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Endomorphism Algebras of Modules with Distinguished Torsion-Free Elements

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1. INTRODUCTION

This paper considers the local case of modules over a fixed but arbitrary discrete valuation ring R. Throughout, p denotes a fixed prime element of R and all modules are unitary R-modules. Although our approach is geared from the outset toward mixed modules (modules M with torsion tM such that $0 \neq tM \neq M$), much of our motivation comes from earlier results in the torsion and torsion-free cases. Let $E(M)$ denote the algebra of R-linear endomorphisms of a module M. If both M and N are torsion, Kaplansky's theorem [5, Theorem 28] asserts that every isomorphism $E(M) \to E(N)$ is induced by an isomorphism of the modules themselves. The abundance of cyclic direct summands of reduced torsion modules plays a key role in Kaplansky's proof of this theorem. In the torsion-free case, Wolfson [13] adapted Kaplansky's method to prove the result for torsion-free modules M and N which possess direct summands isomorphic to R, stating his theorem for torsion-free modules over a complete discrete valuation ring.

For mixed modules M and N, the implications of an isomorphism $E(M) \to E(N)$ are less clear. In many cases Kaplansky's method still provides an isomorphism $tM \to tN$, but the full structure of M and N may not be encoded in the endomorphism algebras in the form of idempotents, if even at all. Isomorphism theorems in this case remain rare, and instead there are far-reaching results stating $E(M) \cong E(N)$ for many nonisomorphic, mixed modules M and N from certain classes [4, 7, 9]. From the short list of affirmative theorems, we state two which are relevant to what we shall prove in this paper. Only in the first theorem is the algebra isomorphism necessarily induced.

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THEOREM [6]. *If M and N have torsion-free rank one and tM is totally projective, then every isomorphism $E(M) \to E(N)$ is induced by an isomorphism $M \to N$.*

THEOREM [2]. *If M and N are Warfield modules and $E(M) \cong E(N)$, then $M \cong N$.*

In Section 2, we introduce the notion of a *stable element* of a mixed module; the properties of such an element are merely some of those it would possess if contained in a direct summand isomorphic to R. For reduced modules M and N with tM totally projective, our main theorem asserts that every isomorphism $E(M) \to E(N)$ is induced by an isomorphism $M \to N$ if each module contains a stable element. We shall obtain several corollaries which reflect back on the results stated above. Let $rk(M)$ denote torsion-free rank.

COROLLARY. *Assume M is a reduced module of finite torsion-free rank such that tM is totally projective, and $\mathrm{rk}(M/p^{\sigma}M) = 1$ for some ordinal σ. If N is a reduced module and $\mathrm{rk}(N) \leq \mathrm{rk}(M)$, then every isomorphism $E(M) \to E(N)$ is induced by an isomorphism $M \to N$.*

COROLLARY. *If M and N are reduced Warfield modules such that tM is totally projective, and the set of height sequences of elements of a decomposition basis for M contains a least element, then every isomorphism $E(M) \to E(N)$ is induced by an isomorphism $M \to N$.*

In regard to the second corollary, let $\underline{\sigma} = (\sigma_0, \sigma_1, \sigma_2, \cdots)$ and $\underline{\tau} = (\tau_0, \tau_1, \tau_2, \cdots)$ denote sequences of ordinals, and recall the partial ordering on all such sequences obtained by defining $\underline{\sigma} \leq \underline{\tau}$ when $\sigma_i \leq \tau_i$ for all $i \geq 0$. If $|x| = |x|_M$ denotes the p-height of x in M, the condition on M in the corollary is met when there is a decomposition basis X for M and an element $x \in X$ such that $|p^i x| \leq |p^i y|$ for all $y \in X$ and $i \geq 0$.

Finally, we state some facts about totally projective modules and cotorsion hulls that will be used in later sections. If P is a nice submodule of M with M/P totally projective, and $\phi\colon P \to N$ a homomorphism that does not decrease p-heights in M and N, then ϕ extends to a homomorphism $M \to N$. To apply this result in certain cases, we use the fact that finitely generated submodules of an R-module M are nice in M when $\mathrm{rk}(M) = 1$, or R is complete.

At a later stage, we adopt the viewpoint that every reduced module M is a submodule of its cotorsion hull $M^\bullet = \mathrm{Ext}^1_R(R(p^\infty), M)$. If $T = tM$ and M/T is divisible, then $M^\bullet = T^\bullet$, hence $E(M)$ may be identified with a subalgebra of $E(T^\bullet)$ by unique extension to endomorphisms of $T^\bullet$. In this case, if $N \subseteq T^\bullet$ and $N^\bullet = T^\bullet$, we have $E(M) = E(N)$ when $\{\phi \in E(T^\bullet)\colon \phi(M) \subseteq M\} = \{\phi \in E(T^\bullet)\colon \phi(N) \subseteq N\}$. The bulk of our proof will be carried out in this setting.

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2. STABLE ELEMENTS

Mixed modules containing elements satisfying the conditions of the following definition play a central role in this paper. If $x \in M$, $E(M)x$ is the submodule $\{\phi(x)\colon \phi \in E(M)\}$ of M.

DEFINITION 1. An element x of a module M is *stable* in M if the following hold.

(1) There exists $\delta \in E(M)$ such that $\delta(x) = x$ and $\delta M/\langle x\rangle$ is torsion.

(2) $M/E(M)x$ is torsion.

