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Ranking of Fuzzy Sets Based on the Concept of Existence

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Abstract—Various approaches have been proposed for the comparison or ranking of fuzzy sets. However, due to the complexity of the problem, a general method which can be used for any situation still does not exist. This paper formalizes the concept of existence for the ranking of fuzzy sets. Many of the existing fuzzy ranking methods are shown to be some application of this concept. An improved fuzzy ranking method is then introduced, based on this concept. This newly introduced method is expanded for treating both normal and nonnormal, convex and nonconvex fuzzy sets. Emphasis is placed on the use of the subjectivity of the decision maker, such as the optimistic or the pessimistic view points. An improved procedure for obtaining linguistic conclusions is also developed. Finally, some numerical examples are given to illustrate the approach.

Keywords—Fuzzy ranking method, Normal/nonnormal fuzzy set, Convex/nonconvex fuzzy set, Linguistic conclusions.

1. INTRODUCTION

Comparison or ranking of fuzzy numbers, or more generally, fuzzy sets, is very important for practical applications. When fuzzy set theory is used to establish mathematical models, manipulations of fuzzy variables or fuzzy parameters invariably involve ranking or comparison problems. But, unfortunately, fuzzy numbers are not in linear order and comparisons of them are not simple. Frequently, overlaps or small separations in the supports of fuzzy sets make comparison a very difficult task.

Various fuzzy ranking methods (FRMs) based on different approaches or different points of view have been proposed in the literature. Several reviews have also appeared [1–3]. In a more recent review [4], the following classification was proposed:

(a) Methods using an α-cut. With this approach, a FRM ranks fuzzy numbers by simply comparing their α-cuts. Often, a method is developed by this approach with the purpose of obtaining fast results.
(b) Methods using the possibility concept. With this approach, a FRM uses the possibility or necessity concepts to rank or to compare fuzzy numbers. The degree of possibility or necessity of a fuzzy number satisfying a fuzzy inequality relation against one or all other fuzzy numbers is determined. In establishing these fuzzy inequality relations, two different situations may be considered. Fuzzy inequality relations may be established, between a fuzzy number and all the remaining fuzzy numbers. Based on these inequality relations, optimal alternatives can be determined. Another approach is to establish inequality relations for each pair of fuzzy numbers. A fuzzy preference relation can then be defined on the entire set.

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Table 1.1. Summary of fuzzy ranking methods [4].

Method	α-cut	Possi.		Integral			Multi-index	Linguistic	[1]Func.	[2]DM
		possi.	max/min	integral	α-cut	max/min				
Adamo [5]	*								R	*
Chow and Chang [6]	*								R	*
Buckley and Chanas [7]	*								R	*
Nanda [8]	*								R	
Bass and Kwakernaak [9]		*							R,C	
Baldwin and Guild [10]		*							C	*
Hannan [11]		*							C	
Ovchinnikov [12]		*							R,C	*
Yager's F_2 [13]		*							C	
Watson *et al.* [14]		*							C	
Dubois and Prade [15]		*							C	*
Dubois and Prade [16]		*					*		R,C	*
Tsukamoto *et al.* [17]		*					*	*	C	*
Buckley [18]		*							C	*
Roubens and Vincke [19]		*							R,C	
Delgado *et al.* [20]		*					*		C	*
Buckley [21]		*							R	
Ammar [22]		*				*			R	
Jain [23, 24]		*	*						R	*
Chen [25]		*	*						R	*
Kim and Park [26]		*	*						R	*

*: indicates the heading. [1]Func.: type of function used by the method, comparison function (C) or ranking function (R). [2]DM: indicates that DM's participation in the decision of comparison is required or allowed.

Table 1.1. Summary of fuzzy ranking methods [4] (cont.).

Method	α-cut	Possi.		Integral			Multi-index	Linguistic	[1]Func.	[2]DM
		possi.	max/min	integral	α-cut	max/min				
Yager's F_1 [13,27,28]				*					R	*
Yager's F_4 [28]				*					R	
Chang [29]				*					R	
Murakami *et al.* [30]				*			*		R	*
Kaufmann [31]				*					R	
Lee and Li [32]				*					R,C	
Tseng and Klein [33]				*					C	
Yuan [34]				*				*	C	
Yager's F_3 [13]				*	*				R	
Mabuchi [35]				*	*			*	C	
Campos and Gonzalez [36,37]				*	*				R	*
Kerre [38]				*		*			R	
Nakamura [39]				*		*			R,C	
McCahon and Lee [40]				*		*			C	*
Hirota and Pedrycz [41]							*			
Efstathiou and Tong [42]								*		*

*: indicates the heading. [1]Func.: type of function used by the method, comparison function (C) or ranking function (R). [2]DM: indicates that DM's participation in the decision of comparison is required or allowed.

(c) Methods by integration. With this approach, an FRM essentially measures a fuzzy number, with or without weighting, by its mean value.
(d) Methods using multiple indices. With this approach, an FRM uses the results of multiple ranking or comparison functions as references to rank fuzzy numbers. Evidently, the developments of multiple-index methods are greatly motivated by the dilemma of occasionally inconsistent outcomes when different FRMs are used.
(e) Linguistic approach. The linguistic approach was developed mainly due to the desire to maintain the fuzzy characteristics of the problem.

A subclass of FRMs within the possibility approach can be established by using the concept of maximizing/minimizing set. Two subclasses, α-cut and fuzzy maximum/minimum, can also be established within the integration approach.

It should be noted that this classification is more based on the characteristics of the FRMs and one FRM could belong to several of the classifications. Most of the FRMs can be classified based on the first three classifications. The integration approach was essentially the probabilistic approach in [2].

Table 1.1 lists the fuzzy ranking methods reviewed in [4]. The type of functions used is also indicated for each FRM. A "comparison function" yields outcomes of comparisons of pairs of fuzzy sets. A "ranking function" yields outcomes of ranks of all the fuzzy sets. A ranking or comparison function is also known as the index. Another important aspect in using the FRMs is the participation of the decision maker. Whether the FRM allows or requires the participation of the decision maker is also indicated in the table.

Most of the methods listed in Table 1.1 seem to suffer from some drawbacks. For example, possibilistic methods frequently suffer from the lack of discrimination. Methods using α-cuts may suffer from the fact that not enough information is used and thus frequently produce biased results. Integration methods appear to be too rigid and tend to defuzzify the intrinsically fuzzy ratings.

In this paper, based on the concept of existence, an improved FRM is proposed. It is shown that several existing fuzzy ranking methods use the central idea of this concept. In developing this approach, the following aspects are emphasized:

(1) comparison of nonconvex fuzzy sets,
(2) comparison of nonnormal fuzzy sets,
(3) linguistic conclusion, and
(4) the subjectivity of decision-maker's opinion such as the optimism and the pessimism view points in the comparison of fuzzy sets.

Finally, in Section 7, two examples are provided to illustrate the approach.

2. THE MEMBERSHIP FUNCTION VERSUS THE UNIVERSE

In order to examine the problems encountered in the ranking of fuzzy numbers, let us look into the basic definition of a fuzzy subset. A fuzzy subset is defined based on the universe and the membership function. The degree of belonging to a certain fuzzy concept of a given element in the universe constitutes the membership function of this element. Thus, when several fuzzy sets are compared, there is a tendency to carry out this comparison based on both the degree of belonging, or the membership function, of the element and the location of the element in the universe. This tendency causes the confusion and, sometimes, causes the undesirable features of the FRM.

Figure 1 helps us to explore this concept further, where w is used to represent the grade of the membership function, and a and b represent the elements in the universe for fuzzy sets A and B, respectively. We conclude that fuzzy set $A <$ fuzzy set B, or $A < B$, if the comparison is carried out between $a(w)$ and $b(w)$, where $a(w)$ and $b(w)$ are based on the same degree of membership function w. This conclusion is possibly correct for the two entire fuzzy sets and is certainly correct at this degree of membership function w. But, on the other hand, if we compare the membership functions based on the same element of the universe, a $(= b)$, such as comparison between $w(a)$ and $w(b)$, a completely erroneous result would be obtained. Thus, fuzzy sets should be compared based on the same degree of membership or the same *existence level*, not based on the same element in the universe. Stated more explicitly: if two fuzzy sets are said to be unequal, then they must have different elements in the universe at least at some membership or existence level. Furthermore, different grades of existence or membership of a given element in the universe play a different role in the comparison of fuzzy sets.

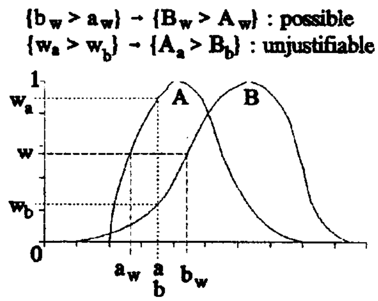

Figure 1.

The concept of existence plays the same role as the level of presumption introduced by Kaufmann and Gupta [43]. For computational purposes, these two concepts are numerically equal to the membership function.

The disadvantage of using these concepts is that we cannot use the concept of possibility. However, as it has been pointed out by various investigators [1,43], the concept of possibility does not take into account sufficiently the shape and the bounds of fuzzy numbers. In fact, many of the inconsistent and against intuition results when FRMs are used are due to the use of the possibility concept. For example, if we define the possibility law on the real line $\mathbb{R}$, as

$$\forall x \in \mathbb{R}: \quad h(x) \in [0,1] \quad \text{and} \quad \bigvee_{x \in \mathbb{R}} h(x) = 1. \tag{1}$$

If A is a fuzzy subset in $\mathbb{R}$, we call the possibility of A for the law $h(x)$

$$\text{poss}_H A = \bigvee_{x \in \mathbb{R}} \left(\mu_A(x) \wedge h(x)\right), \tag{2}$$

which has the meaning "as great as possible." Kaufmann and Gupta [43] compared the possibility law with the agreement index, which is essentially the intersection of $h(x)$ and $\mu_A(x)$ divided by the area of the fuzzy set A. Although the agreement index does not have the property of the possibility concept, from a practical viewpoint, this index is a fairly good one for the ranking of fuzzy numbers.

