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Generalized Lagrange Multipliers in Elastoplastic Torsion

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Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with Lipschitz boundary, and let

$$K = \{v \in H_0^1(\Omega) : |Dv(x)| \leqslant 1 \text{ a.e. in } \Omega\}.$$

Given a measurable function $f\colon \Omega \times \mathbf{R}^n \to [0, +\infty[$ and two constants $0 < \lambda \leqslant \Lambda$, such that

$$f(x, \cdot) \text{ is strictly convex for a.e. } x \in \Omega,\ \forall \xi \in \mathbf{R}^n, \tag{1}$$

$$\lambda\, |\xi|^2 \leqslant f(x, \xi) \leqslant \Lambda(1 + |\xi|^2) \qquad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbf{R}^n, \tag{2}$$

for every $g \in L^2(\Omega)$, there exists a unique solution to the problem

$$\min_{v \in K} \left\{ \int_\Omega f(x, Dv)\, dx - \int_\Omega gv\, dx \right\} \tag{3}$$

(see, for instance, [5]). It is well-known that when $n = 2$, g is constant, and $f(x, \xi) = |\xi|^2$, this problem has a relevant physical meaning, since it describes the elastoplastic torsion of a cylindric bar of cross section Ω (see [6, 8]). In 1972, Brézis [1] proved the existence of a Lagrange multiplier for this particular case. More precisely, he showed that if $f(x, \xi) = |\xi|^2$, g is a positive constant, and u solves problem (3), then there exists a unique function $\lambda \in L^\infty(\Omega)$ such that

$$\begin{cases} \lambda \geqslant 0 & \text{a.e. in } \Omega, \\ \lambda(1 - |Du|) = 0 & \text{a.e. in } \Omega, \\ -\Delta u - \operatorname{div}(\lambda\, Du) = g & \text{in } \mathscr{D}'(\Omega). \end{cases} \tag{4}$$

It is worth noting that, conversely, if $u \in K$ and if there exists λ satisfying (4), then u is a solution to (3) with $f(x, \xi) = |\xi|^2$.

In this paper we prove, by a completely different method, a result analogous to Brézis' when $g \in L^p(\Omega)$ (in general not identically a constant) and f is a more general convex integrand. More precisely, we show the following.

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THEOREM. *Let $f: \Omega \times \mathbf{R}^n \to [0, +\infty[$ be a measurable function satisfying* (1), (2), *and let $g \in L^p(\Omega)$, $p > n$. Assume in addition that one of the following conditions is satisfied*:

$$\begin{cases} f(\cdot, \xi) \in \mathscr{C}^1(\bar{\Omega}) & \text{for every } \xi \in \mathbf{R}^n, \\ f(x, \cdot) \text{ is a quadratic form} & \text{for a.e. } x \in \Omega, \end{cases} \tag{5}$$

$$\begin{cases} \Omega \text{ is convex}, & \\ f(\cdot, \xi) \text{ is constant} & \forall \xi \in \mathbf{R}^n, \\ f(x, \cdot) \in \mathscr{C}^2(\mathbf{R}^n) & \forall x \in \Omega, \\ f_{\xi_i \xi_j}(x, \xi)\, \eta_i \eta_j \geqslant \lambda_0 \, |\eta|^2 & \forall \xi, \eta \in \mathbf{R}^n, \end{cases} \tag{6}$$

with $\lambda_0 > 0$. Then, if u solves problem (3), *there exists a positive Radon measure $\lambda \in M(\Omega; [0, \infty[)$ such that*

$$\int_\Omega (1 - |Du|^2)\, d\lambda = 0, \tag{7}$$

$$-\operatorname{div}(f_\xi(x, Du) + \lambda\, Du) = g \qquad \text{in } \mathscr{D}'(\Omega). \tag{8}$$

We remark that, even if our assumptions are more general, we have not been able to recover Brézis' result, since we just obtain the existence of a measure term λ that plays the role of a Lagrange multiplier. It would be interesting to investigate the uniqueness of λ and its regularity in terms of those of f and g. The proof of the theorem will be given after some preliminary results.

