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A decidable Ehrenfeucht theory with exactly two hyperarithmetic models

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Abstract

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Millar showed that for each $n < \omega$, there is a complete decidable theory having precisely eighteen nonisomorphic countable models where some of these are decidable exactly in the hyperarithmetic set $H(n)$. By combining ideas from Millar's proof with a technique of Peretyat'kin, the author reduces the number of countable models to five. By a theorem of Millar, this is the smallest number of countable models a decidable theory can have if some of the models are not 0″-decidable.

A theory T is *Ehrenfeucht* if it has finitely many countable models up to isomorphism and is not ω-categorical. T is *persistently Ehrenfeucht* if for every n-type $\Gamma(\bar{x})$ of T, the theory $\Gamma(\bar{c})$ in the language $L(T) \cup \{\bar{c}\}$ is Ehrenfeucht.

Let

$$H(0) =_{\text{def}} 0,$$

$$H(2^x) =_{\text{def}} H(x)',$$

$$H(3 \cdot 5^e) =_{\text{def}} \{\langle x, y\rangle : x \in H(\varphi_e(y))\}$$

be representative sets in the hyperarithmetic hierarchy, where x and e are any nonnegative integers. Arbitrarily define $H(n)$ to be 0 for values of n not included above. Given a model $\mathcal{A}$ and a set S, we say that *$\mathcal{A}$ is decidable exactly in S* if $\mathcal{A}$ is decidable relative to S but is not decidable in any Turing degree below that of S. Other standard definitions may be found in [1, 2, 6]. This paper consists of a proof of the following theorem.

Theorem. *For each $n < \omega$ there is a complete, decidable, persistently Ehrenfeucht theory T having exactly five nonisomorphic countable models: three decidable models and two models decidable exactly in $H(n)$.*

In [4], Millar shows that any decidable Ehrenfeucht theory having a countable model which is not $0''$-decidable must have at least five nonisomorphic countable models. Thus the theorem proved in this paper gives an example of a decidable Ehrenfeucht theory having the fewest possible number of countable models where some of these are not $0''$-decidable. It appears to be unknown whether a decidable Ehrenfeucht theory whose undecidable countable models are *exactly* $0''$-decidable can have fewer than five countable models.

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Introduction

There are several examples of decidable Ehrenfeucht theories having undecidable models. Peretyat'kin [5] was the first to find an example having the fewest possible number of nonisomorphic countable models, namely three. We give a brief description of it here, since the main idea is also used in the proof of our theorem.

The language of Peretyat'kin's theory includes infinitely many constant symbols, one for each node of a recursive binary tree D^* having exactly one infinite branch, where that branch is nonrecursive. The theory consists of axioms for a dense binary tree, together with axioms stating that the constants are related by the greatest-lower-bound operator of the dense tree in the same way as the corresponding nodes of D^*. Thus every model of this theory is a dense binary tree in which D^* is embedded. There is a single decidable model, namely the (prime) model having no element above the infinite branch of D^*. One of the other two countable models has a lowest element above the infinite branch of D^*, the other has elements above this branch but no lowest such. It is easy to see that both of these models must be undecidable, for otherwise with the help of a parameter we could tell effectively whether or not an arbitrary element of D^* belonged to the infinite branch: we fix an element x lying above this branch, and for any node d in D^* we need only check to see whether d is below x in the model. In fact, both models are decidable exactly in $0'$, since a node of D^* belongs to the infinite branch if and only if it has infinitely many successors in D^*, and the latter statement can be shown to be Π^0_1.

Millar [3] gave the first example of a decidable Ehrenfeucht theory having a countable model which is not 0′-decidable — in fact, not even arithmetic. He showed that for each $n < \omega$ there is a complete, decidable, persistently Ehrenfeucht theory having three decidable models and fifteen models decidable exactly in $H(n)$.

Like Peretyat'kin's example, Millar's involves coding into the theory a recursive tree Tr having a single infinite branch. In order that this infinite branch be Turing equivalent to $H(n)$, Tr must be a subset of $\omega^{<\omega}$ rather than binary. The underlying ω-categorical theory in Millar's example is not that of a dense tree but is the theory of dense linear order without endpoints, hence the coding of Tr is accomplished through the use of unary relation symbols instead of constants. Millar's theory has a single nonrecursive 1-type whose set of realizations can have order type 0, 1, $1+\eta$, $1+\eta+1$, $\eta+1$ or η, where η is the order type of the rationals. The remaining nonprincipal 1-types are all recursive; the set of elements which realize any one of them can have order type 0, $1+\eta$ or η. The recursive nonprincipal 1-types are tied together by means of binary relation symbols in such a way that, in any given model, all of these 1-types are realized by sets having the same order type; there is no similar connection between these 1-types and the single nonrecursive 1-type. It is shown that every countable model of the theory is characterized by the order type of the realization set for the nonrecursive 1-type and the order type of the realization sets for the remaining nonprincipal 1-types, and therefore there are $6 \times 3 = 18$ different models.

The proof given below combines the techniques used in the two examples just described; the tree Tr is embedded in a dense, infinitely-branching tree, and binary relation symbols are used to tie together the nonprincipal 1-types. We note parenthetically that if $\{f\}$ is a Π^0_1 singleton — that is, $f \in \omega^\omega$ is the unique function satisfying $\forall n\, R(f, n)$ for some recursive relation R in one function variable and one number variable — then there is a recursive tree $\mathrm{Tr} \subset \omega^{<\omega}$ having exactly one infinite branch, where that branch is f. Thus the proof of the theorem will actually show that the two undecidable countable models of T can be made to be decidable exactly in any given Π^0_1 singleton.

Part I

The proof will consist of four parts. In Part I, we show there is an ω-categorical theory T_0 whose countable model is an infinitely-branching dense tree on which two binary relations, $<_{\mathrm{L}}$ and $\leqslant_{\mathrm{H}}$, are defined. T_0 will be the model completion of a theory T_0', the theory of trees with a Kleene–Brouwer ordering $<_{\mathrm{L}}$ and a relation $\leqslant_{\mathrm{H}}$ which is intended to measure the relative 'heights' of nodes in the tree. For example, $\omega^{<\omega}$ is a model of T_0' if we define $a \leqslant_{\mathrm{H}} b$ to mean that $\mathrm{length}(a) \leqslant \mathrm{length}(b)$.

In Part II, the countable model of T_0 will be used to describe the prime model of the theory T. We will then show that T admits effective elimination of quantifiers and is therefore decidable. In Part III, T is shown to have exactly five countable models up to isomorphism, and in Part IV we verify that T is *persistently* Ehrenfeucht. The five models will turn out to be:

(M1) A decidable prime model.

(M2) A decidable nonhomogeneous model which is the reduct of the prime model of a recursive nonprincipal type of T.

(M3) A decidable homogeneous model realizing all of the recursive types of T.

(M4) A nonhomogeneous model decidable exactly in $H(n)$ which is the reduct of the prime model of the single nonrecursive 1-type of T.

(M5) A saturated model decidable exactly in $H(n)$.

The language of T_0 consists of the binary function symbol $\wedge$ and two binary relation symbols, $<_{\mathrm{L}}$ and $\leqslant_{\mathrm{H}}$. We make the following notational conventions:

$$x \leqslant y \ \Leftrightarrow_{\mathrm{def}} \ x \wedge y = x,$$

$$x < y \ \Leftrightarrow_{\mathrm{def}} \ x \leqslant y \ \& \ x \neq y,$$

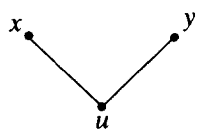
$$x \mid y \ \Leftrightarrow_{\mathrm{def}} \ x \not\leqslant y \ \& \ y \not\leqslant x,$$

$$V(u; x, y) \ \Leftrightarrow_{\mathrm{def}} \ u = x \wedge y \ \& \ x \mid y \ \& \ x <_{\mathrm{L}} y,$$

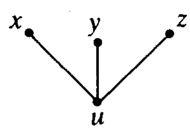
$$W(u; x, y, z) \ \Leftrightarrow_{\mathrm{def}} \ x \wedge y = u = y \wedge z \ \& \ x \mid y \ \& \ y \mid z \ \& \ x <_{\mathrm{L}} y <_{\mathrm{L}} z,$$

$$x =_{\mathrm{H}} y \ \Leftrightarrow_{\mathrm{def}} \ x \leqslant_{\mathrm{H}} y \ \& \ y \leqslant_{\mathrm{H}} x,$$

$$x <_{\mathrm{H}} y \ \Leftrightarrow_{\mathrm{def}} \ x \leqslant_{\mathrm{H}} y \ \& \ x \neq_{\mathrm{H}} y.$$



$V(u; x, y)$



$W(u; x, y, z)$

The axioms of T_0' are:

(1) $x \wedge y = y \wedge x,$

(2) $(x \wedge y) \wedge z = x \wedge (y \wedge z),$

(3) $x \wedge x = x,$

(4) $[x \leqslant z \ \& \ y \leqslant z] \rightarrow [(x \leqslant y) \vee (y \leqslant x)],$

(5) $x \wedge y < y \wedge z \rightarrow x \wedge y = x \wedge z,$

(6) $[x <_{\mathrm{L}} y \ \& \ y <_{\mathrm{L}} z] \rightarrow x <_{\mathrm{L}} z,$

(7) $(x <_{\mathrm{L}} y) \vee (x = y) \vee (y <_{\mathrm{L}} x),$

(8) $x <_{\mathrm{L}} y \rightarrow [x \neq y \ \& \ y \nless_{\mathrm{L}} x],$

(9) $x < y \rightarrow x <_{\mathrm{L}} y,$

(10) $[x <_{\mathrm{L}} y \ \& \ x \nless y \ \& \ x \wedge y < x \wedge z] \rightarrow z <_{\mathrm{L}} y,$

(11) $[x <_{\mathrm{L}} y \,\&\, x \nless y \,\&\, x \wedge y < y \wedge z] \rightarrow x <_{\mathrm{L}} z,$

(12) $x \leqslant_{\mathrm{H}} x,$

(13) $[x \leqslant_{\mathrm{H}} y \,\&\, y \leqslant_{\mathrm{H}} z] \rightarrow x \leqslant_{\mathrm{H}} z,$

(14) $(x \leqslant_{\mathrm{H}} y) \vee (y \leqslant_{\mathrm{H}} x),$

(15) $x < y \rightarrow x <_{\mathrm{H}} y.$

Lemma 1. *Every quantifier-free formula $\varphi(x_0 \cdots x_{n-1})$ in $L(T_0)$ is equivalent in T_0' to a finite Boolean combination of atomic formulas of the following forms:*

$$x_i \wedge x_j = x_k \wedge x_l, \qquad x_i \wedge x_j <_{\mathrm{L}} x_k \wedge x_l, \qquad x_i \wedge x_j \leqslant_{\mathrm{H}} x_k \wedge x_l,$$

where $i, j, k, l < n$.

Sketch of proof. It suffices to prove the lemma for atomic formulas φ. From the axioms in T_0' easily follows

(1) $$w = x \wedge y \wedge z \leftrightarrow [w \leqslant x \,\&\, w \leqslant y \;\&\, [(w = x \wedge z) \vee (w = y \wedge z)]].$$

Next one shows that if $s(x_1 \cdots x_m)$ and $t(y_1 \cdots y_n)$ are any two terms in $L(T_0)$ with exactly the free variables displayed, then in T_0',

(2) $$s(x_1 \cdots x_m) = t(y_1 \cdots y_n) \leftrightarrow \theta(\bar{x}, \bar{y})$$

for some formula $\theta(\bar{x}, \bar{y})$ which is a Boolean combination of atomic formulas of the form $u \wedge v = w \wedge z$, where u, v, w, z are variables from among those occurring in s and t. Without loss of generality, one may assume that s is $x_1 \wedge x_2 \wedge \cdots \wedge x_m$ and t is $y_1 \wedge y_2 \wedge \cdots \wedge y_n$ for some $m \leqslant n$. Then (2) is proved using (1), by induction on m and by induction on n for each m; we omit the details.

Finally, it is easy to show by induction on m that for all $m \geqslant 2$,

$$\bigvee_{1 \leqslant i,j \leqslant m} [x_1 \wedge \cdots \wedge x_m = x_i \wedge x_j].$$

Hence if R is a binary relation symbol, and $s = x_1 \wedge \cdots \wedge x_m$ and $t = y_1 \wedge \cdots \wedge y_n$, then in T_0',

$$R(s, t) \leftrightarrow \bigvee_{\substack{1 \leqslant i,j \leqslant m \\ 1 \leqslant k,l \leqslant n}} [R(x_i \wedge x_j, y_k \wedge y_l) \,\&\, s = x_i \wedge x_j \,\&\, t = y_k \wedge y_l],$$

which completes the proof of the lemma when (2) is applied to the formulas $s = x_i \wedge x_j$ and $t = y_k \wedge y_l$. □

A *basic n-type* $\Gamma(x_0 \cdots x_{n-1})$ of a theory T is a maximal set of quantifier-free formulas in the variables $x_0 \cdots x_{n-1}$ which is consistent with T. From the lemma immediately follows the next corollary.

Corollary 1.1. *Each basic n-type* $\Gamma(x_0 \cdots x_{n-1})$ *of* T_0' *is generated by a single formula of* $L(T_0)$, *which will be denoted by* $\bigwedge \Gamma(x_0 \cdots x_{n-1})$.

Lemma 2. *Let* $\mathscr{A} \models T_0'$, *and let* $a_0 \cdots a_{n-1}$ *be any elements of A closed under* $\wedge$. *If* $a_n \in A$, *then exactly one of the following cases* (I)–(XI) *holds in* $\mathscr{A}$.

(I) $a_n = a_p$, *for some* $p < n$,

(II) $a_n < a_q$,
(III) $a_n \mid a_q \;\&\; a_n <_{\mathrm{L}} a_q$,
(IV) $a_n \mid a_q \;\&\; a_q <_{\mathrm{L}} a_n$,
} *for* $q < n$ *such that* $a_i \nless a_q$, $\forall i < n$,

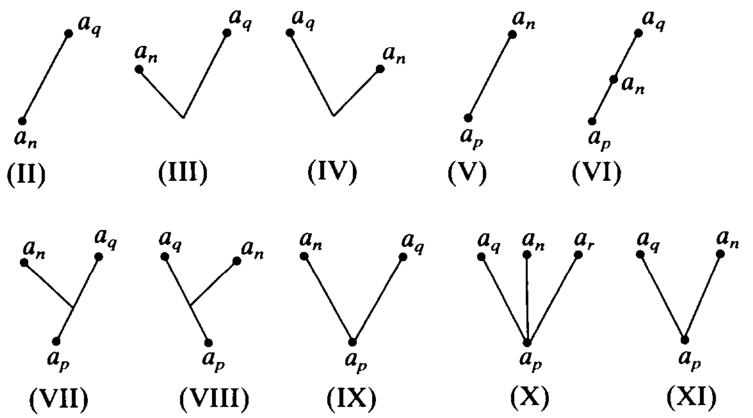
(V) $a_p < a_n$, *for some* $p < n$ *such that* $a_p \nless a_i$, $\forall i < n$,

(VI) $a_p < a_n < a_q$,
(VII) $a_p < a_n \wedge a_q < a_q \;\&\; a_n \mid a_q \;\&\; a_n <_{\mathrm{L}} a_q$,
(VIII) $a_p < a_n \wedge a_q < a_q \;\&\; a_n \mid a_q \;\&\; a_q <_{\mathrm{L}} a_n$,
} *for some* $p, q < n$ *such that* $\neg[a_p < a_i < a_q]$, $\forall i < n$,

(IX) $V(a_p; a_n, a_q)$, *for some* $p, q < n$ *such that* $\neg V(a_p; a_i, a_q)$, $\forall i < n$,

(X) $W(a_p; a_q, a_n, a_r)$, *for some* $p, q, r < n$ *such that* $\neg W(a_p; a_q, a_i, a_r)$, $\forall i < n$,

(XI) $V(a_p; a_q, a_n)$, *for some* $p, q < n$ *such that* $\neg V(a_p; a_q, a_i)$, $\forall i < n$.



