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Jeffreys fluids in forced elongation

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Abstract

In this paper we study existence, uniqueness and regularity of solutions for the equations governing the forced elongation of fluids with differential constitutive law of Jeffreys type. These equations consist of nonlinear first-order hyperbolic equations in one spatial dimension. Forced elongation is imposed through velocity boundary conditions at the domain entry and exit. The existence result is based on the Schauder fixed point theorem and energy methods in the space of boundary-regular functions.

Keywords: Viscoelasticity; Existence theory; Energy estimates; Schauder fixed point theorem

1. Introduction

The mathematical description of thin viscoelastic fluid fibers formed by forced elongation is an analytically challenging issue which promises new insights in the physical mechanisms governing these elementary flows. This description takes the form of a system of nonlinear coupled first-order hyperbolic partial differential equations which are not readily accessible for analysis.

Forced elongation occurs frequently in the formation of filaments both in nature (e.g., spider silk) and in industry (e.g., nylon, manufactured by fiber spinning). The common theme in these flows is that a polymeric fluid, contained in a reservoir, is pressed through a hole (spinneret) and axially stretched by a pulling force to form a thin circular liquid fiber. In industrial processes such as fiber spinning the material is extended by a winder and made to solidify at a fixed point along the axis of elongation. The term “forced elongation”

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refers to the requirement that the "inflow" velocity of the fluid (when it leaves the reservoir to enter the domain of extension) be smaller than the "outflow" velocity imposed by the pulling force.

Simple models of forced elongation are based on two assumptions: thinness of the axisymmetric fiber and dominant viscous/viscoelastic forces. As a result the governing equations are one-dimensional in space and do not account for inertia. In the purely viscous case the governing equations were first posed by Kase and Matsuo [11] and first formally derived by Matovich and Pearson [12]. Recent analytical studies of these equations focused on linear stability and related issues [2,3,7–9]. A few results address existence and uniqueness of solutions, both in viscous and viscoelastic regimes [4–6]. In this article we will treat the viscoelastic case where the viscoelasticity is modeled by the constitutive theory of the upper convected Jeffreys fluid (or Oldroyd-B fluid) [1,10,13]. This constitutive theory is based on a simple macroscopic spring-dash pot model and entails a linear superposition of non-Newtonian and Newtonian stresses. It is well known that the constitutive model of the Jeffreys fluid has major shortcomings in real-world flow predictions, some of them even very serious in certain elongational regimes. However, the Jeffreys fluid is well accepted as a theoretical fluid model that is capable of shedding light on the "qualitatively correct" viscoelastic flow behavior. In addition, the Jeffreys model is the basis for a variety of other important fluid models (among them the Giesekus, Phan-Thien–Tanner and FENE models), thus deserving appropriate attention.

Our main objective in this work is to study (local in time) existence, uniqueness and regularity of solutions in the case of "forced elongation boundary conditions." These boundary conditions are the ones one would naturally hope to impose. Previous studies of viscoelastic fiber flow were restricted to "inflow boundary conditions" [4,6]. Our objective will be tackled by fixed point methods and functional–analytic arguments in an appropriate function space: the space of boundary-regular functions. To the author's knowledge, the fundamental strategy for proving existence of solutions for nonlinear transport equations by means of boundary-regular functions was first published in [5] and later extended in [6]. In this article we will give a brief summary of the most important features of these functions in Section 3. In contrast to related results published in [4–6] the principle ideas for proving existence in this work are geared toward the Schauder fixed point theorem, not the Banach contraction mapping principle. This approach proves shorter and technically far less demanding. In general, fluid models with constitutive equations in differential form that include both Newtonian and non-Newtonian stresses can be analyzed with techniques similar to the ones developed here. Moreover, there is hope to believe that the forced elongation of a Maxwell fluid arising as a singular limit of the Jeffreys fluid with vanishing Newtonian stresses can be discussed in this way as well.