We will refer to x as a stable element of M or simply a stable element when (1) and (2) hold. Every element of a torsion module is stable, and stable elements of nontorsion modules are necessarily torsion-free. We begin with an assortment of mixed modules which possess such elements. In (5) below, the inclusion $T^{\sigma} \subseteq T$ induces an inclusion $T^{\sigma\bullet} \subseteq T^{\bullet}$ because T/T^{σ} is reduced. The purification L_* of L in $T^{\bullet}$ is defined by $L_*/L = t(T^{\bullet}/L)$.

PROPOSITION 1. *A reduced module M has a stable element in the following cases.*

(1) *M has torsion-free rank one.*

(2) *$M = A \oplus B$, where A has a stable element and $B/\mathrm{Hom}(A, B)A$ is torsion.*

(3) *M has countable torsion-free rank, tM is totally projective, and there is an ordinal σ such that $M/p^{\sigma}M$ has torsion-free rank one.*

(4) *M is a Warfield module, and the set of height sequences of elements of a decomposition basis for M contains a least element.*

(5) *$M = (\langle x\rangle + T^{\sigma\bullet})_* \subseteq T^{\bullet}$, where T is totally projective and $x \in T^{\bullet}$ satisfies $p^i x \notin T^{\bullet\sigma}$ for all $i \geq 0$.*

Proof. Part (1) is clear. To prove (2), assume $x \in A$ and $\delta \in E(A)$ satisfy the conditions of Definition 1 for A, and let $\pi\colon M \to A$ be projection. Note $\delta\pi(x) = x$, $\delta\pi M/\langle x\rangle \subseteq \delta A/\langle x\rangle$ is torsion, and that $B/\mathrm{Hom}(A, B)x$ must be torsion. Thus $M/E(M)x$ is torsion, and x is stable in M.

Assume M satisfies (3) and choose $\bar{x} = x + p^{\sigma}M$ torsion-free in $\bar{M} = M/p^{\sigma}M$. Then $\bar{M}/\langle \bar{x} \rangle$ is isomorphic to a countably generated extension of tM, hence is totally projective. Since $p^{\sigma}M$ is nice in M and $|p^i x| < \sigma$ for all $i \geq 0$ we have $|p^i \bar{x}|_{\bar{M}} = |p^i x|_M$ for all $i \geq 0$. Hence, there exists a homomorphism $\phi\colon \bar{M} \to M$ with $\phi(\bar{x}) = x$. By composing ϕ with the natural map $M \to \bar{M}$ we obtain $\delta \in E(M)$ with $\delta(x) = x$ and $\delta M/\langle x \rangle$ torsion. This method also shows $M/E(M)x$ is torsion, finishing (3).

Next assume M is a Warfield module as in (4), and let X be a nice decomposition basis for M with $M/\langle X \rangle$ totally projective. We may assume $X = \{x\} \cup Y$, where $|p^i x| \leq |p^i y|$ for all $y \in Y$ and $i \geq 0$. Then the map $\langle X \rangle \to \langle X \rangle$ induced by $x \mapsto x$ and $\langle Y \rangle \to 0$ extends to $\delta \in E(M)$ such that $\delta(x) = x$ and $\delta M/\langle x \rangle$ is torsion. A similar method shows $M/E(M)x$ is torsion, finishing (4).

Finally, let T, x, and σ be as in (5). We claim x is stable in $M = (\langle x \rangle + T^{\sigma\bullet})_*$. Let $\bar{x}$ be the image of x in $P = \langle x \rangle_*/T^{\sigma}$, and note $|p^i \bar{x}|_P = |p^i x|$ for all $i \geq 0$ because $T^{\bullet\sigma}$ is nice in $T^{\bullet}$. A straightforward argument shows

$$P/\langle \bar{x} \rangle \cong \frac{\langle x \rangle_*/\langle x \rangle}{(T^{\sigma} \oplus \langle x \rangle)/\langle x \rangle} = \frac{\langle x \rangle_*/\langle x \rangle}{(\langle x \rangle_*/\langle x \rangle)^{\sigma}}$$

is totally projective. As in (3), we can obtain $\delta \in E(T^{\bullet})$ with $\delta(T^{\sigma}) = 0$ and $\delta(x) = x$. It follows that $\delta(M) \subseteq M$ and $\delta(M)/\langle x \rangle$ is torsion because δ annihilates $T^{\sigma\bullet}$. For any $z \in T^{\sigma\bullet}$, the same method furnishes $\phi \in E(M)$ with $\phi(x) = z$, finishing the proof that x is stable. ∎

A module M is called Walker-indecomposable if $M = A \oplus B$ implies A or B is torsion; parts (3) and (4) above readily provide examples of such modules of torsion-free rank greater than one which possess stable elements. In Section 5, we use part (5) above to construct a Walker-indecomposable module of uncountable torsion-free rank which has a stable element.

The remainder of this section contains a sequence of lemmas that will figure when we turn to endomorphism algebras in Section 3. We will assume all totally projective modules are reduced. One can use [11, Theorems 1 and 2; 10, Theorem 3.2] to give an inductive proof that totally projective modules are thin; i.e., if T is totally projective, then every homomorphism $C \to T$ is small if C is torsion-complete.

LEMMA 1. *If T is totally projective and $T^{\bullet\sigma}$ is nontorsion, then $(T^{\bullet\sigma})_*$ does not possess a stable element.*

Proof. By way of contradiction, assume $x \in M = (T^{\bullet\sigma})_*$ and $\delta \in E(M)$ satisfy the conditions of Definition 1. Then $\delta(T^{\bullet\sigma})$ is cotorsion of torsion-free rank one, hence has bounded torsion. Thus, we may assume $\delta(T^\sigma) = 0$ and $\delta(p^i x) \neq 0$ for all $i \geq 0$. If $\sigma = 0$, then $\delta(T^\bullet) = 0$ because $\delta(T) = 0$ with $T^\bullet/T$ divisible and $T^\bullet$ reduced. This contradicts $\delta(T^\bullet)$ nontorsion. Hence, we may assume $\sigma > 0$. If $S = T/T^\sigma$, then δ induces an endomorphism f: $S^\bullet \to S^\bullet$ such that $f(S^{\bullet\sigma})$ has torsion-free rank one. We consider two cases for σ.