Proof. It is easy to check that at most one of cases (I)–(XI) can hold. So it remains to show that at *least* one of them must hold. Assume none of cases (I)–(VIII) holds. Then there is $p < n$ such that $a_p < a_n$ and for no $i < n$, $a_p < a_i < a_n$. The element a_p is unique by axiom (4). Since case (V) does not hold, there is at least one $q < n$ such that $a_p < a_q$. For any such q it is clear that $a_p \leqslant a_n \wedge a_q < a_q$; since cases (VI)–(VIII) do not hold we actually have $a_n \mid a_q$ and

$a_p = a_n \wedge a_q$ for all q such that $a_p < a_q$. Axioms (6)–(8) imply that $<_{\mathrm{L}}$ linearly orders the set consisting of a_n and those a_q which are distinct. Depending on where a_n falls in this ordering, one of cases (IX), (X) or (XI) must hold. This completes the proof of Lemma 2. □

Corollary 2.1. *Let $\mathscr{A} \models T_0'$ and let $a_0 \cdots a_{n-1}$ by any n elements of A closed under $\wedge$. If $a_n \in A$, then either $\{a_0 \cdots a_{n-1}, a_n\}$ is closed under $\wedge$, or there is a unique $a \in A$ such that $\{a_0 \cdots a_{n-1}, a\}$ and $\{a_0 \cdots a_{n-1}, a, a_n\}$ are each closed under $\wedge$.*

Proof. If one of cases (I), (II), (V), (VI), (IX)–(XI) holds, then $\{a_0 \cdots a_{n-1}, a_n\}$ is closed. Otherwise, let a be $a_n \wedge a_q$ (in cases (III), (IV), (VII), (VIII)). □

Let us say that an n-type $\Gamma(x_0 \cdots x_{n-1})$ of T_0' is *closed* if for all $i, j < n$ there is $k < n$ such that the formula $x_i \wedge x_j = x_k$ belongs to Γ. Clearly Lemma 2 implies that if $\Gamma(x_0 \cdots x_{n-1}) \subset \Sigma(x_0 \cdots x_{n-1}, x_n)$ are basic n- and $(n+1)$-types of T_0', where Γ is closed, then corresponding to Σ there is a unique case from (I)–(XI) in Lemma 2 which holds for every realization of Σ in a model $\mathscr{A}$ of T_0'.

If $\Gamma(x_0 \cdots x_{n-1}) \subset \Sigma(x_0 \cdots x_{n-1}, x_n)$ are n- and $(n+1)$-types of a theory, T, and $\Phi(x_0 \cdots x_{n-1}, x_n)$ is a set of formulas in the language of T with only the free variables shown, then we say that *Φ generates Σ over Γ* if Σ is the unique $(n+1)$-type of T containing both Γ and Φ. In Lemma 3 we show that every closed basic $(n+1)$-type extension of a closed basic n-type of T_0' is a principal extension, that is, generated over the n-type by a single formula. In fact, for any $m < \omega$, every closed basic $(n+m)$-type extension of such an n-type is principal, but the lemma as stated below will suffice for our purposes.

Lemma 3. *If $\Gamma(x_0 \cdots x_{n-1}) \subset \Sigma(x_0 \cdots x_{n-1}, x_n)$ are closed basic n- and $(n+1)$-types of T_0', then Σ is generated over Γ by the formulas from among $x_n \leqslant_{\mathrm{H}} x_i$ and $x_i \leqslant_{\mathrm{H}} x_n$ (for $i < n$) contained in Σ, and the formula describing the unique case* (I)–(XI) *of Lemma* 2 *corresponding to Σ.*

Proof. This is obvious if Σ contains $x_n = x_i$ for some $i < n$, so assume otherwise. Since Γ and Σ are closed, it follows from Lemma 1 that Σ is generated over Γ by the formulas it contains from among the following, for all $i, j < n$:

$$x_n \wedge x_i = x_j, \qquad x_n <_{\mathrm{L}} x_i, \qquad x_n \leqslant_{\mathrm{H}} x_i,$$
$$x_n \wedge x_i = x_n, \qquad x_i <_{\mathrm{L}} x_n, \qquad x_i \leqslant_{\mathrm{H}} x_n.$$

We prove that membership in Σ of all of these formulas, except for those involving $\leqslant_{\mathrm{H}}$, is decided by Γ and the formula describing the case from (I)–(XI) of Lemma 2 which corresponds to Σ. Since Σ is closed, cases (III), (IV), (VII) and (VIII) of Lemma 2 do not apply. Of the remaining nontrivial cases, the proof of case (IX) is fairly typical.

Case (IX). $\{V(x_p; x_n, x_q)\} \subset \Sigma$, for some $p, q < n$ such that $\{\neg V(x_p; x_i, x_q): i < n\} \subset \Gamma$.

We first show that Σ contains $x_n \wedge x_i = x_p \wedge x_i$, for all $i < n$. This is easy if $x_p \mid x_i$ or $x_i \leqslant x_p$. If $x_p < x_i$, then it suffices to show that $x_n \wedge x_i = x_p$. Because $x_p < x_q$, we have $x_p \leqslant x_i \wedge x_q$. If this inequality is strict, the desired result follows from $x_p = x_n \wedge x_q$ and axiom (5). Suppose $x_p = x_i \wedge x_q$. Note that $x_p \leqslant x_n \wedge x_i$, since $x_p < x_n$. One now checks that $x_q <_{\mathrm{L}} x_i$ and $x_q \nleq x_i$, then uses axiom (11) to derive a contradiction from $x_p < x_n \wedge x_i$.

For all $i < n$, $x_i <_{\mathrm{L}} x_n$ if and only if either $x_i <_{\mathrm{L}} x_p$ or $x_i = x_p$. We leave the verification to the reader. □

Let the axioms of a theory T_0 be those of T_0' together with the following:

(*1) $\forall x\, \exists y\, [x < y]$,

(*2) $\forall x\, \exists y\, [y < x]$,

(*3) $\forall x\, \forall y\, \exists z\, [x < y \rightarrow x < z < y]$,

(*4) $\forall x\, \forall y\, \exists z\, [x < y \rightarrow V(x; z, y)]$,

(*5) $\forall x\, \forall y\, \exists z\, [V(x \wedge y; x, y) \rightarrow W(x \wedge y; x, z, y)]$,

(*6) $\forall x\, \forall y\, \exists z\, [x < y \rightarrow V(x; y, z)]$,

(*7) $\forall x\, \forall y\, \exists z\, [z =_{\mathrm{H}} x \,\&\, [y \leqslant z \vee z \leqslant y]]$.

T_0 is consistent: a countable model $\mathscr{A}$ for T_0 is an infinitely-branching dense tree in which $<_{\mathrm{L}}$ is interpreted to be a Kleene–Brouwer ordering and $\leqslant_{\mathrm{H}}$ has the following interpretation: fix a map $h: A \rightarrow \mathbb{Q}$ onto the rational numbers $\mathbb{Q}$ such that for all $a, b \in A$, if $a < b$, then $h(a) < h(b)$, and such that for all $a \in A$, $q \in \mathbb{Q}$, there is $b \in A$ with $h(b) = q$ and either $a \leqslant b$ or $b \leqslant a$. Define $a \leqslant_{\mathrm{H}} b \Leftrightarrow_{\mathrm{def}} h(a) \leqslant h(b)$.

Lemma 4. *If* $\Gamma(x_0 \cdots x_{n-1}) \subset \Sigma(x_0 \cdots x_{n-1}, x_n)$ *are basic n- and* $(n+1)$*-types of* T_0', *then*

$$T_0 \vdash \forall x_0 \cdots \forall x_{n-1} [\bigwedge \Gamma(x_0 \cdots x_{n-1}) \rightarrow \exists x_n \bigwedge \Sigma(x_0 \cdots x_{n-1}, x_n)].$$

Proof. It suffices to show that, for any model $\mathscr{A}$ of T_0 which realizes Γ, if $\mathscr{A} \models \bigwedge \Gamma(a_0 \cdots a_{n-1})$, then there is a_n in A such that $\mathscr{A} \models \bigwedge \Sigma(a_0 \cdots a_{n-1}, a_n)$. Assume, for the moment, that the lemma holds when Γ and Σ are closed; we prove the lemma for Γ and Σ not necessarily closed. Any type $\Delta(x_0 \cdots x_{n-1})$ of T_0' is contained in a closed type $\Delta_0(x_0 \cdots x_{n-1}, y_0 \cdots y_{p-1})$ such that any model realizing Δ also realizes Δ_0: just expand Δ by adding, for each $i, j < n$, a new variable y_k and the formula $x_i \wedge x_j = y_k$. Fix such a closed basic type $\Gamma_0(\bar{x}, \bar{y})$ containing $\Gamma(\bar{x})$. Then because of the way in which Γ_0 was defined, there is a unique basic type $\Sigma_0(\bar{x}, x_n, \bar{y})$ containing both Γ_0 and $\Sigma(\bar{x}, x_n)$. By Corollary 2.1, either (1) Σ_0 is closed, or (2) there are closed basic types $\Sigma_1(\bar{x}, \bar{y}, y_p) \subset \Sigma_2(\bar{x}, x_n, \bar{y}, y_p)$ such that $\Sigma_0 \subset \Sigma_2$. Let $\mathscr{A}$ be any model of T_0 realizing Γ (and therefore realizing Γ_0), and suppose $\mathscr{A} \models \bigwedge \Gamma_0(\bar{a}, \bar{b})$ for elements $\bar{a} = a_0 \cdots a_{n-1}$ and $\bar{b} = b_0 \cdots b_{p-1}$ of A. If (1) is the case, then by our assumption that the lemma holds for closed types of T_0', there exists $a_n \in A$ such that $\mathscr{A} \models \bigwedge \Sigma_0(\bar{a}, a_n, \bar{b})$. Clearly $\mathscr{A} \models \bigwedge \Sigma(\bar{a}, a_n)$. If, on the other hand, (2) is the case, then the lemma for closed types gives $b_p \in A$ such that $\mathscr{A} \models \bigwedge \Sigma_1(\bar{a}, \bar{b}, b_p)$. Then another application of the lemma gives $a_n \in A$ such that $\mathscr{A} \models \bigwedge \Sigma_2(\bar{a}, a_n, \bar{b}, b_p)$. So again, $\mathscr{A} \models \bigwedge \Sigma(\bar{a}, a_n)$.

It remains to prove the lemma under the assumption that Γ and Σ are closed types. Let $\mathscr{A}$ be any model of T_0 realizing Γ and suppose that $\mathscr{A} \models \bigwedge \Gamma(a_0 \cdots a_{n-1})$. By Lemma 3, it suffices to find $a_n \in A$ such that $a_0 \cdots a_n$ satisfy the formulas from among $x_n \leqslant_{\mathrm{H}} x_i$ and $x_i \leqslant_{\mathrm{H}} x_n$ (for $i < n$) contained in Σ, as well as the formula describing the case from (I)–(XI) of Lemma 2 corresponding to Σ. Since axioms (12)–(14) say that $\leqslant_{\mathrm{H}}$ is a linear ordering of $=_{\mathrm{H}}$-equivalence classes, the cases in the proof are divided into subcases depending on where Σ might say x_n is located in this ordering. We illustrate by giving case (V) in detail.

Case (V). $\{x_p < x_n\} \subset \Sigma$, for some $p < n$ such that $\{x_p \nleqslant x_k : k < n\} \subset \Gamma$.

Subcase (1). $\{x_n =_{\mathrm{H}} x_i\} \subset \Sigma$, for some $i < n$.

By axiom (15) we have $x_p <_{\mathrm{H}} x_n$, hence $x_p <_{\mathrm{H}} x_i \in \Gamma$. Therefore $\mathscr{A} \models a_p <_{\mathrm{H}} a_i$, and now by axiom (*7) there is $a_n \in A$ such that $a_n =_{\mathrm{H}} a_i$ and $a_p < a_n$.

Subcase (2). $\{x_i <_{\mathrm{H}} x_n <_{\mathrm{H}} x_j\} \subset \Sigma$, for some $i, j < n$ such that $\{\neg[x_i <_{\mathrm{H}} x_k <_{\mathrm{H}} x_j] : k < n\} \subset \Gamma$.

Since $x_p < x_n$, we must have $x_p \leqslant_{\mathrm{H}} x_i$ by axiom (15) and the choice of i. So $\mathscr{A} \models a_p \leqslant_{\mathrm{H}} a_i$, and axiom (*7) gives a' and a'' in A such that $a_p \leqslant a' < a''$, $a' =_{\mathrm{H}} a_i$, and $a'' =_{\mathrm{H}} a_j$. By axiom (*3), there is $a_n \in A$ such that $a' < a_n < a''$.

Subcase (3). $\{x_i <_{\mathrm{H}} x_n\} \subset \Sigma$, for all $i < n$.

Fix $j < n$ such that $a_i \leqslant_{\mathrm{H}} a_j$ for all $i < n$. By axiom (*7) there is $a \in A$ such that $a =_{\mathrm{H}} a_j$ and $a_p \leqslant a$. By axiom (*1) there exists $a_n \in A$ such that $a < a_n$. Hence $a_p < a_n$, and $a_i <_{\mathrm{H}} a_n$ for all $i < n$ since $a_j <_{\mathrm{H}} a_n$. □

Lemma 5. *T_0 admits elimination of quantifiers.*

Proof. Let $\varphi(x_0 \cdots x_n)$ be any quantifier-free formula in $L(T_0)$. We show there is quantifier-free $\theta(x_0 \cdots x_{n-1})$ such that

$$T_0 \vdash [\exists x_n\, \varphi(x_0 \cdots x_{n-1}, x_n) \leftrightarrow \theta(x_0 \cdots x_{n-1})].$$

If $T_0' \cup \{\varphi\}$ is not consistent then neither is $T_0 \cup \{\varphi\}$, and in this case we let θ be, say, '$\perp$' (a 0-ary predicate symbol introduced for 'false'). So suppose $T_0' \cup \{\varphi\}$ is consistent. According to Lemma 1, every basic $(n+1)$-type of T_0' in the variables $x_0 \cdots x_n$ is generated by the formulas it contains from among those in the finite set consisting of the following formulas and their negations:

$$x_i \wedge x_j = x_k \wedge x_l, \qquad x_i \wedge x_j <_{\mathrm{L}} x_k \wedge x_l, \qquad x_i \wedge x_j \leqslant_{\mathrm{H}} x_k \wedge x_l$$

for all $i, j, k, l \leqslant n$. So there are just finitely many distinct basic $(n+1)$-types of T_0'. Fix K such that $\Sigma_i(x_0 \cdots x_n)$ for $i < K$ are those basic $(n+1)$-types which contain $\varphi(x_0 \cdots x_n)$. For each $i < K$, let $\Gamma_i(x_0 \cdots x_{n-1})$ be the corresponding n-type projection. Then

$$T_0' \vdash \Big[\varphi(x_0 \cdots x_n) \leftrightarrow \bigvee_{i<K} \bigwedge \Sigma_i(x_0 \cdots x_n)\Big].$$

Using the fact that T_0 extends T_0',

$$T_0 \vdash \Big[\exists x_n\, \varphi(x_0 \cdots x_n) \leftrightarrow \bigvee_{i<K} \exists x_n \bigwedge \Sigma_i(x_0 \cdots x_n)\Big].$$

From Lemma 4 we have,

$$T_0 \vdash [\exists x_n \bigwedge \Sigma_i(x_0 \cdots x_n) \leftrightarrow \bigwedge \Gamma_i(x_0 \cdots x_{n-1})],$$

for all $i < K$. Hence

$$T_0 \vdash \Big[\exists x_n\, \varphi(x_0 \cdots x_n) \leftrightarrow \bigvee_{i<K} \bigwedge \Gamma_i(x_0 \cdots x_{n-1})\Big],$$

and therefore we let $\theta(x_0 \cdots x_{n-1})$ be $\bigvee_{i<K} \bigwedge \Gamma_i(x_0 \cdots x_{n-1})$. This completes the proof of Lemma 5. □

By Lemma 5, every n-type of T_0 is determined by the basic n-type which it contains. Because T_0' is contained in T_0 and the theories are in the same language, every basic n-type of T_0 is a basic n-type of T_0'. Therefore Corollary 1.1 implies that for every n, each n-type of T_0 is principal. Hence T_0 is ω-categorical (see [1, p. 101]).

Since T_0 admits elimination of quantifiers, T_0 is submodel complete. To show that T_0 is the model completion of T_0', it therefore remains to verify that the two theories are mutually model consistent. One direction is immediate, since $T_0' \subset T_0$. For the other direction, it suffices to show that every finitely generated submodel of a model of T_0' can be embedded in some model of T_0. But any such submodel is a model of T_0', since T_0' is universal; and every finitely generated model of T_0' is finite, since it is only necessary to close downward under $\wedge$. Clearly any finite model of T_0' can be embedded in the countable model of T_0.

