JOURNAL OF FUNCTIONAL ANALYSIS 80, 421–450 (1988)

The Abstract Inverse Scattering Problem and the Instability of Completeness of Orthogonal Systems

S. V. KHRUSHCHEV

Leningrad Branch of the V. A. Steklov Mathematical Institute, Fontanka 27, 191011 Leningrad, USSR

Communicated by L. D. Faddeev

Received May 18, 1987

Using the instability of completeness of orthogonal systems we prove that every contractive operator-valued function $S(t)$, $t \in \mathbb{T}$, on a Hilbert space E is the scattering operator of a pair $(U, \mathring{U})$ of unitary operators on $L^2(E)$, where $\mathring{U}$ is the shift $\mathring{U}f = z \cdot f$. A generalization of Weyl's criterion for an operator not to be essentially left invertible is also proved. We apply the result obtained to the general theory of orthogonal systems, to the construction of complete minimal families which are not hereditarily complete, and to the scattering theory. © 1988 Academic Press, Inc.

1. INTRODUCTION

In the classical scattering theory one considers pairs $(U, \mathring{U})$ of unitary operators (or strongly continuous unitary groups) on a Hilbert space H such that the limits

$$W_{\pm} = \operatorname*{s-lim}_{n \to \pm\infty} U^{-n}\mathring{U}^n$$

exist in the strong operator topology. The operator $S = W_+^* W_-$ commuting with $\mathring{U}$ is called the scattering operator of $(U, \mathring{U})$. The abstract inverse scattering problem is stated as follows.

THE INVERSE SCATTERING PROBLEM. Given a unitary operator $\mathring{U}$ and a contraction S commuting with $\mathring{U}$ find all unitary operators U such that S is the scattering operator of $(U, \mathring{U})$.

From the point of view of applications this problem is especially interesting for operators $\mathring{U}$ with absolutely continuous spectral measure $\mathring{\mathscr{E}}$. If we assume in addition that the spectral multiplicity $\mathscr{M}(\mathring{U})$ of $\mathring{U}$ is constant then $\mathring{U}$ is unitarily equivalent to the shift $\mathscr{S}: f \mapsto z \cdot f$ acting on the Hilbert space $L^2(E)$ of E-valued square-summable functions f on the unit

0022-1236/88 $3.00

circle $\mathbb{T}$, E being a Hilbert space with $\dim E = \mathcal{M}(\mathring{U})$ [6]. In the present paper, apart from other considerations, we provide a complete solution to the inverse scattering problem under the assumption that the spectral measure $\mathring{\mathscr{E}}$ is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{T}$ and $\mathcal{M}(\mathring{U}) \equiv \text{const}$. Suppose for simplicity that $\mathring{U} = \mathscr{S}$. Then S coincides with a contractive operator-valued function $S(t)$: $E \to E$, $t \in \mathbb{T}$. An important consequence of our result is that any contractive operator-valued function can be a scattering operator.

The problem stated goes back to 1963 when M. G. Krein raised the question of clarifying the connection between the quantum scattering and the Lax–Phillips approach to the scattering theory [20] (see also [21]). Using the Sz.-Nagy–Foiaş function model [24], Adamjan and Arov [1] obtained a solution of the inverse problem. However, the definition of the wave operators they used was more general than that used in the classical theory. The Adamjan–Arov approach found interesting applications in the system theory [17].

One of the advantages of our approach is that it permits us easily to construct examples of incomplete or asymptotically incomplete scatterings. Recall [34] that the wave operators W_+, W_- are called asymptotically complete if their ranges $\operatorname{ran} W_+$ and $\operatorname{ran} W_-$ coincide. The wave operators W_+, W_- are called complete if $\operatorname{ran} W_- + \operatorname{ran} W_+$ is dense in H.

The importance of these notions is demonstrated by the asymptotic formulae

$$U^n(W_- h) = \mathring{U}^n h + o(1), \quad n \to -\infty;$$

$$U^n(W_- h) = \mathring{U}^n(Sh) + U^n((I - \mathscr{P}_+) W_- h) + o(1), \quad n \to +\infty.$$

Here $\mathscr{P}_+ = W_+ W_+^*$ is the orthogonal projection onto $\operatorname{ran} W_+$. In case of the asymptotically complete scattering, i.e., $\operatorname{ran} W_- = \operatorname{ran} W_+$, the term $(I - \mathscr{P}_+) W_- h$ vanishes and the asymptotic formulae look expecially attractive.

The Friedrichs model is a standard supplier of examples of incomplete scattering (see [9, 31]). The fact that the asymptotic completeness may fail is more difficult. One reason is that in most physical problems studied the scattering operator turns out to be unitary which is equivalent to the asymptotic completeness of the wave operators [34]. The Kato–Rosenblum theorem is another reason since it guarantees the asymptotic completeness together with the existence of the wave operators. However, the asymptotic completeness may fail even in the potential scattering. Such an example was first obtained by Pearson [32].

Our main tools are unitary couplings introduced by Adamjan and Arov in [1], the Sz.-Nagy–Foiaş function model, and approximate orthonormal systems.

DEFINITION. Let $\{e_n\}_{n\geqslant 0}$ be an orthonormal system in a Hilbert space H. An orthogonal system $\{f_n\}_{n\geqslant 0}$ is called approximate to $\{e_n\}_{n\geqslant 0}$ if $\lim_{n\to+\infty}\|e_n-f_n\|=0$.

This notion was first introduced and studied by Nina Bari [2]. We mention here two of her results which are of primary importance for the present paper. The first one is well known.

THEOREM 1.1 [3, 4]. *Let $\{e_n\}_{n\geqslant 0}$ be a complete orthonormal system in H and $\{f_n\}_{n\geqslant 0}$ an orthonormal system satisfying*

$$\sum_{n=0}^{\infty}\|e_n-f_n\|^2<+\infty. \tag{1}$$

Then $\{f_n\}_{n\geqslant 0}$ is complete in H.

The proof can be compressed into a few lines. It follows from (1) that the isometry W defined by $We_n=f_n$, $n=0, 1, \ldots$, can be represented as $W=I+K$, K being a Hilbert–Schmidt operator. Then

$$\operatorname{ind}(W)=\dim(\ker W)-\dim(\operatorname{coker} W)=\operatorname{ind}(I)=0.$$

But $\ker W=\{0\}$ since W is an isometry. It follows that $WH=H$ and $\{f_n\}_{n\geqslant 0}$ is a complete system. ∎

The second important result of [3] was undeservingly forgotten.

