

Hence by Fubini and monotone convergence (see (3.16)),

$$\lim_{a\to-\infty}\int_{-\infty}^{\infty}\frac{e^{-iua}-e^{-iuz}}{iu}\,\phi(-u)\psi(u)\,du$$
$$= \lim_{a\to-\infty}\int_{-\infty}^{\infty}\frac{e^{-iua}-e^{-iuz}}{iu}\,\psi(u)\int_{-\infty}^{\infty}e^{-ius}\,dF_P(s)\,du$$
$$= \lim_{a\to-\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iu(a+s)}-e^{-iu(z+s)}}{iu}\,\psi(u)\,du\,dF_P(s)$$
$$= 2\pi\lim_{a\to-\infty}\int_{-\infty}^{\infty}(H(z+s)-H(a+s))\,dF_P(s)$$
$$= 2\pi\int_{-\infty}^{\infty}H(z+s)\,dF_P(s). \tag{3.54}$$

Now (3.43) follows from (3.15), (3.37), (3.42), (3.46)-(3.48) and (3.51)-(3.54). The expression in (3.44) is just $E[Z_\infty]=\beta_P/\mu_1$.

It only remains to prove (3.45). Divide by $\frac{1}{2}\sigma_U^2|u|$ throughout in (3.42) where $L$ is defined in (3.37). Take absolute values, use the triangle inequality and integrate from $-\infty$ to $\infty$. Then for some constants $c_1$, $c_2$ and $c_3$,

$$\int_{-\infty}^{\infty}|u\psi''(u)|\,du\le c_1\int_{-\infty}^{\infty}|\psi'(u)|\,du+c_2\int_{-\infty}^{\infty}\left|\frac{\nu(u)}{u}\right||\psi(u)|\,du$$
$$+c_3\int_{-\infty}^{\infty}\int_0^{\infty}\left|\frac{\psi(us)-\psi(u)}{u}\right|\,dF_U(s)\,du. \tag{3.55}$$

The first integral on the right is finite by (A2). By (3.21) $|\nu(u)/u|\le K+\frac{1}{2}\sigma_U^2|u|$, hence the second integral is finite by (A1).

We will now prove that the third integral on the right side of (3.55) is finite. From Feller (1971, formula (4.14), p. 514),

$$\int_{-1}^{1}\int_{0}^{\infty}\left|\frac{\psi(us)-\psi(u)}{u}\right| \mathrm{d}F_U(s)\,\mathrm{d}u \leqslant \frac{\beta_P}{\mu_1}\int_{-1}^{1}\int_{0}^{\infty}|s-1|\,\mathrm{d}F_U(s)\,\mathrm{d}u$$
$$\leqslant 2\frac{\beta_P}{\mu_1}(1+E[S_U])<\infty.$$

Furthermore,

$$\int_{1}^{\infty}\int_{0}^{\infty}\left|\frac{\psi(us)-\psi(u)}{u}\right| \mathrm{d}F_U(s)\,\mathrm{d}u \leqslant \int_{1}^{\infty}\int_{0}^{\infty}\left|\frac{\psi(us)}{u}\right| \mathrm{d}F_U(s)\,\mathrm{d}u+\int_{1}^{\infty}\left|\frac{\psi(u)}{u}\right|\mathrm{d}u.$$

The first integral on the left is finite by (3.50) and the second is finite by (A1). The integral from $-\infty$ to $-1$ is similar, hence finiteness of the third integral on the right of (3.55) follows. This finishes the proof of the theorem. □

**Remark 3.6.** If $\sum_{i=1}^{N_{I,t}}\tilde{S}_{I,i}$ and $\sum_{i=1}^{N_{R,t}}\tilde{S}_{R,i}$ are independent, then using (3.41) instead of (3.37) we find that (3.43) takes the form

$$\tfrac{1}{2}(\sigma_U^2z^2+\sigma_P^2)H''(z)+((\alpha_U+\tfrac{1}{2}\sigma_U^2)z-p)H'(z)-(\lambda_I+\lambda_R+\lambda_P)H(z)$$
$$+\lambda_I\int_0^\infty H(z/s)\,\mathrm{d}F_I(s)+\lambda_R\int_0^\infty H(zs)\,\mathrm{d}F_R(s)$$
$$+\lambda_P\int_{-\infty}^{\infty}H(z+s)\,\mathrm{d}F_P(s)=0.$$

In the same way as Theorem 3.4 we can prove:

**Proposition 3.2.** *Assume*:

(B1) *If* $\sigma_U^2>0$ *or* $\sigma_P^2>0$ *then*

$$\int_{-\infty}^{\infty}|u^2\psi(u)|\,\mathrm{d}u<\infty.$$

*Otherwise it is sufficient that*

$$\int_{-\infty}^{\infty}|u\psi(u)|\,\mathrm{d}u<\infty.$$

(B2) $$\int_{-\infty}^{\infty}|u\psi'(u)|\,\mathrm{d}u<\infty.$$

(B3) $E[1/S_U]<\infty.$

*Then* $Z_\infty$ *has a twice continuously differentiable density* $h$ *which is the solution of*

$$\tfrac{1}{2}(\sigma_U^2z^2+\sigma_P^2)h''(z)+((\alpha_U+\tfrac{3}{2}\sigma_U^2)z-p)h'(z)+(\alpha_U+\tfrac{1}{2}\sigma_U^2-\lambda_U-\lambda_P)h(z)$$
$$+\lambda_U\int_0^\infty h(z/s)\,\mathrm{d}F_U(s)+\lambda_P\int_{-\infty}^{\infty}h(z+s)\,\mathrm{d}F_P(s)=0.$$

*With side conditions*

$$\int_{-\infty}^{\infty} h(z)\,\mathrm{d}z = 1 \quad \textit{and} \quad h(z) \geqslant 0 \quad \forall z.$$

*Also*

$$\int_{-\infty}^{\infty} zh(z)\,\mathrm{d}z = \frac{\beta_P}{\mu_1}. \qquad \square$$

We also have:

**Proposition 3.3.** *Assume claims exponentially distributed with expectation* $1/\mu$, *i.e.* $F_P(s) = (1 - \mathrm{e}^{-\mu s}) I_{\{s>0\}}$. *Assume also that* $\sigma_P^2 = 0$ *and that* (A1)-(A3) *in Theorem* 3.9 *are satisfied. Let*

$$V(\mu) = E[H(Y_{T_R}) \mid T_R < \infty] = E[H(S_P)] = \int_0^{\infty} H(z)\mu\,\mathrm{e}^{-\mu z}\,\mathrm{d}z.$$

(*See Theorem* 3.2 *and Corollary* 3.1 *for notation.*) *Then* $V(\mu)$ *is twice continuously differentiable and is the solution of*

$$\tfrac{1}{2}\sigma_U^2 \mu^2 V''(\mu) - (\alpha_U - \tfrac{1}{2}\sigma_U^2 + \lambda_P)\mu V'(\mu) - (p\mu + \lambda_U) V(\mu)$$
$$+ \lambda_U \int_0^{\infty} V(\mu s)\,\mathrm{d}F_U(s) = -p\mu H(0).$$

*Boundary conditions are* $V(0) = 1$ *and* $V(\infty) = H(0)$.

**Proof.** Multiply each term in (3.43) by $\mu\,\mathrm{e}^{-\mu z}$ and integrate from 0 to $\infty$. The calculations are much the same as in the proof of Theorem 3.4 and are omitted. $\square$

We will now return to Theorem 3.4 and find some sufficient conditions for (A1) and (A2) there to hold. We start with the following fairly general result:

**Proposition 3.4.** *Assume* $\sigma_P^2 > 0$ *and that*

$$E[1/S_U^2] < \infty.$$

*Then* (A1)-(A3) *in Theorem* 3.4 *are satisfied.*

**Proof.** By Proposition 3.1 and (3.15),

$$|\psi(u)| \leqslant E[\mathrm{e}^{-u^2 X/2}],$$

and by (3.29) and (3.31),

$$|\psi'(u)| \leqslant c_1 E[Y\,\mathrm{e}^{-u^2 X/2}] + c_2 |u| E[X\,\mathrm{e}^{-u^2 X/2}],$$

where $X=\sigma_P^2\int_0^\infty U_s^2\,\mathrm{d}s$, $Y=\int_0^\infty U_s\,\mathrm{d}s$ and $c_1, c_2$ are constants. By the above inequalities, Fubini and Cauchy-Schwarz inequality,

$$\int_{-\infty}^{\infty}|u\psi(u)|\,\mathrm{d}u\leqslant E\left[\int_{-\infty}^{\infty}|u|\,\mathrm{e}^{-u^2X/2}\,\mathrm{d}u\right]=2E[X^{-1}],$$

$$\begin{aligned}\int_{-\infty}^{\infty}|\psi'(u)|\,\mathrm{d}u&\leqslant c_1E\left[Y\int_{-\infty}^{\infty}\mathrm{e}^{-u^2X/2}\,\mathrm{d}u\right]+c_2E\left[X\int_{-\infty}^{\infty}|u|\,\mathrm{e}^{-u^2X/2}\,\mathrm{d}u\right]\\&\leqslant c_1(E[Y^2])^{1/2}\left(E\left[\left(\int_{-\infty}^{\infty}\mathrm{e}^{-u^2X/2}\,\mathrm{d}u\right)^2\right]\right)^{1/2}+2c_2\\&=\sqrt{2\pi}\,c_1(E[Y^2])^{1/2}(E[X^{-1}])]^{1/2}+2c_2.\end{aligned}$$