For example, Jain's method is essentially based on this concept of possibility and it has been shown [2] that this approach only considers the right hand of the membership function for a triangular fuzzy number. However, the advantage of this approach is that the possibility concept can be used, and thus the decision maker can specify whether more risk should be allowed for certain situations.

Even for methods based on the integration approach, the distinction between the existence level and the elements of the universe is frequently confused. For example, Yager's F_4 method, which is based on the closeness, in term of Hamming distance, between the fuzzy number and the fuzzy "truth value" was shown to produce counter intuitive results [3]. Kerre's method, which is also based on the Hamming distance between the fuzzy number and some goal, has shown to produce inconsistent results when three fuzzy numbers are compared [1]. Both the counter intuitive and the inconsistent results are due to the consideration of the area, which is obtained by considering both the membership function and the elements in the universe.

3. THE CONCEPT OF OVERALL EXISTENCE

For a given existence level w, the inverse image in terms of the membership function, $\mu(x)$, is

$$\mu^{-1}(w) = \{x : \mu(x) = w\}. \tag{3}$$

If the membership function of the fuzzy set A is convex and continuous, then $\{\mu_A^{-1}(w)\}$ always contains two groups of elements in A, for $0 < w < 1$, with $\{\mu_A^{-1}(w = 1)\}$ as the mode. These two groups of elements of $\{\mu^{-1}(w)\}$ are $x_{\max}^w = \max\{x : \mu_A(x) = w\}$ and $x_{\min}^w = \min\{x : \mu_A(x) = w\}$, for $w \in (0, 1]$, and defined as the right and left references, respectively.

Based on the discussions above and the concept of existence, we can formulate the following theorem.

THEOREM. *Consider two fuzzy sets A and B with any kind of membership functions. For any $w \in (0, 1]$, if A is said to be larger than B at w, then it must be that $\{\mu_A^{-1}(w)\} > \{\mu_B^{-1}(w)\}$.*

Notice that the inverse of the above theorem is not generally true. One problem with this approach is that it only considers one point, w, and thus, not enough information is used. The α-cuts approaches of Adamo [5], Buckley and Channas [7], and Nanda [8] are somewhat similar to this approach. For example, in Adamo's method, the concept of existence appears in an index as

$$F_\alpha(A_i) = \max\{x \mid \mu_{A_i}(x) \geq \alpha\}.$$

Graphically, this index is illustrated in Figure 2.

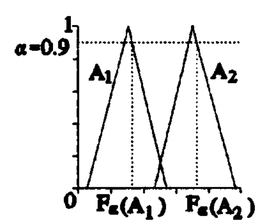

Figure 2. Adamo's method ($\alpha = 0.9$).

A most restrictive definition of $\{\mu_A^{-1}(w)\} > \{\mu_B^{-1}(w)\}$ has been used in Buckley-Chanas' method. Given the α-cuts of A and B, which can be represented by $[(a_{\rm L})^\alpha, (a_{\rm R})^\alpha]$ and $[(b_{\rm L})^\alpha, (b_{\rm R})^\alpha]$, respectively, then

$$A > B, \qquad \text{if} \quad (a_{\rm L})^\alpha > (b_{\rm R})^\alpha .$$

In Nanda's method, the concept of existence seems to appear in a more reasonable manner. Let $X = [x_{\rm L}, x_{\rm R}]$ and $Y = [y_{\rm L}, y_{\rm R}]$ be two closed bounded intervals on the real line $\mathbb{R}$;

$$X \leq Y \qquad \text{if} \quad x_{\rm L} \leq y_{\rm L} \quad \text{and} \quad x_{\rm R} \leq y_{\rm R}.$$

Consider two fuzzy numbers A and B with compact α-cuts A^α, B^α, respectively, then

$$A \leq B \qquad \text{if} \quad A^\alpha \leq B^\alpha \quad \text{for any} \quad \alpha \in [0, 1].$$

Tanaka *et al.* [44] also used a similar approach in modeling the "greater than or equal to" constraints in fuzzy linear programming.

In order to use all the information available, all the existence levels w must be considered. This overall existence index can be defined as follows:

$$I = \int_0^1 g\left(\{\mu_A^{-1}(w)\}\right) dw - \int_0^1 g\left(\{\mu_B^{-1}(w)\}\right) dw, \tag{4}$$

where μ_A and μ_B are the membership functions of the fuzzy sets A and B, respectively, and the function g is a function of the inverse of these membership functions. Based on this index, the following definition can be given.

DEFINITION 1. *Consider two normal fuzzy sets A and B; if A is said to be greater than B, then it must be that $I > \varepsilon$, where ε is a threshold ($\varepsilon \geq 0$) to be satisfied in order for $A > B$ to be true.*

Several existing FRMs, such as the methods of Yager F_3 [13], Kaufmann [31], and Campos and González [36,37] are somewhat similar to this overall existence concept.

Yager's F_3 index is defined as

$$F_3(A_i) = \int_0^{\alpha_{\max}} M\left(A_i^{\alpha}\right) d\alpha, \tag{5}$$

where $\alpha_{\max} = \text{hgt}(A_i)$ is the maximal membership grade in A_i, and $M(A_i^{\alpha})$ represents the mean value of the elements of the α-cut A_i^{α} of A_i. For continuous and convex fuzzy sets, the F_3 index can be obtained by using equation (4) with the function g defined as an averaging operation on all elements of $\left\{\mu_{A_i}^{-1}(w)\right\} = \{x : \mu_{A_i}(x) = w\}$,

$$g\left(\left\{\mu_{A_i}^{-1}(w)\right\}\right) = m\left(\left\{\mu_{A_i}^{-1}(w)\right\}\right).$$

The F_3 index is illustrated in Figure 3. However, if nonconvex fuzzy sets are involved, the Yager F_3 index and equation (4) may give different index values. As is shown in Figure 4, when the F_3 index is used, the elements of the fuzzy set are counted repeatedly for different levels of the α-cuts. This repeated counting can be avoided by the use of equation (4) with the function g defined as an averaging operation on $\left\{\mu_{A_i}^{-1}(w)\right\}$.

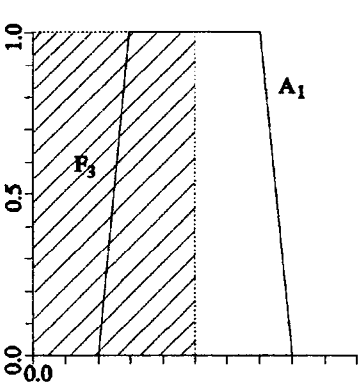

Figure 3. Yager's F_3 index.

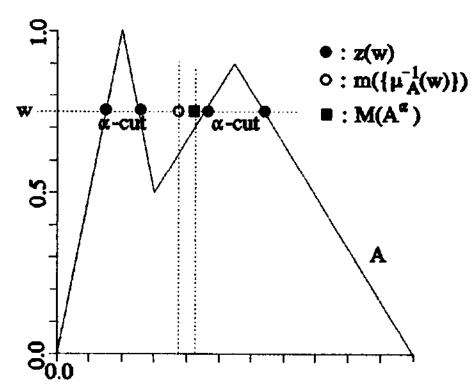

Figure 4.

Kaufmann's $d(A)$ index is defined as

$$d(A) = \frac{1}{2}\left[d_{\mathrm{L}}(A) + d_{\mathrm{R}}(A)\right] = \frac{1}{2}\left[\int_{\alpha=0}^{1} \left|a_{\alpha}^{(1)}(x)\right| dx + \int_{\alpha=0}^{1} \left|a_{\alpha}^{(2)}(x)\right| dx\right], \tag{6}$$

where $d_{\mathrm{L}}(A)$ and $d_{\mathrm{R}}(A)$ represent the Hamming distances from the left and from the right, respectively, of A to the origin. Clearly, Kaufmann's index can be obtained by using equation (4) with the function g defined as the average of $x_{\max}^{w}$ and $x_{\min}^{w}$. Graphically, $d_{\mathrm{L}}(A)$ and $d_{\mathrm{R}}(A)$ are illustrated in Figure 5. With continuous and convex fuzzy sets, Kaufmann's $d(A)$, equation (4), and Yager's F_3 should give the same results.

Campos and González also proposed an index, called the average value AV. It is restricted to convex and semi-continuous fuzzy numbers. If $A_{\alpha} = [a_{\alpha}, b_{\alpha}]$ is the α-cut of A, a closed and bounded real interval, and $Y \subseteq [0,1]$, then the AV of A is

$$V_s^{\lambda}(A) = \int_Y f_A^{\lambda}(\alpha)\, dS(\alpha), \qquad \text{where} \quad f_A^{\lambda}(\alpha) = \lambda\, b_{\alpha} + (1-\lambda)\, a_{\alpha}, \tag{7}$$

where λ is used to adjust the degree of optimism or pessimism and S is an additive measure on Y. When $S(\alpha) = \alpha$ and $dS(\alpha) = d\alpha$, the AV has the form

$$V_L^\lambda(A) = \int_Y [\lambda\, b_\alpha + (1-\lambda)\, a_\alpha]\, d\alpha. \tag{8}$$

For convex and semi-continuous fuzzy numbers, $V_L^\lambda(A)$ corresponds to equation (4) with the function g defined as a weighted average of $x_{\max}^w$ and $x_{\min}^w$. Index $V_L^\lambda(A)$ is illustrated in Figure 6. Different functions of S were also given in [37], such as

(1) $S(x) = x^r$, $r \in \mathbb{R}^+$,
(2) $S(x) = \ln(x+1)/\ln 2$, and
(3) $S(x) = (e^x - 1)/(e-1)$.