THEOREM 1.2 [3]. *Let $0\leqslant d_n\leqslant\sqrt{2}$, $\sum_{n=1}^{\infty}d_n^2=+\infty$. For every complete orthonormal system $\{e_n\}_{n\geqslant 1}$ there exists an incomplete orthonormal system $\{f_n\}_{n\geqslant 1}$ such that $\|f_n-e_n\|=d_n$ for $n=1, 2, \ldots$.*

In modern wording Bari's proof looks as follows. We assume that H is the Hardy class H^2. We recall that H^2 is the closed subspace of $L^2(\mathbb{T})$ spanned by the orthonormal sequence of monomials $\{z^n\}_{n\geqslant 0}$. Let

$$e_n(z)=a_{n0}+a_{n1}z+\cdots+a_{nn}z^n, \quad n=1, 2, \ldots.$$

Clearly, $\|e_n-z^n\|^2=2(1-a_{nn})$. We put $b_n\stackrel{\text{def}}{=}a_{nn}=1-d_n^2/2$, $n=1, 2, \ldots$. The polynomials $\{e_n\}_{n\geqslant 1}$ are defined by induction. Let

$$e_1(z)=(1-b_1^2)^{1/2}+b_1z.$$

If $e_1, \ldots, e_{n-1}$ have already been defined, we set

$$e_n(z)=(1-b_n^2)^{1/2}g_n+b_nz^n,$$

where g_n is the polynomial of degree $n-1$ which is uniquely determined by the conditions

$$g_n(0)>0, \qquad g_n \perp e_1, ..., e_{n-1}, \qquad \|g_n\|^2=1.$$

(Notice that without loss of generality we can assume that $0<d_n<\sqrt{2}$.) It is clear that $\{e_n\}_{n\geqslant 1}$ is an orthonormal system in H^2. To prove that it is complete it suffices to check that $1=z^0$ belongs to the closed span (notationally $1\in \operatorname{span}\{e_n\colon n=1,2,...\}$) of $\{e_n\}_{n\geqslant 1}$. The latter is equivalent to the Parseval identity

$$\sum_{n=1}^{\infty} a_{n0}^2=1.$$

To calculate a_{n0} we compare g_n and g_{n+1}. Since $g_{n+1}\perp e_1, ..., e_{n-1}$ it follows that

$$g_{n+1}=xg_n+y\cdot z^n, \quad x>0, x^2+y^2=1.$$

Taking into account that $g_{n+1}(0)>0$ and $g_{n+1}\perp e_n$, we obtain

$$g_{n+1}=b_n g_n-(1-b_n^2)^{1/2} z^n.$$

Since $g_1\equiv 1$ we obtain successively

$$a_{10}=(1-b_1^2)^{1/2}, a_{20}=b_1(1-b_2^2)^{1/2}, ...,$$
$$a_{n0}=b_1\cdots b_{n-1}(1-b_n^2)^{1/2}, ...$$

Thus

$$\sum_{n=1}^{\infty} a_{n0}^2=(1-b_1^2)+b_1^2(1-b_2^2)+\cdots+b_1^2\cdots b_{n-1}^2(1-b_n^2)+\cdots$$
$$=1-\lim_{n\to+\infty} b_1^2b_2^2\cdots b_n^2=1$$

because $\sum_{n=1}^{\infty}(1-b_n)=\frac{1}{2}\sum_{n=1}^{\infty} d_n^2=+\infty$. We now put $f_n=z^n$, $n=1,2,...$, and observe that the defect $\operatorname{def}\{f_n\}_{n\geqslant 1}$ of $\{f_n\}_{n\geqslant 1}$ in H^2 equals 1. ∎

COROLLARY 1.1. *Let* $0\leqslant d_n\leqslant\sqrt{2}$, $\sum_{n=1}^{\infty} d_n^2=+\infty$, *and* $\alpha\leqslant\aleph_0$. *Then for any complete orthonormal system* $\{e_n\}_{n\geqslant 1}$ *there exists an orthonormal system* $\{f_n\}_{n\geqslant 1}$ *with* $\operatorname{def}\{f_n\}=\alpha$ *such that* $\|f_n-e_n\|=d_n$, $n=1,2,....$

Proof. We decompose $\{d_n\}_{n\geqslant 1}$ into α subsequences $\{d_{n_j}\}$ with $\sum d_{n_j}^2=+\infty$ and apply Theorem 1.2 to each of them. ∎

We can reverse the succession of $\{e_n\}$ and $\{f_n\}$ in Corollary 1.1.

COROLLARY 1.2. *Let* $0 \leqslant d_n \leqslant \sqrt{2}$, $\sum_{n=1}^{\infty} d_n^2 = +\infty$. *Then for any orthonormal system* $\{f_n\}_{n \geqslant 1}$ *in* H *there exists a complete orthonormal system* $\{e_n\}_{n \geqslant 1}$ *such that* $\|f_n - e_n\| = d_n$.

There is another approach to the proof of Theorem 1.2. Leading to a weaker result, however, it reveals important connections with the scattering theory.

A function θ in H^2 is called inner if $|\theta| = 1$ a.e. on $\mathbb{T}$ (see [10] for details). Any inner function determines the invariant subspace θH^2 of the shift $\mathcal{S}$ on $L^2(\mathbb{T})$. We split H^2 into the orthogonal sum

$$H^2 = K_\theta \oplus \theta H^2.$$

It is well known that $\dim K_\theta = \aleph_0$ if and only if θ is not a finite Balschke product [28].

Suppose that $\dim K_\theta = \aleph_0$ and define an auxiliary isometry W: $H^2 \to H^2$ as follows. On θH^2 we put $Wh = h$ while on K_θ we let W be an arbitrary isometry satisfying $\dim(K_\theta \ominus WK_\theta) = \alpha$, $0 \leqslant \alpha \leqslant \aleph_0$. Now we set

$$e_n = z^n, \qquad f_n = We_n, \quad n = 0, 1, \ldots.$$

To estimate the norm $\|e_n - f_n\|$ we observe that the orthogonal projection P_θ onto K_θ is defined by

$$P_\theta f = \theta \mathbf{P}_-(\bar{\theta} f), \qquad f \in H^2,$$

where $\mathbf{P}_-$ denotes the orthogonal projection onto $H^2_- \stackrel{\text{def}}{=} L^2 \ominus H^2$. Since $\{e_n\}_{n \geqslant 0}$ is an orthonormal basis in H^2 every f in H^2 expands into the Fourier series

$$f = \sum_{n=0}^{\infty} \hat{f}(n) z^n,$$

$\hat{f}(n) = \int_{\mathbb{T}} f \cdot \bar{z}^n \, dm$ being the Fourier coefficients of f. Now

$$\begin{aligned} \|f_n - e_n\| &= \|(W - I) P_\theta e_n\| \leqslant 2 \|P_\theta e_n\| \\ &= 2 \|\mathbf{P}_-(\bar{\theta} z^n)\| = 2 \left(\sum_{k<n} |\hat{\theta}(k)|^2 \right)^{1/2} \end{aligned}$$

There exists an infinite Blaschke product B satisfying $\hat{B}(k) = O(1/k)$ [25]. We put $\theta = B$ and obtain

$$\|f_n - e_n\| = O\left(\frac{1}{\sqrt{n}}\right), \qquad n \to +\infty. \quad \blacksquare$$