By (3.5) we have that $E[Y^2]<\infty$ so it remains to prove that $E[X^{-1}]<\infty$, i.e.

$$E\left[\left(\int_0^\infty U_s^2\,\mathrm{d}s\right)^{-1}\right]<\infty.$$

By (2.19), $U_s^2=\exp\{-2\alpha_Ut+2\sigma_UW_{U,t}\}\prod_{i=1}^{N_{U,t}}S_{U,i}^2$, so we define

$$T_1=\inf\{t\colon \exp\{-2\alpha_Ut+2\sigma_UW_{U,t}\}=\exp\{-2\sigma_U\}\},$$

$$T_2=\inf\{t\colon N_{U,t}=2\}.$$

Then

$$\int_0^\infty U_s^2\,\mathrm{d}s\geqslant\exp\{-2\sigma_U\}(T_1\wedge T_2)(1\wedge S_{U,1}^2).$$

By independence of $T_1$, $T_2$ and $S_{U,1}$ we get

$$\begin{aligned}E\left[\left(\int_0^\infty U_s^2\,\mathrm{d}s\right)^{-1}\right]&\leqslant\exp\{2\sigma_U\}E\left[\frac{1}{T_1\wedge T_2}\right]E\left[\frac{1}{1\wedge S_U^2}\right]\\&\leqslant\exp\{2\sigma_U\}\left(E\left[\frac{1}{T_1}\right]+E\left[\frac{1}{T_2}\right]\right)\left(1+E\left[\frac{1}{S_U^2}\right]\right).\end{aligned}\qquad(3.56)$$

By assumption $E[S_U^{-2}]<\infty$. Furthermore,

$$E\left[\frac{1}{T_2}\right]=\int_0^\infty\frac{1}{t}\lambda_U^2t\,\mathrm{e}^{-\lambda_Ut}\,\mathrm{d}t=\lambda_U.$$

Note that $T_1=\inf\{t\colon W_{U,t}-(b/\sigma_U)t=-1\}$, so by Karatzas and Shreve (1988, formula (5.12), p. 197),

$$E\left[\frac{1}{T_1}\right]=\frac{1}{2\sqrt{2\pi}}\int_0^\infty t^{-5/2}\exp\left\{-\frac{(1-(b/\sigma_U)t)^2}{2t}\right\}\mathrm{d}t<\infty.$$

Hence the right side of (3.56) is finite. □

**Remark 3.7.** If we strengthen the assumption to $E[S_U^{-3}]<\infty$, then the assumptions (B1)-(B3) of Proposition 3.2 are satisfied.

We will now consider the more difficult task of verifying (A1) and (A2) in Theorem 3.4 when $\sigma_P^2=0$. Here only a special case is solved. We begin with a lemma.

**Lemma 3.4.** *Assume $\sigma_P^2 = 0$. Let $k(u) = \mathrm{Re}(1-\phi(-u)) \geqslant 0$ and consider the equation*

$$Lf = -\alpha k f, \tag{3.57}$$

*where $L$ is given by (3.37).*

*Let $y(u)$ and $z(u)$ be solutions of (3.57) with $\alpha = \lambda_P$ and $\alpha = 2\lambda_P$ respectively, and such that $0 \leqslant y(u)$, $z(u) \leqslant 1$. Assume:*

(C1) *if $\sigma_U^2 > 0$ then*

$$\int_{-\infty}^{\infty} |uy(u)| \, \mathrm{d}u < \infty.$$

*Otherwise it is sufficient that*

$$\int_{-\infty}^{\infty} y(u) \, \mathrm{d}u < \infty.$$

(C2) $$\int_{-\infty}^{\infty} (z(u))^{1/2} \, \mathrm{d}u < \infty.$$

*Then conditions (A1) and (A2) of Theorem 3.4 are satisfied.*

**Proof.** Let $X(u) = \exp\{\int_0^\infty \nu(uU_s)\,\mathrm{d}s\}$ be as in (3.26). Then since $k$ is real,

$$\begin{aligned}|\psi(u)| = |E[X(u)]| &\leqslant E\left[\left|\exp\left\{-\lambda_P \int_0^\infty k(uU_s)\,\mathrm{d}s\right\}\right|\right] \\ &= E\left[\exp\left\{-\lambda_P \int_0^\infty k(uU_s)\,\mathrm{d}s\right\}\right].\end{aligned}$$