To this aim, we start by fixing some notation. For every $h \in \mathbf{N}$, we set $\alpha_h: \mathbf{R}^n \to \mathbf{R}$, $\alpha_h(\xi) = h(|\xi|^2 - 1)^+$, and we define the functionals F_h, $F: L^2(\Omega) \to [0, +\infty]$ as follows:

$$F_h(u) = \begin{cases} \displaystyle\int_\Omega f(x, Dv)\, dx - \int_\Omega gv\, dx + \int_\Omega \alpha_h(Dv)\, dx & \text{if } \quad v \in H_0^1(\Omega) \\ +\infty & \text{otherwise}, \end{cases} \tag{9}$$

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x, Dv)\, dx - \int_\Omega gv\, dx & \text{if } \quad v \in K \\ +\infty & \text{otherwise}. \end{cases} \tag{10}$$

Moreover, we set

$$m_h = \inf_{v \in L^2(\Omega)} F_h(v), \qquad m = \inf_{v \in L^2(\Omega)} F(v). \tag{11}$$

Since F_h and F are both lower semicontinuous and coercive on $L^2(\Omega)$ for the strong topology, by the direct method of the calculus of variations the infimum is attained in both cases and, if u_h and u denote the corresponding (unique) solutions, i.e.,

$$m_h = F_h(u_h), \qquad m = F(u), \tag{12}$$

the following results hold.

LEMMA 1. *Let f satisfy* (1), (2), *and* $g \in L^2(\Omega)$. *Then, according to* (9)–(12), *we have*

$$m_h \to m, \tag{13}$$

$$u_h \to u \qquad \textit{strongly in } L^2(\Omega), \tag{14}$$

$$\int_\Omega \alpha_h(Du_h)\, dx \to 0, \tag{15}$$

$$\int_\Omega f(x, Du_h)\, dx \to \int_\Omega f(x, Du)\, dx, \tag{16}$$

$$u_h \to u \qquad \textit{strongly in } H^1(\Omega), \tag{17}$$

as $h \to +\infty$.

Remark. Conditions (13) and (14) follow immediately, in the framework of Γ-convergence theory, from the fact that $F = \Gamma(L^2(\Omega)) \lim_{h \to +\infty} F_h$ and that $(F_h)_h$ is equicoercive (see [4]), but for the reader's convenience we derive them directly.

Proof of Lemma 1. Let u_h, u satisfy (12). Then, by the left hand side of (2), it follows that

$$\|u_h\|_{H^1(\Omega)} \leqslant c \tag{18}$$

and hence, up to a subsequence, (u_h) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$, to a function $v \in H_0^1(\Omega)$. If we prove that

$$F(v) \leqslant \liminf_{h \to +\infty} F_h(u_h), \tag{19}$$

then (13) follows immediately from the fact that, being (F_h) monotonically increasing with h, $m_h \leqslant m$ for every h. Moreover, by (13) and (19), $F(v) \leqslant m$ and hence, by the strict convexity of $f(x, \cdot)$, $v = u$, which concludes the proof of (14). To prove (19) we observe that, for every $L^2(\Omega)$-neighbourhood V of v, the following inequality

$$\inf_{w \in V} F_h(w) \leqslant F_h(u_h)$$

holds for h large enough. Hence we have

$$\liminf_{h \to +\infty} \inf_{w \in V} F_h(w) \leqslant \liminf_{h \to +\infty} F_h(u_h);$$

since $(\inf_{w \in V} F_h(w))_h$ increases with h, we can replace the lim inf in the left hand side by a $\sup_h$, thus obtaining

$$\sup_h \inf_{w \in V} F_h(w) \leqslant \liminf_{h \to +\infty} F_h(u_h).$$

Moreover, by taking the supremum over the neighbourhoods of v, we get

$$\sup_V \sup_h \inf_{w \in V} F_h(w) \leqslant \liminf_{h \to +\infty} F_h(u_h).$$