Case 1. Suppose $\sigma = \tau + 1$. By [3, Proposition 56.5] we have $S^{\bullet\sigma} \cong \operatorname{Hom}(K, \overline{S_\tau}/S_\tau)$, where $K = R(p^\infty)$ and $\overline{S_\tau}$ is torsion-completion. On $S^{\bullet\sigma}$, f is induced by $\bar{g}$: $\overline{S_\tau}/S_\tau \to \overline{S_\tau}/S_\tau$, where f first induces g: $\overline{S_\tau} \to \overline{S_\tau}$ with $g(S_\tau) \subseteq S_\tau$. Because $f(S^{\bullet\sigma})$ has torsion-free rank one, it follows from the isomorphism above that $\operatorname{Im} \bar{g} \cong K$. Then $g(\overline{S_\tau}) + S_\tau$ is isomorphic to a countably generated extension of S_τ, hence is totally projective. By the remarks before the lemma, g is small. Hence $g(\overline{S_\tau}) = g(S_\tau) \subseteq S_\tau$ because S_τ is pure in $\overline{S_\tau}$ with divisible quotient. This contradicts $\operatorname{Im} \bar{g} \neq 0$.

Case 2. Suppose σ is a limit ordinal. Then σ has cofinality ω by [8, Lemma 4] because $S^\sigma = 0$ and $S^{\bullet\sigma}$ is nontorsion. By [3, Proposition 56.5] we have $S^{\bullet\sigma} \cong \operatorname{Hom}(K, D)$, where D is the divisible torsion of L/S and $L = \varprojlim S/S^\tau (\tau < \sigma)$. On $S^{\bullet\sigma}$, f is induced by $\bar{g}$: $D \to D$, where f first induces g: $L \to L$ with $g(S) \subseteq S$ in the natural way. As above, we have $\operatorname{Im} \bar{g} \cong K$ because $f(S^{\bullet\sigma})$ has torsion-free rank one. Suppose that for each $k > 0$, there exist $\tau < \sigma$ and $n \geq 0$ such that $(p^n S^\tau)[p^k] \subseteq \ker f$. Let $z \in S^{\bullet\sigma}$. By using a suitably chosen set of roots of multiples of z, one can construct a module $P \subseteq S^\bullet$, containing z, such that P/A is divisible for $A \subseteq \ker f$. We omit the routine details. Then $f(z) = 0$ because $S^\bullet$ is reduced, so that $f(S^{\bullet\sigma}) = 0$. This is a contradiction, hence we conclude there exists $k > 0$ such that $f((p^n S^\tau)[p^k]) \neq 0$ for all $\tau < \sigma$ and $n \geq 0$. Now it follows that we can choose a sequence $\tau_0 < \tau_1 < \tau_2 < \cdots$, together with elements $c_i \in S$, to satisfy the following conditions: (1) $\sup \tau_i = \sigma$; (2) $c_i \in S^{\tau_i}$ and $p^i c_i \in S[p^k]$ for all i; and (3) $f(p^i c_i) \neq 0$ for all i. Let $P = \prod_{i<\omega} \langle c_i \rangle$, and let C be the external direct sum $\bigoplus_{i<\omega} \langle c_i \rangle$. By writing elements of P as formal infinite sums, for each $\tau < \sigma$ we obtain a homomorphism $P \to S/S^\tau$ by taking partial sums. These homomorphisms induce ψ: $P \to L$ with $\psi(C) \subseteq S$. Hence, ψ induces a homomorphism $\bar{\psi}$: $\bar{C}/C \to D$. Note $\bar{g}\bar{\psi}(\bar{C}/C) = (g\psi(\bar{C}) + S)/S \subseteq \operatorname{Im} \bar{g}$ is countably generated, hence $g\psi(\bar{C}) + S$ is totally projective. As above, it follows that $g\psi$ is small on $\bar{C}$. Hence, there exists $n \geq 0$ such that $g\psi(p^n C)[p^k] = 0$. But $p^n c_n \in (p^n C)[p^k]$ and $g\psi(p^n c_n) = f(p^n c_n) \neq 0$, a contradiction. This completes the proof. ▮

If $\underline{\sigma} = (\sigma_i)$ is a sequence of ordinals, $M(\underline{\sigma})$ denotes the submodule $\{x \in M: |p^i x| \geq \sigma_i \text{ for all } i \geq 0\}$ of M.