2. The governing equations

In this section we state the equations governing the forced elongation of a Jeffreys fluid in dimensionless terms. To this end, we denote time by t, the axial variable by z, the cross-sectional fiber area by $a = a(t, z)$ (assumed as circular), and the axial velocity by $v = v(t, z)$. The quantities $T_{rr} = T_{rr}(t, z)$ and $T_{zz} = T_{zz}(t, z)$ denote the radial and axial components of the viscoelastic extra stress tensor. Then the governing equations can be cast in the form (see [14])

$$\partial_t a + \partial_z (va) = 0, \tag{2.1}$$

$$\partial_z\big(3\chi a \partial_z v + (1-\chi)a(T_{zz} - T_{rr})\big) = 0, \tag{2.2}$$

$$\mathrm{We}(\partial_t T_{rr} + v\partial_z T_{rr} + T_{rr}\partial_z v) + T_{rr} = -\partial_z v, \tag{2.3}$$

$$\mathrm{We}(\partial_t T_{zz} + v\partial_z T_{zz} - 2T_{zz}\partial_z v) + T_{zz} = 2\partial_z v. \tag{2.4}$$

The flow domain is $0 \leqslant z \leqslant 1$, $t \geqslant 0$. The positive parameter We, the Weissenberg number, is a dimensionless relaxation time and serves as a measure for the viscoelasticity of the fluid. The quantity $\chi \in [0, 1]$ is a concentration parameter, related to an intrinsic fluid retardation time, and models the contribution of the Newtonian stresses to the momentum balance. The case $\chi = 1$ yields a purely viscous momentum balance where the stress equations decouple from the mass and momentum balances, while the case $\chi = 0$ corresponds to a purely viscoelastic regime. In this latter case the total stresses reduce to what is known as the constitutive theory of the upper convected Maxwell fluid. For all our purposes in this paper we shall assume

$$0 < \chi < 1. \tag{2.5}$$

To close the formulation of the problem we pose the "forced elongation boundary conditions"

$$a(t, 0) = 1, \tag{2.6}$$

$$v(t, 0) = 1, \tag{2.7}$$

$$v(t, 1) = D > 1, \tag{2.8}$$

$$T_{rr}(t, 0) = T^*_{rr}(t), \tag{2.9}$$

$$T_{zz}(t, 0) = T^*_{zz}(t) \tag{2.10}$$

together with initial conditions of the form

$$a(0, z) = a^0(z), \tag{2.11}$$

$$T_{rr}(0, z) = T^0_{rr}(z), \tag{2.12}$$

$$T_{zz}(0, z) = T^0_{zz}(z). \tag{2.13}$$

The quantity $D > 1$, referred to as "draw ratio," is a dimensionless outflow velocity. We remark that the Maxwell regime $\chi = 0$ cannot be treated within the framework above since Eqs. (2.1)–(2.13) would be overdetermined. For this reason previous studies of Eqs. (2.1)–(2.4) with $\chi = 0$ (see [4]) concentrated on the "inflow boundary conditions"

$$a(t, 0) = 1, \tag{2.14}$$

$$v(t, 0) = 1, \tag{2.15}$$

$$T_{rr}(t, 0) = T^*_{rr}(t), \tag{2.16}$$

$$T_{zz}(t, 0) = T^*_{zz}(t). \tag{2.17}$$

Similar conditions were imposed in nonisothermal viscoelastic flow [5]. The more relevant problem of forced elongation boundary conditions (i.e., with the conditions $v(t,0) = 1$, $v(t,1) = D$ and other boundary conditions dropped in (2.14)–(2.17)) has yet to be done for the Maxwell fluid.

3. Boundary-regular functions and the elementary transport equation

In the following, we will need some technical results which are crucial for the existence theory of the governing equations. In the current chapter we briefly summarize these nonstandard results for the reader's convenience.

3.1. Definitions

Let $r_1 < r_2$, $s_1 < s_2$, $t_0 > 0$ and $m, n, k \in \mathbb{N}_0$. We will interpret the following norms and seminorms with respect to (w.r.t.) the entire domain of the particular function. Hence the meaning of t_0, r_1, r_2, s_1, and s_2 will become clear from the context. Throughout we will use the following abbreviations:

(1) $\|\cdot\|_p$ for the norm on the Lebesgue space $L^p(r_1, r_2)$, $1 \leqslant p \leqslant \infty$,
(2) $\|\cdot\|_{H^k}$ for the norm on the Sobolev space $H^k(r_1, r_2)$,
(3) $\|\cdot\|_{m,n}$ for the norm on the Sobolev space $W^{m,\infty}([r_1, r_2]; H^n(s_1, s_2))$,
(4) $\|\cdot\|_{H^{m,n}}$ for the norm on the Sobolev space $H^m([r_1, r_2]; H^n(s_1, s_2))$,
(5) $\|\cdot\|_{m,n,[t]}$ for the seminorms on the space $W^{m,\infty}([0, t_0]; H^n(s_1, s_2))$, defined for $0 \leqslant t \leqslant t_0$ by

$$\|f\|_{m,n,[t]} \stackrel{\text{def}}{=} \|f|_{[0,t]}\|_{m,n}. \tag{3.1}$$

The notion of boundary-regularity will play a prominent role in the following existence theory.