Part II

Fix $n < \omega$ and a recursive tree $\mathrm{Tr} \subset \omega^{<\omega}$ having exactly one infinite branch f^*, where $f^* \equiv_T H(n)$. The existence of such a tree was first remarked on by Kreisel. (See [6, p. 456].) Without loss of generality, assume that if $\eta \in \mathrm{Tr}$ and there is $m < \omega$ such that $\eta^\wedge\langle m\rangle \in \mathrm{Tr}$, then for *all* $m < \omega$, $\eta^\wedge\langle m\rangle$ is in Tr. For if Tr does not have this property we can replace Tr by

$$\mathrm{Tr}' = \{\langle\cdot\rangle\} \cup \{\eta^\wedge\langle m\rangle : \eta \in \mathrm{Tr}, m < \omega\}.$$

The language of T consists of one binary function symbol $\wedge$, infinitely many constant symbols c_η (one for each η in Tr), and infinitely many binary relation symbols: $<_L$, $\leqslant_H$, E^η_ξ, L^η_ξ, H_η, A_η, B_η (for all η, $\xi \in \mathrm{Tr}$).

Define a model $\mathscr{A}$ for this language by letting $\langle A, \wedge, <_L, \leqslant_H\rangle$ be the countable model of T_0 and interpreting the constant symbols and remaining relation symbols as follows.

By the axioms of T_0, fix a map $h : A \to \mathbb{Q}$ onto the rational numbers, such that $h(a) \leqslant h(b) \Leftrightarrow a \leqslant_H b$, for all $a, b \in A$.

Interpretation of the constant symbols c_η

Let $|\eta|$ denote the length of the sequence $\eta \in \mathrm{Tr}$. Fix an embedding $g : \mathrm{Tr} \to A$ such that for all $\eta \in \mathrm{Tr}$, $h(g(\eta)) = |\eta|$, and such that for all nonterminal $\eta \in \mathrm{Tr}$,

$$\neg(\exists a \in A)[g(\eta) < a \;\&\; (\forall m < \omega)[g(\eta^\wedge\langle m\rangle) <_L a]].$$

Define $c_\eta =_{\mathrm{def}} g(\eta)$, for all $\eta \in \mathrm{Tr}$.

From now on, we identify Tr with its image in A under g. We will write 'η' instead of '$g(\eta)$', and Greek letters will be reserved for (the images under g of) elements of Tr, as opposed to arbitrary elements of A. In particular, ρ will denote the root node $\langle\cdot\rangle$ of Tr.

The condition $h(\eta) = |\eta|$ given above implies that no element of A is $\leqslant_H$-higher than every node in Tr (and therefore no element of A lies above the infinite branch f^*). The second condition on g is needed for (3) in the definition of f_η below.

Interpretation of the relation symbols E^η_ξ, etc.

For each $\eta \in \mathrm{Tr}$, let $U_\eta =_{\mathrm{def}} \{a \in A : \eta < a\}$. For each η which is nonterminal in Tr, fix a map f_η from U_η onto $\mathbb{Q}$ such that for all $a, b \in U_\eta$ and all $m < \omega$,

(1) $f_\eta(a) < f_\eta(b) \;\Leftrightarrow\; \eta = a \wedge b$ and $a <_L b$,

(2) $f_\eta(a) = f_\eta(b) \;\Leftrightarrow\; \eta < a \wedge b$.

(3) $f_\eta(\eta^\wedge\langle m\rangle) = m$.

For each terminal node η in Tr, fix a map f_η from U_η onto the *negative* rational numbers such that for all $a, b \in U_\eta$, (1) and (2) above hold. Note that for all

$\eta \in \mathrm{Tr}$ and all $q \in \mathrm{range}(f_\eta)$, clause (2) implies that $f_\eta^{-1}(q)$ is a dense subtree isomorphic to the countable model of T_0. Now define for all η, ξ in Tr and all $a, b \in A$,

$$E_\xi^\eta(a, b) \Leftrightarrow_{\mathrm{def}} \eta < a \text{ and } \xi < b \text{ and } f_\eta(a) = f_\xi(b),$$
$$L_\xi^\eta(a, b) \Leftrightarrow_{\mathrm{def}} \eta < a \text{ and } \xi < b \text{ and } f_\eta(a) < f_\xi(b),$$
$$H_\eta(a, b) \Leftrightarrow_{\mathrm{def}} \eta < b \text{ and } h(a) = f_\eta(b),$$
$$A_\eta(a, b) \Leftrightarrow_{\mathrm{def}} \eta < b \text{ and } h(a) > f_\eta(b),$$
$$B_\eta(a, b) \Leftrightarrow_{\mathrm{def}} \eta < b \text{ and } h(a) < f_\eta(b).$$

This completes the definition of $\mathscr{A}$.

It may be helpful to think of H_η, A_η and B_η as defining a way of measuring height in the tree by means of the relation $<_{\mathrm{L}}$. If b is an element of U_η, define the *η-tree containing* b to be the set $\{a \in A: \eta < a \wedge b\}$. (Each $f_\eta^{-1}(q)$ is therefore an η-tree.) For each η in Tr, the η-trees are ordered by $<_{\mathrm{L}}$ in an obvious way, and because of the density of A this ordering is isomorphic to $\mathbb{Q}$. Fix a nonterminal node η in Tr. To each element $a \in A$ there corresponds a single η-tree in such a way that the higher the element a is with respect to $\leqslant_{\mathrm{H}}$, the further to the right is its corresponding η-tree with respect to $<_{\mathrm{L}}$. Specifically, this correspondence is defined by the relation $H_\eta(a, b)$, which says that the height of element a is 'measured' exactly by the η-tree containing b. The relation $A_\eta(a, b)$ (respectively $B_\eta(a, b)$) says that the height of a is above (below) that measured by the η-tree containing b. By checking through the definitions, it is easy to see that if η is a *terminal* node of Tr, then the η-trees only measure heights less than the height in A of the root node of Tr.

Define $T =_{\mathrm{def}} \mathrm{Th}(\mathscr{A})$. It will turn out that T is the model completion of the universal theory T' whose axioms are listed below. This list is not a minimal set of axioms for T'; several of the axioms can be derived from the others, but have been included in order to simplify later proofs. Axiom schemes (20)–(50) become axioms for all η, ξ, ζ in Tr and $m < \omega$, with the following exceptions: in (29)–(31), η and ξ must be nonterminal, as must be η in (32) and in (36)–(38); in (32), ξ must be terminal. It is easy to check that $\mathscr{A}$ is a model of T'.

The axioms of T' are:

(1)–(15) Axioms of T_0',

(16) $c_\eta \wedge c_\xi = c_\zeta$ (for all η, ξ, ζ such that $\mathrm{Tr} \models \eta \wedge \xi = \zeta$),

(17) $c_\eta \wedge c_\xi \neq c_\zeta$ (for all η, ξ, ζ such that $\mathrm{Tr} \models \eta \wedge \xi \neq \zeta$),

(18) $c_\eta <_{\mathrm{L}} c_\xi$ (for all η, ξ such that $\mathrm{Tr} \models \eta <_{\mathrm{L}} \xi$),

(19) $c_\eta \leqslant_{\mathrm{H}} c_\xi$ (for all $\eta, \xi \in \mathrm{Tr}$ such that $|\eta| \leqslant |\xi|$),

(20) $E_\xi^\eta(x, y) \leftrightarrow c_\eta < x \,\&\, c_\xi < y \,\&\, \neg L_\xi^\eta(x, y) \,\&\, \neg L_\eta^\xi(y, x)$,

(21) $L_\xi^\eta(x, y) \leftrightarrow c_\eta < x \,\&\, c_\xi < y \,\&\, \neg E_\xi^\eta(x, y) \,\&\, \neg L_\eta^\xi(y, x)$,

(22) $L_\eta^\xi(y, x) \leftrightarrow c_\eta < x \,\&\, c_\xi < y \,\&\, \neg E_\xi^\eta(x, y) \,\&\, \neg L_\xi^\eta(x, y)$,

(23) $E^{\eta}_{\xi}(x, y) \,\&\, E^{\xi}_{\zeta}(y, z) \rightarrow E^{\eta}_{\zeta}(x, z)$,

(24) $L^{\eta}_{\xi}(x, y) \,\&\, L^{\xi}_{\zeta}(y, z) \rightarrow L^{\eta}_{\zeta}(x, z)$,

(25) $L^{\eta}_{\xi}(x, y) \,\&\, E^{\xi}_{\zeta}(y, z) \rightarrow L^{\eta}_{\zeta}(x, z)$,

(26) $E^{\zeta}_{\eta}(z, x) \,\&\, L^{\eta}_{\xi}(x, y) \rightarrow L^{\zeta}_{\xi}(z, y)$,

(27) $E^{\eta}_{\eta}(x, y) \leftrightarrow c_\eta < x \wedge y$,

(28) $L^{\eta}_{\eta}(x, y) \leftrightarrow c_\eta < x \,\&\, c_\eta < y \,\&\, c_\eta = x \wedge y \,\&\, x <_{\mathrm{L}} y$,

(29) $E^{\eta}_{\xi}(c_{\eta^\wedge\langle m\rangle}, x) \leftrightarrow c_\xi < x \wedge c_{\xi^\wedge\langle m\rangle}$,

(30) $L^{\eta}_{\xi}(c_{\eta^\wedge\langle m\rangle}, x) \leftrightarrow c_\xi < x \,\&\, c_\xi = x \wedge c_{\xi^\wedge\langle m\rangle} \,\&\, c_{\xi^\wedge\langle m\rangle} <_{\mathrm{L}} x$,

(31) $L^{\xi}_{\eta}(x, c_{\eta^\wedge\langle m\rangle}) \leftrightarrow c_\xi < x \,\&\, c_\xi = x \wedge c_{\xi^\wedge\langle m\rangle} \,\&\, x <_{\mathrm{L}} c_{\xi^\wedge\langle m\rangle}$,

(32) $L^{\xi}_{\eta}(x, c_{\eta\wedge\langle 0\rangle}) \leftrightarrow c_\xi < x$ (for ξ terminal in Tr),

(33) $H_\eta(x, y) \leftrightarrow c_\eta < y \,\&\, \neg A_\eta(x, y) \,\&\, \neg B_\eta(x, y)$,

(34) $A_\eta(x, y) \leftrightarrow c_\eta < y \,\&\, \neg H_\eta(x, y) \,\&\, \neg B_\eta(x, y)$,

(35) $B_\eta(x, y) \leftrightarrow c_\eta < y \,\&\, \neg H_\eta(x, y) \,\&\, \neg A_\eta(x, y)$,

(36) $H_\eta(c_\xi, x) \leftrightarrow E^{\eta}_{\eta}(x, c_{\eta^\wedge\langle|\xi|\rangle})$,

(37) $A_\eta(c_\xi, x) \leftrightarrow L^{\eta}_{\eta}(x, c_{\eta^\wedge\langle|\xi|\rangle})$,

(38) $B_\eta(c_\xi, x) \leftrightarrow L^{\eta}_{\eta}(c_{\eta^\wedge\langle|\xi|\rangle}, x)$,

(39) $H_\eta(x, y) \rightarrow [H_\xi(x, z) \leftrightarrow E^{\eta}_{\xi}(y, z)]$,

(40) $H_\eta(x, y) \rightarrow [A_\xi(x, z) \leftrightarrow L^{\xi}_{\eta}(z, y)]$,

(41) $H_\eta(x, y) \rightarrow [B_\xi(x, z) \leftrightarrow L^{\eta}_{\xi}(y, z)]$,

(42) $A_\eta(x, y) \,\&\, B_\xi(x, z) \rightarrow L^{\eta}_{\xi}(y, z)$,

(43) $A_\eta(x, y) \,\&\, [E^{\eta}_{\xi}(y, z) \vee L^{\xi}_{\eta}(z, y)] \rightarrow A_\xi(x, z)$,

(44) $B_\eta(x, y) \,\&\, [E^{\eta}_{\xi}(y, z) \vee L^{\eta}_{\xi}(y, z)] \rightarrow B_\xi(x, z)$,

(45) $H_\eta(x, y) \rightarrow [H_\eta(z, y) \leftrightarrow x =_{\mathrm{H}} z]$,

(46) $H_\eta(x, y) \rightarrow [A_\eta(z, y) \leftrightarrow x <_{\mathrm{H}} z]$,

(47) $H_\eta(x, y) \rightarrow [B_\eta(z, y) \leftrightarrow z <_{\mathrm{H}} x]$,

(48) $A_\eta(x, y) \,\&\, B_\eta(z, y) \rightarrow z <_{\mathrm{H}} x$,

(49) $A_\eta(x, y) \,\&\, x \leqslant_{\mathrm{H}} z \rightarrow A_\eta(z, y)$,

(50) $B_\eta(x, y) \,\&\, z \leqslant_{\mathrm{H}} x \rightarrow B_\eta(z, y)$.

The goal for the remainder of this part is to show that T is decidable. For $N > 0$, define S_N to be the finite tree $\mathrm{Tr} \cap N^{<N}$. Let $L(T) \restriction N$ denote the restriction of $L(T)$ to the following symbols:

$\wedge$, $<_{\mathrm{L}}$, $\leqslant_{\mathrm{H}}$,

c_η, E^{η}_{ξ}, L^{η}_{ξ}, H_η, A_η, B_η, for $\eta, \xi \in S_N$.

Let $T' \upharpoonright N$ denote the sentences of T' which are in the language $L(T) \upharpoonright N$. Thus $T' \upharpoonright N$ is a finite subset of T'.

Lemma 6. *Let $N > 0$. Every quantifier-free formula φ in $L(T) \upharpoonright N$ is equivalent in $T' \upharpoonright N$ to a finite Boolean combination of atomic formulas of the following forms:*

$$u \wedge v = w \wedge z, \qquad u <_{\mathrm{L}} w, \qquad E^{\eta}_{\xi}(u, w), \qquad H_{\eta}(u \wedge v, w),$$
$$u \wedge v \leqslant_{\mathrm{H}} w \wedge z, \qquad L^{\eta}_{\xi}(u, w), \qquad A_{\eta}(u \wedge v, w),$$

where each u, v, w, z is either a variable from among the variables occurring in φ, or a constant in $L(T) \upharpoonright N$.

Proof. Since $T'_0 \subset T' \upharpoonright N$, it follows by a proof just like that of Lemma 1 that if $s(x_1 \cdots x_m)$ and $t(y_1 \cdots y_n)$ are any two terms in $L(T) \upharpoonright N$ with exactly the free variables displayed, and R is a binary relation symbol, then

(1) $\qquad T' \upharpoonright N \vdash [s(x_1 \cdots x_m) = t(y_1 \cdots y_n) \leftrightarrow \psi(\bar{x}, \bar{y})],$

(2) $\qquad T' \upharpoonright N \vdash [R(s(x_1 \cdots x_m), t(y_1 \cdots y_n)) \leftrightarrow \theta(\bar{x}, \bar{y})],$

for some formula $\psi(\bar{x}, \bar{y})$ which is a Boolean combination of atomic formulas of the form $u \wedge v = w \wedge z$, and some formula $\theta(\bar{x}, \bar{y})$ which is a Boolean combination of atomic formulas of the forms $R(u \wedge v, w \wedge z)$ and $u \wedge v = w \wedge z$, where in both cases u, v, w, z are variables or constants from among those occurring in the terms s and t. Thus every quantifier-free formula φ in $L(T) \upharpoonright N$ is equivalent in $T' \upharpoonright N$ to a Boolean combination of atomic formulas of the following forms:

$$u \wedge v = w \wedge z, \qquad E^{\eta}_{\xi}(u \wedge v, w \wedge z), \qquad H_{\eta}(u \wedge v, w \wedge z),$$
$$u \wedge v \leqslant_{\mathrm{H}} w \wedge z, \qquad L^{\eta}_{\xi}(u \wedge v, w \wedge z), \qquad A_{\eta}(u \wedge v, w \wedge z),$$
$$u \wedge v <_{\mathrm{L}} w \wedge z,$$

where u, v, w, z are variables or constants from among those occurring in φ. (Note that, by the axioms in (35), any formula containing the relation symbol B_{η} is equivalent to a formula not involving B_{η}.) To complete the proof of the lemma, it therefore remains to prove (3)–(7) below in $T' \upharpoonright N$, and apply (1) to the right-hand sides of each of these where necessary. The proofs are fairly straightforward using the axioms, and will be omitted here.