Now we can treat a very special case of the inverse scattering problem. Namely, put $E=\mathbb{C}$, $S=\theta^*=\bar{\theta}$, where θ is an inner function. The orthonormal system $\{f_n\}_{n\in\mathbb{Z}}$ in $H=L^2(\mathbb{T})$ is defined by

$$f_n = z^n \text{ for } n<0; \qquad f_n=\theta z^n \text{ for } n\geqslant 0.$$

Clearly, $f_n \perp K_\theta$ for $n\in\mathbb{Z}$. Applying either the Bari theorem or the above arguments we can construct a complete orthonormal system $\{e_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{T})$ satisfying

$$\lim_{|n|\to+\infty} \|e_n - f_n\| = 0. \tag{2}$$

We consider the unitary operator $\mathscr{F}$ defined by $\mathscr{F}z^k=e_k$, $k\in\mathbb{Z}$, and put $\mathring{U}=\mathscr{S}$, $U=\mathscr{F}^*\mathscr{S}\mathscr{F}$. We fix $m\in\mathbb{Z}$ and using (2) obtain

$$\begin{aligned} W_+z^m &= \lim_{n\to+\infty} U^{-n}\mathring{U}^n z^m = \lim_{n\to+\infty} \mathscr{F}^*\mathscr{S}^{-n}e_{n+m} \\ &= \lim_{n\to+\infty} \mathscr{F}^*\mathscr{S}^{-n}f_{n+m} = \mathscr{F}^*(\theta z^m). \end{aligned}$$

Similarly

$$\begin{aligned} W_-z^m &= \lim_{n\to-\infty} U^{-n}\mathring{U}^n z^m = \lim_{n\to-\infty} \mathscr{F}^*\mathscr{S}^{-n}e_{n+m} \\ &= \lim_{n\to-\infty} \mathscr{F}^*\mathscr{S}^{-n}f_{n+m} = \mathscr{F}^*z^m. \end{aligned}$$

It follows that $W_+^*W_- = \theta^*\mathscr{F}\mathscr{F}^* = \theta^*$. ∎

In the general case S is an operator-valued function which is not necessarily unitary-valued, while S^* may not be analytic. To overcome these difficulties we need more information about approximate orthonormal systems. In Section 2 we prove Theorem 2.2, which is used in Section 4 for the solution of the inverse scattering problem.

Given S there may exist different U such that S is the scattering operator of $(U, \mathring{U})$. Such nonuniqueness is related with the existence of isometries W of the form $W=I+A$, where A satisfies

$$\lim_{|n|\to+\infty} \|Ae_n\| = 0$$

for some orthonormal system $\{e_n\}_{n\in\mathbb{Z}}$ in H. By Weyl's criterion (see Section 2) this can happen if and only if the operator A is not essentially left invertible (see the definition in Section 2). In Section 2 we consider in detail the operators which are not essentially left invertible. We prove a generalization of Weyl's criterion, Theorem 2.1. The rest of Section 2 except for Theorem 2.2 is not required in what follows. We discuss here applications of Theorem 2.1. One important application, Theorem 2.4, concerns the hereditary completeness property. We show how Theorem 2.1 can be used to obtain the main result of [7].

In Section 3 we treat general aspects of the inverse scattering problem. We specify the contractions which can be scattering operators and present an algebraic model which describes all pairs $(U, \mathring{U})$ with given scattering operator S. Here we do not assume that the spectral type $[\mathring{\mathscr{E}}]$ of the spectral measure $\mathring{\mathscr{E}}$ of $\mathring{U}$ is the type of the Lebesgue measure m. Such a general setting reveals an interesting connection with a class of thin sets in harmonic analysis, the so-called U_0-sets. The Banach space $M(\mathbb{T})$ of finite complex Borel measures contains a closed subspace $M_0(\mathbb{T})$ consisting of measures whose Fourier coefficients vanish at infinity. The spectral measure $\mathring{\mathscr{E}}$ decomposes into the orthogonal sum $\mathring{\mathscr{E}} = \mathring{\mathscr{E}}_0 \oplus \mathring{\mathscr{E}}_1$, where $[\mathring{\mathscr{E}}_0] \in M_0(\mathbb{T})$ and $[\mathring{\mathscr{E}}_1]$ is singular to $M_0(\mathbb{T})$. We show that if the wave operators of $(U, \mathring{U})$ exist then $U = \mathring{U}$ on $\mathring{\mathscr{E}}_1 H$. In particular, if the spectrum $\sigma(\mathring{U})$ of $\mathring{U}$ is a U_0-set then the existence of the wave operators yields $U = \mathring{U}$.

In Section 4 we deal with the inverse problem for the homogeneous Lebesgue spectrum. We prove (Theorem 4.1) that any contraction which is unitarily equivalent to an operator-valued function on $L^2(E)$ is the scattering operator of a pair $(U, \mathring{U})$. We also discuss a function model for general scattering with homogeneous Lebesgue spectrum. In case S^* is analytic in $\mathbb{D}$ the function S^* coincides with the characteristic function of a contraction and therefore any known function model can be used for the purposes of the scattering theory. In [26, 27] one can find different transcriptions of function models. The most important among them are the Sz.-Nagy–Foiaş model, the de Branges–Rovnyak model, and the Pavlov model.

2. A Generalization of Weyl's Criterion, Approximate Orthonormal Systems and Their Applications

Definition. A closed operator A with dense domain $\mathscr{D}(A)$ in H is called essentially left invertible if there exists a bounded operator B such that $BA = I + K$ on $\mathscr{D}(A)$, where K is compact.

Every linear operator A: $H \to H'$ determines a linear subspace $G(A) \stackrel{\text{def}}{=} \{h \oplus Ah : h \in \mathscr{D}(A)\}$ in $H \oplus H'$. The subspace $G(A)$ is called the graph of A. Recall that A is closed if and only if $G(A)$ is a closed subspace of $H \oplus H'$ [19].

Theorem (H. Weyl). *A necessary and sufficient condition that a bounded operator A not be essentially left invertible is that there exists an orthonormal sequence $\{f_n\}$ such that* $\lim_{n \to +\infty} \|Af_n\| = 0$.

See [15, p. 89] for a proof of Weyl's criterion. Here we prove a generalization of Weyl's criterion.

THEOREM 2.1. *Let A be a closed densely defined operator in H which is not essentially left invertible. Let $d_n > 0$, $n > 0, 1, \ldots, \sum_{n=0}^{\infty} d_n^2 = \infty$. Then there exists a complete orthonormal system $\{e_n\}_{n \geqslant 0}$, $e_n \in \mathscr{D}(A)$, $n = 0, 1, \ldots$, such that $\|Ae_n\| \leqslant d_n$, $n = 0, 1, \ldots$, and $\{e_n \oplus Ae_n\}_{n \geqslant 0}$ spans the graph $G(A)$ of A.*

In other words Theorem 2.1 says that every such operator "almost" belongs to the Hilbert–Schmidt class.

The result looks interesting even for Hilbert–Schmidt operators. Then, of course, $\sum_{n=0}^{\infty} \|Ae_n\|^2 < +\infty$ for every complete orthonormal system $\{e_n\}_{n \geqslant 0}$, but the inequality $\|Ae_n\| > d_n$ may occur infinitely often in general.