Finally, by interchanging the two sup and by taking into account that F_h is $L^2(\Omega)$-lower semicontinuous, we get

$$F(v) = \sup_h F_h(v) \leqslant \liminf_{h \to +\infty} F_h(u_h)$$

which is (19). To prove (15) we note that, by (13) and (14),

$$\lim_{h \to +\infty} \left(\int_\Omega f(x, Du_h)\, dx + \int_\Omega \alpha_h(Du_h)\, dx \right) = \int_\Omega f(x, Du)\, dx. \tag{20}$$

Since that functional

$$u \mapsto \int_\Omega f(x, Du)\, dx$$

is $L^2(\Omega)$-lower semicontinuous on $H_0^1(\Omega)$, we have

$$\limsup_{h \to +\infty} \int_\Omega \alpha_h(Du_h)\, dx \leqslant 0.$$

α_h being nonnegative, we immediately get (15). Condition (16) follows directly from (20) and (15). Finally, in order to prove (17) when f satisfies (5) it is enough to note that by (2) conditions (16) and (17) are actually equivalent. Otherwise, under the general assumptions (1), (2), the strong convergence of (u_h) to u follows from a result by Visintin (see Theorem 3 in [9]). ∎

LEMMA 2. *Let f satisfy* (1), (2), *and one of the conditions* (5) *and* (6). *If u solves problem* (3) *with $g \in L^p(\Omega)$, $p > n$, then $u \in \mathscr{C}^{1,\alpha}(\bar{\Omega})$, $\alpha = 1 - n/p$.*

The proof of Lemma 2 is due to Brézis and Stampacchia [2].

LEMMA 3. *Let* $\psi_h: \mathbf{R}^n \to \mathbf{R}$ *be defined by*

$$\psi_h(\xi) = \begin{cases} 0 & \text{if } |\xi| < 1 \\ 2h & \text{if } |\xi| \geqslant 1, \end{cases} \tag{21}$$

and let $u_h \in H_0^1(\Omega)$ *satisfy* (12). *If* f *satisfies* (1), (2), *and* $g \in L^2(\Omega)$, *then we have*

$$0 \leqslant \int_\Omega \psi_h(Du_h)\, |Du_h|^2\, dx \leqslant c_1, \tag{22}$$

$$0 \leqslant \int_\Omega \psi_h(Du_h)\, dx \leqslant c_2, \tag{23}$$

with c_1, c_2 *positive constants depending on* g.

Proof. Let $u_h \in H_0^1(\Omega)$ be the solution to the minimum problem in (12) related to F_h, and let ψ_h be defined by (21). It is easy to see that u_h satisfies the Euler equation

$$-\operatorname{div}[f_\xi(x, Du_h) + \psi_h(Du_h)\, Du_h] = g, \tag{24}$$

in the weak sense of $H^{-1}(\Omega)$; i.e.,

$$\int_\Omega (f_\xi(x, Du_h), Dv)\, dx + \int_\Omega (\psi_h(Du_h)\, Du_h, Dv)\, dx = \int_\Omega gv\, dx$$

for every $v \in H_0^1(\Omega)$, where the symbol (,) denotes the usual inner product of $\mathbf{R}^n$. By choosing $v = u_h$ and by taking into account that $f_\xi: \mathbf{R}^n \to \mathbf{R}^n$ is monotone and hence

$$\int_\Omega (f_\xi(x, Du_h) - f_\xi(x, 0), Du_h)\, dx \geqslant 0,$$

we have

$$\int_\Omega \psi_h(Du_h)\, |Du_h|^2\, dx \leqslant \int_\Omega (f_\xi(x, 0), Du_h)\, dx + \int_\Omega gu_h\, dx. \tag{25}$$