(3) $\qquad u \wedge v <_{\mathrm{L}} w \wedge z \leftrightarrow [u \wedge v < w \wedge z \vee [u <_{\mathrm{L}} w \,\&\, (u \wedge v) \wedge (w \wedge z) < (w \wedge z)]],$

(4) $\qquad E^{\eta}_{\xi}(u \wedge v, w \wedge z) \leftrightarrow c_{\eta} < u \wedge v \,\&\, c_{\xi} < w \wedge z \,\&\, E^{\eta}_{\xi}(u, w),$

(5) $\qquad L^{\eta}_{\xi}(u \wedge v, w \wedge z) \leftrightarrow c_{\eta} < u \wedge v \,\&\, c_{\xi} < w \wedge z \,\&\, L^{\eta}_{\xi}(u, w),$

(6) $\qquad H_{\eta}(u \wedge v, w \wedge z) \leftrightarrow c_{\eta} < w \wedge z \,\&\, H_{\eta}(u \wedge v, w),$

(7) $\qquad A_{\eta}(u \wedge v, w \wedge z) \leftrightarrow c_{\eta} < w \wedge z \,\&\, A_{\eta}(u \wedge v, w). \qquad \square$

Corollary 6.1. *Any basic n-type $\Gamma(x_0 \cdots x_{n-1})$ of $T' \restriction N$ is generated by a single formula of $L(T) \restriction N$, which will be denoted by $\bigwedge \Gamma(x_0 \cdots x_{n-1})$.*

Proof. Immediate from the lemma, since $L(T) \restriction N$ is finite. □

At this point it is convenient to have the following notation:

$$c_\eta \blacktriangleleft x \Leftrightarrow_{\text{def}} c_\eta < x \text{ and } \neg(\exists \xi \in \text{Tr})[c_\eta < c_\xi < x].$$

Lemma 7. *Let $N > 0$ and $n > 0$. If $\Gamma(x_0 \cdots x_{n-1}) \subset \Sigma(x_0 \cdots x_{n-1}, x_n)$ are basic n- and $(n+1)$-types of $T' \restriction N$, then*

$$T \vdash \forall x_0 \cdots \forall x_{n-1} [\bigwedge \Gamma(x_0 \cdots x_{n-1}) \rightarrow \exists x_n \bigwedge \Sigma(x_0 \cdots x_{n-1}, x_n)].$$

Proof. Since $T = \text{Th}(\mathscr{A})$, we assume $\mathscr{A} \models \bigwedge \Gamma(a_0 \cdots a_{n-1})$ for some $a_0 \cdots a_{n-1}$ in A and show there is $a_n \in A$ such that $\mathscr{A} \models \bigwedge \Sigma(a_0 \cdots a_{n-1}, a_n)$. By the first part of the proof of Lemma 4, it suffices to prove this under the assumption that Γ and Σ are both closed. To further simplify matters, we may also assume without loss of generality that for all $\xi \in S_N$, the formula $x_i = c_\xi$ belongs to Γ for some $i < n$. Therefore it is enough to find a_n so that $\mathscr{A} \models \sigma(a_0 \cdots a_{n-1}, a_n)$ for every atomic formula $\sigma \in \Sigma$ whose only terms are single variables.

Fix $\eta \in S_N$ such that $c_\eta \blacktriangleleft x_n$ is in Σ. If no such η exists, then the proof is considerably simpler; it is like that given below when one ignores those statements containing references to η or r. In the following, h and f_ξ (for ξ in S_N) are the functions that were used in defining the model $\mathscr{A}$.

Claim 1. *There are rational numbers q and r such that* (1)–(5) *hold*:

$$(1)\quad \left.\begin{aligned} q < h(a_i) &\Leftrightarrow x_n <_{\mathrm{H}} x_i \in \Sigma, \\ q = h(a_i) &\Leftrightarrow x_n =_{\mathrm{H}} x_i \in \Sigma, \\ h(a_i) < q &\Leftrightarrow x_i <_{\mathrm{H}} x_n \in \Sigma, \end{aligned}\right\} \quad \forall i < n,$$

$$(2)\quad \left.\begin{aligned} q < f_\xi(a_j) &\Leftrightarrow B_\xi(x_n, x_j) \in \Sigma, \\ q = f_\xi(a_j) &\Leftrightarrow H_\xi(x_n, x_j) \in \Sigma, \\ q > f_\xi(a_j) &\Leftrightarrow A_\xi(x_n, x_j) \in \Sigma, \end{aligned}\right\} \quad \begin{aligned} &\forall \xi \in S_N \text{ and} \\ &\forall j < n \text{ such that} \\ &c_\xi < x_j \in \Gamma, \end{aligned}$$

$$(3)\quad \begin{aligned} q < r &\Leftrightarrow B_\eta(x_n, x_n) \in \Sigma, \\ q = r &\Leftrightarrow H_\eta(x_n, x_n) \in \Sigma, \\ q > r &\Leftrightarrow A_\eta(x_n, x_n) \in \Sigma, \end{aligned}$$

$$(4)\quad \left.\begin{aligned} h(a_i) < r &\Leftrightarrow B_\eta(x_i, x_n) \in \Sigma, \\ h(a_i) = r &\Leftrightarrow H_\eta(x_i, x_n) \in \Sigma, \\ h(a_i) > r &\Leftrightarrow A_\eta(x_i, x_n) \in \Sigma, \end{aligned}\right\} \quad \forall i < n,$$

$$(5)\quad \left.\begin{aligned} r < f_\xi(a_j) &\Leftrightarrow L_\xi^\eta(x_n, x_j) \in \Sigma, \\ r = f_\xi(a_j) &\Leftrightarrow E_\xi^\eta(x_n, x_j) \in \Sigma, \\ f_\xi(a_j) < r &\Leftrightarrow L_\eta^\xi(x_j, x_n) \in \Sigma, \end{aligned}\right\} \quad \begin{aligned} &\forall \xi \in S_N \text{ and} \\ &\forall j < n \text{ such that} \\ &c_\xi < x_j \in \Gamma. \end{aligned}$$

Proof of Claim 1. Because $\mathscr{A} \models \bigwedge \Gamma(a_0 \cdots a_{n-1})$ and Σ is consistent with axioms (12)–(14), it is certainly possible to find $q \in \mathbb{Q}$ satisfying (1). By the consistency of Σ with the axioms in T', q can be found satisfying both (1) and (2). For example, suppose that $n > 2$ and that (1) says q must satisfy

$$(*) \qquad h(a_1) < q < h(a_2),$$

where there is no $k < n$ such that $h(a_1) < h(a_k) < h(a_2)$. Let $\xi \in S_N$ and $j < n$ be such that $c_\xi < x_j \in \Gamma$. We show that q can be chosen satisfying (2) for this ξ and j as well as satisfying (*). If $h(a_2) \leq f_\xi(a_j)$, then by the definition of $\mathscr{A}$, $\mathscr{A} \models [B_\xi(a_2, a_j) \vee H_\xi(a_2, a_j)]$, hence Γ contains $B_\xi(x_2, x_j) \vee H_\xi(x_2, x_j)$. Also $x_n <_{\mathrm{H}} x_2 \in \Sigma$ by (*) and (1), so $B_\xi(x_n, x_j) \in \Sigma$ by axioms (47) and (50). Thus any q satisfying (*) will satisfy (2) for ξ and j, since $q < f_\xi(a_j)$. Similarly, (2) can be satisfied if $f_\xi(a_j) \leq h(a_1)$, using (46) and (49). Suppose $h(a_1) < f_\xi(a_j) < h(a_2)$. Axioms (33)–(35) imply that Σ contains exactly one of $B_\xi(x_n, x_j)$, $H_\xi(x_n, x_j)$ or $A_\xi(x_n, x_j)$. If $B_\xi(x_n, x_j) \in \Sigma$, choose q so that $h(a_1) < q < f_\xi(a_j)$; if $H_\xi(x_n, x_j) \in \Sigma$, let $q = f_\xi(a_j)$; if $A_\xi(x_n, x_j) \in \Sigma$, choose any q satisfying $f_\xi(a_j) < q < h(a_2)$.

Suppose q has been found satisfying (1) and (2). Since Σ is consistent with axioms (12)–(14), (33)–(35) and (45)–(50), it is clear that $r \in \mathbb{Q}$ can be found satisfying (3) and (4). By an argument similar to that just given above, r can be chosen so that (5) holds as well. □ Claim 1

Claim 2. *If q and r satisfy Claim 1, then there is a_n in A such that $h(a_n) = q$, $f_\eta(a_n) = r$, and $a_0 \cdots a_{n-1}, a_n$ realize the restriction of $\Sigma(x_0 \cdots x_{n-1}, x_n)$ to $L(T_0)$.*

Assuming this claim for the moment, let $q, r \in \mathbb{Q}$ and $a_n \in A$ satisfy Claims 1 and 2. To show that $\langle a_0 \cdots a_n \rangle$ realizes Σ, it remains to show that, for all $i, j \leq n$ and all $\xi, \zeta \in S_N$,

$$(**) \qquad \mathscr{A} \models R(a_i, a_j) \Leftrightarrow R(x_i, x_j) \in \Sigma,$$

where R can be B_ξ, H_ξ, A_ξ, E_ξ^ζ or L_ξ^ζ. If both $i < n$ and $j < n$, (**) follows immediately from the fact that $\mathscr{A} \models \bigwedge \Gamma(a_0 \cdots a_{n-1})$. Let R be B_ξ, and suppose $i < n$ and $j = n$. If $c_\xi \nless x_n \in \Sigma$, then $\mathscr{A} \models c_\xi \nless a_n$ by Claim 2; hence $\mathscr{A} \models \neg B_\xi(a_i, a_n)$ and $\neg B_\xi(x_i, x_n) \in \Sigma$ by axiom (35). If $\xi = \eta$, then (**) follows from (4), since

$$\mathscr{A} \models B_\eta(a_i, a_n) \Leftrightarrow h(a_i) < f_\eta(a_n) \Leftrightarrow h(a_i) < r \Leftrightarrow B_\eta(x_i, x_n) \in \Sigma.$$

We are left with the case where $c_\xi < c_\eta \blacktriangleleft x_n \in \Sigma$. Then $c_\xi < c_\eta \wedge x_n \in \Sigma$, thus $\mathscr{A} \models c_\xi < c_\eta \wedge a_n$ by Claim 2, hence $\mathscr{A} \models E_\xi^\xi(c_\eta, a_n)$ and $E_\xi^\xi(c_\eta, x_n) \in \Sigma$, by axiom (27). So

$$\mathscr{A} \models B_\xi(a_i, a_n) \Leftrightarrow \mathscr{A} \models B_\xi(a_i, c_\eta) \Leftrightarrow B_\xi(x_i, c_\eta) \in \Gamma \Leftrightarrow B_\xi(x_i, x_n) \in \Sigma,$$

by axiom (44). The cases where $i = n$ are proved similarly, as are the cases involving substitution of the other relation symbols for R in (**).

Proof of Claim 2. By (1) and Lemma 3, it suffices to find $a_n \in A$ such that $h(a_n) = q$, $f_\eta(a_n) = r$, and $\langle a_0 \cdots a_{n-1}, a_n \rangle$ satisfies the formula describing the case (I)–(XI) of Lemma 2 corresponding to Σ. The cases to be considered are the same as those in the proof of Lemma 4, without the subcases needed there. We sketch the proof of case (V), which is typical.

Case (V). $\{x_i < x_n\} \subset \Sigma$, for some $i < n$ such that $\{x_i \nless x_k : k < n\} \subset \Gamma$.

Since by assumption $c_\eta \blacktriangleleft x_n \in \Sigma$, we have $c_\eta \leq x_i \in \Gamma$. First assume $c_\eta < x_i \in \Gamma$. Let $a \in A$ be an element such that $h(a) = q$. Since $x_i <_{\mathrm{H}} x_n \in \Sigma$ by axiom (15), $\mathscr{A} \vDash a_i <_{\mathrm{H}} a$ by (1). Axiom (*7) gives $a_n \in A$ such that $h(a_n) = h(a)$ and $c_\eta < a_i < a_n$. Since the formula $c_\eta < x_i < x_n$ is in Σ, we obtain $f_\eta(a_n) = f_\eta(a_i) = r$ from (5) and axiom (27).

Now assume $c_\eta = x_i \in \Gamma$. Note that if η is a terminal node of Tr, then $r < 0$ by (5) since in that case $L^\eta_\rho(x_n, c_{\langle 0 \rangle}) \in \Sigma$ by axiom (32), and since $f_\rho(c_{\langle 0 \rangle}) = 0$. So, whether or not η is a terminal node, r is in the range of f_η and we can let $D = f_\eta^{-1}(r)$, a dense subtree of $\mathscr{A}$ growing above c_η. Fix $a \in A$ such that $h(a) = q$, and let a' be an element of D. As above, $\mathscr{A} \vDash a_i <_{\mathrm{H}} a$, and there is $a_n \in A$ such that $h(a_n) = h(a)$ and a_n and a' are comparable. This implies $c_\eta = a_i < a_n$, hence $c_\eta < a_n \wedge a'$, so $f_\eta(a_n) = f_\eta(a') = r$ by axiom (27) and the choice of a'. □

Define $\mathscr{C}^M_N$ to be the set of all models of $L(T) \restriction N$ having for their universe the complete finite tree $M^{<M}$, with $\wedge$ and $<_{\mathrm{L}}$ interpreted as usual. Since $L(T) \restriction N$ is finite, so is the set $\mathscr{C}^M_N$, and clearly the models in $\mathscr{C}^M_N$ can be constructed uniformly effectively given M and N.

Lemma 8. *Fix $N > 0$ and let $\varphi(x_0 \cdots x_{n-1})$ by any quantifier-free formula in $L(T) \restriction N$ with the n free variables shown. If $T' \restriction N \cup \{\varphi\}$ has a model, then it has a model in $\mathscr{C}^M_N$, where $M = \frac{1}{2}n^2 + |S_N| \cdot n + |S_N|$.*

Proof. Suppose $\mathscr{B}$ is a model of $T' \restriction N$ and $\mathscr{B} \vDash \varphi(b_0 \cdots b_{n-1})$ for some $b_0 \cdots b_{n-1}$ in B. Let

$$B' =_{\mathrm{def}} \{b_i \wedge b_j : i, j < n\} \cup \{b_i \wedge c_\eta : i < n,\ \eta \in S_N\} \cup S_N.$$

Note that $|B'| \leq \frac{1}{2}n^2 + |S_N| \cdot n + |S_N|$. From (1) in the proof of Lemma 1, we have

$$T' \restriction N \vdash \forall w\, \forall x\, \forall y\, \forall z\, [w = x \wedge y \wedge z \rightarrow [(w = x \wedge z) \vee (w = y \wedge z)]].$$

So B' is closed under $\wedge$. Therefore let $\mathscr{B}'$ be the submodel of $\mathscr{B}$ with universe B'. $\mathscr{B}'$ is a model of $T' \restriction N$ because T' is universal, and $\mathscr{B}' \vDash \varphi(b_0 \cdots b_{n-1})$ because φ is quantifier-free. Let $\mathscr{B}_0$ be the reduct of $\mathscr{B}'$ to the language $\{\wedge, <_{\mathrm{L}}\}$. It suffices to prove the following claim.

Claim. *If* $\mathcal{B}_0 = \langle B_0, \wedge, <_L\rangle$ *is a finite model of axioms* (1)–(11) *of* T', *then for any* M *such that* $|B_0| \leq M$, *there is an isomorphic embedding* $\mathcal{B}_0 \subsetapprox M^{<M}$.

Proof of Claim. By induction on $|B_0|$. The claim obviously holds if $|B_0| = 1$. Assume $|B_0| > 1$ and fix $M \geq |B_0|$. Because B_0 is finite, axioms (6)–(8) imply there is $d_0 \in B_0$ which is $<_L$-minimal in B_0. For all $d \in B_0$, $\mathcal{B}_0 \models d_0 \leq d$. (If there were $d \in B_0$ with $d_0 \nleq d$, then $d_0 \wedge d < d_0$, contradicting the $<_L$-minimality of d_0.) Map d_0 to the root of $M^{<M}$.