A bounded operator with infinite-dimensional kernel is another example of an operator which is not essentially left invertible. In fact Atkinson's theorem (see [15, p. 87]) claims that a bounded operator A is essentially left invertible if and only if ker A is a finite-dimensional subspace and ran A is closed.

The class of Hankel operators provides further examples of such operators [33]. Recall that the Hankel operator H_φ with symbol $\varphi \in L^\infty(\mathbb{T})$ is defined on the Hardy class H^2 by

$$H_\varphi f = \mathbf{P}_-(\varphi f), \qquad f \in H^2.$$

Clearly, $\lim_{n \to +\infty} \|\mathbf{P}_-(\varphi z^n)\| = 0$, which implies that H_φ is not essentially left invertible. Similarly, $\lim_{n \to -\infty} \|\mathbf{P}_+(\bar{\varphi} z^n)\| = 0$, which means that H_φ^* is not essentially left invertible. It follows that a bounded Hankel operator is not essentially left, as well as right, invertible. Recall that the class of bounded operators which are not essentially right invertible coincides with the class of operators which are unitarily equivalent to bounded Carleman integral operators on $L^2(\mathbb{T})$ [15].

For the sake of completeness we give a proof to the following lemma.

LEMMA 2.1. *Let A be a closed operator with dense domain $\mathscr{D}(A)$ in H. The following are equivalent:*

(1) *A is essentially left invertible;*

(2) *there is no orthonormal sequence $\{f_n\}$ in $\mathscr{D}(A)$ such that $\|Af_n\| \to 0$;*

(3) *the range of A is closed and* $\dim \ker A < \aleph_0$.

Proof. We consider the polar factorization $A = V|A|$ of A [6]. Recall that $|A|$ is a selfadjoint operator with domain $\mathscr{D}(A)$ and V is a partial isometry with the initial space clos(ran A^*) and the final space clos(ran A). Since $V^*A = |A|$ we see that A satisfies (1), (2), or (3) if and only if $|A|$ satisfies (1), (2), or (3). Next, since $|A|$ is selfadjoint we can split H into an orthogonal sum $H = H_0 \oplus H_1$ of reducing subspaces of $|A|$ so that the restriction of $|A|$ to H_1 is left invertible and $|A| \,|\, H_0$ is bounded. It remains to observe that $|A| \,(\mathscr{D}(|A|) \cap H_1) = H_1$ and to apply Atkinson's theorem and Weyl's criterion to the bounded operator $|A| \,|\, H_0$. ∎

Proof of Theorem 2.1. Suppose first that A is bounded. By Weyl's criterion there exists an orthogonal sequence $\{f_n\}_{n \geq 0}$ satisfying $\|Af_n\| \leq d_n/2$. By Corollary 1.2 there exists a complete orthonormal system $\{e_n\}_{n \geq 0}$ in H such that $\|e_n - f_n\| \leq d_n \cdot (2\,\|A\|)^{-1}$, $n = 0, 1, 2, \ldots$. It follows that

$$\|Ae_n\| \leq \|Af_n\| + \|A\| \cdot \|e_n - f_n\| \leq d_n/2 + d_n/2 = d_n.$$

If now $x \oplus Ax \perp e_n \oplus Ae_n$, $n = 0, 1, \ldots$, then $(I + A^*A)x \perp e_n$, $n = 0, 1, \ldots$, and therefore $x = 0$ since $A^*A \geq 0$.

In the general case using the polar factorization we can assume without loss of generality that $A = |A|$ is a nonnegative selfadjoint operator. Indeed, we have $\|Ae_n\| = \|\,|A|\, e_n\|$ and

$$\begin{aligned}(e_n \oplus Ae_n, x \oplus Ax) &= (e_n, x) + (V\,|A|\, e_n, V\,|A|\, x) \\ &= (e_n \oplus |A|\, e_n, x \oplus |A|\, x).\end{aligned}$$

Let $\mathscr{E}$ be the spectral measure of A, $A = \int_0^{+\infty} \lambda \, d\mathscr{E}(\lambda)$. Since A is not essentially left invertible it follows (from Weyl's criterion, see Lemma 2.1) that $\dim \mathscr{E}(\Delta_\varepsilon)H = \aleph_0$ for every interval $\Delta_\varepsilon = [0, \varepsilon)$, $\varepsilon > 0$. Thus

$$\mathscr{E}(\Delta_1)H = \bigoplus_{i=0}^{\infty} K_i,$$

where K_i reduces A and $A \,|\, K_i$ is not essentially left invertible. We now put $H_i = K_i \oplus \mathscr{E}([i+1, i+2))H$, $i = 0, 1, 2, \ldots$. Then $H = \bigoplus_{i=0}^{\infty} H_i$, H_i reduces A, and $A \,|\, H_i$ is bounded. Let σ: $\mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ be a one-to-one mapping such that

$$\sum_{j=0}^{\infty} d_{\sigma(i,j)}^2 = +\infty$$

for every $i \geqslant 0$. Since $A \mid H_i$ is bounded and is not essentially left invertible, there exists a complete orthonormal system $\{\varphi_{ij}\}_{j \geqslant 0}$ in H_i such that $\|A\varphi_{ij}\| \leqslant d_{\sigma(i,j)}$. Then $\{e_n\}_{n \geqslant 0}$, $e_n =^{\text{def}} \varphi_{\sigma^{-1}n}$, $n \in \mathbb{Z}_+$, is a complete orthonormal system in $\mathscr{D}(A)$ satisfying $\|Ae_n\| \leqslant d_n$.

Since $e_n \in \mathscr{D}(A^*A)$ it follows that $\operatorname{span}\{e_n \oplus Ae_n\colon n \geqslant 0\} = G(A)$ if $\operatorname{span}\{(I + A^*A)e_n\colon n = 0, 1, \ldots\} = H$. But H_i reduces A, $A \mid H_i$ is bounded, and $A^* = A$. Therefore $(I + A^*A)$ maps H_i isomorphically onto itself. Thus $\operatorname{span}\{(I + A^*A)\varphi_{ij}\colon j = 0, 1, \ldots\} = H_i$ and it remains to notice that $H = \bigoplus_{i=0}^{\infty} H_i$. ∎

We now consider an application of Corollary 1.1 to approximate orthonormal systems lying in a given closed subspace. Corollary 1.1 says that for every complete orthonormal system $\{e_n\}_{n \geqslant 0}$ in H there exists a closed subspace F with $\operatorname{def} F = \dim(H \ominus F) = \aleph_0$ such that $\lim_{n \to +\infty} \operatorname{dist}(e_n, F) = 0$. Indeed, $F = \operatorname{span}\{f_n\colon n = 0, 1, \ldots\}$ satisfies the conditions stated.

Suppose now that we are given an orthonormal system $\{e_n\}_{n \geqslant 0}$ in H and a closed subspace F satisfying $\lim_{n \to \infty} \operatorname{dist}(e_n, F) = 0$. The following question naturally arises. Is it possible to construct a complete orthonormal system $\{f_n\}_{n \geqslant 0}$ in F such that $\lim_{n \to +\infty} \|e_n - f_n\| = 0$?