Since by (1), (2) there exists a positive constant c such that

$$|f_\xi(x, \xi)| \leqslant c(1 + |\xi|) \tag{26}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^n$, then by applying Hölder's inequality to the right-hand side of (25) we obtain (22). The proof of (23) follows directly from (22) by noting that

$$0 \leqslant \psi_h(\xi) \leqslant \psi_h(\xi)\, |\xi|^2$$

for every $\xi \in \mathbf{R}^n$. ∎

In the following we shall denote by $M(\Omega; [0, +\infty[)$ the set of finite positive Radon measures on Ω and by $M(\Omega; \mathbf{R}^n)$ the space of all vector-valued measures with finite variation on Ω. We recall that the variation $|\mu|$ of a measure $\mu \in M(\Omega; \mathbf{R}^n)$ is defined by

$$|\mu(B)| = \sup \left\{ \sum_{h=1}^{\infty} |\mu(B_h)| : \bigcup_{h=1}^{\infty} B_h \subseteq B,\ B_h \text{ pairwise disjoint} \right\}$$

for every Borel set $B \subseteq \Omega$; moreover, if $\mu \in M(\Omega; [0, +\infty[)$, then $|\mu| = \mu$. We denote by $\mathscr{C}_c(\Omega; \mathbf{R}^n)$ the space of all continuous functions $\varphi: \Omega \to \mathbf{R}^n$ with compact support, and by $\mathscr{C}_0(\Omega; \mathbf{R}^n)$ the space of all continuous functions $\varphi: \Omega \to \mathbf{R}^n$ vanishing on the boundary. It is well-known (see, for instance [7]) that the space $\mathscr{C}_0(\Omega; \mathbf{R}^n)$ equipped with the sup norm is a separable Banach space, having $\mathscr{C}_c(\Omega; \mathbf{R}^n)$ as a dense subset, and that $M(\Omega; \mathbf{R}^n)$ can be identified with the dual space of $\mathscr{C}_0(\Omega; \mathbf{R}^n)$ by the duality

$$\langle \mu, \varphi \rangle = \int_\Omega \varphi \, d\mu$$

for every $\varphi \in \mathscr{C}_0(\Omega; \mathbf{R}^n)$ and every $\mu \in M(\Omega; \mathbf{R}^n)$. Moreover, it turns out that

$$|\mu(\Omega)| = \sup\{ |\langle \mu, \varphi \rangle| : \varphi \in \mathscr{C}_0(\Omega; \mathbf{R}^n),\ \|\varphi\|_{\mathscr{C}_0(\Omega; \mathbf{R}^n)} \leq 1 \}$$

for all $\mu \in M(\Omega; \mathbf{R}^n)$. In the following the space $M(\Omega; \mathbf{R}^n)$ will be endowed with the weak* topology deriving from the duality between $M(\Omega; \mathbf{R}^n)$ and $\mathscr{C}_0(\Omega; \mathbf{R}^n)$. More precisely, a sequence (μ_h) in $M(\Omega; \mathbf{R}^n)$ will be said to w^*-converge to $\mu \in M(\Omega; \mathbf{R}^n)$, i.e., $\mu_h \rightharpoonup \mu$, if

$$\langle \mu_h, \varphi \rangle \to \langle \mu, \varphi \rangle \qquad \forall \varphi \in \mathscr{C}_0(\Omega; \mathbf{R}^n)$$

or equivalently if

$$\sup_{h \in \mathbf{N}} |\mu_h|(\Omega) < +\infty \qquad \text{and} \qquad \langle \mu_h, \varphi \rangle \to \langle \mu, \varphi \rangle \quad \forall \varphi \in \mathscr{C}_c(\Omega; \mathbf{R}^n).$$

Finally, we recall that, given a function $f \in L^1(\Omega; \mathbf{R}^n)$, the functional

$$\mu_f : \varphi \mapsto \int_\Omega (\varphi, f)\, dx \qquad \forall \varphi \in \mathscr{C}_0(\Omega; \mathbf{R}^n)$$

is an element of $M(\Omega; \mathbf{R}^n)$ and $|\mu_f|(\Omega) = \|f\|_{L^1(\Omega; \mathbf{R}^n)}$. Analogous considerations can be made for $M(\Omega; [0, +\infty[)$ considered as a subset of $M(\Omega; \mathbf{R})$.