Since B_0 is finite, there is a maximal set $\{d_1 \cdots d_k\}$ of distinct elements of B_0 such that for all $1 \leq i \leq k$, $d_i \neq d_0$ and there is no $d \in B_0$ with $d_0 < d < d_i$. For each such i, define $B_i =_{\text{def}} \{d \in B_0 : d_i \leq d\}$. Clearly

$$B_0 = \{d_0\} \cup \bigcup_{1 \leq i \leq k} B_i.$$

By axiom (4), $B_i \cap B_j = \emptyset$ whenever $1 \leq i \neq j \leq k$. From axioms (1)–(3) it also follows that each B_i is closed under $\wedge$, so for each i there is a submodel $\mathcal{B}_i$ of $\mathcal{B}_0$ having universe B_i, and $\mathcal{B}_i$ is a model of axioms (1)–(11), since these axioms are universal. Now for each $1 \leq i \leq k$, we have $|B_i| < |B_0|$ and $|B_i| \leq M - 1$, so by the induction hypothesis there is an isomorphic embedding $\mathcal{B}_i \subsetapprox (M-1)^{<M-1}$ sending d_i to the root of $(M-1)^{<M-1}$. For each such i, there is also an obvious isomorphism $(M-1)^{<M-1} \subsetapprox M^{<M}$ sending the root of $(M-1)^{<M-1}$ to the node $\langle i\rangle$ in $M^{<M}$. Without loss of generality assume $d_1 <_L d_2 <_L \cdots <_L d_k$. Map each B_i to the isomorphic copy of $(M-1)^{<M-1}$ having root $\langle i\rangle$ in $M^{<M}$.

This completes the definition of the isomorphic embedding $\mathcal{B}_0 \subsetapprox M^{<M}$, proving the Claim and Lemma 8. □

Lemma 9. *T admits effective elimination of quantifiers.*

Proof. We show that uniformly effectively in quantifier-free $\varphi(x_0 \cdots x_n)$ there is quantifier-free $\theta(x_0 \cdots x_{n-1})$ such that

$$T \vdash [\exists x_n\, \varphi(x_0 \cdots x_{n-1}, x_n) \leftrightarrow \theta(x_0 \cdots x_{n-1})].$$

Fix such $\varphi(x_0 \cdots x_n)$ and let N be least such that φ is in $L(T) \restriction N$. By Lemma 8, we can effectively determine whether or not $T' \restriction N \cup \{\varphi\}$ is consistent. If it is *not* consistent, then neither is $T \cup \{\varphi\}$, and in that case we just let $\theta(x_0 \cdots x_{n-1})$ be, say, $c_\rho \neq c_\rho$. If $T' \restriction N \cup \{\varphi\}$ *is* consistent, then by Lemma 8 we can effectively determine all basic $(n+1)$-types of $T' \restriction N$ containing φ, as follows. By Corollary 6.1, each basic $(n+1)$-type of $T' \restriction N$ is generated by a single formula which is the conjunction of certain formulas from a fixed finite set of atomic formulas. This fixed set can be effectively obtained uniformly in n and N, so by Lemma 8 we can effectively determine all of the formulas $\bigwedge \Sigma(x_0 \cdots x_n)$ which generate different basic $(n+1)$-types $\Sigma(x_0 \cdots x_n)$ of $T' \restriction N$. For each one of these formulas, we use Lemma 8 again to determine whether or not the conjunction with φ is consistent with $T' \restriction N$.

Therefore fix K such that $\Sigma_i(x_0 \cdots x_n)$ for $i < K$ are those basic $(n+1)$-types of $T' \restriction N$ which contain φ, and let $\bigwedge \Sigma_i(x_0 \cdots x_n)$ denote the formula which generates the type. The proof is completed as in Lemma 5, using Lemma 7 and the fact that T extends $T' \restriction N$. □

Corollary 9.1. *Every formula φ in $L(T)$ is equivalent in T to a finite Boolean combination of atomic formulas of the following forms:*

$$u \wedge v = w \wedge z, \qquad u <_{\mathrm{L}} w, \qquad E^{\eta}_{\xi}(u, w), \qquad H_\eta(u \wedge v, w),$$
$$u \wedge v \leqslant_{\mathrm{H}} w \wedge z, \qquad L^{\eta}_{\xi}(u, w), \qquad A_\eta(u \wedge v, w),$$

where each u, v, w, z is either a variable from among the free variables occurring in φ, or a constant in $L(T)$.

Proof. By Lemma 9, φ is equivalent to a quantifier-free formula ψ having the same free variables. Letting N be least such that ψ is in $L(T) \restriction N$, the corollary now follows from Lemma 6. □

Corollary 9.2. *T is decidable.*

Proof. It suffices to show that any quantifier-free sentence σ in $L(T)$ is decidable. This follows from the axioms in T' and the fact that Tr is recursive. □

Since T admits quantifier elimination, T is submodel complete. The verification that T is the model completion of T' is similar to that which showed T_0 to be the model completion of T'_0 (at the end of Part I). In particular, a finitely generated model of T' has for its universe the tree Tr with finitely many additional nodes attached, and any such model can be embedded in the model $\mathscr{A}$ defined at the beginning of Part II. We do not give further details since we will not make explicit use of the fact that T is the model completion of T' in what follows.

Part III

In this part we show that T has exactly five countable models up to isomorphism: three decidable models and two models decidable exactly in $H(n)$. We first characterize all 1-types of T. Fix a recursive map $l : \omega \to \mathrm{Tr}$ such that $|l(k)| = k$ for all $k < \omega$.

Lemma 10. *Each* 1*-type $\Gamma(x)$ of T contains exactly one of the sets* (A1)–(A12) *below (for unique η in* Tr *and $m < \omega$ where applicable, and for a unique choice of $H_\eta(x, x)$, $A_\eta(x, x)$ or $B_\eta(x, x)$ if this set is* (A10) *or* (A11)), *and contains exactly one of the sets* (B1)–(B4) *below (for a unique $k < \omega$ if this set is* (B2) *or* (B3)). *Moreover, Γ is generated by these two sets.*

(A1) $\{x = c_\eta\}$,

(A2) $\{x < c_\rho\}$,

(A3) $\{x \mid c_\rho, x <_{\mathrm{L}} c_\rho\}$,

(A4) $\{x \mid c_\rho, c_\rho <_{\mathrm{L}} x\}$,

(A5) $\{c_\eta < x\}$, *where* η *is terminal in* Tr,

(A6) $\{c_\eta < x < c_{\eta^\wedge\langle m\rangle}\}$,

(A7) $\{c_\eta < x \wedge c_{\eta^\wedge\langle m\rangle} < c_{\eta^\wedge\langle m\rangle}, x \mid c_{\eta^\wedge\langle m\rangle}, x <_{\mathrm{L}} c_{\eta^\wedge\langle m\rangle}\}$,

(A8) $\{c_\eta < x \wedge c_{\eta^\wedge\langle m\rangle} < c_{\eta^\wedge\langle m\rangle}, x \mid c_{\eta^\wedge\langle m\rangle}, c_{\eta^\wedge\langle m\rangle} <_{\mathrm{L}} x\}$,

(A9) $\{V(c_\eta; x, c_{\eta\langle 0\rangle})\}$,

(A10) $\{W(c_\eta; c_{\eta^\wedge\langle m\rangle}, x, c_{\eta^\wedge\langle m+1\rangle})\} \cup \{?\}$,

(A11) $\{c_\eta < x, c_{\eta^\wedge\langle m\rangle} <_{\mathrm{L}} x : m < \omega\} \cup \{?\}$,

(A12) $\{c_\xi < x : \xi \in f^*\}$,

$$? = \begin{cases} H_\eta(x, x) \\ A_\eta(x, x), \\ B_\eta(x, x), \end{cases}$$

(B1) $\{x <_{\mathrm{H}} c_\rho\}$,

(B2) $\{x =_{\mathrm{H}} c_{l(k)}\}$,

(B3) $\{c_{l(k)} <_{\mathrm{H}} x <_{\mathrm{H}} c_{l(k+1)}\}$,

(B4) $\{c_\xi <_{\mathrm{H}} x : \xi \in \mathrm{Tr}\}$.

Proof. Let $\Gamma(x)$ be any 1-type of T. Clearly Γ contains a unique set from (B1)–(B4), since axioms (12)–(14) make $\leqslant_{\mathrm{H}}$ a linear ordering of $=_{\mathrm{H}}$-equivalence classes. A proof like that of Lemma 2 shows that Γ must contain a unique set from (A1)–(A12). To prove that Γ is generated by these two sets, it suffices to show that they decide all the atomic formulas listed in Corollary 9.1 (for all η, $\xi \in \mathrm{Tr}$) having the single free variable x. That all such formulas of the form $u \wedge v = w \wedge z$ are decided is easily checked with the help of the following claim.

Claim. *If* $\mu \in \mathrm{Tr}$, *then* $\Gamma(x)$ *contains at least one of the following four formulas*:

$x \wedge c_\mu = x$,

$x \wedge c_\mu = c_\nu$, *for some* $\nu \in \mathrm{Tr}$,

$x \wedge c_\mu = x \wedge c_\rho$,

$x \wedge c_\mu = x \wedge c_{\eta^\wedge\langle m\rangle}$, *for some* η *and* m *such that* Γ *contains* (A7) *or* (A8).

Proof of Claim. The Claim is immediate if Γ contains (A1). If Γ contains (A2), then clearly $x \wedge c_\mu = x$ is in Γ; if (A3) or (A4), then $x \wedge c_\mu = x \wedge c_\rho$; and if (A5) for some terminal $\eta \in \mathrm{Tr}$, then $x \wedge c_\mu = c_\eta \wedge c_\mu$. Suppose (A6) is in Γ for some $\eta \in \mathrm{Tr}$ and $m < \omega$. It is not hard to show that Γ then contains $x \wedge c_\mu = x$ if $c_{\eta^\wedge\langle m\rangle} \leqslant c_\mu$, and contains $x \wedge c_\mu = c_{\eta^\wedge\langle m\rangle} \wedge c_\mu$ otherwise. Cases (A7) and (A8) are similar. If Γ contains (A9), (A10) or (A11) for some η, then $x \wedge c_\mu = c_\eta \wedge c_\mu$. Finally, suppose (A12) is in Γ. Then there is $\xi \in f^*$ such that $c_\xi \nleqslant c_\mu$, and Γ contains $x \wedge c_\mu = c_\xi \wedge c_\mu$ for any such ξ. This completes the proof of the Claim.
□ Claim

Next we consider formulas of the form $u <_{\mathrm{L}} w$. Fix any $\mu \in \mathrm{Tr}$. If Γ contains (A2) or (A3), then clearly $x <_{\mathrm{L}} c_\mu$ is in Γ; if (A4), then $c_\mu <_{\mathrm{L}} x$. In the remaining cases (A5)–(A12), one can show using axioms (1)–(11) that $x <_{\mathrm{L}} c_\mu$ is equivalent in Γ to a formula involving only constants. For example, if Γ contains (A5), then the equivalent sentence is $c_\eta <_{\mathrm{L}} c_\mu$; if (A8), it is $[c_{\eta^\wedge\langle m\rangle} <_{\mathrm{L}} c_\mu \,\&\, c_{\eta^\wedge\langle m\rangle} \nleq c_\mu]$; if (A12), then $x <_{\mathrm{L}} c_\mu \leftrightarrow c_\xi <_{\mathrm{L}} c_\mu$ for any $\xi \in f^*$ such that $c_\xi \nleq c_\mu$. With axioms (7), (8) and (16)–(18) this shows that membership in Γ of all formulas of the form $u <_{\mathrm{L}} w$ is determined by the set from (A1)–(A12).

It is easy to see that $x \leqslant_{\mathrm{H}} c_\mu \in \Gamma$ if and only if Γ contains either (B1), or (B2) for $k \leqslant |\mu|$, or (B3) for $k < |\mu|$. A similar result holds for the formula $c_\mu \leqslant_{\mathrm{H}} x$. Other formulas of the form $u \wedge v \leqslant_{\mathrm{H}} w \wedge z$ are equivalent in Γ to formulas whose membership in Γ we already know is decided. In showing this, one makes extensive use of the Claim to reduce the number of cases that need to be considered.

Axioms (20)–(50) can be used to show that the remaining formulas also are equivalent to formulas previously dealt with. For example, consider $A_\xi(x, c_\mu)$. This formula is clearly not in Γ if $c_\xi \nleq c_\mu$, by axiom (34). Otherwise there is $k < \omega$ such that $c_{\xi^\wedge\langle k\rangle} \leqslant c_\mu$, hence $E^\xi_\xi(c_{\xi^\wedge\langle k\rangle}, c_\mu)$ by (27), so

$$A_\xi(x, c_\mu) \leftrightarrow A_\xi(x, c_{\xi^\wedge\langle k\rangle}) \leftrightarrow c_{l(k)} <_{\mathrm{H}} x,$$

using (43), (46) and $H_\xi(c_{l(k)}, c_{\xi^\wedge\langle k\rangle})$ (axiom (36)).

The proofs for formulas of the forms $E^\zeta_\xi(u, w)$ and $L^\zeta_\xi(u, w)$ tend to split into two cases depending on whether or not ξ is a terminal node of Tr. □

As in Part I, we say that an s-type $\Sigma(x_0 \cdots x_{s-1})$ of T is *closed* if for all $i, j < s$ there is $k < s$ such that $x_i \wedge x_j = x_k$ is in Σ.

Lemma 11. *Each closed type of T is generated by its 1-type projections and finitely many formulas of $L(T)$.*

Proof. Let $\Sigma(x_0 \cdots x_{s-1})$ be a closed s-type of T, and let $\Gamma_i(x_i)$ for $i < s$ be its 1-type projections. Fix N least such that for each $i < s$,

$\eta \in S_N$, if Γ_i contains (A1), (A5), (A9), (A10) or (A11) for η,
$\eta, \eta^\wedge\langle m\rangle \in S_N$, if Γ_i contains (A6), (A7) or (A8) for η and m.

Define

$$\begin{aligned} G^s_N =_{\mathrm{def}} \{ & u \wedge v = w \wedge z,\ u \wedge v \leqslant_{\mathrm{H}} w \wedge z,\ u <_{\mathrm{L}} w, \\ & E^\eta_\xi(u, w),\ L^\eta_\xi(u, w),\ H_\eta(u \wedge v, w),\ A_\eta(u \wedge v, w): \\ & \eta, \xi \in S_N,\ u, v, w, z \in \{x_i : i < s\} \cup \{c_\eta : \eta \in S_N\}\}, \end{aligned}$$

$$\begin{aligned} \mathrm{Gen}_\Sigma(x_0 \cdots x_{s-1}) =_{\mathrm{def}} & \{\varphi(x_0 \cdots x_{s-1}): \varphi \in G^s_N \cap \Sigma\} \\ & \cup \{\neg\varphi(x_0 \cdots x_{s-1}): \varphi \in G^s_N - \Sigma\}. \end{aligned}$$

We show that Σ is generated by its 1-type projections Γ_i and the finite set Gen_Σ.

Since Σ is closed, and since the Claim in the proof of Lemma 10 is true of the 1-type projections Γ_i, it follows that each formula in Σ of the forms $u \wedge v = w \wedge z$ or $u \wedge v \leqslant_{\mathrm{H}} w \wedge z$ is equivalent to either a formula in a single variable or a formula in G_N^s. Membership in Σ of formulas of the form $u <_{\mathrm{L}} w$ is obviously determined by the 1-type projections if u and/or w are constants. Otherwise we have a formula $x_i <_{\mathrm{L}} x_j$ which occurs in G_N^s.

Next we consider the formulas $E_\nu^\mu(x_i, x_j)$ and $L_\nu^\mu(x_i, x_j)$ where $i, j < s$ and $\mu, \nu \in \mathrm{Tr}$. These formulas do not belong to Σ if $c_\mu \nless x_i \in \Gamma_i$ or $c_\nu \nless x_j \in \Gamma_j$ (axioms (20) and (21)). Recall that if $c_\mu \blacktriangleleft x$, then by definition $c_\mu < x$ and there is no $\eta \in \mathrm{Tr}$ such that $c_\mu < c_\eta < x$. If both $c_\mu \blacktriangleleft x_i \in \Gamma_i$ and $c_\nu \blacktriangleleft x_j \in \Gamma_j$, then by the choice of N both μ and ν belong to S_N, hence membership in Σ of the formulas under consideration is determined by Gen_Σ. Now suppose there is $\eta \in \mathrm{Tr}$ such that $c_\mu < c_\eta < x_i \in \Gamma_i$. Then $E_\mu^\mu(c_\eta, x_i)$ by axiom (27), hence

$$E_\nu^\mu(x_i, x_j) \leftrightarrow E_\nu^\mu(c_\eta, x_j), \qquad L_\nu^\mu(x_i, x_j) \leftrightarrow L_\nu^\mu(c_\eta, x_j)$$

by (23) and (26), respectively, so membership in Σ of the formulas on the left is determined by Γ_j. Similarly, if Γ_j contains $c_\nu < c_\eta < x_j$ for some η, then Γ_i determines whether or not these formulas belong to Σ.