It is worth mentioning that a complete orthonormal system $\{e_n\}$ cannot approximate a subspace F, $\operatorname{def} F = \aleph_0$, too fast. For such pairs $\{e_n\}$, F we always have

$$\sum_{n=0}^{\infty} \operatorname{dist}^2(e_n, F) = +\infty.$$

Otherwise we could find a sequence $\{f_n\}_{n \geqslant 0}$ in F with $\sum_{n=0}^{\infty} \|e_n - f_n\|^2 < +\infty$, which would contradict the hypothesis $\operatorname{def} F = \aleph_0$ [11].

The following lemma is the basic geometric tool in the construction of approximate orthogonal systems lying in a fixed subspace.

LEMMA 2.2. *Let $\{e_n\}_{n \geqslant 0}$ be an orthonormal system approximating a closed subspace F, i.e., $\lim_{n \to +\infty} \operatorname{dist}(e_n, F) = 0$. Let G be an infinite-dimensional subspace of F such that*

$$\lim_{n \to +\infty} \|\mathscr{P}_G e_n\| = 0, \tag{1}$$

$\mathscr{P}_G$ being the orthogonal projection onto G. Then there exists an isometry W: $H \to F$ such that $\lim_{n \to +\infty} \|We_n - e_n\| = 0$.

Proof. We put $K = H \ominus F$, $F_1 = F \ominus G$. Then $H = K \oplus G \oplus F_1$. For $h \in F_1$ we put $Wh = h$. Since $\dim G = \aleph_0$ we can choose W so that W maps $K \oplus G$ isometrically onto G. We have

$$\begin{aligned}\| We_n - e_n \| &= \| (W - I)\, \mathscr{P}_{K\oplus G} e_n + (W - I)\, \mathscr{P}_{F_1} e_n \| \\ &= \| (W - I)\, \mathscr{P}_{K\oplus G} e_n \| \leqslant 2\, \| \mathscr{P}_K e_n \oplus \mathscr{P}_G e_n \| \\ &= 2(\| \mathscr{P}_K e_n \|^2 + \| \mathscr{P}_G e_n \|^2)^{1/2}. \end{aligned} \tag{2}$$

Since $\|\mathscr{P}_K e_n\| = \operatorname{dist}(e_n, F)$, this implies that $\lim_{n\to+\infty} \| We_n - e_n \| = 0$. ∎

The following lemma is the main analytic tool in our construction.

LEMMA 2.3. *Let F be a closed subspace of H and $\{e_n\}_{n\geqslant 0}$ an orthonormal sequence satisfying $\lim_{n\to+\infty} \operatorname{dist}(e_n, F) = 0$. Then there exists an infinite-dimensional subspace G in F satisfying* (1).

Proof. We construct G as $\operatorname{span}\{g_k\colon k = 0, 1, \ldots\}$, where $\{g_k\}_{k\geqslant 0}$ is an orthonormal sequence in G. We construct $\{g_k\}_{k\geqslant 0}$ in turn by induction using an auxiliary family

$$h_{kl} = 2^{-k/2} \sum_{j=0}^{2^k - 1} e_{j+l}. \tag{3}$$

It is clear that $\|h_{kl}\| = 1$ and $\lim_{l\to+\infty} \operatorname{dist}(h_{kl}, F) = 0$ for every k. Putting here $k = 0$, we see that there exist an integer l_0 and a unit vector g_0 in F such that

$$\|h_{0l_0} - g_0\| < 1.$$

Suppose that we have already constructed integers $l_0 < l_1 < \cdots < l_{k-1}$ and an orthonormal family $g_0, g_1, \ldots, g_{k-1}$ of vectors of F such that

(1) $\{h_{sl_s}\}_{s=0}^{k-1}$ is an orthonormal family:

(2) $\|h_{sl_s} - g_s\| < 2^{-s}$, $s = 0, 1, \ldots, k-1$.

Clearly $h_{kl} \perp h_{sl_s}$ for $s = 0, 1, \ldots, k-1$ if $l \geqslant l_{k-1} + 2^{k-1}$. Since $\{e_n\}_{n\geqslant 0}$ is an orthonormal sequence we obtain that

$$\lim_{n\to+\infty} (g_s, e_n) = 0 \tag{4}$$

for $s = 0, 1, \ldots, k$. It follows that the norm

$$\left(\sum_{s=0}^{k} |(h_{kl}, g_s)|^2 \right)^{1/2}$$

of the orthogonal projection of h_{kl} onto $G_k =^{\text{def}} \text{span}\{g_0, ..., g_{k-1}\}$ tends to zero as $l \to +\infty$. In addition $\lim_{l \to +\infty} \text{dist}(h_{kl}, F) = 0$. Hence the norm of the orthogonal projection onto G_k of the vector in F of the best approximation to h_{kl} tends to zero too. It follows that there exist $l_k > l_{k-1} + 2^{k-1}$ and a unit vector g_k in F such that $\|h_{kl_k} - g_k\| < 2^{-k}$ and $g_k \perp G_k$.

We claim that the space $G = \text{span}\{g_k: k = 0, 1, ...\}$ constructed satisfies (1). Indeed,

$$\|\mathcal{P}_G e_n\|^2 = \sum_{k=0}^{\infty} |(e_n, g_k)|^2.$$

Letting $h_s =^{\text{def}} h_{sl_s}$ for brevity and taking into account the inequality $\|h_s - g_s\| < 2^{-s}$ we obtain

$$|(e_n, g_s)| \leqslant 2^{-s} + |(e_n, h_s)|. \tag{5}$$

According to (3) either $(e_n, h_s) = 0$ or $(e_n, h_s) = 2^{-s/2}$. Hence $|(e_n, g_s)| \leqslant 2 \cdot 2^{-s/2}$ and therefore

$$\|\mathcal{P}_G e_n\|^2 \leqslant \sum_{k=0}^{N} |(e_n, g_k)|^2 + 4 \sum_{s<N} 2^{-s}.$$

Using (4) we obtain (1). ∎

THEOREM 2.2. *Let $\{e_n\}_{n \geqslant 0}$ be an orthonormal system in H and F a closed subspace of H. Then there exists a complete orthonormal system $\{f_n\}_{n \geqslant 0}$ in F such that*

$$\lim_{k \to +\infty} \|e_{n_k} - f_{n_k}\| = 0$$

for every subsequence $\{n_k\}_{k \geqslant 0}$ satisfying $\lim_{k \to +\infty} \text{dist}(e_{n_k}, F) = 0$.