And as in Theorem 3.3,

$$y(u) = E\left[\exp\left\{-\lambda_P \int_0^\infty k(uU_s)\,\mathrm{d}s\right\}\right] \leqslant 1$$

is the solution of (3.57) with $\alpha = \lambda_P$. Hence (C1) implies (A1). Furthermore by (3.22), (3.28) and (3.31), for some constant $c$,

$$\begin{aligned}|\psi'(u)| \leqslant E[|X'(u)|] &\leqslant KE\left[\left|X(u)\int_0^\infty U_s\,\mathrm{d}s\right|\right] \\ &\leqslant K\left(E\left[\left(\int_0^\infty U_s\,\mathrm{d}s\right)^2\right]\right)^{1/2}(E[|X(u)|^2])^{1/2} \\ &\leqslant c\left(E\left[\left|\exp\left\{-2\lambda_P\int_0^\infty k(uU_s)\,\mathrm{d}s\right\}\right|\right]\right)^{1/2} = c(z(u))^{1/2}.\end{aligned}$$

Since by (3.5) $E[(\int_0^\infty U_s\,\mathrm{d}s)^2] < \infty$. Again as in Theorem 3.3,

$$z(u) = E\left[\exp\left\{-2\lambda_P \int_0^\infty k(uU_s)\,\mathrm{d}s\right\}\right] \leqslant 1$$

is the solution of (3.57) with $\alpha = 2\lambda_P$. Hence (C2) implies (A2). □

We will use Lemma 3.4 to prove the following:

**Proposition 3.5.** *Assume* $\sigma_P^2 = \lambda_U = 0$ *and that there exist positive constants* $k$, $c$ *and* $\varepsilon$ *such that when* $|u| \geq K$, $\mathrm{Re}(\phi(u)) \leq cu^{-\varepsilon}$. *Assume*

$$\lambda_P > 2\alpha_U + 2\sigma_U^2 = 2(r - \bar{\imath}) + 3\sigma_I^2 + \sigma_R^2 - 4\rho\sigma_I\sigma_R. \tag{3.58}$$

*Then conditions* (A1) *and* (A2) *of Theorem* 3.4 *are satisfied.*

**Proof.** The equation $Ly = -\lambda_P ky$ in Lemma 3.4 now takes the form $\tilde{L}y = ry$, where $r(u) = -\lambda_P \, \mathrm{Re}(\phi(-u))$ and $\tilde{L}$ is the differential operator

$$\tilde{L} = \tfrac{1}{2}\sigma_U^2 u^2 \frac{\mathrm{d}^2}{\mathrm{d}u^2} - (\alpha_U - \tfrac{1}{2}\sigma_U^2) u \frac{\mathrm{d}}{\mathrm{d}u} - \lambda_P.$$

First we solve $\tilde{L}w = 0$. This is just the Euler equation, and two independent solutions are given by

$$w_1(u) = |u|^{\beta_1} \quad \text{and} \quad w_2(u) = |u|^{\beta_2},$$

where $\beta_1$ and $\beta_2$ are solutions of the equation $\frac{1}{2}\sigma_U^2\beta(\beta-1) - (\alpha_U - \frac{1}{2}\sigma_U^2)\beta - \lambda_P$. The solution is

$$\beta = \frac{\alpha_U}{\sigma_U^2} \pm \sqrt{\left(\frac{\alpha_U}{\sigma_U^2}\right)^2 + 2\frac{\lambda_P}{\sigma_U^2}}. \tag{3.59}$$

If we let $\beta_1$ denote the negative solution, direct calculation and use of (3.58) give that $\beta_1 < -2$. This also implies that $\beta_2 > 2$.