Formulas of the forms $H_\nu(u \wedge v, w)$ and $A_\nu(u \wedge v, w)$ are treated in a similar manner. Note that if we have a formula like $H_\nu(x_i \wedge c_\mu, x_j)$ for some $i, j < s$, and if $c_\nu \blacktriangleleft x_j \in \Gamma_j$, then one can use the Claim from the proof of Lemma 10 to make a substitution for $x_i \wedge c_\mu$ which results in a formula which either is in one variable or belongs to G_N^s. $\square$

Let

$$\Delta(x) =_{\mathrm{def}} \{c_\xi <_{\mathrm{H}} x \colon \xi \in \mathrm{Tr}\},$$
$$\Gamma^*(x) =_{\mathrm{def}} \{c_\xi < x \colon \xi \in f^*\},$$
$$\Phi_\eta(x) =_{\mathrm{def}} \{c_\eta < x,\ c_{\eta\frown\langle m\rangle} <_{\mathrm{L}} x \colon m < \omega\}.$$

By Lemma 10, a 1-type of T is nonprincipal if and only if it contains at least one of these sets. It may contain at most two, namely $\Delta(x)$ together with either $\Gamma^*(x)$ or $\Phi_\eta(x)$ for some η. Note that of these sets of formulas only $\Gamma^*(x)$ generates a complete type, since any 1-type containing (A12) of Lemma 10 contains (B4). This type is the single nonrecursive 1-type of T. Also note that, by the axioms in (28), $\Phi_\eta(x)$ is equivalent to $\{L_\eta^\eta(c_{\eta\langle m\rangle}, x) \colon m < \omega\}$.

In the next three lemmas, we show that realization of any nonprincipal 1-type (in particular the nonrecursive 1-type) in some model of T forces *all* the recursive 1-types of T to be realized in that model. We first note the following.

Every model $\mathscr{A}$ of T satisfies one option from (Ia)–(Ic) and one option from (IIa)–(IIc) below.

(Ia) Δ is omitted,

(Ib) $(\exists a' \in A)[\Delta(a') \,\&\, (\forall a \in A)[\Delta(a) \Leftrightarrow \mathscr{A} \models a' \leqslant_{\mathrm{H}} a]]$, (Ic) Otherwise; that is, Δ is realized and

$$(\forall a' \in A)[\Delta(a') \Rightarrow (\exists a \in A)[\Delta(a) \,\&\, \mathcal{A} \models a <_{\mathrm{H}} a']],$$

(IIa) Γ^* is omitted,
(IIb) $(\exists a' \in A)[\Gamma^*(a') \,\&\, (\forall a \in A)[\Gamma^*(a) \Leftrightarrow \mathcal{A} \models a' \leqslant a]]$,
(IIc) Otherwise; that is, Γ^* is realized and

$$(\forall a' \in A)[\Gamma^*(a') \Rightarrow (\exists a \in A)[\Gamma^*(a) \,\&\, \mathcal{A} \models a < a']].$$

Lemma 12. *Let $\mathcal{A}$ be any model of T. If $\mathcal{A}$ satisfies option* (IIb), *then $\mathcal{A}$ satisfies option* (Ib). *If $\mathcal{A}$ satisfies option* (IIc), *then $\mathcal{A}$ satisfies option* (Ic).

Proof. (IIb)$\Rightarrow$(Ib) Suppose that $a' \in A$ realizes Γ^*, and that for all $a \in A$, $\Gamma^*(a) \Leftrightarrow \mathcal{A} \models a' \leqslant a$. Clearly a' realizes Δ. We show that if $a \in A$, then $\Delta(a)$ if and only if $\mathcal{A} \models a' \leqslant_{\mathrm{H}} a$. One direction is immediate: if $\mathcal{A} \models a' \leqslant_{\mathrm{H}} a$, then clearly $\Delta(a)$. For the other direction assume $\mathcal{A} \models a <_{\mathrm{H}} a'$ in order to prove the contrapositive. Then by axiom (*7), there is $b \in A$ such that $b =_{\mathrm{H}} a$ and $b < a'$. By choice of a', b does not realize Γ^*, so there is $\eta \in f^*$ such that $\mathcal{A} \models b < c_\eta$. Hence $\mathcal{A} \models b <_{\mathrm{H}} c_\eta$ by axiom (15), and therefore $\mathcal{A} \models a <_{\mathrm{H}} c_\eta$; so a does not realize Δ.

(IIc)$\Rightarrow$(Ic) Any element realizing Γ^* realizes Δ, so (IIc) implies Δ is realized. Suppose $\Delta(a')$ for some a' in A. We want to show there is $a \in A$ such that $\Delta(a)$ and $\mathcal{A} \models a <_{\mathrm{H}} a'$. Fix any element $a_0 \in A$ realizing Γ^*. If $\mathcal{A} \models a_0 \leqslant_{\mathrm{H}} a'$, then using (IIc) we take $a \in A$ to be an element realizing Γ^* such that $\mathcal{A} \models a < a_0$, hence $\mathcal{A} \models a <_{\mathrm{H}} a'$ by axioms (13) and (15). So assume that $\mathcal{A} \models a' <_{\mathrm{H}} a_0$. There is $b \in A$ such that $b =_{\mathrm{H}} a'$ and $b < a_0$ (axiom (*7)). Necessarily b realizes Γ^*, since otherwise there would be $\eta \in f^*$ such that $\mathcal{A} \models b < c_\eta$, and we would have $\mathcal{A} \models a' <_{\mathrm{H}} c_\eta$, which contradicts $\Delta(a')$. By (IIc), there is $a \in A$ realizing Γ^* such that $\mathcal{A} \models a < b$. Clearly $\Delta(a)$ and $\mathcal{A} \models a <_{\mathrm{H}} a'$. This completes the proof of Lemma 12. □

In Lemma 15, we will prove that any two countable models of T satisfying the same options from (Ia)–(Ic) and (IIa)–(IIc) are isomorphic, so Lemma 12 will imply that T has at most five countable models up to isomorphism. The existence and Turing degree of each of these models will be verified following the proof of Lemma 15. Referring to the list at the beginning of Part I, (M1) will be the model satisfying (Ia) and (IIa), (M2) the model satisfying (Ib) and (IIa) (the reduct of the prime model of $\Gamma(a)$, where $\Gamma(x)$ is any recursive 1-type of T containing $\Delta(x)$), (M3) the model satisfying (Ic) and (IIa), (M4) the model satisfying (Ib) and (IIb) (the reduct of the prime model of $\Gamma^*(a)$), (M5) the model satisfying (Ic) and (IIc).

Lemma 13. *Let $\mathcal{A}$ be a model of T, let η, ξ be any nonterminal nodes of* Tr, *and let $a, b \in A$.*

(1) *If* $[H_\eta(a, b) \vee B_\eta(a, b)]$ *and* $\Delta(a)$, *then* $\Phi_\eta(b)$.
(2) *If* $[H_\eta(a, b) \vee A_\eta(a, b)]$ *and* $\Phi_\eta(b)$, *then* $\Delta(a)$.
(3) *If* $[E^\eta_\xi(a, b) \vee L^\eta_\xi(a, b)]$ *and* $\Phi_\eta(a)$, *then* $\Phi_\xi(b)$.

Proof. (1) Let $m < \omega$. Fix $\zeta \in \mathrm{Tr}$ such that $|\zeta| = m$. Then $H_\eta(c_\zeta, c_{\eta^\wedge\langle m\rangle})$ by axiom (36). Now $\Delta(a)$ implies $c_\zeta <_{\mathrm{H}} a$, hence axiom (46) yields $A_\eta(a, c_{\eta^\wedge\langle m\rangle})$. If $H_\eta(a, b)$, then $L^\eta_\eta(c_{\eta^\wedge\langle m\rangle}, b)$ by (40). If $B_\eta(a, b)$, use (42) instead of (40). The proofs of (2) and (3) are similar. □

The following lemma implies that if $\mathscr{A}$ is a model of T and $\mathscr{A}$ realizes Δ or realizes Φ_η for some η in Tr, then $\mathscr{A}$ realizes Φ_η for *every* nonterminal η in Tr. Statement (b) in this lemma says that if $\mathscr{A}$ satisfies option (Ib), then growing above any nonterminal node η of Tr there is a $<_{\mathrm{L}}$-leftmost subtree of A every element of which realizes Φ_η, namely the subtree $\{a \in A: E^\eta_\eta(b', a)\}$.

Lemma 14. *Let* $\mathscr{A}$ *be a model of* T *and let* η *be any nonterminal node of* Tr.

(a) *If* $\mathscr{A}$ *satisfies option* (Ia), *then* $\mathscr{A}$ *omits* Φ_η.

(b) *If* $\mathscr{A}$ *satisfies option* (Ib) *and* $a' \in A$ *is as in* (Ib), *then*

$$(\exists b' \in A)[\Phi_\eta(b') \text{ and } \mathscr{A} \models H_\eta(a', b') \text{ and}$$
$$(\forall b \in A)[\Phi_\eta(b) \Leftrightarrow \mathscr{A} \models [E^\eta_\eta(b', b) \vee L^\eta_\eta(b', b)]]].$$

(c) *If* $\mathscr{A}$ *satisfies option* (Ic), *then* $\mathscr{A}$ *realizes* Φ_η *and*

$$(\forall b' \in A)[\Phi_\eta(b') \Rightarrow (\exists b \in A)[\Phi_\eta(b) \,\&\, \mathscr{A} \models L^\eta_\eta(b, b')]].$$

Proof. (a) To prove the contrapositive, assume there is b in A realizing Φ_η. Because the map h defined in Part II is onto, $T \vdash \forall x\, \exists y\, [c_\eta < x \rightarrow H_\eta(y, x)]$, so there is $a \in A$ such that $\mathscr{A} \models H_\eta(a, b)$. Then $\Delta(a)$ by Lemma 13(2).

(b) Suppose that a' in A realizes Δ and that, for all $a \in A$, $\Delta(a) \Leftrightarrow \mathscr{A} \models a' \leqslant_{\mathrm{H}} a$, as asserted in (Ib). Since η is nonterminal, the map f_η defined in Part II is onto $\mathbb{Q}$, hence $T \vdash \forall x\, \exists y\, H_\eta(x, y)$. So there is b' in A such that $H_\eta(a', b')$, and this b' realizes Φ_η by Lemma 13(1). If some $b \in A$ realizes Φ_η, then, as in the proof of (a) above, there is $a \in A$ realizing Δ such that $H_\eta(a, b)$. By choice of a', $\mathscr{A} \models a' \leqslant_{\mathrm{H}} a$, hence either $H_\eta(a, b')$ or $A_\eta(a, b')$ by (45) and (46), so $\mathscr{A} \models [E^\eta_\eta(b', b) \vee L^\eta_\eta(b', b)]$ by (39) and (40). The converse follows from Lemma 13(3).

(c) Since $\mathscr{A}$ satisfies (Ic), $\mathscr{A}$ realizes Δ, hence $\mathscr{A}$ realizes Φ_η by Lemma 13(1) because $T \vdash \forall x\, \exists y\, H_\eta(x, y)$. So suppose that b' realizes Φ_η in A. Then there is a' in A realizing Δ such that $\mathscr{A} \models H_\eta(a', b')$. Because $\mathscr{A}$ satisfies option (Ic), there is $a \in A$ realizing Δ such that $a <_{\mathrm{H}} a'$, hence $\mathscr{A} \models B_\eta(a, b')$ by axiom (47). Finally, there is $b \in A$ such that $\Phi_\eta(b)$ and $\mathscr{A} \models H_\eta(a, b)$, so by axiom (41) we have $\mathscr{A} \models L^\eta_\eta(b, b')$, as was to be shown.

This completes the proof of Lemma 14. □

Corollary 14.1. *Any model of T satisfying option* (Ia) *is prime.*

Proof. If $\mathcal{A}$ is a model of T satisfying option (Ia), then $\mathcal{A}$ clearly omits any 1-type containing (B4) of Lemma 10. By Lemma 14, $\mathcal{A}$ also omits any 1-type containing (A11), and by Lemma 12, $\mathcal{A}$ cannot realize (A12). Therefore Lemmas 10 and 11 imply that the only closed types realized by $\mathcal{A}$ are principal. Since any nonprincipal type of T is contained in a nonprincipal closed type realized by exactly the same models of T, it follows that $\mathcal{A}$ realizes only principal types. □

Lemma 15. *If two models $\mathcal{A}$ and $\mathcal{B}$ of T satisfy the same option from* (Ia)–(Ic) *and the same option from* (IIa)–(IIc), *then $\mathcal{A} \cong \mathcal{B}$.*

Proof. If $\mathcal{A}$ and $\mathcal{B}$ satisfy option (Ia), then by Corollary 14.1 they are prime models of T and therefore isomorphic. So in the proof below we assume that $\mathcal{A}$ and $\mathcal{B}$ satisfy either (Ib) or (Ic).

We construct the isomorphism $f: A \to B$ in stages, defining finite partial functions f_s so that $f =_{\text{def}} \bigcup_s f_s$. At each stage $s > 0$, elements $a_s \in A$ and $b_s \in B$ are chosen so that

$$f_s =_{\text{def}} f_{s-1} \cup \{\langle a_s, b_s \rangle\}$$

is a partial isomorphism. The construction is a modification of the usual 'back-and-forth' method, with the added complexity of the construction resulting primarily from the necessity of guaranteeing that the type of $\langle a_0 \cdots a_s \rangle$ is closed at every stage s.

Suppose that at the beginning of a 'forth' stage $s > 0$ we have s-tuples $\langle a_0 \cdots a_{s-1} \rangle$ in A^s and $\langle b_0 \cdots b_{s-1} \rangle$ in B^s such that the partial function f_{s-1}, defined by $f_{s-1}(a_i) =_{\text{def}} b_i$ for all $i < s$, is an isomorphism. How do we find an image $b_s \in B$ for a given element $a_s \in A$? If $\Gamma(x_0 \cdots x_{s-1}) \subset \Sigma(x_0 \cdots x_{s-1}, x_s)$ are the types realized in A by $\langle a_0 \cdots a_{s-1} \rangle$ and $\langle a_0 \cdots a_{s-1}, a_s \rangle$, respectively, then $\langle b_0 \cdots b_{s-1} \rangle$ realizes Γ and we need only find $b_s \in B$ such that $\langle b_0 \cdots b_{s-1}, b_s \rangle$ realizes Σ. To do this it will suffice to find a formula $\theta(x_0 \cdots x_s)$ in $\Sigma(x_0 \cdots x_s)$ which generates Σ over Γ, that is, such that

$$\Gamma(d_0 \cdots d_{s-1}) \vdash \theta(d_0 \cdots d_{s-1}, x_s) \to \sigma(d_0 \cdots d_{s-1}, x_s)$$

for all $\sigma \in \Sigma$. For since Γ contains $\exists x_s\, \theta(x_0 \cdots x_{s-1}, x_s)$, we have $\mathcal{B} \models \exists x_s\, \theta(b_0 \cdots b_{s-1}, x_s)$, and we can let b_s be any element of B such that $\mathcal{B} \models \theta(b_0 \cdots b_{s-1}, b_s)$. Usually it will be possible to define θ with the help of Lemmas 10 and 11; this is where we need the restriction that Γ and Σ be closed. When it is not possible to define such θ, it will be necessary to find b_s by more direct means.