LEMMA 2. *If $\underline{\sigma} = (\sigma_i)$ is a strictly increasing sequence of ordinals and C is a reduced cotorsion module, then $C(\underline{\sigma})$ is cotorsion.*

Proof. For each i we obtain a composition of homomorphisms $C \to p^iC \to p^iC/p^{\sigma_i}C$ in the natural way, and these induce a homomorphism $\theta: C \to \prod_{i<\omega} p^iC/p^{\sigma_i}C$ with kernel $C(\underline{\sigma})$. Since Im θ is reduced, $C(\underline{\sigma})$ is cotorsion by [3, Sect. 54]. ▮

If $x \in M$, we use $\|x\|$ or $\|x\|_M$ to denote the height sequence $(|x|, |px|, |p^2x|, \cdots)$ of x in M. If x is stable, we may assume $\|x\|$ contains infinitely many gaps or else none at all because nonzero multiples of x remain stable. For such an element, it will be convenient to define

$$\mu_x = \begin{cases} \sup|p^ix|, & \text{if } \|x\| \text{ has infinitely many gaps} \\ |x|, & \text{if } \|x\| \text{ is gapless.} \end{cases}$$

LEMMA 3. *Assume M is a reduced module with totally projective torsion T and M/T is divisible. If $\underline{\sigma} = (\sigma_i)$ is a strictly increasing sequence of ordinals such that $T^\bullet(\underline{\sigma}) \subseteq M$, and x is a stable torsion-free element of M, then $\sup \sigma_i > \mu_x$.*

Proof. Suppose $\|x\|$ contains infinitely many gaps, and let $\delta \in E(M) \subseteq E(T^\bullet)$ be as in Definition 1. Then $\delta(T^\bullet(\underline{\sigma})) \subseteq \langle x \rangle_*$ is cotorsion of torsion-free rank one by Lemma 2, hence has bounded torsion. Thus $p^k\delta(T(\underline{\sigma})) = 0$ for some k, and it follows that $(p^{\sigma_0+k}T)[p] \subseteq \ker \delta$. If $\sup \sigma_i \leq \mu_x$, we may choose m so that $|p^mx| \geq \sigma_0 + k$ and $|p^{m+1}x| > |p^mx| + 1$. Then $p^{m+1}x = py$ for $y \in M$ with $|y| > |p^mx|$. Thus $p^mx = \delta(p^mx) = \delta(y)$ because $y - p^mx \in (p^{\sigma_0+k}T)[p]$. This contradicts our choice of y, hence $\sup \sigma_i > \mu_x$ in this case.

Now assume $\|x\|$ is gapless, and note $M \subseteq (T^{\bullet\tau})_*$ if τ is the smallest ordinal such that $x \notin T^{\bullet\tau+1}$. If $\sup \sigma_i \leq \mu_x = |x|$, then $T^{\bullet\tau} \subseteq T^\bullet(\underline{\sigma})$. It follows that $M = (T^{\bullet\tau})_*$. This contradicts Lemma 1, hence $\sup \sigma_i > \mu_x$ as desired. ▮

In the next lemma, $\ell(S)$ denotes p-length.

LEMMA 4. *Assume M is reduced with totally projective torsion T and M/T is divisible. If x is a stable, torsion-free element of M, we may regard $T^\bullet$ as an isotype submodule of $\tilde{T}^\bullet$, where $\tilde{T} = S \oplus S'$ is reduced torsion, $\ell(S) \leq \mu_x$, and $x \in S^\bullet$.*

Proof. Denote $\mu = \mu_x$. Let $S = T/p^\mu T$, a totally projective module of length at most μ. We claim there exists $x' \in S^\bullet = T^\bullet/(p^\mu T)^\bullet$ such that $\|x'\|_{S^\bullet} = \|x\|_M$. If $\mu = \sup|p^ix|$, it is easily seen that the image x' of x in $S^\bullet$ has this property because $(p^\mu T)^\bullet \subseteq p^\mu T^\bullet$ and the latter is nice in $T^\bullet$. If $\mu = |x|$, such an x' exists if $\ell(S^\bullet) = \mu + \omega$, which will be the case if the image of x in $S^\bullet$ is torsion-free. If instead $p^k x \in (p^\mu T)^\bullet$ for some k, we obtain $p^{k+n}x = p^{k+m}y$ for some $y \in (p^\mu T)^\bullet$ and integers $m > n$ because $(p^\mu T)^\bullet / p^\mu T$ is divisible. Note $|p^{k+n}x| = \mu + k + n$ because $\|x\|$ is gapless. Then $|p^{k+m}y| \geq \mu + k + m > |p^{k+n}x|$, a contradiction.

Now let $x' \in S^\bullet$ be as guaranteed above and define $\tilde{T} = S \oplus S'$, where $S' \cong T$. If $\langle x' \rangle_*$ denotes purification in $S^\bullet$ and $\langle x \rangle_*$ purification in $T^\bullet$, there exists an isomorphism $\langle x \rangle_* \oplus S \to \langle x' \rangle_* \oplus S'$ taking x to x'. Hence, there is an isomorphism ϕ: $T^\bullet \oplus S^\bullet \to \tilde{T}^\bullet$ with $\phi(x) = x'$. The restriction of ϕ to $T^\bullet$ is the desired embedding in $\tilde{T}^\bullet$. ▮

3. ENDOMORPHISM ALGEBRAS

This section leads to the proof of our first isomorphism theorem for the endomorphism algebras of modules with totally projective torsion and stable elements. We focus first on the case where M and N are embedded in a common cotorsion hull with $E(M) = E(N)$. The next definition and lemma first appeared in [8].

DEFINITION 2. If M is reduced, the *core* $C(M)$ of M is the maximal $E(M^\bullet)$-module contained in M.

Clearly, $C(M)$ is a pure submodule of M containing tM.

LEMMA 5. *If T is a reduced torsion module and M and N are pure submodules of $T^\bullet$ containing T such that $E(M) = E(N)$, then* $\mathrm{Hom}(M, C(M)) = \mathrm{Hom}(N, C(N))$.

Proof. See [8, p. 488].

LEMMA 6. *Assume M and N are pure submodules of $T^\bullet$ containing T such that $E(M) = E(N)$. If $x \in M$ and $\delta \in E(M)$ satisfy the conditions of Definition* 1, *then* $\mathrm{Hom}(\langle x \rangle_*, C(M)) = \mathrm{Hom}((\delta N)_*, C(N))$.