Let $u > K$. By the method of variation of parameters, a general solution is given as

$$y(u) = a_1 u^{\beta_1} + a_2 u^{\beta_2} + \int_K^u G(u, t) r(t) y(t) \, \mathrm{d}t,$$

where $G(u, t) = -(1/(\beta_2 - \beta_1))(u^{\beta_1}/t^{\beta_1+1} - u^{\beta_2}/t^{\beta_2+1})$ is the one-sided Green's function. This gives

$$\begin{aligned} y(u) = a_1 u^{\beta_1} &- \frac{1}{\beta_2 - \beta_1} u^{\beta_1} \int_K^u t^{-(\beta_1+1)} r(t) y(t) \, \mathrm{d}t \\ &+ u^{\beta_2}\left(a_2 + \frac{1}{\beta_2 - \beta_1} \int_K^u t^{-(\beta_2+1)} r(t) y(t) \, \mathrm{d}t\right). \end{aligned}$$

By assumption $r(t) \leq ct^{-\varepsilon}$ when $t > K$. Also $0 \leq y(u) \leq 1$ (see Lemma 3.4), hence for some constants $c_1$ and $M$,

$$\begin{aligned} u^{\beta_1} \int_K^u t^{-(\beta_1+1)} r(t) y(t) \, \mathrm{d}t &\leq c u^{\beta_1} \int_K^u t^{-(\beta_1+1+\varepsilon)} \, \mathrm{d}t \\ &\leq c_1(u^{\beta_1} + u^{-\varepsilon}) < M. \end{aligned} \tag{3.60}$$

Therefore since $\beta_2 > 0$ we must have

$$a_2 = -\frac{1}{\beta_2 - \beta_1} \int_K^\infty t^{-(\beta_2+1)} r(t) y(t)\, \mathrm{d}t.$$

which implies

$$y(u) = a_1 u^{\beta_1} - \frac{1}{\beta_2 - \beta_1} u^{\beta_1} \int_K^u t^{-(\beta_1+1)} r(t) y(t)\, \mathrm{d}t$$
$$- \frac{1}{\beta_2 - \beta_1} u^{\beta_2} \int_u^\infty t^{-(\beta_2+1)} r(t) y(t)\, \mathrm{d}t. \tag{3.61}$$

Using the upper bounds of $r(t)$ and $y(t)$ gives

$$u^{\beta_2} \int_u^\infty t^{-(\beta_2+1)} r(t) y(t)\, \mathrm{d}t \leq c u^{\beta_2} \int_u^\infty t^{-(\beta_2+1+\varepsilon)}\, \mathrm{d}t \leq c_2 u^{-\varepsilon} \tag{3.62}$$

for some constant $c_2$.

Inserting (3.60) and (3.62) into (3.61) and using the triangle inequality gives for some constant $c_3$,

$$0 \leq y(u) \leq c_3 (u^{\beta_1} + u^{-\varepsilon}). \tag{3.63}$$

If $\varepsilon > 2$ then $\int_K^\infty uy(u)\, \mathrm{d}u < \infty$ (since $\beta_1 < -2$), and we can stop the argument. Otherwise by (3.63), $0 \leq y(u) \leq c_4 u^{-\varepsilon}$ for some constant $c_4$, and inserting this inequality into the left sides of (3.60) and (3.62) gives for some constants $c_5$ and $c_6$,

$$u^{\beta_1} \int_K^u t^{-(\beta_1+1)} r(t) y(t)\, \mathrm{d}t \leq c_5 (u^{\beta_1} + u^{-2\varepsilon}),$$

$$u^{\beta_2} \int_u^\infty t^{-(\beta_2+1)} r(t) y(t)\, \mathrm{d}t \leq c_6 u^{-2\varepsilon}.$$

Inserting these inequalities into (3.61) and using the triangle inequality gives for some constant $c_7$,

$$0 \leq y(u) \leq c_7 (u^{\beta_1} + u^{-2\varepsilon}).$$

Like this we may continue $N$ steps until $N\varepsilon > -\beta_1$. Then for some constant $c_8$, $0 \leq y(u) \leq c_8 u^{\beta_1}$. Hence we get that

$$\int_K^\infty uy(u)\, \mathrm{d}u < \infty.$$

The case $\int_{-\infty}^{-K} |uy(u)|\, \mathrm{d}u < \infty$ is treated similarly. Since $0 \leq y(u) \leq 1$, we thus have

$$\int_{-\infty}^\infty |uy(u)|\, \mathrm{d}u < \infty. \tag{3.64}$$

Obviously $2\lambda_P > 2\alpha_U + 2\sigma_U^2$, hence as above (see Lemma 3.4) when $u > K$, $0 \leq z(u) \leq c_9 u^{\beta_1}$ where $c_9$ is some constant. But $\beta_1 < -2$ and we get as above,

$$\int_{-\infty}^{\infty} (z(u))^{1/2} \, du < \infty. \tag{3.65}$$

The result now follows from (3.64), (3.65) and Lemma 3.4. □

**Remark 3.8.** If in Proposition 3.5 we instead of (3.58) assume that

$$\lambda_P > 3\alpha_U + \tfrac{9}{2}\sigma_U^2 = 3(r - \bar{i}) + 6\sigma_I^2 + 3\sigma_R^2 - 9\rho\sigma_I\sigma_R,$$

then $\beta_1 < -3$ (see (3.59)). It can be proved along the same lines as above that this implies that both (B1) and (B2) in Proposition 3.2 are satisfied.