LEMMA 4. *Let f satisfy* (1) *and* (2), *and let $g \in L^2(\Omega)$. If u_h is defined by* (12), *then up to a subsequence*

$$\psi_h(Du_h)\, Du_h \rightharpoonup \mu \in M(\Omega; \mathbf{R}^n), \tag{27}$$

$$\psi_h(Du_h) \rightharpoonup \lambda \in M(\Omega; [0, +\infty[), \tag{28}$$

as $h \to +\infty$.

Proof. Since

$$\int_\Omega |\psi_h(Du_h)\, Du_h|\, dx \leqslant \int_\Omega \psi_h(Du_h)\, |Du_h|^2\, dx,$$

the result follows from estimates (22), (23), and the fact that, by the Banach–Alaoglu theorem, bounded sets in $M(\Omega; \mathbf{R}^n)$ are weakly* compact. ∎

LEMMA 5. *Let $\mu \in M(\Omega; \mathbf{R}^n)$ be defined by* (27). *Then*

$$|\mu|\,(A) \leqslant \liminf_{h \to +\infty} \int_A \psi_h(Du_h)\, |Du_h|\, dx \tag{29}$$

for every open set $A \subseteq \Omega$.

Proof. Let A be an open subset of Ω. By (27) and by the definition of weak* convergence in $M(\Omega; \mathbf{R}^n)$ we have that

$$\int_\Omega \psi_h(Du_h)(Du_h, \varphi)\, dx \to \langle \mu, \varphi \rangle$$

for every $\varphi \in \mathscr{C}_0(\Omega; \mathbf{R}^n)$ and in particular for every $\varphi \in \mathscr{C}_0(A; \mathbf{R}^n)$. Then the convergence (27) holds also in $M(A; \mathbf{R}^n)$ and hence, by the weak* lower semicontinuity of the norm in $M(A; \mathbf{R}^n)$, we immediately obtain (29). ∎

LEMMA 6. *Let $\mu \in M(\Omega; \mathbf{R}^n)$ and $\lambda \in M(\Omega; [0, +\infty[)$ be defined by* (27) *and* (28), *respectively. Then there exists a function $\Phi \in L^1(\Omega, \lambda; \mathbf{R}^n)$ such that*

$$|\Phi(x)| \leqslant 1 \qquad \lambda - a.e. \text{ in } \Omega, \tag{30}$$

$$\frac{d\mu}{d\lambda} = \Phi. \tag{31}$$

Proof. To prove the lemma it is enough to show that

$$|\mu(B)| \leqslant \lambda(B) \tag{32}$$

for every Borel set $B \subseteq \Omega$. In fact, this would imply that $|\mu| \ll \lambda$ and $0 \leqslant d|\mu|/d\lambda \leqslant 1$, λ-a.e. in Ω, from which (30), (31) follow directly. Let us start by noting that, by (29),

$$|\mu(A)| \leqslant \limsup_{h \to +\infty} \int_A \psi_h(Du_h)\, |Du_h|^2\, dx \tag{33}$$

for every open set $A \subseteq \Omega$. Moreover, we have

$$\limsup_{h \to +\infty} \int_A \psi_h(Du_h)\, |Du_h|^2\, dx = \limsup_{h \to +\infty} \int_A \psi_h(Du_h)\, dx, \tag{34}$$

since

$$\int_A \psi_h(Du_h)(|Du_h|^2 - 1)\, dx = \int_A \alpha_h(Du_h)\, dx$$

which tends to 0 by Lemma 1. We recall that for every sequence $(\lambda_h) \in M(\Omega; [0, +\infty[)$ that w^*-converges to λ and for every closed set $C \subseteq \Omega$ we have

$$\limsup_{h \to +\infty} \lambda_h(C) \leqslant \lambda(C)$$

(see, for instance, (3.1.4) in [3]). By applying these results with $C = \bar{A}$, and by taking into account (33) and (34), we get

$$|\mu|\,(A) \leqslant \lambda(\bar{A}) \tag{35}$$

for every open set $A \subseteq \Omega$. Now, since both $|\mu|$ and λ are positive, regular, Radon measures, condition (32) will follow directly, if we prove that

$$|\mu|\,(K) \leqslant \lambda(K) \tag{36}$$

for every compact set $K \subseteq \Omega$. To this aim, let us fix a compact set $K \subseteq \Omega$, and let us define $K_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, K) < \varepsilon\}$, for every $\varepsilon > 0$. Then K_ε is open, $\bigcap_{\varepsilon > 0} K_\varepsilon = \bigcap_{\varepsilon > 0} \bar{K}_\varepsilon = K$ and, by (35),

$$|\mu|\,(K_\varepsilon) \leqslant \lambda(\bar{K}_\varepsilon)$$

for every $\varepsilon > 0$. By taking the limit as $\varepsilon > 0$ we immediately obtain (36). ∎