Proof. Since $x \in \delta M \subseteq \langle x \rangle_*$ we have $\mathrm{Hom}((\delta M)_*, C(M)) = \mathrm{Hom}(\langle x \rangle_*, C(M))$. If $\phi \in \mathrm{Hom}(\langle x \rangle_*, C(M))$, then $\phi\delta \in \mathrm{Hom}(M, C(M)) = \mathrm{Hom}(N, C(N))$ by Lemma 5, hence $\phi \in \mathrm{Hom}((\delta N)_*, C(N))$. The reverse inclusion is similar. ▮

LEMMA 7. *Assume M is a reduced module with totally projective torsion T and M/T is divisible. If x is a stable torsion-free element of M, then $x \notin C(M)$.*

Proof. Denote $\|x\|$ by $\underline{\sigma}$. First, assume $\underline{\sigma}$ contains infinitely many gaps. For any $z \in T^\bullet(\underline{\sigma})$, there exists $\phi \in E(T^\bullet)$ with $\phi(x) = z$ because $\langle x \rangle$ is nice in $\langle x \rangle_*$ with totally projective quotient. If $x \in C(M)$, it follows that $T^\bullet(\underline{\sigma}) \subseteq M$. But then $\sup |p^i x| > \mu_x = \sup |p^i x|$ by Lemma 3, a contradiction. Thus $x \notin C(M)$ in this case. Now suppose $\underline{\sigma}$ is gapless. If $x \in C(M)$, we have $M = (E(T^\bullet)x)_*$ because x is stable in M. It follows that $M = (T^{\bullet\tau})_*$ if τ is the smallest ordinal such that $x \notin T^{\bullet\tau+1}$. This contradicts Lemma 1, finishing the proof. ∎

Let $\hat{R}$ denote the p-adic completion of R. To prove the next lemma, we will need to consider $\hat{R}$-submodules of $T^\bullet$ and use a consequence of [8, Lemma 5]: if L is a reduced $\hat{R}$-module and $x \in L$ is torsion-free, then $\ell(t(L/\hat{R}x)) < \ell(tL) + \omega$.

LEMMA 8. *Assume T is totally projective, and M and N are pure submodules of $T^\bullet$ containing T such that $E(M) = E(N)$. If x is a stable torsion-free element of M and $\hat{R}x \cap N = 0$, then there exists a strictly increasing sequence of ordinals $\underline{\sigma} = (\sigma_i)$ such that $T^\bullet(\underline{\sigma}) \subseteq C(N)$ and* $\sup \sigma_i \leq \mu_x$.

Proof. Suppose $x \in M \setminus T$ and $\delta \in E(M)$ satisfy the conditions of Definition 1, and $\hat{R}x \cap N = 0$. Then $\operatorname{Hom}(\langle x \rangle_*, C(M)) = \operatorname{Hom}((\delta N)_*, C(N))$ by Lemma 6, and Lemma 7 implies the existence of $y \in \delta N$ such that $y \notin C(N)$ (because $x \notin C(M)$). Let L be the $\hat{R}$-module $(\hat{R}\langle x, y\rangle)_*/\hat{R}x$ of torsion-free rank one, where the purification is taken in $T^\bullet$. Define $\sigma_i = |p^i y + \hat{R}x|_L$ for $i \geq 0$, and $\underline{\sigma} = (\sigma_i)$. If $z \in T^\bullet(\underline{\sigma})$, there exists ϕ: $L \to T^\bullet$ with $\phi(y + \hat{R}x) = z$ because tL is totally projective. Hence, by composing ϕ with the natural map $(\hat{R}\langle x, y\rangle)_* \to L$ and extending to $T^\bullet$, we obtain $\psi \in E(T^\bullet)$ such that $\psi(y) = z$ and $\psi(x) = 0$. It follows that $z \in C(N)$ because $\psi \in \operatorname{Hom}(\langle x \rangle_*, C(M))$ and $y \in \delta N$. Thus, $T^\bullet(\underline{\sigma}) \subseteq C(N)$. Note that since $\hat{R}x$ is nice, for each i there exists $r_i \in \hat{R}$ such that $\sigma_i = |p^i y + r_i x|_{T^\bullet}$.

It remains to be shown that $\sup \sigma_i \leq \mu_x$. If $\underline{\sigma}$ contains only finitely many gaps, then $\sigma_{n+i} = \sigma_n + i$ $(i \geq 0)$ for some n, hence $p^n y + r_n x \in T^\bullet(\underline{\sigma}) \subseteq C(N)$. Thus $0 \neq r_n x \in N$ because $y \notin C(N)$, contradicting $\hat{R}x \cap N = 0$. Therefore $\underline{\sigma}$ contains infinitely many gaps. By Lemma 4, we may regard $T^\bullet \subseteq \tilde{T}^\bullet$, where $\tilde{T} = S \oplus S'$, $\ell(S) \leq \mu_x$, and $x \in S^\bullet$. Let π: $\tilde{T}^\bullet \to S^\bullet$ be projection and define $\tilde{L} = (\hat{R}\langle \pi(y), x\rangle)_*/\hat{R}x$, where the purification is taken in $S^\bullet$. Let $\tilde{\sigma}_i = |p^i \pi(y) + \hat{R}x|_{\tilde{L}}$ for $i \geq 0$, and note $\sigma_i \leq \tilde{\sigma}_i$ for all i.