**Remark 3.9.** The assumption $\lambda_P > 2\alpha_U + 2\sigma_U^2$ is normally very weak. On a yearly basis typically $2\alpha_U + 2\sigma_U^2 < 0.5$, while $\lambda_P \gg 1$.

**Example 3.1.** Assume $\lambda_U = \lambda_P = 0$ and that $\sigma_P^2 > 0$. Then

$$Z_\infty = \int_0^\infty \exp\{-\alpha_U t + \sigma_U W_{U,t}\} \, dP_t \tag{3.66}$$

where $P_t = pt + \sigma_P W_{P,t}$. Here $W_U$ and $W_P$ are independent Brownian motions.

If $\sigma_U^2 = 0$, it follows directly from (3.66) and isometric properties of the stochastic integral that $Z_\infty$ is normally distributed with expectation $p/\alpha_U$ and variance $\sigma_P^2/2\alpha_U$.

If $\sigma_U^2 > 0$, it follows from Theorem 3.4 and Proposition 3.4 that the density $h$ of $Z_\infty$ is given as the solution of

$$\tfrac{1}{2}(\sigma_U^2 z^2 + \sigma_P^2)h'(z) = (p - (\alpha_U + \tfrac{1}{2}\sigma_U^2)z)h(z). \tag{3.67}$$

The solution is easily found to be

$$h(z) = \frac{h_0}{(\sigma_P^2 + \sigma_U^2 z^2)^{1/2+\alpha_U/\sigma_U^2}} \exp\left\{\frac{2p}{\sigma_U\sigma_P} \arctan\left(\frac{\sigma_U}{\sigma_P} z\right)\right\} \tag{3.68}$$

where $h_0$ is a normalizing constant. Note that the solution of (3.67) gives the above mentioned distribution when $\sigma_U^2 = 0$ as well.

We see that $h(z)$ has a finite first moment iff $\frac{1}{2} + \alpha_U/\sigma_U^2 > 1$, i.e. iff $\alpha_U - \frac{1}{2}\sigma_U^2 = \mu_1 > 0$ (see (3.2)). Similarly $h(z)$ has a finite second moment iff $\alpha_U - \sigma_U^2 = \frac{1}{2}\mu_2 > 0$. This is in accordance with Theorem 3.1.

On the other hand we only know that $h(z)$ is the density of $Z_\infty$ when $\mu_2 > 0$. In our derivation of (3.68) we made use of Theorem 3.3 which involves the second derivative of $\psi(u) = E[\exp\{iuZ_\infty\}]$, and $\psi''(0) < \infty$ iff $E[Z_\infty^2] < \infty$ (Feller, 1971, Corollary, p. 512). It therefore looks difficult to verify whether $h(z)$ is the density of $Z_\infty$ under the weaker assumption $\mu_1 > 0$.

By Corollary 3.1, the probability of eventual ruin is

$$R(y) = H(-y)/H(0). \tag{3.69}$$

Substituting $v = \arctan((\sigma_U/\sigma_P)z)$ in the integral $H(x) = \int_{-\infty}^{x} h(z)\,dz$, and cancelling common constants in the nominator and denominator in (3.69), we find that

$$R(y) = \frac{G(-\arctan((\sigma_U/\sigma_P)y))}{G(0)}$$

where

$$G(x) = \int_{-\pi/2}^{x} \cos^{\alpha} v \cdot e^{\beta v}\, dv.$$

Here $\alpha = 2\alpha_U/\sigma_U^2 - 1$ and $\beta = 2p/\sigma_U\sigma_P$.

Table 1 gives $R(y)$ when $\bar{I}_t \equiv 1$, $r = 0.1$, $\sigma_R = 0$, 0.1, 0.2 and 0.3, $p = 1$, $\sigma_P = 1$ and $y = 0.2, 0.4, \ldots, 4.0$. We see that the impact of a stochastic interest rate is fairly small when the probability of ruin is large, but becomes increasingly important as the probability of ruin decreases. For large values of $y$ we see that the uncertainty in return on investments may increase the probability of eventual ruin several times. This impression has been confirmed by making other choices of $r$, $\sigma_R$, $p$ and $\sigma_P$.