Proof. We fix any subsequence $\{n_k\}_{k \geqslant 0}$ with the property stated in the theorem and apply Lemmas 2.2 and 2.3 to $\{e_{n_k}\}$ and F. We obtain a closed subspace $G \subset F$ and an isometry W: $H \to F$. By Lemma 2.3 we have $\lim_{k \to \infty} \|\mathcal{P}_G e_{n_k}\| = 0$. Suppose now that $m \neq n_k$, $k = 0, 1, ...$ Then $e_m \perp e_{n_k}$ for $k = 0, 1, ...$ In particular, we have $e_m \perp h_k$, $k = 0, 1, ...,$ where $\{h_k\}$ are the vectors constructed in Lemma 2.3. It follows from (5) that $|(e_m, g_s)| \leqslant 2^{-s}$ and therefore

$$\|\mathcal{P}_G e_m\|^2 \leqslant \sum_{k=0}^{N} |(e_m, g_k)|^2 + \sum_{s<N} 2^{-2s},$$

which yields $\lim_{m\to+\infty}\|\mathscr{P}_G e_m\|=0$. It is clear from (2) that $\{f_n\}_{n\geqslant 0}$, $f_n=We_n$ satisfies the conditions required except, perhaps, the completeness. Applying Corollary 1.2 we can approximate $\{f_n\}_{n\geqslant 0}$ by a complete system in F. ∎

The following obvious corollary of Theorem 2.2 will be used in the proof of Theorem 4.1 below.

COROLLARY 2.1. *Let $\{e_{nj}: n=0, 1, ..., j\in J\}$ be an orthonormal system in H and F a closed subspace of H satisfying*

$$\lim_{n\to+\infty}\operatorname{dist}(e_{nj}, F)=0$$

for every $j\in J$. Then there exists a complete orthonormal system $\{f_{nj}\}$ in F such that $\lim_{n\to+\infty}\|e_{nj}-f_{nj}\|=0$, $j\in J$.

As another application of Corollary 1.1 we mention one result first obtained in [8].

THEOREM 2.3. *Let $\{e_n\}_{n\geqslant 0}$ be a complete orthonormal system in H and $\{d_n\}_{n\geqslant 0}$ be a sequence of nonnegative numbers with $\sum_{n=0}^{\infty} d_n^2=+\infty$. Then there exists an infinite-dimensional subspace G in H such that*

$$(g, e_n)=O(d_n), \qquad n\to+\infty,$$

for every g in G.

Proof. We apply Corollary 1.1 to $\{e_n\}_{n\geqslant 0}$ with $\alpha=\aleph_0$. Let $\{f_n\}_{n\geqslant 0}$ be the orthonormal system obtained and $G=\{g\in H\colon (g, f_n)=0, n=0, 1, 2, ...\}$. Then $\dim G=\aleph_0$ and for $g\in G$ we have

$$|(g, e_n)|=|(g, e_n-f_n)|\leqslant\|g\|\cdot\|e_n-f_n\|\leqslant\|g\|\cdot d_n. \quad ∎$$

Remark [8]. If $\sum_{n=0}^{\infty} d_n^2<+\infty$ then the subspace G with the properties stated in Theorem 2.3 cannot exist. Indeed, for every $\varepsilon>0$ the convex set $K(\varepsilon)=\{h\in H\colon |(h, e_n)|\leqslant\varepsilon\, d_n\}$ is compact. Thus the unit ball of such a G must be compact in the norm-topology of H which yields $\dim G<\aleph_0$.

Now we indicate an application of Theorem 2.1 to the Schatten–von Neumann classes. We recall that a bounded operator A belongs to $\mathfrak{S}_p$, $0<p<\infty$, if A is compact and the sequence $\{S_n(A)\}_{n\geqslant 0}$ of the eigenvalues of $(A^*A)^{1/2}$ counted with their multiplicities satisfies

$$\|A\|_{\mathfrak{S}_p}^p \overset{\text{def}}{=} \sum_{n=0}^{\infty} S_n^p(A)<+\infty.$$

It has been shown in [12, 13] (see also [11]) that for $p \leqslant 2$ and for every bounded operator A the inequality $\|A\|_{\mathfrak{S}_p}^p \leqslant \sum_{n=0}^{\infty} \|Ae_n\|^p$ holds for every orthonormal basis in H. For $p > 2$ the opposite inequality holds. In that case the convergence of $\sum_{n=0}^{\infty} \|Ae_n\|^p$ cannot provide the inclusion $A \in \mathfrak{S}_\infty$. In fact, as Theorem 2.1 shows, we cannot even assert that A is bounded. Indeed, let $d_n = (n+1)^{-1/2}$, $n = 0, 1, \ldots$. Then for any A which is not essentially left invertible there exists a complete orthonormal system $\{e_n\}_{n \geqslant 0}$ such that $\sum_{n=0}^{\infty} \|Ae_n\|^p < +\infty$ for every $p > 2$.

Theorem 2.1 can also be applied to a construction of complete orthonormal systems with curious properties (see [29, p. 54]).

Let w be an arbitrary nonnegative function on $[0, 1]$ finite a.e. and such that $w^{-1} \notin L^\infty[0, 1]$ but $w^{-1} \in \bigcap_{p < +\infty} L^p[0, 1]$. We consider the operator A, $Af = \sqrt{w}\, f$ on $L^2[0, 1]$. Clearly A is not essentially left invertible. It follows that there exists a complete orthonormal system $\{e_n\}_{n \geqslant 0}$ in L^2 (for a given sequence $\{d_n\}_{n \geqslant 0}$, $d_n > 0$, $\sum_{n=0}^{\infty} d_n^2 = +\infty$) such that

$$\int_0^1 |e_n|^2 w\, dx \leqslant d_n^2.$$

For every $p < 2$, using the Hölder inequality, we obtain

$$\int_0^1 |e_n|^p\, dx = \int_0^1 |e_n|^p w^{p/2} w^{-p/2}\, dx$$
$$\leqslant \left(\int_0^1 |e_n|^2 w\, dx \right)^{p/2} \cdot \left(\int_0^1 w^{-p/(2-p)}\, dx \right)^{(2-p)/2}.$$

This yields $\|e_n\|_p \leqslant c_p d_n$ for every p, $p < 2$.

We turn to an application of Theorem 2.1 to the geometry of Hilbert spaces. The application to the hereditary completeness requires some preliminaries.

A family $\{x_n\}_{n \in \mathbb{Z}}$ of vectors of H is called minimal if $x_n \notin \operatorname{span}\{x_k : k \neq n\}$. If, in addition, $\{x_n\}_{n \in \mathbb{Z}}$ is complete then there exists a unique biorthogonal family $\{x_n'\}_{n \in \mathbb{Z}}$, i.e., $(x_n, x_m') = \delta_{nm}$. It follows that every vector $x \in H$ can be expanded into the formal Fourier series

$$x \sim \sum_{n \in \mathbb{Z}} (x, x_n') x_n. \tag{6}$$

If the series in (6) converges in any reasonable sense then obviously

$$x \in \operatorname{span}\{x_n : (x, x_n') \neq 0\}. \tag{7}$$

Definition. A complete minimal family $\{x_n\}_{n \in \mathbb{Z}}$ is called hereditarily complete if (7) holds for every x in H.