We claim $\tilde{L}$ has torsion-free rank one. If $\tilde{L}$ is torsion, then $p^k \pi(y) + rx = 0$ for some $r \in \hat{R}$ and k, hence $p^k y = z - rx$ for some $z \in T^\bullet \cap S'^\bullet$. Now $\sigma_{i+k} = |p^{i+k} y + r_{i+k} x|_{\tilde{T}^\bullet} = |r_{i+k} x - p^i rx + p^i z|_{\tilde{T}^\bullet} \leq |p^i z|_{S'^\bullet}$ for $i \geq 0$, so that $z \in T^\bullet(\underline{\sigma}) \subseteq C(N)$. Thus $rx = z - p^k y \in \hat{R}x \cap N = 0$, so that $z = p^k y$. This implies $y \in C(N)$, a contradiction. Therefore $\operatorname{rk}(\tilde{L}) = 1$ as claimed.

Finally, note that $\ell(t\tilde{L}) < \ell(S) + \omega \leq \mu_x + \omega$ by the remark before the lemma. Hence, $(\tilde{\sigma}_i)$ will be gapless from some point on if $\sup \tilde{\sigma}_i > \mu_x$. This implies (σ_i) can have only finitely many gaps if $\sup \sigma_i > \mu_x$. Thus $\sup \sigma_i \leq \mu_x$, as desired. ∎

LEMMA 9. *Assume T is totally projective, M and N are pure submodules of $T^\bullet$ containing T such that $E(M) = E(N)$, and that M has a stable element. If $C(N) = T$ or N has a stable element, then there is a unit $\alpha \in \hat{R}$ such that $\alpha M \subseteq N$.*

Proof. If $M = T$, then $N = C(N)$. In this case Lemma 7 applied to N shows $N = T$, hence $M = N$. Therefore assume M is nontorsion, and let x be a stable element of M. If $\hat{R}x \cap N \neq 0$, then $\alpha x \in N$ for some unit $\alpha \in \hat{R}$, hence $\alpha M \subseteq N$ because $M/E(M)x$ is torsion and $E(M) = E(N)$. For the sake of contradiction, assume $\hat{R}x \cap N = 0$ and obtain $\underline{\sigma} = (\sigma_i)$ such that $T^\bullet(\underline{\sigma}) \subseteq C(N)$ and $\sup \sigma_i \leq \mu_x$ by Lemma 8. If $\|x\|$ is gapless, then $x \in C(N)$ because $x \in T^\bullet(\underline{\sigma})$, a contradiction. Therefore, assume $\|x\|$ contains infinitely many gaps. We consider two cases for N.

Case 1. Assume $C(N) = T$. Because of the gaps in $\|x\|$, we have $\ell(T) \geq \mu_x$ and a standard argument shows that $T^\bullet(\underline{\sigma})$ contains torsion-free elements. This contradicts $T^\bullet(\underline{\sigma}) \subseteq C(N)$.

Case 2. Assume N has a stable element y, and note y is torsion-free since $N \neq T$ (otherwise $M = T$ by Lemma 7). We have $\mu_x \geq \sup \sigma_i > \mu_y$ by Lemma 3. If $\hat{R}y \cap M \neq 0$, then $\beta N \subseteq M$ for a unit $\beta \in \hat{R}$, hence $T^\bullet(\underline{\sigma}) \subseteq C(N) \subseteq C(M)$. Thus $\sup \sigma_i > \mu_x$ by Lemma 3. This contradiction shows $\hat{R}y \cap M = 0$, hence by Lemma 8 there exists a sequence $\underline{\tau} = (\tau_i)$ such that $T^\bullet(\underline{\tau}) \subseteq C(M)$ and $\sup \tau_i \leq \mu_y$. But $\sup \tau_i > \mu_x$ by Lemma 3, contradicting $\mu_y < \mu_x$. ∎

Before turning to the theorem, we prepare for the case where M is a reduced module with a stable element and M/tM is not divisible. Write $T = tM$, and assume that $x \in M$ and $\delta \in E(M)$ satisfy the conditions of Definition 1. Given $y \in M$, there exist nonzero $r, s \in R$ and $\phi \in E(M)$ such that $\phi(rx) = sy$. If $\langle x \rangle_* / T$ is divisible, then ϕ induces an epimorphism $\langle x \rangle_* / T \to \langle y \rangle_* / T$. Hence, $\langle x \rangle_* / T \cong R$ if M/T is not divisible. In this case, δ composed with the natural map $M \to M/T$ has image isomorphic to R, from which it follows that M has a direct summand isomorphic to R.

THEOREM 1. *Assume M and N are reduced modules and the torsion of M is totally projective. If each module possesses a stable element, then every isomorphism $E(M) \to E(N)$ is induced by an isomorphism $M \to N$.*

Proof. Let Φ: $E(M) \to E(N)$ be an R-algebra isomorphism. First, assume M/tM is not divisible. Then $M = M_0 \oplus M_1$, where $M_0 \cong R$. Let $N = N_0 \oplus N_1$ be the corresponding decomposition of N induced by Φ. Since $R \cong E(M_0) \cong E(N_0)$, N_0 is torsion-free and indecomposable. We claim $N_0 \cong R$. Since N/tN is not divisible and N contains a stable element, we have $N = N_2 \oplus N_3$, where $N_2 \cong R$. Let $M = M_2 \oplus M_3$ be the corresponding decomposition of M induced by Φ^{-1}. Since $E(M_2) \cong R$,

M_2 is nonzero. Thus $\mathrm{Hom}(M_0, M_2) \cong M_2$ is nonzero. Φ induces an isomorphism $\mathrm{Hom}(M_0, M_2) \cong \mathrm{Hom}(N_0, N_2)$, hence there exists a nonzero homomorphism $\gamma: N_0 \to N_2$. Since $N_2 \cong R$, γ splits. Therefore $N_0 = \mathrm{Im}\,\gamma \cong R$ because N_0 is indecomposable. This proves the claim. Using the corresponding summands M_0 and N_0, Kaplansky's method can be applied to construct an isomorphism $\phi: M \to N$ which induces Φ.