Definition 3.1. The space $BR(t_\alpha, t_\omega; a, b)$ of boundary-regular functions consists of all functions $g = g(t, x)$ on $[t_\alpha, t_\omega] \times [a, b]$ such that

$$g \in W^{1,\infty}\big([t_\alpha, t_\omega]; H^1(a, b)\big) \cap L^\infty\big([t_\alpha, t_\omega]; H^2(a, b)\big), \tag{3.2}$$

$$\partial_x g(\cdot, a), \partial_x g(\cdot, b) \in H^1(t_\alpha, t_\omega). \tag{3.3}$$

The space $BR(t_\alpha, t_\omega; a, b)$ is endowed with the energy norm

$$\mathcal{E}(g) \stackrel{\text{def}}{=} \big(\|g\|_{0,2}^2 + \|g\|_{1,1}^2 + \big\|\partial_x g(\cdot, a)\big\|_{H^1}^2 + \big\|\partial_x g(\cdot, b)\big\|_{H^1}^2\big)^{1/2}. \tag{3.4}$$

3.2. The general transport equation

The importance of the notion of "boundary-regularity" lies in the following theorem and its corollary. For details and proofs we refer to the comprehensive account in [5].

Theorem 3.2. *Let f and p be functions on $[0, t_0] \times [0, 1]$, u^0 on $[0, 1]$, and u^α on $[0, t_0]$ such that*

$$p \text{ and } f \text{ are boundary-regular,} \tag{3.5}$$
$$p > 0 \quad \text{on } [0, t_0] \times [0, 1], \tag{3.6}$$
$$u^0 \in H^2(0, 1), \tag{3.7}$$
$$u^\alpha \in H^2(0, t_0), \tag{3.8}$$
$$u^0(0) = u^\alpha(0), \tag{3.9}$$
$$\partial_t u^\alpha(0) + p(0, 0)\partial_x u^0(0) = f(0, 0). \tag{3.10}$$

Then the boundary-initial value problem

$$\partial_t u(t, x) + p(t, x)\partial_x u(t, x) = f(t, x), \quad t \in [0, t_0],\ x \in [0, 1], \tag{3.11}$$
$$u(0, x) = u^0(x), \quad x \in [0, 1], \tag{3.12}$$
$$u(t, 0) = u^\alpha(t), \quad t \in [0, t_0], \tag{3.13}$$

has a boundary-regular solution u such that

$$u \in C^1\big([0, t_0]; H^1(0, 1)\big) \cap C\big([0, t_0]; H^2(0, 1)\big), \tag{3.14}$$
$$u \text{ is unique in } W^{1,\infty}\big([0, t_0]; L^2(0, 1)\big) \cap L^\infty\big([0, t_0]; H^1(0, 1)\big). \tag{3.15}$$

Corollary 3.3. *Let the function $u^\alpha \in H^2(0, t^*)$ be given for some $t^* > 0$. For $t_0 \in (0, t^*]$, suppose that the functions f, p, u^0 and u^α satisfy the conditions* (3.5)–(3.10). *Then, for $0 \leqslant t \leqslant t_0$, there exist continuous, nonnegative functions $E = E(t)$, $F = F(t)$ and $G = G(t)$ which depend on $\mathcal{E}(p)$, $\mathcal{E}(f)$, $\|u^0\|_{H^2}$, $\|u^\alpha\|_{H^2}$ and t^* such that*

$$E(0) = \|u^0\|^2_{H^2}, \tag{3.16}$$
$$F(0) = \|u^0\|^2_{H^1} + \big\| p(0, \cdot)\partial_x u^0 + f(0, \cdot)\big\|^2_{H^1}, \tag{3.17}$$
$$G(0) = 0 \tag{3.18}$$

and such that the solution u of the boundary-initial value problem (3.11)–(3.13) *obeys the estimates*

$$\|u\|^2_{0,2,[t]} \leqslant E(t) \quad \text{for } 0 \leqslant t \leqslant t_0, \tag{3.19}$$
$$\|u\|^2_{1,1,[t]} \leqslant F(t) \quad \text{for } 0 \leqslant t \leqslant t_0, \tag{3.20}$$
$$\big\|\partial_x u(\cdot, 1)\big\|^2_{H^1} \leqslant F(t_0), \tag{3.21}$$
$$\big\|\partial_x u(\cdot, 0)\big\|^2_{H^1} \leqslant G(t_0). \tag{3.22}$$