When $S(x) = x^r$ is used, the value of r has the effect of a concentration or a dilution operation.

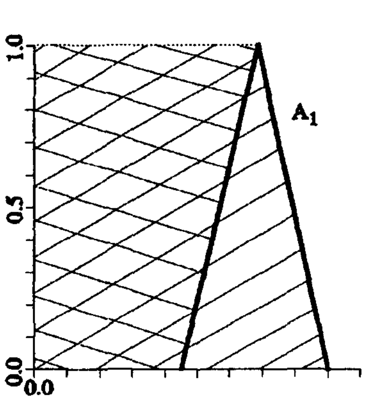

Figure 5. Kaufmann's $d(A)$ index.

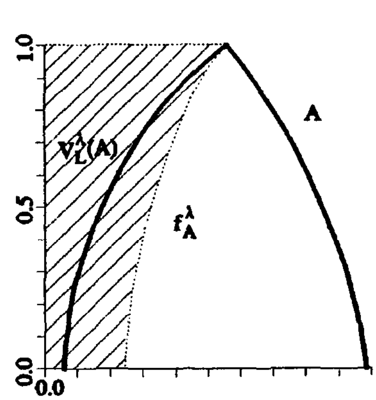

Figure 6. Campos and González's method: $V_L^\lambda(A)$.

As was shown above, various different forms for the function g in equation (4) can be proposed. A simple and useful form appears to be the weighted summation of the inverse of the membership function. The index can now be expressed as

$$I_1 = \int_0^1 \omega \left(\sum x(w)\right) dw - \int_0^1 \omega \left(\sum y(w)\right) dw, \tag{9}$$

where ω defines a weighting function of w; and $x(w)$ denotes the elements of $\{\mu_A^{-1}(w)\}$, $y(w)$ denotes that of $\{\mu_B^{-1}(w)\}$, and A and B are continuous, convex, and normal fuzzy sets.

Since it has been proven [28] that given two fuzzy numbers, A and B, if $C = A \ominus B$, then $C_\alpha = A_\alpha \ominus B_\alpha$, thus equation (9) can be rewritten as

$$I_1 = \int_0^1 \omega \left(\sum x(w) - \sum y(w)\right) dw = \int_0^1 \omega \left(\sum z(w)\right) dw, \tag{10}$$

where $\{\mu_C^{-1}(w)\} = \{z : \mu_C(z) = w\}$.

The method of Mabuchi [35] may be compared with the above index. Let D_{ij} denote the fuzzy difference set of A_i and A_j and $(D_{ij})^\alpha = [z_{\mathrm{L}}(\alpha), z_{\mathrm{R}}(\alpha)]$ denote the α-cut of D_{ij}. Let $m^+(D_{ij})^\alpha$ and $m^-(D_{ij})^\alpha$ be the length of positive and negative regions, respectively, of $[z_{\mathrm{L}}(\alpha), z_{\mathrm{R}}(\alpha)]$. Mabuchi defines the function $J_{ij}(\alpha)$ as

$$J_{ij}(\alpha) \triangleq \frac{m^+(D_{ij})^\alpha - m^-(D_{ij})^\alpha}{m^+(D_{ij})^\alpha + m^-(D_{ij})^\alpha}, \tag{11}$$

and $J_{ij}(\alpha) \in [-1,1]$ is interpreted as the degree of dominance of A_i over A_j at the level of α. $J_{ij}(\alpha)$ is zero if the length of the interval $([z_{\mathrm{L}_0}(\alpha),\, z_{\mathrm{R}}(\alpha)])$ is zero. Based on $J_{ij}(\alpha)$, the index of comparison J^0_{ij} between A_i and A_j is defined as

$$J^0_{ij}(\alpha) = 2\int_0^{\mathrm{hgt}(D_{ij})} \alpha\, J_{ij}(\alpha)\, d\alpha \tag{12}$$

and $J^0_{ij} \in [-1,1]$.

If the fuzzy sets are continuous and convex, the index I and J^0_{ij} are essentially equivalent, since $\{\mu_C^{-1}(w)\}$ contains only two points, and $m^+(D_{ij})^\alpha$ and $m^-(D_{ij})^\alpha$ measure the positive and negative portions of the line between these two points. The denominator of $J_{ij}(\alpha)$ is just a normalization factor. Moreover, it can be seen that in this case, I_1 and Mabuchi's index, as well as Campos and González's AV index are essentially equivalent, when the AV index is used with $S(\alpha) = \frac{1}{2}\alpha^2$ and $\lambda = \frac{1}{2}$.

However, when nonconvex fuzzy sets are involved, the results of the indexes J^0_{ij} and I, may be different. An example of this difference is shown in Figure 7, where D_{ij} represents a nonconvex fuzzy difference set.

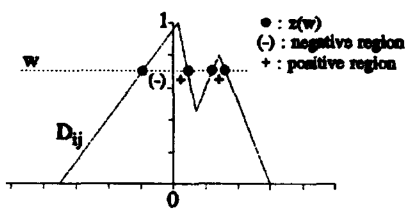

Figure 7.

Yuan [34] also proposed a method related to the above discussion. This method improves the method of Nakamura [39] by using the fuzzy difference set $(A - B)$ and a crisp 0 set instead of the two fuzzy sets A and B originally used in Nakamura's method. In order to introduce Yuan's method, let first recall the method proposed by Nakamura.

Nakamura's method uses a fuzzy comparison function to produce fuzzy preference relations. This fuzzy preference relation, denoted as P, between two fuzzy sets is defined as

$$\mu_P(A,B) = \begin{cases} \dfrac{1}{\Delta_\alpha}\left[\alpha D\left(A_{\mathrm{R}},\, \widetilde{\min}(A_{\mathrm{R}},B_{\mathrm{R}})\right) + (1-\alpha)\, D\left(A_{\mathrm{L}},\, \widetilde{\min}(A_{\mathrm{L}},B_{\mathrm{L}})\right)\right], & \text{if } \Delta_\alpha \neq 0, \\ \dfrac{1}{2}, & \text{if } \Delta_\alpha = 0, \end{cases}$$

for $\alpha \in [0,1]$, where

$$\Delta_\alpha = \alpha\left[D\left(A_{\mathrm{R}},\, \widetilde{\min}(A_{\mathrm{R}},B_{\mathrm{R}})\right) + D\left(B_{\mathrm{R}},\, \widetilde{\min}(A_{\mathrm{R}},B_{\mathrm{R}})\right)\right] + (1-\alpha)\left[D\left(A_{\mathrm{L}},\, \widetilde{\min}(A_{\mathrm{L}},B_{\mathrm{L}})\right) + D\left(B_{\mathrm{L}},\, \widetilde{\min}(A_{\mathrm{L}},B_{\mathrm{L}})\right)\right];$$

D represents the Hamming distance defined as $D(A,B) = \int_S |\mu_\lambda(x) - \mu_B = (x)|\, dx$, and

$$\mu_{A_{\mathrm{L}}}(r) = \sup_{(x)|x\geq r} \mu_A(x) \quad \mu_{A_{\mathrm{R}}}(r) = \sup_{(x)|x\leq r} \mu_A(x) \qquad \forall r \in \mathbb{R}. \tag{13}$$

When $A \succ$ (strictly dominates) B, we should have $\mu_P(A,B) = 1$. On the other hand, if $A \succeq$ (weakly dominates) B, then $\mu_P(A,B) \geq \frac{1}{2}$.

Yuan improved Nakamura's method in the following manner. In addition to the fuzzy difference set $(A_1 - A_j)$, the real number 0 is defined with a membership function,

$$\mu_{Z_0}(z) = \begin{cases} 1, & z = 0, \\ 0, & z \neq 0. \end{cases}$$

Using $(A_i - A_j)$ and Z_0, a fuzzy preference relation between A_i and A_j, denoted as $Q(A_i, A_j)$, is defined with the membership function

$$\mu_Q(A_i, A_j) = \mu_P(A_i - A_j, Z_0),$$

which represents the degree of preference of A_i over A_j; $\mu_P(A_i - A_j, Z_0)$ is the preferability of $(A_i - A_j)$ over Z_0 in Nakamura's method, with $\alpha = 0.5$.

4. NORMAL AND NONNORMAL FUZZY SETS

For nonnormal fuzzy sets, the level of existence is less than one and thus direct comparison between normal and nonnormal fuzzy sets cannot be made. One way to overcome this difficulty is to normalize the nonnormal fuzzy sets. However, this is often undesirable because the original meaning of the problem is lost. Since equation (4), which is the definition of the index for overall existence, is based on fuzzy difference, a more reasonable approach can be obtained by examining the definition of the fuzzy difference based on the extension principle. Figure 8 illustrates the similarity of the results when fuzzy differences are obtained between two nonnormal fuzzy sets and between one normal and one nonnormal fuzzy sets. This has been called "top flattening" by Dubois and Prade [15].

Based on the concept of top flattening, the overall index of existence can be modified for nonnormal fuzzy sets as follows: let $w^*_{\text{hgt}} = \min[\text{hgt}(A), \text{hgt}(B)]$. Suppose $\text{hgt}(A) > \text{hgt}(B)$ and A is top-flattened to be A^T, then

$$I_{\text{nonnorm}} = \int_0^{w^*_{\text{hgt}}} g\left(\{\mu^{-1}_{A^T}(w)\}\right) dw - \int_0^{w^*_{\text{hgt}}} g\left(\{\mu^{-1}_{B}(w)\}\right) dw. \tag{14}$$

Fuzzy maximum or fuzzy minimum based on the extension principle has the same top flattening effect. This is illustrated in Figure 9 for fuzzy maximum. Similar results can be obtained for fuzzy minimum. Thus, fuzzy maximum or fuzzy minimum based on the extension principle also only consider the smallest maximum membership grade of the fuzzy sets. Fuzzy maximum or fuzzy minimum can be incorporated into FRMs. Indirect and successful applications of the fuzzy maximum or fuzzy minimum in FRMs appeared in [22,38–40]. The reader is referred to the literature for more detail [1–4].