Now denote $T = tM$, and assume M/T is divisible. Then N/T is divisible, and by [8, Proposition 3] we may assume M and N are pure submodules of $T^{\bullet}$ containing T such that $E(M) = E(N)$. According to this proposition, Φ is induced by an isomorphism of the original modules precisely when $N = \alpha M$ for a unit $\alpha \in \hat{R}$. By Lemma 9, there exist units $\alpha, \beta \in \hat{R}$ such that $\alpha M \subseteq N$ and $\beta N \subseteq M$, so that $\alpha\beta M \subseteq M$. If $x \in M$ and $\delta \in E(M)$ satisfy the conditions of Definition 1, then $\langle x, \alpha\beta x\rangle \subseteq \delta M \subseteq \langle x \rangle_*$ because $\delta(\alpha\beta x) = \alpha\beta\delta(x) = \alpha\beta x$. Thus $\alpha\beta \in R$ and $\alpha\beta M = M$. Therefore $\alpha M = N$, and the theorem is proved. ▮

When R is complete, the example $E(R) \cong R \cong E(R(p^{\infty}))$ shows the necessity of assuming N is reduced in Theorem 1. In Section 5, we show that the hypothesis that N have a stable element cannot be dropped in the case of infinite torsion-free rank. In the finite-rank case, though, it is possible to do so if $\mathrm{rk}(N) \leq \mathrm{rk}(M)$ and $tM \neq 0$. This situation is considered in the next section.

The first corollary to Theorem 1 is immediate.

COROLLARY 1. *If M is a reduced module with a stable element and totally projective torsion, then every automorphism of $E(M)$ is inner.*

COROLLARY 2. *Assume M and N are reduced modules of countable torsion-free rank, tM is totally projective, and there are ordinals σ and τ such that $M/p^{\sigma}M$ and $N/p^{\tau}N$ have torsion-free rank one. Then every isomorphism $E(M) \to E(N)$ is induced by an isomorphism $M \to N$.*

Proof. See Theorem 1 and Proposition 1(3). ▮

COROLLARY 3. *Assume M and N are reduced Warfield modules, tM is totally projective, and the set of height sequences of elements of a decomposition basis for M contains a least element. Then every isomorphism $E(M) \to E(N)$ is induced by an isomorphism $M \to N$.*

Proof. Let $\Phi: E(M) \to E(N)$ be an isomorphism. By [2, Theorem 4.3], M is isomorphic to N. Therefore N contains a stable element. By Theorem 1, Φ is induced by an isomorphism of M with N. ▮

4. FINITE TORSION-FREE RANK

If the rank of $\hat{R}$ as an R-module is infinite, and M is any reduced R-module of finite, positive torsion-free rank, then [9, Theorem 7] implies there are uncountably many pairwise nonisomorphic R-modules N with $E(N) \cong E(M)$. One consequence of the next theorem is that this occurs only for N of larger torsion-free rank than M when the latter has totally projective torsion and contains a stable element.

THEOREM 2. *Assume M and N are reduced modules with $\mathrm{rk}(N) \leq \mathrm{rk}(M) < \aleph_0$ and $tM \neq 0$ is totally projective. If M possesses a stable element, then every isomorphism $E(M) \to E(N)$ is induced by an isomorphism $M \to N$.*

Proof. Let Φ: $E(M) \to E(N)$ be an isomorphism. To begin, assume M/tM is not divisible. Then $M = M_0 \oplus M_1$, where $M_0 \cong R$. Let $N = N_0 \oplus N_1$ be the corresponding decomposition of N induced by Φ. Since $E(N_0) \cong R$, N_0 is torsion-free. As in the proof of Theorem 1, it suffices to show $N_0 \cong R$. Because $tM \neq 0$, [1, Corollary 3.4] can be applied to show that $\hat{N}_0 \cong \hat{M}_0 \cong \hat{R}$. Φ induces an isomorphism $\mathrm{Hom}(M_0, M_1) \cong \mathrm{Hom}(N_0, N_1)$. Since $\mathrm{Hom}(M_0, M_1) \cong M_1$, the torsion-free rank of $\mathrm{Hom}(N_0, N_1)$ equals $\mathrm{rk}(M_1)$. Fix $0 \neq z \in N_0$, and let $\phi \in \mathrm{Hom}(N_0, N_1)$. Because $\hat{N}_0 \cong \hat{R}$ and ϕ extends to a homomorphism $\hat{N}_0 \to N_1^\bullet$, it follows that ϕ is a torsion element of $\mathrm{Hom}(N_0, N_1)$ if $\phi(z) \in tN_1$. Hence, the torsion-free rank of $\mathrm{Hom}(N_0, N_1)$ is at most that of the submodule $\{\phi(z) : \phi \in \mathrm{Hom}(N_0, N_1)\}$ of N_1. Therefore $\mathrm{rk}(M_1) \leq \mathrm{rk}(N_1)$. Since $\mathrm{rk}(M) = \mathrm{rk}(M_1) + 1$ and $\mathrm{rk}(N) \leq \mathrm{rk}(M)$, we obtain $\mathrm{rk}(N_0) = 1$. Therefore $N_0 \cong R$, as desired.