The proof of Theorem 3.2 proceeds as follows: first one establishes the existence results and estimates for the boundary-initial value problem (3.11)–(3.13) assuming sufficient smoothness of the coefficient functions p and f; then one shows that boundary-regular coefficients can be approximated by smooth coefficients. Finally one applies weak and weak* convergence arguments to deduce the necessary estimates for the given problem.

4. Existence and uniqueness of solutions

Our principal strategy for proving (local in time) existence of solutions is the Schauder fixed point theorem and a discussion of uniqueness of solutions. As it turns out this approach is more elegant and less technical than related discussions employing the Banach contraction mapping principal.

4.1. Statement of the main result

Definition 4.1. A vector field (a, v, T_{rr}, T_{zz}), defined on $[0, t_0] \times [0, 1]$, is a solution of the boundary-initial value problem (2.1)–(2.13) if

$$a, v, T_{rr}, T_{zz} \in W^{1,\infty}\big([0, t_0]; H^1(0,1)\big) \cap L^\infty\big([0, t_0]; H^2(0,1)\big), \tag{4.1}$$

$$a, v, T_{rr}, T_{zz} \text{ satisfy Eqs. (2.1)–(2.4)}, \tag{4.2}$$

$$a \text{ satisfies Eqs. (2.6), (2.11) and } a > 0, \tag{4.3}$$

$$T_{rr} \text{ satisfies Eqs. (2.9) and (2.12)}, \tag{4.4}$$

$$T_{zz} \text{ satisfies Eqs. (2.10) and (2.13)}, \tag{4.5}$$

$$v \text{ satisfies Eqs. (2.7), (2.8) and } v > 0. \tag{4.6}$$

The requirement $v > 0$ is physically plausible and would certainly be expected. The following existence theory hinges on this assumption to be valid at least initially. Equations (2.2), (2.7) and (2.8) imply the relation

$$\begin{aligned} v(t,z) = 1 &+ \frac{D-1}{\int_0^1 a(t,x)^{-1}\,dx} \int_0^z \frac{1}{a(t,x)}\,dx \\ &- \frac{1-\chi}{3\chi} \int_0^z \big(T_{zz}(t,x) - T_{rr}(t,x)\big)\,dx \\ &+ \frac{(1-\chi)\int_0^1 (T_{zz}(t,x) - T_{rr}(t,x))\,dx}{3\chi \int_0^1 a(t,x)^{-1}\,dx} \int_0^z \frac{1}{a(t,x)}\,dx. \end{aligned} \tag{4.7}$$

For the initial velocity v^0 we obtain

$$\begin{aligned} v^0(z) = v(0,z) = 1 &+ \frac{D-1}{\int_0^1 a^0(x)^{-1}\,dx} \int_0^z \frac{1}{a^0(x)}\,dx \\ &- \frac{1-\chi}{3\chi} \int_0^z \big(T^0_{zz}(x) - T^0_{rr}(x)\big)\,dx \\ &+ \frac{(1-\chi)\int_0^1 (T^0_{zz}(x) - T^0_{rr}(x))\,dx}{3\chi \int_0^1 a^0(x)^{-1}\,dx} \int_0^z \frac{1}{a^0(x)}\,dx. \end{aligned} \tag{4.8}$$

Theorem 4.2. *Let the initial values a^0, T_{rr}^0, T_{zz}^0 and the boundary values T_{rr}^*, T_{zz}^* be given such that*

$$a^0, T_{rr}^0, T_{zz}^0 \in H^2(0,1), \tag{4.9}$$

$$a^0 > 0 \quad \textit{on } [0,1], \tag{4.10}$$

$$T_{rr}^*, T_{zz}^* \in H^2(0,t^*) \quad \textit{for some } t^* > 0. \tag{4.11}$$