LEMMA 7. *Let $u \in H_0^1(\Omega)$ be the solution to problem* (3) *with f, g satisfying the assumptions of Lemma* 2. *Moreover, let λ, μ, Φ be as in Lemma* 4 *and Lemma* 6. *Then*

$$\operatorname{supp} \lambda \subseteq \{x \in \Omega : |Du(x)| = 1\} \tag{37}$$

$$\Phi = Du \qquad \lambda - a.e. \text{ in } \Omega. \tag{38}$$

Proof. Let us remark that, under the assumptions of Lemma 2, the solution $u \in H_0^1(\Omega)$ to problem (3) is actually in $\mathscr{C}^{1,\alpha}(\bar{\Omega})$, $\alpha > 0$, and hence $Du \in \mathscr{C}(\bar{\Omega}; \mathbf{R}^n)$ and $|Du(x)| \leqslant 1$ for every $x \in \bar{\Omega}$. Now, in order to prove (37), (38), it is enough to show that

$$\lambda(\Omega) \leqslant \int_\Omega Du\, d\mu. \tag{39}$$

In fact, this implies that, by Lemma 6,

$$\int_\Omega (1 - (Du, \Phi))\, d\lambda \leqslant 0,$$

from which, since $(Du, \Phi) \leqslant 1$, λ-a.e. in Ω, we have

$$\int_\Omega (1 - (Du, \Phi))\, d\lambda = 0.$$

Hence, $\operatorname{supp} \lambda \subseteq \{x \in \Omega : (Du(x), \Phi(x)) = 1\}$, but $(Du, \Phi) \leqslant |Du|$ λ-a.e. in Ω, and this implies immediately (37), (38). To prove (39), let $u_h \in H_0^1(\Omega)$ be defined by (12). We recall that u_h satisfies the Euler equation (24), i.e.,

$$\int_\Omega (f_\xi(x, Du_h), D\varphi)\, dx + \int_\Omega (\psi_h(Du_h)\, Du_h, D\varphi)\, dx = \int_\Omega g\varphi\, dx \tag{40}$$

for every $\varphi \in \mathscr{C}_0^1(\Omega)$. By (17), (26), and (27), we can pass to the limit as $h \to +\infty$ and we obtain

$$\int_\Omega (f_\xi(x, Du), D\varphi)\, dx + \int_\Omega \varphi\, d\mu = \int_\Omega g\varphi\, dx$$

for every $\varphi \in \mathscr{C}_0^1(\Omega)$, i.e.,

$$-\operatorname{div}[f_\xi(x, Du) + \mu] = g \qquad \text{in } \mathscr{D}'(\Omega). \tag{41}$$

Moreover, by choosing $\varphi = u_h$ in (40) we get

$$\int_\Omega (f_\xi(x, Du_h), Du_h)\, dx + \int_\Omega \psi_h(Du_h)\, |Du_h|^2\, dx = \int_\Omega g u_h\, dx. \tag{42}$$

Now, by (28), it follows that

$$\lambda(\Omega) \leqslant \liminf_{h \to +\infty} \int_\Omega \psi_h(Du_h)\, dx \leqslant \liminf_{h \to +\infty} \int_\Omega \psi_h(Du_h)\, |Du_h|^2\, dx,$$

and if we express the right-hand side by means of (42), (41), we immediately obtain (39). ∎

Proof of the Theorem. The result follows directly from (41) and the fact that, by (31) and (38), $\mu = Du\lambda$ on supp λ, and hence on supp μ. ∎

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