Before describing the construction in detail, we give an informal outline. We start things off at stage 0 by simply letting $a_0 =_{\text{def}} c_\rho$, $b_0 =_{\text{def}} c_\rho$, and defining $f_0 =_{\text{def}} \{\langle a_0, b_0 \rangle\}$. At stage 1, we find elements $a_1 \in A$ and $b_1 \in B$ realizing Δ if the models satisfy option (IIa) and realizing Γ^* otherwise. If the models satisfy (Ib) (respectively (IIb)), these elements are chosen so that they witness the existential statement in (Ib) (respectively (IIb)). (The proof of Lemma 12 shows that any element witnessing (IIb) also witnesses (Ib).) The elements a_1 and b_1 will make it easier to define f_s at stages $s>1$ should a_s and b_s happen to realize a nonprincipal type. At the stages $s=12k+2$, for $k\geqslant 0$, we first choose $a\in A$ least (with respect to some well-ordering of A fixed throughout the construction) such that $a\notin \mathrm{dom}(f_{s-1})$; this element will enter $\mathrm{dom}(f)$ at stage $12k+7$, ensuring that $\mathrm{dom}(f)$ is A. In case $\{a_0\cdots a_{s-1}, a\}$ is closed under $\wedge$, f is trivially extended at stages $12k+2$, $12k+3$ and $12k+4$; then at stages $p=12k+5$ and $q=12k+6$, f is extended, if necessary, so that a_p and a_q satisfy certain conditions involving the element a which will make it easy to find the image $b_r\in B$ of $a_r =_{\mathrm{def}} a$ at stage $r=12k+7$. But if $\{a_0\cdots a_{s-1}, a\}$ is *not* closed, then by Corollary 2.1 there is $a'\in A$ such that $\{a_0\cdots a_{s-1}, a'\}$ and $\{a_0\cdots a_{s-1}, a', a\}$ are each closed; in this case a' must enter $\mathrm{dom}(f)$ before a does, and we make sure this happens at stage $12k+4$, if necessary with the help of elements a_p and a_q chosen at stages $p=12k+2$ and $q=12k+3$ to satisfy certain conditions related to a'. In both cases, the elements a_p and a_q are chosen in such a way that $\{a_0\cdots a_s\}$ is closed under $\wedge$ for every s, $12k+2\leqslant s\leqslant 12k+7$. Thus six consecutive stages are used in order to define f so that $\mathrm{dom}(f)$ includes the element $a\in A$ chosen at the beginning of stage $12k+2$.

At stage $s=12k+8$, we will choose $b\in B$ to be the least element (with respect to some fixed well-ordering of B) such that $b\notin \mathrm{rng}(f_{s-1})$. This element will enter $\mathrm{rng}(f)$ at stage $12k+13$, ensuring that $\mathrm{rng}(f)$ is B. Stages $12k+8$ through $12k+13$ are just the 'back' versions of stages $12k+2$ through $12k+7$, respectively. This concludes the informal outline of the construction.

In the description below, to *extend f trivially* at stage s will mean to define $a_s =_{\mathrm{def}} a_{s-1}$, $b_s =_{\mathrm{def}} b_{s-1}$.

The construction

Stage 0. Let $a_0 =_{\mathrm{def}} c_\rho$, $b_0 =_{\mathrm{def}} c_\rho$, $f_0 =_{\mathrm{def}} \{\langle a_0, b_0\rangle\}$.

Stage 1. Fix a terminal node τ in Tr. Depending on which options are satisfied by $\mathscr{A}$, choose any $a_1\in A$ such that

(Ib) & (IIa): $c_\tau < a_1$, $\Delta(a_1)$, $(\forall a\in A)[\Delta(a)\leftrightarrow a_1 \leqslant_{\mathrm{H}} a]$,

(Ic) & (IIa): $c_\tau < a_1$, $\Delta(a_1)$,

(Ib) & (IIb): $\Gamma^*(a_1)$, $(\forall a\in A)[\Gamma^*(a)\leftrightarrow a_1\leqslant a]$,

(Ic) & (IIc): $\Gamma^*(a_1)$.

Similarly choose b_1 in B. According to Lemma 10, a_1 and b_1 realize the same 1-type $\Gamma(x_1)$, namely the type generated by the sets (A5) and (B4) if the models satisfy option (IIa), and generated by (A12) and (B4) otherwise. Clearly the

2-type $\Sigma(x_0, x_1)$ generated by this 1-type and the formula $x_0 = c_\rho$ is closed and is realized by both $\langle a_0, a_1\rangle$ and $\langle b_0, b_1\rangle$, so we can define f_1 to be $f_0 \cup \{\langle a_1, b_1\rangle\}$.

Stage $s = 12k + 2$ $(k \geq 0)$. Choose the least $a \in A$ such that $a \notin \mathrm{dom}(f_{s-1})$. If $\{a_0 \cdots a_{s-1}, a\}$ is not closed, let $a' \in A$ be the unique element such that $\{a_0 \cdots a_{s-1}, a'\}$ and $\{a_0 \cdots a_{s-1}, a', a\}$ are each closed (Corollary 2.1); otherwise a' is undefined.

If a' is undefined, or if there is $i < s$ such that $H_\eta(a', a_i)$ for some η, then extend f trivially. Otherwise choose a_s to be the least element in A such that

(1) $c_\rho < a_s$.

(2) $a_s =_{\mathrm{H}} c_{\langle 0\rangle}$.

(3) $H_\rho(a', a_s)$.

(4) $E^\rho_\rho(a_s, a) \rightarrow [a_s \leq a \vee a \leq a_s]$.

It is routine to check that a_s satisfying (1)–(4) always exists, using $T \vdash \forall x\, \exists y\, H_\rho(x, y)$ and axioms (*7) and (39). Let $\Gamma(x_0 \cdots x_{s-1}) \subset \Sigma(x_0 \cdots x_{s-1}, x_s)$ be the s- and $(s+1)$-types of $\langle a_0 \cdots a_{s-1}\rangle$ and $\langle a_0 \cdots a_{s-1}, a_s\rangle$. Σ is closed (as is easily verified) and is therefore generated over Γ by the 1-type of a_s and the finite set Gen_Σ (Lemma 11).

If the 1-type of a_s is principal, then, supposing this 1-type to be generated by $\varphi(x_s)$, we take

$$\theta(x_0 \cdots x_s) =_{\mathrm{def}} \varphi(x_s) \,\&\, \bigwedge \mathrm{Gen}_\Sigma(x_0 \cdots x_s)$$

for the formula generating Σ over Γ.

Assume that the 1-type of a_s is nonprincipal. Because of (2), this 1-type cannot contain $\Gamma^*(x_s)$ or $\Delta(x_s)$, and therefore Lemma 10 implies that it is generated by $\Phi_\rho(x_s)$ and one other formula $\varphi(x_s)$. (It can be shown that $\varphi(x)$ is the formula $B_\rho(x, x) \,\&\, x =_{\mathrm{H}} c_{\langle 0\rangle}$.) If there is $i < s$ such that $\Delta(a_i)$ and either $H_\rho(a_i, a_s)$ or $B_\rho(a_i, a_s)$, then let

$$\theta(x_0 \cdots x_s) =_{\mathrm{def}} [H_\rho(x_i, x_s) \vee B_\rho(x_i, x_s)] \,\&\, \varphi(x_s) \,\&\, \bigwedge \mathrm{Gen}_\Sigma(x_0 \cdots x_s),$$

since $H_\rho(a_i, x) \vee B_\rho(a_i, x)$ generates $\Phi_\rho(x)$ in $\mathrm{Th}(\mathcal{A}, a_i)$ by Lemma 13(1). If there is no $i < s$ as above, then we make a second attempt at defining θ: if there is $i < s$ such that $\Phi_\eta(a_i)$ and $L^\eta_\rho(a_i, a_s)$ for some η, we let

$$\theta(x_0 \cdots x_s) =_{\mathrm{def}} L^\eta_\rho(x_i, x_s) \,\&\, \varphi(x_s) \,\&\, \bigwedge \mathrm{Gen}_\Sigma(x_0 \cdots x_s).$$

For $L^\eta_\rho(a_i, x)$ generates $\Phi_\rho(x)$ in $\mathrm{Th}(\mathcal{A}, a_i)$ by Lemma 13(3). If both attempts at defining θ fail we are forced to find b_s directly. The failure of the first attempt implies that $\mathcal{A} \vDash A_\rho(a_i, a_s)$ for all $i < s$ such that $\Delta(a_i)$ (axiom (34)). It also implies that the models satisfy option (Ic), for otherwise option (Ib) together with Lemma 14(b) and the choice of a_1 to witness (Ib) would give the existence of a'' in A such that $H_\rho(a_1, a'')$ and $E^\rho_\rho(a'', a_s) \vee L^\rho_\rho(a'', a_s)$, hence $H_\rho(a_1, a_s) \vee B_\rho(a_1, a_s)$

via axioms (39) and (41). The failure of the second attempt means that $\mathscr{A} \vDash L^\rho_\eta(a_s, a_i)$ for all $i<s$ and $\eta \in \mathrm{Tr}$ such that $\Phi_\eta(a_i)$. This follows from axiom (21); $E^\rho_\eta(a_s, a_i)$ is ruled out because otherwise we would have $H_\eta(a', a_i)$ by (3) and axiom (39), contradicting the choice of a_s since we did not extend f trivially at this stage. Therefore $\langle a_0 \cdots a_{s-1}, a_s \rangle$ satisfies all of the formulas in

$$\begin{aligned} &\Gamma(x_0 \cdots x_{s-1}), \qquad \Phi_\rho(x_s), \qquad x_s =_{\mathrm{H}} c_{\langle 0 \rangle}, \\ (*) \qquad &A_\rho(x_i, x_s), \quad \text{for all } i<s \text{ such that } \Delta(x_i) \subset \Gamma, \\ &L^\rho_\eta(x_s, x_i) \quad \text{for all } i<s \text{ and } \eta \text{ in Tr such that } \Phi_\eta(x_i) \subset \Gamma. \end{aligned}$$

It can be shown that (*) generates a unique $(s+1)$-type of T. One verifies that for any formula $\psi(x_0 \cdots x_s)$ in $L(T)$, either ψ or $\neg\psi$ is a consequence of (*), and as usual it suffices to consider formulas ψ having the forms given in Corollary 9.1. In particular, it is easy to check that (*) implies $B_\rho(x_i, x_s)$ for all $i<s$ such that $\Delta(x_i) \not\subset \Gamma$, and $L^\eta_\rho(x_i, x_s)$ for all $i<s$ and $\eta \in \mathrm{Tr}$ such that $c_\eta < x_i$ and $\Phi_\eta(x_i) \not\subset \Gamma$. Other formulas ψ may first be simplified by observing that Γ is closed and that (*) implies $x_s \wedge x_i = c_\rho \wedge x_i$ and $x_s \wedge c_\mu = c_\rho$ for all $i<s$ and $\mu \in \mathrm{Tr}$. The easiest way to treat formulas whose only free variable is x_s is to verify that $\Phi_\rho(x_s)$ and $x_s =_{\mathrm{H}} c_{\langle 0 \rangle}$ generate a unique 1-type.

It remains to show that there is b_s in B such that $\langle b_0 \cdots b_{s-1}, b_s \rangle$ satisfies (*). Of course, we already have $\Gamma(b_0 \cdots b_{s-1})$. By the definition of the prime model,

$$\begin{aligned} &T \vdash \forall x\, \exists y\, H_\rho(x, y), \\ &T \vdash \forall x\, \exists y\, [c_\eta < x \rightarrow E^\eta_\rho(x, y)] \end{aligned}$$

for any η in Tr. So for each $i<s$, choose $b(i) \in B$ such that $\mathscr{B} \vDash H_\rho(b_i, b(i))$, and, if $c_\eta \blacktriangleleft b_i$, choose $b'(i) \in B$ so that $\mathscr{B} \vDash E^\eta_\rho(b_i, b'(i))$. By Lemma 13, we have $\Phi_\rho(b(i))$ if $\Delta(b_i)$, and $\Phi_\rho(b'(i))$ if $\Phi_\eta(b_i)$. Since $\mathscr{B}$ satisfies (Ic), there is $b \in B$ satisfying $\Phi_\rho(b)$ as well as

$$\begin{aligned} &L^\rho_\rho(b, b(i)) \quad \text{for all } i<s \text{ such that } \Delta(b_i), \text{ and} \\ &L^\rho_\rho(b, b'(i)) \quad \text{for all } i<s \text{ such that } \Phi_\eta(b_i) \text{ for some } \eta. \end{aligned}$$

Let b_s be any element in B such that $b_s =_{\mathrm{H}} c_{\langle 0 \rangle}$ and $b_s \leqslant b \vee b \leqslant b_s$ (axiom (*7)). It is easy to check that $E^\rho_\rho(b_s, b)$, hence $\Phi_\rho(b_s)$. Now for each $i<s$ such that $\Delta(b_i)$, we have

$$L^\rho_\rho(b, b(i)) \rightarrow L^\rho_\rho(b_s, b(i)) \rightarrow A_\rho(b_i, b_s)$$

by axioms (26) and (40). For each $i<s$ such that $\Phi_\eta(b_i)$,

$$L^\rho_\rho(b, b'(i)) \rightarrow L^\rho_\rho(b_s, b'(i)) \rightarrow L^\rho_\eta(b_s, b_i)$$

by axioms (26) and (25).

Stage $s = 12k+3$. If a' is undefined, or if there is no ξ in Tr such that $\Phi_\xi(a')$, or if there is such ξ and there is $i<s$ such that $E^\eta_\xi(a_i, a')$ for some η, then extend f

trivially. Otherwise, fix ξ such that $\Phi_\xi(a')$, and choose a_s to be the least element in A such that

(1) $c_\rho < a_s$,

(2) $a_s =_{\mathrm{H}} c_{\langle 0 \rangle}$,

(3) $E^\rho_\xi(a_s, a')$,

(4) $E^\rho_\rho(a_s, a) \rightarrow [a_s \leq a \vee a \leq a_s]$.

Continue as in stage $12k+2$.

Stage $s = 12k+4$. If a' is undefined, extend f trivially. Otherwise set $a_s =_{\mathrm{def}} a'$. Let $\Gamma(x_0 \cdots x_{s-1}) \subset \Sigma(x_0 \cdots x_{s-1}, x_s)$ be the s- and $(s+1)$-types of $\langle a_0 \cdots a_{s-1} \rangle$ and $\langle a_0 \cdots a_{s-1}, a_s \rangle$. Σ is closed (as will be verified shortly) and is therefore generated over Γ by Gen_Σ and the 1-type of a_s. We show that this 1-type is principal over Γ, so that if it is generated over Γ by $\varphi(x_0 \cdots x_s)$ we just let

$$\theta(x_0 \cdots x_s) =_{\mathrm{def}} \varphi(x_0 \cdots x_s) \,\&\, \bigwedge \mathrm{Gen}_\Sigma(x_0 \cdots x_s).$$

By the remark following the proof of Lemma 11 it suffices to show that $\Delta(x_s)$, $\Phi_\xi(x_s)$ and $\Gamma^*(x_s)$ are principal over Γ whenever these sets happen to be in Σ. By previous stages, we have $p \leq 12k+2$ such that $H_\mu(a_s, a_p)$ for some μ, and if $\Phi_\xi(a_s)$, $q \leq 12k+3$ such that $E^\nu_\xi(a_q, a_s)$ for some ν.

If $\Delta(a_s)$, then $\Phi_\mu(a_p)$ by Lemma 13(1), hence Lemma 13(2) implies that $\Delta(x)$ is generated by $H_\mu(x, a_p)$ in $\mathrm{Th}(\mathscr{A}, a_p)$.

If $\Phi_\xi(a_s)$, then $\Phi_\nu(a_q)$ by Lemma 13(3), hence $\Phi_\xi(x)$ is generated by $E^\nu_\xi(a_q, x)$ in $\mathrm{Th}(\mathscr{A}, a_q)$.

Finally, suppose $\Gamma^*(a_s)$. Then $\Delta(a_s)$, so $\Phi_\mu(a_p)$ by Lemma 13(1). Since $\mathscr{A}$ realizes Γ^*, we must have $\Gamma^*(a_1)$. Then there is $r < s$ such that $\Gamma^*(a_r)$ and either $a_s \leq a_r$ or $a_r \leq a_s$: namely $r = 1$ if $a_s \wedge a_1 = a_s$, otherwise r is such that $a_s \wedge a_1 = a_r$ (using the fact that Σ is closed). We leave it to the reader to show that $\Gamma^*(x)$ is generated by

$$H_\mu(x, a_p) \,\&\, [x \leq a_r \vee a_r \leq x]$$

in $\mathrm{Th}(\mathscr{A}, a_p, a_r)$.