Assume that the initial velocity v^0, defined by (4.8)*, is positive and that the compatibility conditions*

$$a^0(0) = 1, \qquad T_{rr}^0(0) = T_{rr}^*(0), \qquad T_{zz}^0(0) = T_{zz}^*(0), \tag{4.12}$$

$$\partial_z a^0|_{z=0} + \partial_z v^0|_{z=0} = 0, \tag{4.13}$$

$$\mathrm{We}\big(\partial_t T_{rr}^*|_{t=0} + \partial_z T_{rr}^0|_{z=0} + T_{rr}^0|_{z=0}\partial_z v^0|_{z=0}\big) + T_{rr}^0|_{z=0} = -\partial_z v^0|_{z=0}, \tag{4.14}$$

$$\mathrm{We}\big(\partial_t T_{zz}^*|_{t=0} + \partial_z T_{zz}^0|_{z=0} - 2T_{zz}^0|_{z=0}\partial_z v^0|_{z=0}\big) + T_{zz}^0|_{z=0} = 2\partial_z v^0|_{z=0} \tag{4.15}$$

hold true. Then there exists $t_0 \in (0,t^*]$ such that the boundary-initial value problem (2.1)–(2.13) *has a unique solution (a, v, T_{rr}, T_{zz}) on* $[0,t_0] \times [0,1]$*. This solution (a, v, T_{rr}, T_{zz}) has the properties*

$$a, T_{rr}, T_{zz} \in \bigcap_{k=0}^{2} C^k\big([0,t_0]; H^{2-k}(0,1)\big), \tag{4.16}$$

$$v \in \bigcap_{k=0}^{2} C^k\big([0,t_0]; H^{3-k}(0,1)\big), \tag{4.17}$$

$$a, v, T_{rr}, T_{zz} \textit{ are boundary-regular.} \tag{4.18}$$

The conditions imposed on the initial and boundary values can easily be satisfied, e.g., by assuming a Newtonian-like regime:

$$\gamma = \ln D, \qquad a^0(z) = \exp(-\gamma z), \tag{4.19}$$

$$T_{rr}^0(z) = -\gamma \exp(\gamma z), \qquad T_{zz}^0(z) = 2\gamma \exp(\gamma z). \tag{4.20}$$

The remaining conditions on the boundary values are readily determined. For the following it will be understood without further reference that the assumptions made in Theorem 4.2 are fulfilled.

4.2. Proof of the main result

Definition 4.3. For $L > 0$ and $t' \in (0,t^*]$, let $\mathbb{S}(t', L)$ be the set of functions $(b, S, T)^T$ on $[0,t'] \times [0,1]$ such that

$$b, S, T \in BR(0,t'; 0,1), \tag{4.21}$$

$$\mathcal{E}(b)^2 + \mathcal{E}(S)^2 + \mathcal{E}(T)^2 \leqslant L^2, \tag{4.22}$$

$$b(0,z) = a^0(z) \quad \text{and} \quad b(t,0) = 1, \tag{4.23}$$

$$S(0,z)=T_{rr}^0(z) \quad \text{and} \quad S(t,0)=T_{rr}^*(t), \tag{4.24}$$

$$T(0,z)=T_{zz}^0(z) \quad \text{and} \quad T(t,0)=T_{zz}^*(t). \tag{4.25}$$

Lemma 4.4. *For all sufficiently small $t' \in (0,t^*]$ and sufficiently large $L>0$, the set $\mathbb{S}(t',L)$ is nonempty such that, for each $(b,S,T)^T \in \mathbb{S}(t',L)$, the conditions $b>0$ and*

$$1+\frac{D-1}{\int_0^1 b(t,x)^{-1}\,dx}\int_0^z \frac{1}{b(t,x)}\,dx-\frac{1-\chi}{3\chi}\int_0^z \big(T(t,x)-S(t,x)\big)\,dx \\ +\frac{(1-\chi)\int_0^1 (T(t,x)-S(t,x))\,dx}{3\chi\int_0^1 b(t,x)^{-1}\,dx}\int_0^z \frac{1}{b(t,x)}\,dx>0 \tag{4.26}$$

hold true.