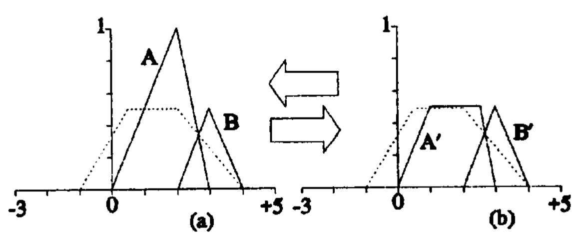

Figure 8. Fuzzy difference.

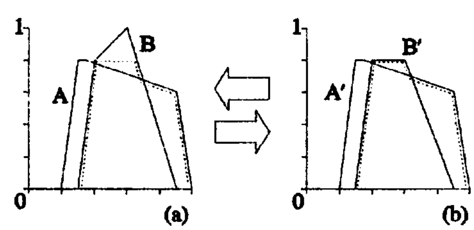

Figure 9. Fuzzy maximum.

Since nonnormalities of fuzzy sets are frequently due to the lack of information at the current moment, and more information may become available later, the comparison results should be treated tentatively and the nonnormalities of the fuzzy sets should be reported to the DM. Mabuchi [35] has proposed a linguistic procedure to take this tentative aspect into consideration. The information of nonnormality in fuzzy sets compared is reported linguistically, based on the height of the fuzzy difference set. Four linguistic terms were used:

(a) the height h nearly 1—credible,
(b) h over 0.5—moderate credible,
(c) h under 0.5—little credible, and (d) h nearly 0—not credible.

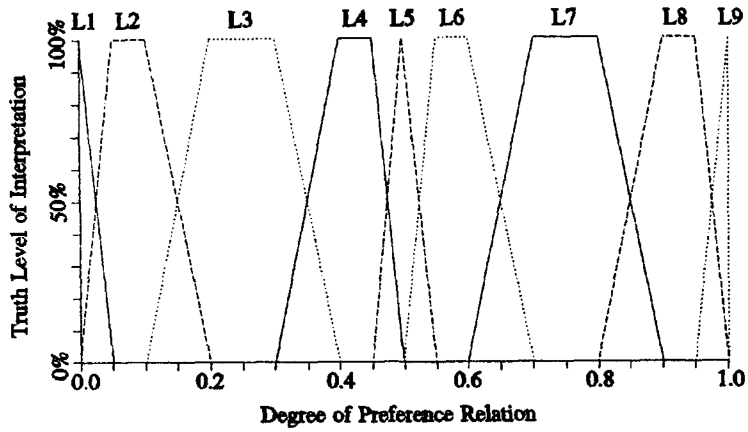

Figure 10. An illustration of linguistic interpretation of $u_Q(A_i, A_j)$ in Yuan's method. L1: A_j absolutely better than A_i. L2: A_j strongly better than A_i. L3: A_j moderately better than A_i. L4: A_j weakly better than A_i. L5: Indifference between A_i and A_j. L6: A_i weakly better than A_j. L7: A_i moderately better than A_j. L8: A_i strongly better than A_j. L9: A_i absolutely better than A_j.

5. LINGUISTIC APPROACHES

Linguistic approaches avoid the tendency to defuzzify an intrinsically fuzzy rating and thus are preferred from the standpoint of retaining the fuzzy nature. Several linguistic procedures have been proposed. The procedures of Mabuchi [35] and Yuan [34] are briefly summarized in the following. Yuan's approach appears to be closely related to the suggestions of Freeling [45]. The linguistic interpretation of Yuan of a fuzzy preference relation between two fuzzy sets is shown in Figure 10. If a fuzzy preference relation, $Q(A_i, A_j) = 0.5375$, then it may be concluded that A_i is weakly better than A_j with truth level 75% and A_i is indifferent from A_j with truth level 25%.

In Mabuchi's approach, a linguistic conclusion is drawn from the curve of α vs. $J_{ij}(\alpha)$ and considers the following two factors:

(a) The range or distribution of the curve over the J_{ij} axis is used to indicate the complication of the conclusion:
   (1) impulse type—no complication;
   (2) one sided—slight complication;
   (3) one sided, narrow—very slight complication;
   (4) two sided but biased to one side—moderate complication; and
   (5) widely or equally distributed on both sides—much complication.

(b) The average position is used to indicate dominance: if the average position is
   (1) nearly 1—definite dominance;
   (2) over 0.5—strong dominance;
   (3) about 0.5—moderate dominance;
   (4) under 0.5—slight dominance; and
   (5) nearly 0—no dominance.

For example, if the curve of $\alpha - J_{ij}(\alpha)$ of the fuzzy difference of the two fuzzy sets A_i and A_j is widely and nearly equally distributed on both sides of the origin on the J_{ij} axis, and its average position is about 0.5, then it is concluded that A_i is moderately dominant to A_j with much complication.

One of the problems of Mabuchi's approach is that the results are difficult to interpret. In fact, even determining a linguistic description based on factors (a) and (b) is sometimes difficult. Furthermore, even if such a linguistic description is obtained, the interpretation may be still too vague to be of any use.

Yuan's index $Q(A_i, A_j)$, which is a fuzzy preference relation in [0, 1], can be modified easily for use for the index defined in equation (4). This modification is shown in Figure 11. Linguistic

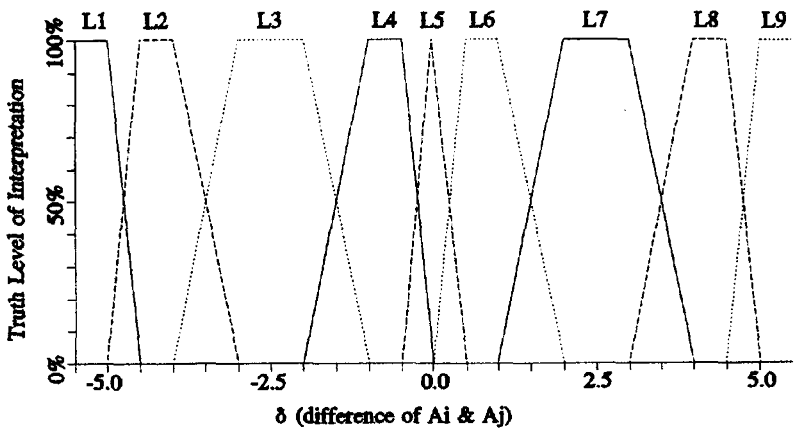

Figure 11. An illustration of linguistic interpretation of $\mathrm{CL}(A_i, A_j)$ for equation (4), L1: A_j absolutely better than A_i. L2: A_j strongly better than A_i. L3: A_j moderately better than A_i. L4: A_j weakly better than A_i. L5: Indifference between A_i and A_j. L6: A_i weakly better than A_j. L7: A_i moderately better than A_j. L8: A_i strongly better than A_j. L9: A_i absolutely better than A_j.

preference relations can now be obtained by the combined use of Figure 11 and the index defined by equation (4).

6. A FUZZY RANKING METHOD WITH DECISION MAKER'S PARTICIPATION

One important aspect of a fuzzy set is its subjectivity. This is especially true when very similar fuzzy sets need to be ranked. In fact, from a purely mathematical or objective viewpoint, there is no difference between the fuzzy numbers A and B in Figure 12 or 13, provided that the degree of fuzziness of a fuzzy number is not considered. However, from a practical or subjective standpoint, the decision maker may feel some differences from his intuition or past experiences. The FRM index to be introduced emphasizes this subjective viewpoint with decision maker's participation.

DEFINITION 2. *Let A and B be two fuzzy sets with membership functions $\mu_A(x)$ and $\mu_B(y)$, respectively, and let $\{\mu_A^{-1}(w)\}$ and $\{\mu_B^{-1}(w)\}$ be two ordinary subsets, denoting the inverse images of the membership functions with $w \in (0,1]$, i.e.,*

$$\{\mu_A^{-1}(w)\} = \{x : \mu_A(x) = w\} \quad \text{and} \quad \{\mu_B^{-1}(w)\} = \{y : \mu_B(y) = w\},$$

with $x,\, y \in \mathbb{R}$. The difference between A and B is defined as

$$d(A,B) = \int_0^{w^*_{\text{hgt}}} g_A\left(\{\mu_A^{-1}(w)\}\right)\, dw - \int_0^{w^*_{\text{hgt}}} g_B\left(\{\mu_B^{-1}(w)\}\right)\, dw,$$
$$w^*_{\text{hgt}} = \min v\left[\text{hgt}(A), \text{hgt}(B)\right], \qquad (15)$$

where the function g can be defined as follows.

Let $x_i''(w) = \max\{x_i : \mu_{A_i}(x_i) = w\}$ and $x_i'(w) = \min\{x_i : \mu_{A_i}(x_i) = w\}$, then:

$$g_i\left(\{\mu_{A_i}^{-1}(w)\}\right) = \omega(w)\,\left[\chi_1(w)\, x_i'(w) + \chi_2(w)\, x_i''(w)\right].$$

Definition 2 can be expanded for n fuzzy sets A_i ($i = 1, \dots, n$) and obtains individual measurements for each fuzzy set.

For notational convenience, let us define:

$$\text{OM}(A_i) = \int_0^{w^*_{\text{hgt}}} g_i\left(\{\mu_A^{-1}(w)\}\right)\, dw, \qquad \text{where} \quad w^*_{\text{hgt}} = \min_i\left[\text{hgt}(A_i)\right]. \qquad (16)$$

It should be emphasized that Definition 2 applies to all kinds of membership functions such as convex, nonconvex, normal, nonnormal, continuous, or piecewise-continuous.