For this special case we see that $\mu_2 > 0$ iff $\sigma_R < \sqrt{0.2/3} \approx 0.258$. Therefore we do not know whether Table 1 is valid for the case $\sigma_R = 0.3$. On the other hand $\mu_1 > 0$ iff $\sigma_R < \sqrt{0.1} \approx 0.316$. Calculating $R(y)$ for various values of $\sigma_R$ in the vicinity of 0.258 there was no evidence of discontinuity.

Table 1

| $y$ | $\sigma_R$ | | | |
|---|---|---|---|---|
| | 0.00 | 0.10 | 0.20 | 0.30 |
| 0.20 | 0.65559 | 0.65695 | 0.66119 | 0.66896 |
| 0.40 | 0.42651 | 0.42873 | 0.43567 | 0.44841 |
| 0.60 | 0.27534 | 0.27803 | 0.28645 | 0.30201 |
| 0.80 | 0.17639 | 0.17923 | 0.18819 | 0.20489 |
| 1.00 | 0.11213 | 0.11490 | 0.12369 | 0.14034 |
| 1.20 | 0.07073 | 0.07328 | 0.08144 | 0.09723 |
| 1.40 | 0.04427 | 0.04651 | 0.05377 | 0.06823 |
| 1.60 | 0.02750 | 0.02939 | 0.03565 | 0.04856 |
| 1.80 | 0.01695 | 0.01849 | 0.02375 | 0.03507 |
| 2.00 | 0.01036 | 0.01160 | 0.01591 | 0.02571 |
| 2.20 | 0.00629 | 0.00725 | 0.01073 | 0.01914 |
| 2.40 | 0.00379 | 0.00452 | 0.00729 | 0.01446 |
| 2.60 | 0.00226 | 0.00281 | 0.00499 | 0.01109 |
| 2.80 | 0.00134 | 0.00174 | 0.00344 | 0.00863 |
| 3.00 | 0.00079 | 0.00108 | 0.00240 | 0.00680 |
| 3.20 | 0.00046 | 0.00067 | 0.00168 | 0.00543 |
| 3.40 | 0.00027 | 0.00041 | 0.00119 | 0.00439 |
| 3.60 | 0.00015 | 0.00025 | 0.00085 | 0.00360 |
| 3.80 | 0.00009 | 0.00016 | 0.00062 | 0.00297 |
| 4.00 | 0.00005 | 0.00010 | 0.00045 | 0.00249 |

**Example 3.2.** Assume $\lambda_U = \sigma_P^2 = 0$ and that $F_P(s) = (1 - e^{-\mu s}) I_{\{s \geq 0\}}$. Then $Z_\infty$ is as in (3.66), but where

$$P_t = pt + \sum_{i=1}^{N_{P,t}} S_{P,i}.$$

By Theorem 3.4 and Proposition 3.5, the assumption $\lambda_P > 2\alpha_U + 2\sigma_U^2$ implies that the distribution $H$ of $Z_\infty$ satisfies the integro-differential equation $\mathscr{L}H = 0$ where

$$\mathscr{L}H(z) = \tfrac{1}{2}\sigma_U^2 z^2 H''(z) + ((\alpha_U + \tfrac{1}{2}\sigma_U^2)z - p)H'(z)$$
$$- \lambda_P H(z) + \mu\lambda_P e^{\mu z} \int_z^\infty H(v) e^{-\mu v} \, dv.$$

Assuming $\lambda_P > 3\alpha_U + \frac{9}{2}\sigma_U^2$, by Remark 3.8, $H$ is three times continuously differentiable. The equation

$$\frac{d}{dz} \mathscr{L}H(z) - \mu \mathscr{L}H(z) = 0$$

can then be written as

$$\tfrac{1}{2}\sigma_U^2 z^2 h''(z) - (\tfrac{1}{2}\mu\sigma_U^2 z^2 - \alpha_U z + p)h'(z)$$
$$- (\mu(\alpha_U + \tfrac{1}{2}\sigma_U^2)z + (\lambda_P - \alpha_U - \tfrac{1}{2}\sigma_U^2 - \mu p))h(z) = 0$$

where $h$ is the density of $Z_\infty$. Side conditions are

$$\int_{-\infty}^\infty h(z) \, dz = 1 \quad \text{and} \quad \int_{-\infty}^\infty z h(z) \, dz = \frac{\beta_P}{\mu_1}.$$

This is a rather complicated second order differential equation with unpleasant side conditions, making it less attractive for numerical solutions.