Proof. For all sufficiently large L and sufficiently small t', $\mathbb{S}(t',L)$ contains the initial values $(a^0,T_{rr}^0,T_{zz}^0)^T$. Now there exists a constant $C=C(L)$ such that, for each $(b,S,T)^T \in \mathbb{S}(t',L)$,

$$\big|b(t,z)-a^0(z)\big| \leqslant \int_0^t \big|\partial_t b(s,z)\big|\,ds \leqslant Ct', \tag{4.27}$$

$$\big|S(t,z)-T_{rr}^0(z)\big| \leqslant \int_0^t \big|\partial_t S(s,z)\big|\,ds \leqslant Ct', \tag{4.28}$$

$$\big|T(t,z)-T_{zz}^0(z)\big| \leqslant \int_0^t \big|\partial_t T(s,z)\big|\,ds \leqslant Ct'. \tag{4.29}$$

Hence the claim follows. $\square$

In the following we will tacitly assume that the set $\mathbb{S}(t',L)$ is nonempty such that the conclusions of Lemma 4.4 are true. Note that this assumption remains automatically correct for $\mathbb{S}(t,L)$, $0<t<t'$.

Theorem 4.5. $\mathbb{S}(t',L)$ *is convex and compact in* $(L^2([0,t']\times[0,1]))^3$.

Proof. Convexity is clear by definition of $\mathbb{S}(t',L)$. Since $\mathbb{S}(t',L)$ is contained in $(H^1([0,t']\times[0,1]))^3$ and since the latter space is compactly embedded in $(L^2([0,t']\times[0,1]))^3$, $\mathbb{S}(t',L)$ is relatively compact in $(L^2([0,t']\times[0,1]))^3$. Now let $(p_n)=(b_n,S_n,T_n)^T$ be a sequence in $\mathbb{S}(t',L)$ that is Cauchy in the space $(L^2([0,t']\times[0,1]))^3$ with limit p^*. We want to show that $p^* \in \mathbb{S}(t',L)$, thus proving closedness of $\mathbb{S}(t',L)$ in $(L^2([0,t']\times[0,1]))^3$. Since $\mathcal{E}(b)^2+\mathcal{E}(S)^2+\mathcal{E}(T)^2 \leqslant L^2$, there exists a subsequence of (p_n), say (q_n), with the following properties:

(1) (q_n) is weak* convergent in $L^\infty([0,t'];H^2(0,1)) \cap W^{1,\infty}([0,t'];H^1(0,1))$ with the unique weak* limit q^*,
(2) $(\partial_z q_n(\cdot,0))$, $(\partial_z q_n(\cdot,1))$ are weakly convergent in $H^1(0,t')$ with weak limits d^0, d^1, respectively.

Since the weak* convergence implies strong convergence in $(L^2([0,t'] \times [0,1]))^3$, we have $q^* = p^*$. Next we note that the sequences $(\partial_z q_n(\cdot,0))$, $(\partial_z q_n(\cdot,1))$ converge strongly in $L^2(0,t')$ to their respective weak limits. On the other hand the sequence $(\partial_z q_n)$ is weakly convergent in $L^2([0,t'];H^1(0,1))$ with weak limit $\partial_z p^*$. Hence the sequences $(\partial_z q_n(\cdot,0)), (\partial_z q_n(\cdot,1))$ converge weakly in $L^2(0,t')$ to $\partial_z p^*(\cdot,0)$, $\partial_z p^*(\cdot,1)$, thus implying $\partial_z p^*(\cdot,0) = d^0$, $\partial_z p^*(\cdot,1) = d^1$, respectively. Hence p^* is boundary-regular and belongs to $\mathbb{S}(t',L)$. $\square$

Definition 4.6. The operator Σ is defined on $\mathbb{S}(t',L)$ by

$$\Sigma : \begin{pmatrix} b \\ S \\ T \end{pmatrix} \mapsto \begin{pmatrix} c \\ U \\ V \end{pmatrix}, \tag{4.30}$$

where $c = c(t,z)$, $U = U(t,z)$, $V = V(t,z)$ solve the boundary-initial value problem for $0 \leqslant t \leqslant t'$, $0 \leqslant z \leqslant 1$,

$$\partial_t \begin{pmatrix} c \\ U \\ V \end{pmatrix} + w(b,S,T)\partial_z \begin{pmatrix} c \\ U \\ V \end{pmatrix} = \begin{pmatrix} -w'(b,S,T)b \\ -w'(b,S,T)(S + \mathrm{We}^{-1}) - \mathrm{We}^{-1} S \\ 2w'(b,S,T)(T + \mathrm{We}^{-1}) - \mathrm{We}^{-1} T \end{pmatrix}, \tag{4.31}$$

$$\begin{pmatrix} c \\ U \\ V \end{pmatrix}\Bigg|_{z=0} = \begin{pmatrix} 1 \\ T^*_{rr} \\ T^*_{zz} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c \\ U \\ V \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} a^0 \\ T^0_{rr} \\ T^0_{zz} \end{pmatrix}. \tag{4.32}$$