The above definition can be simplified for the L-R type of fuzzy numbers [15]. Let the left side of a L-R fuzzy number A be denoted as A_{L} and the right side as A_{R}, then

$$x'(w) = \mu_{A_{\mathrm{L}}}^{-1}(w) \quad \text{and} \quad x''(w) = \mu_{A_{\mathrm{R}}}^{-1}(w).$$

That is, $x'(w)$ and $x''(w)$ are the inverse images of the left reference (L) and the right reference (R), respectively, of the membership function of A. Definition 2 now becomes:

$$\mathrm{OM}(A) = \int_0^1 \omega(w) \left[\chi_1(w)\, \mu_{A_{\mathrm{L}}}^{-1}(w) + \chi_2(w)\, \mu_{A_{\mathrm{R}}}^{-1}(w)\right] dw. \tag{17}$$

The weighting measures $\omega(w)$ and $\chi_1(w)$, $\chi_2(w)$ must be determined subjectively by the decision maker. These measures are used to emphasize different aspects at different existence levels and they are examined in detail in the following.

Let us consider the weighting measure $\omega(w)$ first. This measure can be used as a pure existence-level weighting function and is defined as follows:

$$\omega(w) = \frac{w}{\frac{1}{2}\left(w_{\mathrm{hgt}}^*\right)^2}, \qquad \text{with} \quad \frac{1}{2}\left(w_{\mathrm{hgt}}^*\right)^2 = \int_0^{w_{\mathrm{hgt}}^*} w\, dw. \tag{18}$$

We shall call it the *pure $\omega(w)$-weighting*, which is equivalent to the α-weighting in the Mabuchi's method and $S(\alpha) = \frac{1}{2}\alpha^2$ in the AV index of Campos and González. Alternatively, different weights can be added to different levels of existence. For example, the DM may decide that the elements at the middle existence levels should weigh more. Therefore, $\omega(w)$ can be defined as

$$\omega(w) = \frac{w\,(1-w)}{\Lambda^*}, \qquad \text{where} \quad \Lambda^* = \int_0^{w_{\mathrm{hgt}}^*} w\,(1-w)\, dw. \tag{19}$$

Apparently, this *subjective $\omega(w)$-weighting* reflects a more conservative attitude of the DM by weighing less the elements at higher and lower existence levels and weighing more at middle levels. Other subjective $\omega(w)$-weighting may also be defined, such as $\omega(w) = w(w^r)/\Lambda*$, $r \in \mathbb{R}$.

The subjective weighting measures of $\chi_1(w)$ and $\chi_2(w)$, which normally require $\chi_1(w) + \chi_2(w) = 1$, can play even more important roles in the decision maker's preference. To illustrate these situations, let us look at some examples. Consider the situation illustrated in Figure 12. If the degree of fuzziness is not considered and a simple comparison method is used, $A_1 = A_2$ may be concluded. However, the comparison between A_1 and A_2 in Figure 12 could be very controversial and subtle if one considers the actual situation of the problem and the requirements of the subjective DM. For example, for a maximizing decision, when the DM is very optimistic, A_1 should be chosen, since A_1 contains larger elements. On the other hand, if the DM is very pessimistic and is always looking for the worst situation, A_2 should be chosen since its smallest values are larger than that of A_1. Similar situations apply to Figures 13 and 14.

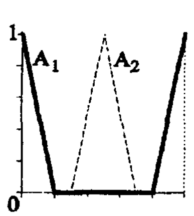

Figure 12.

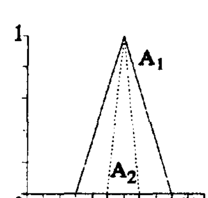

Figure 13.

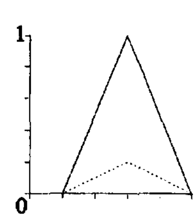

Figure 14.

Different functions for $\chi_1(w)$ and $\chi_2(w)$ can be conceived for different situations. The simplest functions are:

$$\chi_1(w) = \chi_1, \quad \chi_2(w) = \chi_2, \qquad \chi_1,\ \chi_2 \in [0,1] \quad \text{and} \quad \chi_1 + \chi_2 = 1,$$

which will be called the *fixed-$\chi(w)$ type weighting*. This type of weighting can be optimism or pessimism depending upon whether the decision is to maximize or to minimize. Figure 15 illustrates three fixed-$\chi(w)$ type weightings for $\chi_1 = 0.8$, 0.5, and 0.2. If $\chi_1 = \chi_2 = 0.5$, it will be called *indifference fixed-$\chi(w)$ weighting*. On the other hand, if $\chi_1 \neq 0.5$, it is called *difference fixed-$\chi(w)$ weighting*.

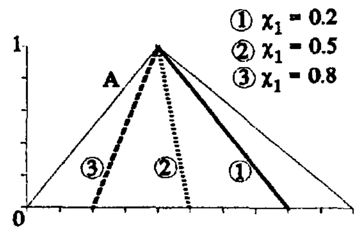

Figure 15. Illustrations of the fixed-χ weighting: $[\chi_1\, x'(w) + \chi_2\, x''(w)]$.

A more realistic type of $\chi(w)$ weighting is the *linear-$\chi(w)$ weighting* defined below. Let $\chi_1(w)$ and $\chi_2(w)$ be (piecewise) linear functions of w and $\chi_1(w) + \chi_2(w) = 1$, for all $w \in (0,1]$. Assume that there exists a w_e^f $(0 < w_e^f \le w_{\text{hgt}}^*)$ such that for $w \in (w_e^f, w_{\text{hgt}}^*]$, $\chi_1(w)$ (likewise $\chi_2(w)$) is linearly increasing or decreasing with w. For $w \in (0, w_e^f]$, $\chi_1(w)$ and $\chi_2(w)$ constitute a fixed-$\chi(w)$ weighting, i.e., $\chi_1(w) = \chi_1^e$, $\chi_2(w) = \chi_2^e$, $\chi_1^e + \chi_2^e = 1$. The linear-$\chi(w)$ weighting can be defined as follows.

(i) If $\chi_1(w_{\text{hgt}}^*) = \chi_1^s > \chi_1^e$, or, $\chi_1(w)$ is an increasing function of w from w_e^f to w_{hgt}^*; therefore, $\chi_2(w)$ is a decreasing function of w during the same interval; we have

$$\chi_1(w) = \begin{cases} \chi_1^e & \text{for} \quad 0 < w \le w_e^f, \\ \chi_1^s - \dfrac{(\chi_1^s - \chi_1^e)\left(w_{\text{hgt}}^* - w\right)}{\left(w_{\text{hgt}}^* - w_e^f\right)}, & \text{for} \quad w_e^f < w \le w_{\text{hgt}}^*, \end{cases} \tag{20}$$

and $\chi_2(w) = 1 - \chi_1(w)$.

(ii) If $\chi_1(w_{\text{hgt}}^*) = \chi_1^s < \chi_1^e$, or, $\chi_1(w)$ is a decreasing function of w from w_e^f to w_{hgt}^*; therefore, $\chi_2(w)$ is an increasing function of w during the same interval; we have

$$\chi_1(w) = \begin{cases} \chi_1^e, & \text{for} \quad 0 < w \le w_e^f, \\ \chi_1^s + \dfrac{(\chi_1^e - \chi_1^s)\left(w_{\text{hgt}}^* - w\right)}{\left(w_{\text{hgt}}^* - w_e^f\right)}, & \text{for} \quad w_e^f < w \le w_{\text{hgt}}^*, \end{cases} \tag{21}$$

and $\chi_2(w) = 1 - \chi_1(w)$.

(iii) If $\chi_1\left(w_{\text{hgt}}^*\right) = \chi_1^s = \chi_1^e$ or $w_e^f = w_{\text{hgt}}^*$, then $\chi_1(w) = \chi_1^e$ and $\chi_2(w) = 1 - \chi_1^e = \chi_2^e$ for all $w \in \left(0, w_{\text{hgt}}^*\right]$.

Obviously, the fixed-$\chi(w)$ type weighting is a special case of the linear-$\chi(w)$ type weighting.

Similarly, *nonlinear-$\chi(w)$ type weighting* can also be defined. One simple nonlinear-$\chi(w)$ type weighting can be obtained directly from the above definitions of the linear-$\chi(w)$ type weighting as follows:

(i) If

$$\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) > \chi_1^e,$$

$$\chi_1(w) = \chi_1^s - \frac{(\chi_1^s - \chi_1^e)\left(w_{\text{hgt}}^* - w\right)^r}{\left(w_{\text{hgt}}^* - w_e^f\right)^r}, \qquad r \in \mathbb{R}^+, \quad \text{for} \quad w_e^f < w \le w_{\text{hgt}}^*; \tag{22}$$

(ii) if

$$\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) < \chi_1^e,$$
$$\chi_1(w) = \chi_1^s + \frac{\left(\chi_1^e - \chi_1^s\right)\left(w_{\text{hgt}}^* - w\right)^r}{\left(w_{\text{hgt}}^* - w_e^f\right)^r}, \qquad r \in \mathbb{R}^+, \quad \text{for} \quad w_e^f < w \le w_{\text{hgt}}^*. \tag{23}$$

To illustrate the meaning of these weighting functions, let us consider the following four typical examples for the linear-$\chi(w)$ case.