On the other hand we may use Theorem 3.3 which gives us the characteristic function $\psi$ of $Z_\infty$ as the solution of

$$\tfrac{1}{2}\sigma_U^2 u^2 \psi''(u) - (\alpha_U - \tfrac{1}{2}\sigma_U^2) u \psi'(u) + \left( ipu - \lambda_P \frac{u}{u - i\mu} \right) \psi(u) = 0 \tag{3.70}$$

with the more pleasant initial conditions

$$\psi(0) = 0 \quad \text{and} \quad \psi'(0) = i\beta_P / \mu_1.$$

For computations it is probably easiest to solve (3.70) numerically and then numerically invert the solution to obtain $H(z)$ for various values of $z$. Then Corollary 3.1 may be invoked to find numerical values for the probability of eventual ruin. Note that in this case we do not have to make any assumptions about $\lambda_P$.

If we assume $\lambda_P > 2\alpha_U + 2\sigma_U^2$, by Proposition 3.3 the denominator of $R(y)$ in Corollary 3.1, i.e. $V(\mu) = E[H(S_P)]$, is given as the solution of

$$\tfrac{1}{2}\sigma_U^2 \mu^2 V''(\mu) - (\alpha_U - \tfrac{1}{2}\sigma_U^2 + \lambda_P)\mu V'(\mu) - p\mu V(\mu) = -p\mu H(0) \tag{3.71}$$

with boundary conditions $V(0)=1$ and $V(\infty)=H(0)$. It is easy to see that a particular solution of this equation is $V(\mu)=H(0)$. Hence if we can find a solution of

$$\tfrac{1}{2}\sigma_U^2\mu^2 g''(\mu)-(\alpha_U-\tfrac{1}{2}\sigma_U^2+\lambda_P)\mu g'(\mu)-p\mu g(\mu)=0 \tag{3.72}$$

with boundary conditions $g(0)=1-H(0)$ and $g(\infty)=0$, the solution of (3.71) is given by $V(\mu)=g(\mu)+H(0)$. By the method of Frobenius, straightforward calculations show that two linearly independent solutions of (3.72) are

$$g_1(\mu)=\sum_{n=0}^{\infty}\frac{b^n}{n!\prod_{i=0}^{n-1}(\mathrm{i}-a)}\mu^n=\Gamma(-a)(b\mu)^{(1+a)/2}I_{-(1+a)}(2\sqrt{b\mu}),$$

$$g_2(\mu)=\mu^{1+a}\sum_{n=0}^{\infty}\frac{b^n}{n!\prod_{i=0}^{n-1}(\mathrm{i}+2+a)}\mu^n=\Gamma(2+a)(b\mu)^{(1+a)/2}I_{(1+a)}(2\sqrt{b\mu}).$$

Here $a=2(a_U+\lambda_P)/\sigma_U^2-1$ must be a noninteger, $b=2p/\sigma_U^2$ and $I_\alpha(z)$ is the Bessel function with purely imaginary argument. We see that $g_1(0)=1$ and $g_2(0)=0$, hence by using the asymptotic expansion of $I_\alpha(z)$ when $z$ becomes large (Whittaker and Watson, 1958, Section 17.7), $V(\mu)$ is readily found to be

$$\begin{aligned}V(\mu)=H(0)+(1-H(0))\Gamma(-a)(b\mu)^{(1+a)/2}\\ \times(I_{-(1+a)}(2\sqrt{b\mu})-I_{1+a}(2\sqrt{b\mu})).\end{aligned} \tag{3.73}$$

Another way of solving (3.72) is to use contour integration. Trying a solution of the form

$$g(\mu)=\int_0^\infty \mathrm{e}^{-\mu t}P(t)\,\mathrm{d}t$$

gives that $P(t)=ct^{-(2+a)}\,\mathrm{e}^{-b/t}$, i.e.

$$g(\mu)=c\int_0^\infty t^{-(2+a)}\,\mathrm{e}^{-(\mu t+b/t)}\,\mathrm{d}t \tag{3.74}$$

where $a$ and $b$ are as above. Monotone convergence gives that $\lim_{\mu\to\infty} g(\mu)=0$ and it is easy to verify that $g(0)=1-H(0)$ implies that

$$c=\frac{1-H(0)}{\Gamma(1+a)}b^{1+a}.$$

Substituting $u=\sqrt{(b/\mu)}/t$ in (3.74) then gives

$$V(\mu)=H(0)+\frac{1-H(0)}{\Gamma(1+a)}(b\mu)^{(1+a)/2}\int_0^\infty u^a\,\mathrm{e}^{-\sqrt{b\mu}(u+1/u)}\,\mathrm{d}u. \tag{3.75}$$

Now the probability of eventual ruin is by Corollary 3.1,

$$R(y)=H(-y)/V(\mu).$$

Therefore we only need to calculate numerically $H(-y)$ and $H(0)$ and then use either (3.73) or (3.75).

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