The operators w and w' are defined for $(b,S,T)^T \in \mathbb{S}(t',L)$ by

$$\begin{aligned} w(b,S,T)(t,z) \overset{\text{def}}{=} 1 &+ \frac{D-1}{\int_0^1 b(t,x)^{-1}\,dx} \int_0^z \frac{1}{b(t,x)}\,dx \\ &- \frac{1-\chi}{3\chi} \int_0^z \bigl(T(t,x) - S(t,x)\bigr)\,dx \\ &+ \frac{(1-\chi)\int_0^1 (T(t,x) - S(t,x))\,dx}{3\chi \int_0^1 b(t,x)^{-1}\,dx} \int_0^z \frac{1}{b(t,x)}\,dx, \end{aligned} \tag{4.33}$$

$$\begin{aligned} w'(b,S,T)(t,z) \overset{\text{def}}{=} &\frac{D-1}{\int_0^1 b(t,x)^{-1}\,dx} \frac{1}{b(t,z)} - \frac{1-\chi}{3\chi}\bigl(T(t,z) - S(t,z)\bigr) \\ &+ \frac{(1-\chi)\int_0^1 (T(t,x) - S(t,x))\,dx}{3\chi \int_0^1 b(t,x)^{-1}\,dx} \frac{1}{b(t,z)}. \end{aligned} \tag{4.34}$$

Lemma 4.7. *The operator Σ is well defined on $\mathbb{S}(t', L)$.*

Proof. The initial and boundary values and all the coefficient functions appearing in problem (4.31), (4.32) are such that the regularity and compatibility conditions (3.5)–(3.10) of Theorem 3.2 hold true. Hence the claim follows. □

Theorem 4.8. *There exist $L > 0$ and $t_0 \in (0, t^*]$ such that, for all $t' \in (0, t_0]$, the operator Σ maps $\mathbb{S}(t', L)$ into $\mathbb{S}(t', L)$ continuously w.r.t. the topology of $(L^2([0, t] \times [0, 1]))^3$.*

Proof. It is an immediate consequence of Corollary 3.3 that there are $L > 0$ large and $t_0 > 0$ small such that $\Sigma(\mathbb{S}(t', L)) \subset \mathbb{S}(t', L)$ for all $0 < t' \leqslant t_0$. For $(b, S, T)^T, (b', S', T')^T \in \mathbb{S}(t', L)$, let

$$(c, U, V)^T \stackrel{\text{def}}{=} \Sigma\big((b, S, T)^T\big), \tag{4.35}$$

$$(c', U', V')^T \stackrel{\text{def}}{=} \Sigma\big((b', S', T')^T\big). \tag{4.36}$$

By the Sobolev embedding theorem, there exists a constant $C = C(L)$ such that

$$\big|w'(b, S, T) - w'(b', S', T')\big| \leqslant C\Bigg(|b - b'| + \int_0^1 |b - b'|\,dx + |T - T'| + |S - S'| + \int_0^1 |T - T'|\,dx + \int_0^1 |S - S'|\,dx\Bigg). \tag{4.37}$$

We also obtain

$$\big|w(b, S, T) - w(b', S', T')\big| \leqslant C\Bigg(\int_0^1 |b - b'|\,dx + \int_0^1 |T - T'|\,dx + \int_0^1 |S - S'|\,dx\Bigg). \tag{4.38}$$