(1) $\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) \ll \frac{1}{2}$, $w_e^f = 0$, and $\chi_1^e = \frac{1}{2}$. From the definition, we know that $\chi_2^s \gg \frac{1}{2}$. In this case, $\chi(w)$ weighs very heavily for $x_i''(w)$ at w_{hgt}^* and gradually decreases to the indifference level of $\chi_1^e = \chi_2^e = \frac{1}{2}$ at $(w)_e^f = 0$. This linear weighting case is illustrated in Figure 16, marked with (1). Applying this weighting to the fuzzy set A shown in Figure 15, the curve marked (1) in Figure 17 is obtained. Note that even though the $\chi(w)$ weighting is linear, the outcome of $[\chi_1(w)\, x_i'(w) + \chi_2(w)\, x_i''(w)]$ is nonlinear.

(2) $\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) \ll \frac{1}{2}, \chi_1^e = \frac{1}{2}$, and $w_e^f > 0$. From the definition, we know that $\chi_2^s \gg \frac{1}{2}$. This is a modification of Case (1) and is illustrated in Figures 16 and 17, marked with (2). The values of $\chi_1^s = 0.9$, $w_e^f = \frac{1}{2}$, and $\chi_1^e = \frac{1}{2}$ are used in Figure 17.

(3) $\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) < \frac{1}{2}$, $w_e^f = 0$, and $\chi_1^s < \chi_1^e < \frac{1}{2}$. By definition, we must have $\chi_2^s = \chi_2(w_{\text{hgt}}^*) > \frac{1}{2}$ and $\chi_2^s > \chi_2^e > \frac{1}{2}$. The functions $\chi_1(w), \chi_2(w)$ for this case are shown in Figure 16 and marked as (3). The outcome of the construction for this case is shown in Figure 17 marked as (3) with $\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) = 1$, $w_e^f = 0$, and $\chi_1^e = 0.7$.

(4) $\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) < \frac{1}{2}$, $\chi_1^s < \chi_1^e < \frac{1}{2}$, and $w_e^f > 0$. By definition, we must have $\chi_2^s = \chi_2\left(w_{\text{hgt}}^*\right) > \frac{1}{2}$ and $\chi_2^s > \chi_2^e > \frac{1}{2}$.

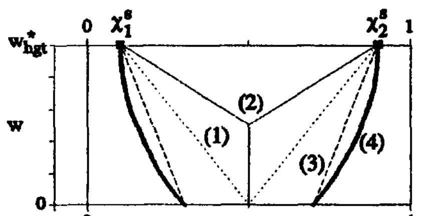

Figure 16. Illustrations of typical (non-) linear-χ weighting functions. (1), (2), (3): linear-χ; (4): nonlinear-χ.

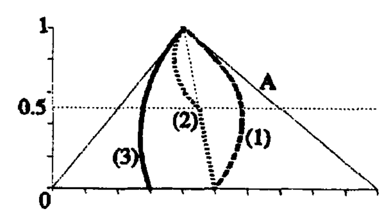

Figure 17. Illustrations of linear-χ weighting: $[\chi_1(w)\, x'(w) + \chi_2(w)\, x''(w)]$.

An example for the functions of nonlinear-$\chi(w)$ weighting is also shown in Figure 16, marked as (4), which corresponds with Case (3) of the linear-$\chi(w)$ weighting with $r = 2$.

7. NUMERICAL EXAMPLES WITH EMPHASIS ON LINGUISTIC CONCLUSIONS

Two examples, which possess certain subtleness and controversy, are solved to illustrate the approach. Emphasis is placed on the subjective weighting of the decision maker and the resulting linguistic conclusions. The first example is a comparison between two normal continuous convex fuzzy numbers and the second example is a comparison between convex and nonconvex, and normal and nonnormal fuzzy sets. In order to construct meaningful linguistic conclusions, an improved method for reaching a final linguistic interpretation is introduced.

EXAMPLE 1. Consider the comparison of two fuzzy numbers A_1 and A_2, illustrated in Figure 18. The membership functions of these two fuzzy numbers are

$$\mu_1(x) = \begin{cases} x-5, & \text{for } 5 \le x \le 6, \\ 7-x, & \text{for } 6 \le x \le 7, \\ 0, & \text{elsewhere}, \end{cases} \qquad \mu_2(x) = \begin{cases} \frac{x}{7}, & \text{for } 0 \le x \le 7, \\ 8-x, & \text{for } 7 \le x \le 8, \\ 0, & \text{elsewhere}. \end{cases}$$

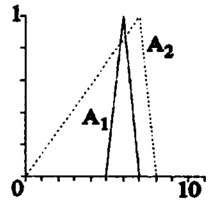

Figure 18.

Therefore, the inverse images of $\mu_1(x)$ and $\mu_2(x)$ for $0 < w \le 1$ are

$$\{\mu_1^{-1}(w)\} = \{(5+w), (7-w)\} \quad \text{and} \quad x_1'(w) = 5+w, \quad x_1''(w) = 7-w, \quad \text{for } A_1;$$
$$\{\mu_2^{-1}(w)\} = \{(7w), (8-w)\} \quad \text{and} \quad x_2'(w) = 7w, \quad x_2''(w) = 8-w, \quad \text{for } A_2.$$

(1) Indifference fixed-$\chi(w)$ weighting with $\omega(w) = w$, $\chi_1(w) = \chi_2(w) = 0.5$, for $0 < w \le 1$ and $w_{\text{hgt}}^* = 1$;

$$\text{OM}(A_1) = \frac{\int_0^1 w\,\{0.5\,(5+w) + 0.5\,(7-w)\}\,dw}{\frac{1}{2}\left(w_{\text{hgt}}^*\right)^2} = 6,$$
$$\text{OM}(A_2) = \frac{\int_0^1 w\,\{0.5\,(7w) + 0.5\,(8-w)\}\,dw}{\frac{1}{2}\left(w_{\text{hgt}}^*\right)^2} = 6, \qquad \text{and}$$
$$d(A_1, A_2) = 0.$$

(2) Difference fixed-$\chi(w)$ weightings with $\omega(w) = w, w_{\text{hgt}}^* = 1$. The results are summarized in Table 7.1. The calculation of one case is illustrated in the following. For $\chi_1 = 0.9$, $\chi_2 = 0.1$, we have

$$\text{OM}(A_1) = \frac{\int_0^1 w\,\{0.9\,(5+w) + 0.1\,(7-w)\}\,dw}{\frac{1}{2}\left(w_{\text{hgt}}^*\right)^2} = 5.73,$$
$$\text{OM}(A_2) = \frac{\int_0^1 w\,\{0.9\,(7w) + 0.1\,(8-w)\}\,dw}{\frac{1}{2}\left(w_{\text{hgt}}^*\right)^2} = 4.93, \qquad \text{and}$$
$$d(A_1, A_2) = 0.8.$$

(3) Linear-$\chi(w)$ weightings with $\omega(w) = w$, $w_{\text{hgt}}^* = 1$. The results for this case are again summarized in Table 7.1. The calculation of one case is illustrated in the following. For

$\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) = 0.9$, $w_e^f = 0$, and $\chi_1^e = 0.5$, we have

$$\chi_1(w) = \chi_1^s - \frac{(\chi_1^s - \chi_1^e)\,(1-w)}{\left(1 - w_e^f\right)} = 0.9 - \frac{(0.9-0.5)\,(1-w)}{(1-0)} = 0.5 + 0.4\,w,$$

$$\chi_2(w) = 1 - \chi_1(w) = 0.5 - 0.4\,w,$$

$$\text{OM}(A_1) = \frac{\int_0^1 w\ \{(0.5+0.4\,w)\,(5+w) + (0.5-0.4\,w)(7-w)\}\ dw}{\frac{1}{2}(w_{\text{hgt}}^*)^2} = 5.87,$$

$$\text{OM}(A_2) = \frac{\int_0^1 w\ \{(0.5+0.4\,w)\,(7w) + (0.5-0.4\,w)\,(8-w)\}\ dw}{\frac{1}{2}\left(w_{\text{hgt}}^*\right)^2} = 5.47, \quad \text{and}$$

$$d(A_1, A_2) = 0.4.$$

(4) Nonlinear-$\chi(w)$ weighting with $\omega(w) = w$, $r = 2$, $w_{\text{hgt}}^* = 1$. For $\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) = 0.9$, $w_e^f = 0$, and $\chi_1^e = 0.5$, we have

$$\chi_1(w) = \chi_1^s - \frac{(\chi_1^s - \chi_1^e)\,(1-w)^2}{\left(1 - w_e^f\right)^2} = 0.9 - \frac{(0.9-0.5)\,(1-w)^2}{(1-0)^2} = 0.5 + 0.8\,w - 0.4\,w^2,$$

$$\chi_2(w) = 1 - \chi_1(w) = 0.5 - 0.8\,w + 0.4\,w^2,$$

$$\text{OM}(A_1) = \frac{\int_0^1 w\ \left\{\left(0.5+0.8\,w-0.4\,w^2\right)\,(5+w) + \left(0.5-0.8\,w+0.4\,w^2\right)(7-w)\right\}\ dw}{\frac{1}{2}(w_{\text{hgt}}^*)^2} = 5.81,$$

$$\text{OM}(A_2) = \frac{\int_0^1 w \left\{\left(0.5+0.8\,w-0.4\,w^2\right)\,(7w) + \left(0.5-0.8w+0.4\,w^2\right)\,(8-w)\right\} dw}{\frac{1}{2}\left(w_{\text{hgt}}^*\right)^2} = 5.25, \text{ and}$$

$$d(A_1, A_2) = 0.56.$$

The other results are summarized in Table 7.1.

Table 7.1. Summary of results for Example 1.