Verification that Σ is closed. Γ is closed, and by the choice of a' at stage $12k+2$, $\{a_0 \cdots a_{12k+1}, a_s\}$ is closed, so it only remains to show that $a_s \wedge a_i$ is in $\{a_0 \cdots a_s\}$ for $i = 12k+2$ and $i = 12k+3$. This is fairly easy if $\neg E^\rho_\rho(a_s, a_i)$, in which case one verifies that $a_s \wedge a_i = a_s \wedge c_\rho = a_s \wedge a_0$. So assume $E^\rho_\rho(a_s, a_i)$. We have $E^\rho_\rho(a_s, a)$ since $c_\rho < a' < a$ and $a_s = a'$, thus $E^\rho_\rho(a_i, a)$ by axiom (23). So (4) at stage i guarantees that a and a_i are comparable, hence so are a_s and a_i (use axiom (4)). Therefore either $a_s \wedge a_i = a_s$ or $a_s \wedge a_i = a_i$.

Stage $s = 12k + 5$. If there is $i < s$ such that $H_\eta(a, a_i)$ for some η, then extend f trivially. Otherwise choose a_s to be the least element of A such that

(1) $c_\rho < a_s$,

(2) $a_s =_{\mathrm{H}} c_{\langle 0 \rangle}$,

(3) $H_\rho(a, a_s)$,

(4) $E^\rho_\rho(a_s, a) \rightarrow [a_s \leqslant a \vee a \leqslant a_s]$.

Continue as in stage $12k + 2$.

Stage $s = 12k + 6$. If there is no ξ in Tr such that $\Phi_\xi(a)$, or if there is such ξ and there is $i < s$ such that $E^\eta_\xi(a_i, a)$ for some η, then extend f trivially. Otherwise, fix ξ such that $\Phi_\xi(a)$, and choose a_s to be least in A such that

(1) $c_\rho < a_s$,

(2) $a_s =_{\mathrm{H}} c_{\langle 0 \rangle}$,

(3) $E^\rho_\xi(a_s, a)$,

(4) $E^\rho_\rho(a_s, a) \rightarrow [a_s \leqslant a \vee a \leqslant a_s]$.

Continue as in stage $12k + 2$.

Stage $s = 12k + 7$. Let $a_s =_{\mathrm{def}} a$, and let $\Gamma(x_0 \cdots x_{s-1}) \subset \Sigma(x_0 \cdots x_{s-1}, x_s)$ be the s- and $(s+1)$-types of $\langle a_0 \cdots a_{s-1} \rangle$ and $\langle a_0 \cdots a_{s-1}, a_s \rangle$. Continue as in stage $12k + 4$, with the help of $p \leqslant 12k + 5$ such that $H_\mu(a_s, a_p)$ for some μ, and if $\Phi_\xi(a_s)$, using $q \leqslant 12k + 6$ such that $E^\nu_\xi(a_q, a_s)$ for some ν.

Stages $12k + 8$ through $12k + 13$ are the 'back' versions of stages $12k + 2$ through $12k + 7$, respectively: simply replace 'dom' by 'rng', and 'a' by 'b', everywhere. This completes the description of the construction and the proof of Lemma 15. □

Corollary 15.1. *T has exactly five countable models up to isomorphism: three decidable models and two models decidable exactly in $H(n)$.*

Proof. By the remark following the proof of Lemma 12, T has at most five nonisomorphic countable models.

The decidability of the prime model (M1) follows from the effective version of the omitting-types theorem and the fact that T has only finitely many countable models up to isomorphism. (See, for example, [5].) By results in [2], T must also have a homogeneous decidable model (M3) realizing all of the recursive types of T.

Let $\Gamma(x)$ be a recursive 1-type of T containing $\Delta(x)$. Then (M3) can be expanded to a decidable model of the theory $\Gamma(d)$, and the expanded model realizes all of the recursive types of $\Gamma(d)$. Thus there exists an r.e. list of all of the recursive types of $\Gamma(d)$, and so by the effective version of the omitting-types theorem there is a decidable model of $\Gamma(d)$ omitting all of the nonprincipal types in this list — i.e., a decidable prime model of $\Gamma(d)$. The reduct of this model to $L(T)$ must be (M2) since it is not homogeneous. (The prime model of $\Gamma(d)$ omits any type containing $\Delta(x) \cup \{x <_{\mathrm{H}} d\}$, so the element d is $\leqslant_{\mathrm{H}}$-minimal realizing Δ.)

Models (M4) and (M5) exist and are decidable in $H(n)$ by relativizations of the arguments given above. If either of these models were decidable in a set S of degree below that of $H(n)$, then with the help of a parameter we could tell effectively in S whether or not an arbitrary element $\eta \in \mathrm{Tr}$ belonged to f^*: we fix an element x lying above f^* in the model, and simply check to see if η is below x. Thus (M4) and (M5) are decidable *exactly* in $H(n)$. □

Part IV

We prove that T is *persistently* Ehrenfeucht, namely that whenever $\Gamma(x_0 \cdots x_{n-1})$ is an n-type of T, then the theory $\Gamma(d_0 \cdots d_{n-1})$ in the language $L(T) \cup \{d_0 \cdots d_{n-1}\}$ has finitely many countable models up to isomorphism. It is enough to prove this for closed n-types. For suppose $\bar{\Gamma}(x_0 \cdots x_{n-1}, x_n \cdots x_{p-1})$ is the closure of $\Gamma(x_0 \cdots x_{n-1})$ (so that $\bar{\Gamma}$ says $x_n \cdots x_{p-1}$ are precisely all $x_i \wedge x_j$ for $i, j < n$). Then clearly any model of $\Gamma(d_0 \cdots d_{n-1})$ can be expanded to a model of $\bar{\Gamma}(d_0 \cdots d_{p-1})$ in only one way, and any isomorphism of models of $\bar{\Gamma}(d_0 \cdots d_{p-1})$ induces an isomorphism of models of $\Gamma(d_0 \cdots d_{n-1})$.

Suppose that $\langle \mathscr{A}, a_0 \cdots a_{n-1} \rangle$ and $\langle \mathscr{B}, b_0 \cdots b_{n-1} \rangle$ are models of $\Gamma(d_0 \cdots d_{n-1})$ such that the reducts $\mathscr{A}$ and $\mathscr{B}$ are isomorphic homogeneous models of T. If $f : \mathscr{A} \to \mathscr{B}$ is the isomorphism, then $\langle fa_0 \cdots fa_{n-1} \rangle$ also realizes Γ. Therefore, since $\mathscr{B}$ is homogeneous, there is an automorphism $g : \mathscr{B} \to \mathscr{B}$ sending fa_i to b_i for each $i < n$, and so $g \circ f$ is an isomorphism from $\langle \mathscr{A}, a_0 \cdots a_{n-1} \rangle$ to $\langle \mathscr{B}, b_0 \cdots b_{n-1} \rangle$.

That T is persistently Ehrenfeucht will be an immediate consequence of this observation, Lemma 16 below, and the fact that T is Ehrenfeucht. In order to state Lemma 16, we first note that if $\mathscr{A}$ is any model of T, then each element a' in A satisfies exactly one of eleven '1-type options', namely: either a' is minimal realizing Γ^* (that is, a' witnesses (IIb)); or a' realizes Γ^* but is not the minimal such element; or a' does not realize Γ^*, in which case there are nine possibilities depending on whether or not a' realizes Δ (and if a' does, whether or not it witnesses (Ib)), and whether or not a' simultaneously realizes Φ_η for some (necessarily unique) $\eta \in \mathrm{Tr}$ (and if a' does, whether or not it satisfies

$$(\forall a \in A)[\Phi_\eta(a) \Leftrightarrow E_\eta^\eta(a', a) \vee L_\eta^\eta(a', a)]).$$

Lemma 16. *Let $\mathscr{A}$ and $\mathscr{B}$ be isomorphic nonhomogeneous models of T, and let $\Gamma(x_0 \cdots x_{n-1})$ be any closed n-type of T. If $\langle a_0 \cdots a_{n-1} \rangle$ and $\langle b_0 \cdots b_{n-1} \rangle$ realize*

Γ in $\mathscr{A}$ and $\mathscr{B}$ respectively, and if for each $i < n$, a_i and b_i satisfy the same '1-type options' indicated above, then

$$\langle \mathscr{A}, a_0 \cdots a_{n-1} \rangle \cong \langle \mathscr{B}, b_0 \cdots b_{n-1} \rangle.$$

Proof. We may assume without loss of generality that Γ contains $x_0 = c_\rho$. We wish to use the construction from the proof of Lemma 15 to obtain the isomorphism. The difficulty here is in finding 'nonhomogeneous' elements $a_n \in A$ and $b_n \in B$ to play the roles of the elements that were chosen at stage 1 in the earlier construction. Suppose we have found these elements, so that $\langle a_0 \cdots a_n \rangle$ and $\langle b_0 \cdots b_n \rangle$ realize the same closed $(n+1)$-type of T. Define $f_n =_{\text{def}} \{\langle a_i, b_i \rangle : i \leqslant n\}$, and let k be least such that $n + 1 \leqslant 12k + 2$. If $n + 1 < 12k + 2$, then let f_s be a trivial extension of f_{s-1} for each s such that $n + 1 \leqslant s < 12k + 2$. Now starting from stage $12k + 2$, the construction duplicates that used in the proof of Lemma 15, except that all references to a_1 (respectively b_1) in the earlier construction are replaced by references to a_n (respectively b_n). The resulting map f is actually an isomorphism from $\langle \mathscr{A}, a_0 \cdots a_{n-1} \rangle$ to $\langle \mathscr{B}, b_0 \cdots b_{n-1} \rangle$ because of the definition of f_n. It remains to show how to obtain the elements a_n and b_n. Since $\mathscr{A}$ and $\mathscr{B}$ are nonhomogeneous models of T they must satisfy option (Ib), and therefore there are just two cases.

Case 1. $\mathscr{A}$ and $\mathscr{B}$ satisfy options (Ib) and (IIa).

We need $a_n \in A$ and $b_n \in B$ satisfying the first condition listed in stage 1 of the earlier construction. Fix a terminal node $\tau \in \mathrm{Tr}$ such that $\mathscr{A} \vDash \neg E^\rho_\rho(c_\tau, a_i)$ for all $i < n$. Without loss of generality we may assume that Γ contains $x_j = c_\tau$ for some $j < n$. Since $\mathscr{A}$ satisfies (Ib), axiom (*7) gives the existence in A of elements above c_τ which are $\leqslant_{\mathrm{H}}$-minimal realizing Δ. Because the map f_τ used in defining the prime model of T is *onto* the negative rational numbers, we have for all $\xi \in \mathrm{Tr}$,

$$T \vdash \forall x\, \exists y\, [c_\xi < x \rightarrow L^\tau_\xi(y, x)],$$

$$T \vdash \forall x\, \exists y\, A_\tau(x, y).$$

Thus it is possible to choose a_n such that

(1) $\quad c_\tau < a_n, \quad \Delta(a_n), \quad (\forall a \in A)[\Delta(a) \Leftrightarrow a_n \leqslant_{\mathrm{H}} a],$

(2) $\quad c_\xi < a_i \rightarrow L^\tau_\xi(a_n, a_i), \quad$ for all $i < n$ and ξ in Tr,

(3) $\quad A_\tau(a_i, a_n), \quad$ for all $i < n$.

It may not be obvious that we can choose a_n to satisfy (2) if there is $i < n$ such that $c_\xi < a_i$ for infinitely many ξ in Tr. But in that case $\Gamma^*(a_i)$, and for each such ξ there is $m < \omega$ such that $E^\xi_\xi(c_{\xi^\wedge\langle m \rangle}, a_i)$. Since τ is terminal we have $L^\tau_\xi(a_n, c_{\xi^\wedge\langle m \rangle})$ by axiom (32). Now apply axiom (25).

Using the same τ, choose $b_n \in B$ to satisfy (1)–(3) in $\mathscr{B}$. It remains to show that $\langle a_0 \cdots a_n \rangle$ and $\langle b_0 \cdots b_n \rangle$ realize the same closed $(n+1)$-type of T. It suffices to check that for each atomic formula $\psi(x_0 \cdots x_n)$ listed in Corollary 9.1, satisfaction of ψ by $\langle a_0 \cdots a_n \rangle$ depends only on the following properties: (1)–(3), the options satisfied by $\mathscr{A}$ and by $a_0 \cdots a_n$, and the fact that $\langle a_0 \cdots a_{n-1} \rangle$ realizes Γ. We omit the details. Since the same properties hold for $\mathscr{B}$ and $b_0 \cdots b_n$, it follows that $\langle b_0 \cdots b_n \rangle$ realizes the same type as $\langle a_0 \cdots a_n \rangle$. In particular, these properties imply $a_n \wedge a_i = c_\tau \wedge a_i$ for all $i < n$, hence the type of $\langle a_0 \cdots a_n \rangle$ is closed, since Γ is closed and contains $c_\tau = x_j$ for some $j < n$ by assumption.

Case 2. $\mathscr{A}$ and $\mathscr{B}$ satisfy options (Ib) and (IIb).

Choose $a_n \in A$ and $b_n \in B$ minimal elements realizing Γ^*. Without loss of generality we may assume $a_n \neq a_i$ for all $i < n$, since otherwise we are done. We may also assume there is $j < n$ such that $x_j = c_\zeta \in \Gamma$ for some ζ in f^* having the property that, for all $i < n$, $c_\zeta < a_i$ implies $\Gamma^*(a_i)$. Then $a_n \wedge a_i = a_n$ if a_i realizes Γ^*, and $a_n \wedge a_i = c_\zeta \wedge a_i$ otherwise (use axiom (5)), so the $(n+1)$-type of $\langle a_0 \cdots a_n \rangle$ is closed. As in Case 1, one shows that $\langle b_0 \cdots b_n \rangle$ also realizes this type by showing that satisfaction of a formula $\psi(x_0 \cdots x_n)$ depends only on the following properties: the options satisfied by $\mathscr{A}$ and by $a_0 \cdots a_n$, and the fact that $\langle a_0 \cdots a_{n-1} \rangle$ realizes Γ. □

Corollary 16.1. *T is persistently Ehrenfeucht.*

Proof. Let $\Gamma(x_0 \cdots x_{n-1})$ be an n-type of T. As noted earlier, we can assume without loss of generality that Γ is closed. Since T has three isomorphism types of countable homogeneous models, the observation made at the beginning of Part IV implies that the theory $\Gamma(d_0 \cdots d_{n-1})$ has at most three isomorphism types of countable models whose reducts to $L(T)$ are homogeneous. Since T has two isomorphism types of countable nonhomogeneous models, and since any element of a model of T satisfies exactly one of the eleven options listed prior to Lemma 16, that lemma implies that $\Gamma(d_0 \cdots d_{n-1})$ has at most $2 \cdot 11^n$ isomorphism types of countable models whose reducts to $L(T)$ are nonhomogeneous.

Finally, the fact that T is not ω-categorical prevents $\Gamma(d_0 \cdots d_{n-1})$ from being ω-categorical. (See, for example, [1, Theorem 2.3.13].) □

Corollary 16.2. *There is a* 1*-type* $\Gamma(x)$ *of* T *such that the theory* $\Gamma(d)$ *has fewer than five countable models up to isomorphism.*

Proof. Let $\Gamma(x)$ be the type generated by $\Gamma^*(x)$. The theory $\Gamma(d)$ has only three nonisomorphic countable models: one countable model whose reduct to $L(T)$ satisfies options (Ic) and (IIc), and two countable models whose reducts to $L(T)$ satisfy (Ib) and (IIb). In one of the latter two, the interpretation of d is minimal realizing Γ^*, while in the other it is not. □

Corollary 16.3. *There is a* 1*-type* $\Gamma(x)$ *of* T *such that the theory* $\Gamma(d)$ *has more than five nonisomorphic countable models.*

Proof. Let $\Gamma(x)$ be the type generated by $c_\tau < x$ and $\Delta(x)$, for some terminal node $\tau \in \mathrm{Tr}$. The theory $\Gamma(d)$ has six countable models up to isomorphism. It has one model whose reduct to $L(T)$ satisfies (Ic) and (IIa), and one model whose reduct satisfies (Ic) and (IIc). There are two models whose reducts to $L(T)$ satisfy (Ib) and (IIa): one in which the interpretation of d is $\leqslant_H$-minimal realizing Δ, and one in which it is not. Similarly there are two models whose reducts satisfy (Ib) and (IIb). □

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