As we take the difference of the governing equations (4.31) for $(c, U, V)^T$ and $(c', U', V')^T$, multiply the components by $c - c'$, $U - U'$ and $V - V'$, respectively, and integrate over the spatial domain $[0, 1]$, the estimates (4.37), (4.38) lead to an inequality of the form

$$\begin{aligned} &\frac{d}{dt}\big(\|c(t) - c'(t)\|_2^2 + \|U(t) - U'(t)\|_2^2 + \|V(t) - V'(t)\|_2^2\big) \\ &\quad \leqslant \kappa\big(\|b(t) - b'(t)\|_2^2 + \|c(t) - c'(t)\|_2^2 + \|S(t) - S'(t)\|_2^2 \\ &\qquad + \|U(t) - U'(t)\|_2^2 + \|T(t) - T'(t)\|_2^2 + \|V(t) - V'(t)\|_2^2\big) \end{aligned} \tag{4.39}$$

with $\kappa = \kappa(L)$ constant. A straightforward application of Gronwall's lemma yields the estimate

$$\|c(t) - c'(t)\|_2^2 + \|U(t) - U'(t)\|_2^2 + \|V(t) - V'(t)\|_2^2$$
$$\leqslant \kappa \int_0^t \exp(\kappa t - \kappa s)\big(\|b(s) - b'(s)\|_2^2 + \|S(s) - S'(s)\|_2^2 + \|T(s) - T'(s)\|_2^2\big)\,ds. \tag{4.40}$$

Hence the claim follows. □

The preceding proof implies the following important corollary.

Corollary 4.9. *The operator Σ has at most one fixed point in $\mathbb{S}(t', L)$.*

Proof. Suppose there are $(b, S, T)^T$, $(b', S', T')^T \in \mathbb{S}(t', L)$ such that

$$(b, S, T)^T = \Sigma\big((b, S, T)^T\big), \tag{4.41}$$
$$(b', S', T')^T = \Sigma\big((b', S', T')^T\big). \tag{4.42}$$

For these values estimate (4.39) reads

$$\frac{d}{dt}\big(\|b(t) - b'(t)\|_2^2 + \|S(t) - S'(t)\|_2^2 + \|T(t) - T'(t)\|_2^2\big)$$
$$\leqslant 2\kappa\big(\|b(t) - b'(t)\|_2^2 + \|S(t) - S'(t)\|_2^2 + \|T(t) - T'(t)\|_2^2\big). \tag{4.43}$$

However, this inequality implies

$$\|b(t) - b'(t)\|_2^2 + \|S(t) - S'(t)\|_2^2 + \|T(t) - T'(t)\|_2^2 \leqslant 0. \tag{4.44}$$

Hence the claim is proved. □

We conclude this section with the proof of Theorem 4.2.

Proof of Theorem 4.2. For t_0 sufficiently small, the Schauder fixed point theorem applies to the operator Σ on $\mathbb{S}(t_0, L)$ by Theorems 4.5 and 4.8. Hence Σ has a fixed point $(a, T_{rr}, T_{zz})^T$ in $\mathbb{S}(t_0, L)$. By Corollary 4.9, this is the only fixed point. The regularity results for a, T_{rr} and T_{zz} stated in (4.16) and (4.18) are immediate consequences of Theorem 3.2 when the velocity v is defined by Eq. (4.7). The regularity (4.17) of v is clear as well. By definition of Σ and v, (a, v, T_{rr}, T_{zz}) solves the governing equations (2.1)–(2.4). On the other hand, if (a, v, T_{rr}, T_{zz}) is a solution of Eqs. (2.1)–(2.4), then v satisfies Eq. (4.7) and $(a, T_{rr}, T_{zz})^T$ is a fixed point of the operator Σ on some space $\mathbb{S}(t_0, L)$. Hence the proof is finished. □

Finally we remark that the developments in this work do not require any novel ideas to cover the more general situation of nonconstant boundary conditions for a at $z = 0$ and v at $z = 0$, $z = 1$.

5. Conclusion

Theorem 4.2 is the first instance of an existence result for the forced elongation of a viscoelastic fluid where velocity boundary conditions are prescribed at the inflow and outflow boundaries. We have focused on the important constitutive theory of Jeffreys fluids. The class of boundary-regular functions proved an essential tool in the analysis of the governing equations. In particular, the compact embedding of the space of boundary-regular functions $BR(0, t'; 0, 1)$ in $L^2([0, t'] \times [0, 1])$ allowed an elegant study by means of the Schauder fixed point theorem. Forced elongation of Maxwell fluids which arise as a singular limit of the constitutive theory of Jeffreys fluids has yet to be discussed.

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