Any basis of a Hilbert space is a hereditarily complete family. The simplest example of a family which is not hereditarily complete is given by a complete minimal family $\{x_n\}_{n\in\mathbb{Z}}$ whose biorthogonal family $\{x'_n\}_{n\in\mathbb{Z}}$ is not complete. Then (7) is violated for vectors x such that $(x, x'_n)=0$, $n\in\mathbb{Z}$. A more difficult example with complete biorthogonal family was first obtained in [23]. In [14, 23] V. I. Gurarii developed a method of quasi-complementary subspaces which allowed him to construct many examples of such type.

For $\sigma\subset\mathbb{Z}$ let $H_\sigma=\operatorname{span}\{x_n\colon\ n\in\sigma\}$ and $H^\sigma=\{x\in H\colon\ (x, x'_n)=0,\ n\in\mathbb{Z}\backslash\sigma\}$. Then $H_\sigma\subset H^\sigma$ and $\{x_n\}$ is hereditarily complete if and only if $H_\sigma=H^\sigma$ for every $\sigma\subset\mathbb{Z}$ (see [7] for details). If we suppose that both $\{x_n\}_{n\in\mathbb{Z}}$ and $\{x'_n\}_{n\in\mathbb{Z}}$ are complete then $H_\sigma\cap H_{\mathbb{Z}\backslash\sigma}=\{0\}$ and $\operatorname{clos}(H_\sigma+H_{\mathbb{Z}\backslash\sigma})=H$ for every σ. If we suppose further that $H_\sigma\neq H^\sigma$ then $H_{\mathbb{Z}\backslash\sigma}+H_\sigma\neq H$ [7].

DEFINITION [14]. Subspaces X, Y of H are called quasi-complementary if $X\cap Y=\{0\}$ and $\operatorname{clos}(X+Y)=H$. If in addition $X+Y\neq H$ then we write $H=X\,\hat{+}\,Y$.

V. I. Gurarii observed that every pair X, Y of subspaces with $H=X\,\hat{+}\,Y$ leads to a complete minimal family which is not hereditarily complete. In [7] his construction was extended to obtain families required, which approximate orthogonal bases in H. Here we show that Theorem 2 of [7] is a corollary of Theorem 2.1.

There is a one-to-one correspondence between pairs (X, Y) of quasi-complementary subspaces and closed densely defined operators $\Gamma\colon X^\perp\to X$. The correspondence is given by the equality

$$Y=G(\Gamma)\overset{\text{def}}{=}\{x^\perp\oplus\Gamma x^\perp\colon x^\perp\in\mathscr{D}(\Gamma)\}.$$

Indeed, $X\cap Y=\{0\}$ means that Y is the graph of a closed operator Γ. The domain $\mathscr{D}(\Gamma)$ of Γ is dense in $X^\perp$ since $\mathscr{D}(\Gamma)=\mathscr{P}_{X^\perp}Y$ and $\operatorname{clos}(X+Y)=H$ ($\mathscr{P}$ stands for the orthogonal projection onto the corresponding subspace).

Since Γ is closed the domain $\mathscr{D}(\Gamma^*)=\mathscr{P}_XY^\perp$ of the adjoint operator Γ^* is dense in X [19]. The operator $\Gamma^*\colon X\to X^\perp$ is defined by

$$\Gamma^*(\mathscr{P}_X y^\perp)=-\mathscr{P}_{X^\perp}y^\perp,\qquad y^\perp\in Y^\perp.$$

It follows that $G(-\Gamma^*)=Y^\perp$.

The core of the method, as it is presented in [7], is illustrated by the following construction. Let $\Gamma_0\subset\Gamma_1$ be a pair of closed densely defined operators, $\Gamma_i\colon X^\perp\to X$, $i=0, 1$, such that $\dim\mathscr{D}(\Gamma_1)/\mathscr{D}(\Gamma_0)\geqslant 1$. To obtain an example of such a pair we suppose that $H=X\,\hat{+}\,Y_0$. Then there exists a vector y, $y \in H \backslash (X + Y_0)$, and we can put $Y_1 = \text{span}\{Y_0, y\}$. We now define Γ_0 and Γ_1 by $G(\Gamma_i) = Y_i$, $i = 0, 1$.

Let $\{\varphi_n\}_{n \geqslant 0}$ be a complete orthonormal system in $X^\perp$ satisfying $\varphi_n \in \mathscr{D}(\Gamma_0)$, $n = 0, 1, \ldots$, and $\{\varphi_n\}_{n<0}$ be a complete orthonormal system in X satisfying $\varphi_n \in \mathscr{D}(\Gamma_1^*)$, $n = -1, -2, \ldots$. Then

$$x_n = \begin{cases} \varphi_n \oplus \Gamma_0 \varphi_n, & n \geqslant 0 \\ 0 \oplus \varphi_n, & n < 0; \end{cases} \qquad x'_n = \begin{cases} \varphi_n \oplus 0, & n \geqslant 0 \\ (-\Gamma_1^* \varphi_n) \oplus \varphi_n, & n < 0 \end{cases}$$

are complete biorthogonal families in H. Clearly, $H_+ =^{\text{def}} \text{span}\{x_n: n \geqslant 0\} \subset G(\Gamma_0) = Y_0$, while $\text{span}\{x'_n: n < 0\} \subset G(-\Gamma_1^*) = G(\Gamma_1)^\perp$. It follows that $H^+ =^{\text{def}} [\text{span}\{x'_n: n<0\}]^\perp \supset G(\Gamma_1)^{\perp\perp} = G(\Gamma_1) = Y_1 \not\subseteq Y_0 \supset H_+$. Hence $\{x_n\}_{n \in \mathbb{Z}}$ is not hereditarily complete.

Since

$$\|x_n - \varphi_n\| = \|\Gamma_0 \varphi_n\|, n \geqslant 0, \quad \|x'_n - \varphi_n\| = \|\Gamma_1^* \varphi_n\|, n < 0,$$

the choice of $\{\varphi_n\}_{n \in \mathbb{Z}}$ can be specified by Theorem 2.1 to force $\{x_n\}_{n \in \mathbb{Z}}$ and $\{x'_n\}_{n \in \mathbb{Z}}$ to approximate a complete orthonormal system $\{\varphi_n\}_{n \in \mathbb{Z}}$.

THEOREM 2.4 [7]. *Let $\{d_n\}_{n \in \mathbb{Z}}$ be a sequence of positive numbers such that $\sum_{n \geqslant 0} d_n^2 = \sum_{n<0} d_n^2 = \infty$. Suppose that $H = X \dotplus Y$ and that either $\mathscr{P}_X(Y)$ is not closed or $\dim(X^\perp \cap Y) = \dim(X \cap Y^\perp) = \aleph_0$. Then there exist a complete orthonormal system $\{\varphi_n\}_{n \in \mathbb{Z}}$ in H and a complete family $\{x_n\}_{n \in \mathbb{Z}}$ in H with total biorthogonal family $\{x'_n\}_{n \in \mathbb{Z}}$ such that*

(1) $\text{span}\{x_n: n<0\} = X$, $\text{span}\{x_n: n \geqslant 0\} = Y$;

(2) *$\{x_n\}_{n \in \mathbb{Z}}$ is not hereditarily complete;*

(3) $\|x_n - \varphi_n\| \leqslant d_n$, $\|x'_n - \varphi_n\| \leqslant d_n$, $n \in \mathbb{Z}$.