Now let $T = tM$ and assume M/T is divisible. By [8, Lemma 2], N/tN is divisible. As in the proof of Theorem 1, we may assume M and N are pure submodules of $T^\bullet$ containing T such that $E(M) = E(N)$. Since N has finite torsion-free rank we have $C(N) = T$ (see [8, Lemma 7]), hence $\alpha M \subseteq N$ for a unit $\alpha \in \hat{R}$ by Lemma 9. Thus $N = \alpha M$ because $N/\alpha M$ is torsion and αM is pure in $T^\bullet$, proving the theorem. ∎

The next two corollaries follow from Theorem 2 and Proposition 1. The case $tM = 0$ is easily handled by other means.

COROLLARY 4. *Assume M and N are reduced modules with $\mathrm{rk}(N) \leq \mathrm{rk}(M) < \aleph_0$ and tM is totally projective. If $\mathrm{rk}(M/p^\sigma M) = 1$ for some ordinal σ, then every isomorphism $E(M) \to E(N)$ is induced by an isomorphism $M \to N$.*

COROLLARY 5. *Assume M is a reduced Warfield module of finite torsion-free rank, tM is totally projective, and the set of height sequences of a* *decomposition basis of M contains a least element. If N is a reduced module with* $\mathrm{rk}(N) \le \mathrm{rk}(M)$, *then every isomorphism $E(M) \to E(N)$ is induced by an isomorphism $M \to N$.*

The algebra isomorphism in the next corollary need not be induced; the example in [2, Sect. 5] demonstrates that possibility for modules of torsion-free rank two.

COROLLARY 6. *Assume $M = \oplus M_i$ is a reduced module of finite torsion-free rank, $tM \neq 0$ is totally projective, and each module M_i possesses a stable element. If N is a reduced module and* $\mathrm{rk}(N) \le \mathrm{rk}(M)$, *then $E(M) \cong E(N)$ implies $M \cong N$.*

Proof. We may assume $M = \bigoplus_{1 \le i \le k} M_i$, where each M_i contains a stable element and tM_i is totally projective. Using the isomorphism $E(M) \to E(N)$, decompose $N = \bigoplus_{1 \le i \le k} N_i$ so that $E(M_i) \cong E(N_i)$ for all i. Because $\mathrm{rk}(N) \le \mathrm{rk}(M), \mathrm{rk}(N_j) \le \mathrm{rk}(M_j)$ for some j. We claim $M_j \cong N_j$. This follows from Theorem 2 if $tM_j \neq 0$. If M_j is torsion-free, then there exist nonzero, torsion direct summands S of M and S' of N such that $E(M_j \oplus S) \cong E(N_j \oplus S')$. Therefore $M_j \oplus S \cong N_j \oplus S'$ by Theorem 2, so that $M_j \cong N_j$. Since $\mathrm{rk}(N/N_j) \le \mathrm{rk}(M/M_j)$, we have $\mathrm{rk}(N_{j'}) \le \mathrm{rk}(M_{j'})$ for some $j' \neq j$. By repeating the above argument and continuing in this fashion, we obtain $M_i \cong N_i$ for all $i = 1, \ldots, k$. ∎

5. EXAMPLES

For the first of two examples, we construct a Walker-indecomposable module M of uncountable rank which contains a stable element. Let T be a totally projective module of length $\omega 2$ with Ulm invariants $U_T(2i) = 1$ and $U_T(2i+1) = 0$ for $i < \omega$. By [3, Sect. 103], there is a module P of torsion-free rank one with $tP \cong T$ that contains an element with height sequence $(0, 2, 4, \cdots)$. Then P embeds in $T^\bullet$, so we may assume $x \in T^\bullet$ satisfies $|p^i x| = 2i$ for $i \ge 0$. By Proposition 1, x is stable in $M = (\langle x \rangle + T^{1\bullet})_* \subseteq T^\bullet$. Since T^1 is unbounded, $T^{1\bullet}$ and hence M has torsion-free rank at least $2^{\aleph_0}$. If $M = A \oplus B$, we claim A or B is bounded. Writing $x = a + b$ in this decomposition, we may assume $a \notin (T^{\bullet 1})_*$. Since M/M^1 has torsion-free rank one, there exist $r \in R$ and k such that $p^k x + ra \in M^1$, hence $|p^{k+i} x|_M = |p^i ra|_A$ for $i \ge 0$. Thus $U_A(2i) = 1$ for almost all $i < \omega$, and it follows that B is bounded because $U_B(i) = 0$ for almost all $i < \omega$.

For the second example, we construct reduced modules that show the main hypothesis on N in Theorem 1 cannot be replaced by a condition on ranks. Specifically, we show isomorphism can fail when $\mathrm{rk}(M) = \mathrm{rk}(N) =$

$\aleph_0$, $tM \neq 0$, and only M contains a stable element. For convenience, let $R = \mathbb{Z}_{(p)}$ (localization at p) and define $X = \bigoplus_{\aleph_0} R$. By [4, Proposition 4.3], we may regard X as embedded in a reduced R-module $G(X)$ such that $X = G(X)^1$, $tG(X)$ is a direct sum of cyclics, $G(X)/X$ is torsion, and every endomorphism of X has an extension to one of $G(X)$. Then $G(X)$ satisfies the hypotheses for M in Theorem 1 because every nonzero element of X is stable in X, hence in $G(X)$. Let Y be a pure submodule of $\hat{X}$ of rank $\aleph_0$ not isomorphic to X, such that $\hat{Y} = \hat{X}$ and $E(Y) = E(X)$ in $E(\hat{X})$. Then $E(G(X)) \cong E(G(Y))$ by the proposition in [4], but $G(X)$ is not isomorphic to $G(Y)$ because $G(X)^1$ and $G(Y)^1 = Y$ are not isomorphic.

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