		[1]Fixed type weighting$-\chi$			[2] Linear type weighting$-\chi$			[3]Nonlinear type weighting$-\chi$		
χ_1^s	χ_2^s	OM(A_1)	OM(A_2)	$d(A_1,A_2)$	OM(A_1)	OM(A_2)	$d(A_1,A_2)$	OM(A_1)	OM(A_2)	$d(A_1,A_2)$
0.9	0.1	5.73	4.93	0.8	5.87	5.47	0.4	5.81	5.25	0.56
0.7	0.3	5.87	5.47	0.4	5.93	5.73	0.2	5.91	5.63	0.28
0.5	0.5	6.0	6.0	0.0						
0.3	0.7	6.13	6.53	−0.4	6.07	6.27	−0.2	6.09	6.37	−0.28
0.1	0.9	6.27	7.07	−0.8	6.13	6.53	−0.4	6.19	6.75	−0.56

[1]Fixed-χ type weighting: $w_e^f = w_{\text{hgt}}^*$. [2]Linear-χ type weighting: $w_e^f = 0$, $\chi_1^e = \chi_2^e = \frac{1}{2}$.

[3]Nonlinear-χ type weighting: $r = 2$, $w_e^f = 0$, $\chi_1^e = \chi_2^e = \frac{1}{2}$.

Obviously, the results for Example 1, which are summarized in Table 7.1, need some explanation. In the following, we shall draw some linguistic conclusions from Table 7.1, based on the subjective view point of the decision maker and for a maximizing decision problem. Since it is a maximizing situation, the degree of optimism refers to the degree of emphasis placed on the right-hand side of the fuzzy numbers, and the degree of pessimism refers to the left-hand side.

(1) When the indifference fixed-$\chi(w)$ weighting ($\chi_1^s = \chi_2^s = 0.5$) is assumed, the numerical results give an outcome of indifference between the two fuzzy numbers, i.e., $A_1 = A_2$. However, if a pessimistic comparison is requested, e.g., $\chi_1^s = 0.9$, the numerical outcome is $A_1 > A_2$. This outcome is equivalent to choosing a fuzzy number with a smaller spread

as the preferred. Conversely, if an optimistic comparison is requested, e.g., $\chi_2^s = 0.9$, the numerical outcome is $A_1 < A_2$, which is equivalent to preferring a fuzzy number with a larger spread.

(2) Obviously, any set of results of Table 7.1 can be used as the final results of the comparison. Suppose the results of the nonlinear-$\chi(w)$ weighting ($r = 2$, $\chi_1^s = \chi_1(w_{\text{hgt}}^*) = 0.7$, $w_s^f = 0$, and $\chi_1^e = 0.5$) is adopted. According to Figure 11 and due to $d(A_1, A_2) = 0.28$, we obtain a linguistic conclusion as: "A_1 and A_2 are indifferent with truth level 44% and A_1 is weakly better than A_2 with truth level 56%" with "degree of optimism of 30% or degree of pessimism of 70%."

(3) In many practical situations, the DM's attitude can be very fuzzy and is very difficult to obtain a crisp level. Suppose the DM gives a range $[p', p'']$ regarding the degree of optimism or pessimism, then from Table 7.1 for $[p', p''] \ni \chi_1^s$, we can obtain the range of values for $d(A_1, A_2)$ with a given type of weighting. From Figure 11, we can obtain the corresponding truth level for L_k. It should be noted that the range of $d(A_1, A_2)$ may satisfy several truth levels of L_k. A natural tendency would be to select that L_k which is most suited to this range. In order to define this "most suited," we will use the following integration approach:

$$T_k = \int_{d'_{1,2}}^{d''_{1,2}} \mu_{L_k}(\delta_{1,2})\, d\delta_{1,2}, \tag{24}$$

where $[d'_{1,2}, d''_{1,2}]$ corresponds to the lower and upper limits, respectively, of $d(A_1, A_2), T_k$ represents the integrated truth level of L_k, and the integrant represents the membership curve of L_k. The most appropriate L_k is that one which gives the largest value of T_k.

Suppose the DM gives the range of pessimistic degrees as from 0.7 to 0.9, then from Table 7.1, corresponding to $\chi_1^s \in [p', p''] = [0.7, 0.9]$, we obtain the value of $d(A_1, A_2) = [0.2, 0.4]$. In this case, since $[d'_{1,2}, d''_{1,2}] = [0.2, 0.4]$, we have

$$T_5 = \int_{0.2}^{0.4} 2\,(0.5 - \delta)\, d\delta = 0.08, \quad T_6 = \int_{0.2}^{0.4} 2\,\delta\, d\delta = 0.12, \tag{25}$$

and $T_k = 0$, for $k \neq 5$, 6. Thus

$$T_6 = \max_{k=1,\dots,9} [T_k]. \tag{26}$$

Therefore, L_6 is the most appropriate linguistic interpretation. A conclusion may then be stated as: "A_1 is weakly better than A_2" with "degree of pessimism in [0.7, 0.9] or [70%, 90%]."

EXAMPLE 2. Consider the comparison of two fuzzy sets A_1 and A_2 in Figure 19. A_1 is normal but nonconvex while A_2 is convex but nonnormal. Numerically, A_1 and A_2 are defined with the membership functions $\mu_1(x)$ and $\mu_2(x)$, respectively, as

$$\mu_1(x) = \begin{cases} 1 - x, & \text{for } 0 \le x \le 1, \\ \frac{1}{2}(x - 6), & \text{for } 6 \le x \le 8, \\ 0, & \text{elsewhere,} \end{cases} \qquad \mu_2(x) = \begin{cases} x - 3, & \text{for } 3 \le x \le 3.5, \\ 0.5, & \text{for } 3.5 \le x \le 4.5, \\ 5 - x, & \text{for } 4.5 \le x \le 5, \\ 0, & \text{elsewhere.} \end{cases}$$

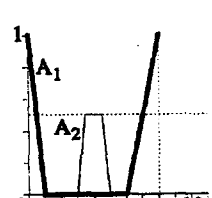

Figure 19.

The decision is for maximization and the DM's attitude reflects some degree of optimism with (χ_2^s) between 50% and 70%, that is, between indifference and moderate optimism. The inverse images of $\mu_1(x)$ and $\mu_2(x)$ are

$$\{\mu_1^{-1}(w)\} = \{(1-w),(6+2w)\}, \text{ and } x_1'(w) = 1-w, \ x_1''w = 6+2w, \text{ for } 0 < w \le 1 \text{ for } A_1;$$

$$\{\mu_2^{-1}(w)\} = \begin{cases} (3+w),(5-w)\}, & \text{for } 0 < w < 0.5 \\ [3.5, 4.5], & \text{for } w = 0.5, \end{cases} \quad \text{for } A_2,$$

i.e., a compact continuous real interval, and

$$x_2'(w) = 3+w, \quad x_2''(w) = 5-w, \qquad \text{for } 0 < w \le 0.5.$$

The height is: $w_{\text{hgt}}^* = \min[\text{hgt}(A_1), \text{hgt}(A_2)] = \min[1, 0.5] = 0.5$. Let us assume that linear-$\chi(w)$ type weighting and $\omega(w) = w$ would be appropriate to reflect the DM's degree of optimism and pessimism. The calculations are carried out as follows

(1) Indifference fixed-$\chi(w)$ weighting with $\chi_1 = \chi_2 = 0.5$:

$$\text{OM}(A_1) = \frac{\int_0^{0.5} w \ \{0.5\,(1-w) + 0.5\,(6+2w)\}\, dw}{\frac{1}{2}\,(0.5)^2} = 3.67,$$

$$\text{OM}(A_2) = \frac{\int_0^{0.5} w \ \{0.5\,(3+w) + 0.5\,(5-w)\}\, dw}{\frac{1}{2}\,(0.5)^2} = 4.00, \qquad \text{and}$$

$$d(A_1, A_2) = 0.33.$$

(2) Linear-$\chi(w)$ weighting with $w_e^f = 0$ and $\chi_1^e = 0.5$.

(a) $\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) = 0.3$, (or, $\chi_2^s = \chi_2\left(w_{\text{hgt}}^*\right) = 0.7$)

$$\chi_1(w) = \chi_1^s + \frac{(\chi_1^e - \chi_1^s)\left(w_{\text{hgt}}^* - w\right)}{\left(w_{\text{hgt}}^* - w_e^f\right)} = 0.3 + \frac{(0.5-0.3)\,(0.5-w)}{(0.5-0)} = 0.5 - 0.4\,w,$$

$$\chi_2(w) = 1 - \chi_1(w) = 0.5 + 0.4\,w,$$

$$\text{OM}(A_1) = \frac{\int_0^{0.5} w \ \{(0.5-0.4\,w)\,(1-w) + (0.5+0.4\,w)\,(6+2w)\}\, dw}{\frac{1}{2}\,(0.5)^2} = 4.48,$$

$$\text{OM}(A_2) = \frac{\int_0^{0.5} w \ \{(0.5-0.4\,w)\,(3+w) + (0.5+0.4\,w)\,(5-w)\}\, dw}{\frac{1}{2}\,(0.5)^2} = 4.17, \qquad \text{and}$$

$$d(A_1, A_2) = 0.31.$$

(b)$\chi_1^s = \chi_1\left(w_{\text{hgt}}^*\right) = 0.1$, (or, $\chi_2^s = \chi_2\left(w_{\text{hgt}}^*\right) = 0.9$)

$$\chi_1(w) = \chi_1^s + \frac{(\chi_1^e - \chi_1^s)\left(w_{\text{hgt}}^* - w\right)}{\left(w_{\text{hgt}}^* - w_e^f\right)} = 0.1 + \frac{(0.5-0.1)\,(0.5-w)}{(0.5-0)} = 0.5 - 0.8\,w,$$

$$\chi_2(w) = 1 - \chi_1(w) = 0.5 + 0.8\,w,$$

$$\text{OM}(A_1) = \frac{\int_0^{0.5} w\,\{(0.5-0.8\,w)\,(1-w) + (0.5+0.8\,w)\,(6+2w)\}\, dw}{\frac{1}{2}\,(0.5)^2} = 5.30,$$

$$\text{OM}(A_2) = \frac{\int_0^{0.5} w \ \{(0.5-0.8\,w)\,(3+w) + (0.5+0.8\,w)\,(5-w)\}\, dw}{\frac{1}{2}\,(0.5)^2} = 4.33, \qquad \text{and}$$

$$d(A_1, A_2) = 0.97.$$

Thus, the corresponding range of $d(A_1, A_2)$ to $[0.5, 0.7] \ni \chi_2^s$ appears as $[-0.33, 0.31]$, and

$$T_4 = \int_{-0.33}^{0} -2\delta\, d\delta = 0.109, \quad T_5 = \int_{-0.33}^{0} (1 + d\delta)\, d\delta + \int_{0}^{0.31} (1 - 2\delta)\, d\delta = 0.531,$$

$$T_6 = \int_{0}^{0.31} 2\delta\, d\delta = 0.096, \tag{27}$$

and $T_k = 0$, for $k \neq 4,\ 5,\ 6$. Thus,

$$T_5 = \max_{k=1,\ldots,9} [T_k].$$

Hence, we may conclude that "A_1 and A_2 are indifferent" with "degree of optimism from 0.5 to 0.7 or 50% to 70%" and "degree of nonnormality $(1 - w^*_{\text{hgt}}) = 0.5$ or 50%."

Suppose the degrees of optimism are [0.7, 0.9] instead of [0.5, 0.7], then $[d'_{1,2}, d''_{1,2}]$ corresponding to [0.7, 0.9] for χ_2^s is [0.31, 0.97], and

$$T_5 = \int_{0.31}^{0.5} (1 - 2\delta)\, d\delta = 0.036, \quad T_6 = \int_{0.31}^{0.5} 2\delta\, d\delta + \int_{0.5}^{0.97} d\delta = 0.624,$$

and $T_k = 0$, for $k \neq 5,\ 6$. Thus,

$$T_6 = \max_{k=1,\ldots,9} [T_k].$$

We can now conclude that "A_1 is weakly better than A_2" with "degree of optimism from 70% to 90%" and "degree of nonnormality 50%."

REFERENCES

1. G. Bortolan and R. Deganin, A review of some methods for ranking fuzzy subsets, *Fuzzy Sets and Systems* **15**, 1–19 (1985).
2. R.J. Li and E.S. Lee, Ranking fuzzy numbers—A comparison, In *Proceedings of North American Fuzzy Information Processing Society Workshop*, pp. 169–204, West Lafayette, IN, (1987).
3. Q. Zhu and E.S. Lee, Comparison and ranking of fuzzy numbers, In *Fuzzy Regression Analysis* (Edited by J. Kacprzyk and M. Fedrizzi), Omnitech Press, Warsaw, Poland, (1992).
4. P.-T. Chang and E.S. Lee, Fuzzy arithmetics and comparison of fuzzy numbers, In *Fuzzy Optimization, Recent Advances* (Edited by M. Delgado, J. Kacprzyk, L. Verdegay and M.A. Vila), Physica-Verlag, Heidelberg (to appear).
5. J.M. Adamo, Fuzzy decision tree, *Fuzzy Sets and Systems* **4**, 207–219 (1980).
6. L.R. Chow and W. Chang, A new ranking technique of fuzzy alternatives and its applications to decision making, *Policy and Information* **7**, 31–48 (1983).
7. J.J. Buckley and S. Chanas, A fast method of ranking alternatives using fuzzy numbers, *Fuzzy Sets and Systems* **30**, 337–338 (1989).
8. S. Nanda, On sequences of fuzzy numbers, *Fuzzy Sets and Systems* **33**, 123–126 (1989).
9. S.M. Baas and H. Kwakernaak, Rating and ranking of multiple-aspect alternatives using fuzzy sets, *Automatica* **13**, 47–58 (1977).
10. J.F. Baldwin and N.C.F. Guild, Comparison of fuzzy sets on the same decision space, *Fuzzy Sets and Systems* **2**, 213–231 (1979).
11. E.L. Hannan, Fuzzy decision making with multiple objectives and discrete membership functions, *Int. J. Man-Machine Studies* **18**, 49–54 (1983).
12. S. Ovchinnikov, Transitive fuzzy orderings of fuzzy numbers, *Fuzzy Sets and Systems* **30**, 283–295 (1989).
13. R.R. Yager, Ranking fuzzy subsets over the unit interval, In *Proceedings of the 1978 CDC*, 1435–1473, (1978).
14. S.R. Watson, J.J. Weiss and M.L. McDonnell, Fuzzy decision analysis, *IEEE Trans. Systems Man Cybernet* **9**, 1–9 (1979).
15. D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, (1980).
16. D. Dubois and H. Prade, Ranking fuzzy numbers in the setting of possibility theory, *Information Sciences* **30**, 183–224 (1983).
17. Y. Tsukamoto, P.N. Nikiforuk and M.M. Gupta, On the comparison of fuzzy sets using fuzzy chopping, In *Proc. 8th IFA World Congress,* Vol. 5, 46–52, (1981).
18. J.J. Buckley, Ranking alternatives using fuzzy numbers, *Fuzzy Sets and Systems* **15**, 21–31 (1985).

19. M. Roubens and P. Vincke, Fuzzy possibility graphs and their application to ranking fuzzy numbers, In *Non-Conventional Preference Relations in Decision Making* (Edited by J. Kacprzyk and M. Roubens), pp. 119–128, Springer-Verlag, Berlin, (1988).
20. M. Delgado, J.L. Verdegay and M.A. Vila, A procedure for ranking fuzzy numbers using fuzzy relations, *Fuzzy Sets and Systems* **26**, 49–62 (1988).
21. J.J. Buckley, A fuzzy ranking of fuzzy numbers, *Fuzzy Sets and Systems* **33**, 119–121 (1989).
22. S. Ammar, Determining the 'best' decision in the presence of imprecise information, *Fuzzy Sets and Systems* **29**, 293–302 (1989).
23. R. Jain, Decision-making in the presence of fuzzy variables, *IEEE Trans. Systems Man Cybernet* **6**, 698–703 (1976).
24. R. Jain, A procedure for multiple-aspect decision making using fuzzy sets, *Int. J. Systems Science* **8** (1), 1–7 (1977).
25. S.H. Chen, Ranking fuzzy numbers with maximizing set and minimizing set, *Fuzzy Sets and Systems* **17**, 113–129 (1985).
26. K. Kim and K.S. Park, Ranking fuzzy numbers with index of optimism, *Fuzzy Sets and Systems* **35**, 143–150 (1990).
27. R.R. Yager, On choosing between fuzzy subsets, *Kybernetes* **9**, 151–154 (1980).
28. R.R. Yager, A procedure for ordering fuzzy subsets of the unit interval, *Information Sciences* **24**, 143–161 (1981).
29. W. Chang, Ranking of fuzzy utilities with triangular membership functions, In *Proc. Int. Conf. on Policy Anal. and Inf. Systems*, 263–272, (1981).
30. S. Murakami, H. Maeda and S. Imamura, Fuzzy decision analysis on the development of centralized regional energy control system, Preprints of the *IFAC Conference on Fuzzy Information, Knowledge Representation and Decision Analysis*, pp. 353–358, (1983).
31. A. Kaufmann, Hybrid data—Various associations between fuzzy subsets and random variables, In *Fuzzy Sets Theory and Applications* (Edited by A. Jones *et al.*), pp. 171–211, Reidel, Boston, MA, (1986).
32. E.S. Lee and R.J. Li, Comparison of fuzzy numbers based on the probability measure of fuzzy events, *Computers Math. Applic.* **15** (10), 887–896 (1988).
33. T.Y. Tseng and C.M. Klein, New algorithm for the ranking procedure in fuzzy decision making, *IEEE Trans. Systems Man Cybernet.* **19** (5), 1289–1296 (1989).
34. Y. Yuan, Criteria for evaluating fuzzy ranking methods, *Fuzzy Sets and Systems* **44**, 139–157 (1991).
35. S. Mabuchi, An approach to the comparison of fuzzy subsets with an α-cut dependent index, *IEEE Trans. Systems Man Cybernet.* **18** (2), 264–272 (1988).
36. L.M. Campos and A. González, A subjective approach for ranking fuzzy numbers, *Fuzzy Sets and Systems* **29**, 145–163 (1989).
37. A. González, A study of the ranking function approach through mean values, *Fuzzy Sets and Systems* **35**, 29–41 (1990).
38. E.E. Kerre, The use of fuzzy sets theory in cardiological diagnostics, In *Approximate Reasoning in Decision Analysis* (Edited by M.M. Gupta and E. Sanchez), pp. 277–282, North-Holland, New York, (1982).
39. K. Nakamura, Preference relations on a set of fuzzy utilities as a basis for decision making, *Fuzzy Sets and Systems* **20**, 147–162 (1986).
40. C.S. McCahon and E.S. Lee, Comparing fuzzy numbers: The proportion of the optimal method, *Int. J. Approx. Reasoning* **4**, 159–163 (1990).
41. K. Hirota and W. Pedrycz, Interpretation of results of ranking methods with the aid of probabilistic sets, *Fuzzy Sets and Systems* **32**, 263–274 (1989).
42. J. Efstathiou and R.M. Tong, Ranking fuzzy sets: A decision theoretic approach, *IEEE Trans. Systems Man Cybernet* **12**, 655–659 (1982).
43. A. Kaufmann and M. M. Gupta, *Introduction to Fuzzy Arithmetic*, Van Nostrand Reinhold Co., New York, (1985).
44. H. Tanaka, H. Ichihashi and K. Asai, A formulation of fuzzy linear programming problem based on comparison of fuzzy numbers, *Control and Cybernetics* **13** (3), 185–194 (1984).
45. A.N.S. Freeling, Fuzzy sets and decision analysis, *IEEE Trans. Systems Man Cybernet.* **10**, 341–354 (1980).