Proof. We put $Y_0 = Y$ in the above construction and define Y_1 by $Y_1 = \text{span}\{Y, y\}$, where $y \in H \backslash (X + Y)$. The operators Γ_0 and $\mathscr{P}_X | Y$ have common kernel and range. It follows that either $\Gamma_0(X^\perp)$ is not closed or $\dim \ker \Gamma_0 = \aleph_0$. If $\Gamma_0(X^\perp)$ is not closed then the range of Γ_0^* is not closed too [19, Chap. IV, Theorem 5.13]. Since $\dim(\mathscr{D}(\Gamma_1)/\mathscr{D}(\Gamma_0)) = 1$, we see that $\Gamma_1^*(X)$ is not closed. Thus both Γ_0 and Γ_1^* are not essentially left invertible and we can apply Theorem 2.1 to them. It follows that there exists a complete orthonormal system $\{\varphi_n\}_{n \in \mathbb{Z}}$ in H such that

$$\text{span}\{\varphi_n \oplus \Gamma_0 \varphi_n : n \geqslant 0\} = Y, \quad \text{span}\{(-\Gamma_1^* \varphi_n) \oplus \varphi_n : n < 0\} = Y^\perp,$$

and $\|\Gamma_0 \varphi_n\| \leqslant d_n$, $n \geqslant 0$, $\|\Gamma_1^* \varphi_n\| \leqslant d_n$, $n < 0$. The sequence $\{x_n\}_{n \in \mathbb{Z}}$ is not hereditarily complete since $H_+ = G(\Gamma_0) = Y$ and $H^+ = G(-\Gamma_1^*)^\perp = G(\Gamma_1) = Y_1 \neq Y$.

If $\mathscr{P}_X(Y)$ is closed then $\dim\ker \Gamma_0 = \dim\ker \Gamma_1^* = \aleph_0$ and again both Γ_0 and Γ_1^* are not essentially left invertible. ∎

3. The Inverse Problem of Abstract Scattering Theory

We say that a pair $(U, \mathring{U})$ of unitary operators admits scattering (notationally $(U, \mathring{U}) \in \Omega$) if the limits

$$W_+ = \operatorname*{s-lim}_{n\to+\infty} U^{-n}\mathring{U}^n, \qquad W_- = \operatorname*{s-lim}_{n\to-\infty} U^{-n}\mathring{U}^n$$

exist in the strong operator topology. It is well known [19] and can easily be proved that the isometries W_-, W_+ satisfy

$$UW_\pm = W_\pm \mathring{U} \tag{1}$$

$$\operatorname*{s-lim}_{n\to\pm\infty} (W_\pm - I)\,\mathring{U}^n = 0. \tag{2}$$

On the contrary, if a pair $(U, \mathring{U})$ of unitary operators and a pair (W_-, W_+) of isometries satisfy (1), (2) then W_- and W_+ are the wave operators of $(U, \mathring{U})$. Indeed,

$$\operatorname*{s-lim}_{n\to\pm\infty} U^{-n}\mathring{U}^n = \operatorname*{s-lim}_{n\to\pm\infty} U^{-n}(I - W_\pm)\,\mathring{U}^n + \operatorname*{s-lim}_{n\to\pm\infty} U^{-n}W_\pm\,\mathring{U}^n$$

$$= 0 + \operatorname*{s-lim}_{n\to\pm\infty} W_\pm\,\mathring{U}^{-n}\mathring{U}^n = W_\pm.$$

It follows from (1) that the scattering operator $S = W_+^* W_-$ commutes with $\mathring{U}$.

We observe that every contraction S on H admits a representation $S = V_+^* V_-$, where V_+ and V_- are isometries. Indeed, there exists a unitary operator $\mathscr{U}$ on $H \oplus H$ such that $S = P\mathscr{U} \mid H$, where P denotes the orthogonal projection onto $H \oplus \{0\}$ (see [16, Problem 222]). Consider now any isometry J of $H \oplus H$ onto H. Since $S = PJ^*J\mathscr{U} \mid H$ we can put $V_+ h = J(h \oplus 0)$, $V_- h = J\mathscr{U}h$, $h \in H$.

The factorizations $S = V_+^* V_-$ can easily be classified.

Lemma 3.1. *Let* (W_-, W_+) *be a pair of isometries such that* $S = W_+^* W_-$. *If* (V_-, V_+) *is a pair of isometries then* $S = V_+^* V_-$ *if and only if there exists an isometry* $\mathscr{U}$ *of* $\operatorname{clos}(W_- H + W_+ H)$ *onto* $\operatorname{clos}(V_- H + V_+ H)$ *such that* $V_\pm = \mathscr{U} W_\pm$.

Proof. We consider an auxiliary map $W\colon H \oplus H \to H$ defined by

$$W(h_- \oplus h_+) = W_- h_- + W_+ h_+.$$

The map $V=(V_-, V_+)$ is defined similarly. Since the matrix of W^*W in $H\oplus H$ is given by

$$W^*W=\begin{pmatrix} W_-^* \\ W_+^* \end{pmatrix}(W_-\ W_+)=\begin{pmatrix} I & S^* \\ S & I \end{pmatrix} \tag{3}$$

we see that $V^*V=W^*W$. It follows that the map $W_-h_- + W_+h_+ \mapsto V_-h_- + V_+h_+$ extends to an isometry $\mathscr{U}$: $\operatorname{clos}(W_-H+W_+H)\to \operatorname{clos}(V_-H+V_+H)$. Clearly, $V_\pm = \mathscr{U}W_\pm$.

If now $V_\pm = \mathscr{U}W_\pm$ then $V_+^*V_- = W_+^*\mathscr{U}^*\mathscr{U}W_- = W_+^*W_- = S$. ∎

COROLLARY 3.1. *Let (W_-, W_+) be a pair of isometries with $S=W_+^*W_-$ and $\mathring{U}$ be a unitary operator. In order that there exist a unitary operator U satisfying* (1) *it is necessary and sufficient that S commute with $\mathring{U}$.*

Proof. It has already been mentioned that $S\mathring{U}=\mathring{U}S$ if such a U exists. Let now $S=\mathring{U}^*S\mathring{U}$. We put $V_\pm = W_\pm\mathring{U}$. Clearly, $S=V_+^*V_-$. By Lemma 3.1 there exists a unitary operator $\mathscr{U}$: $\operatorname{clos}(W_-H+W_+H)\to \operatorname{clos}(V_-H+V_+H)$ such that $V_\pm = \mathscr{U}W_\pm$. Taking U to be equal $\mathscr{U}$ on $\operatorname{clos}(W_-H+W_+H)$ and to be arbitrary on the orthogonal complement of this subspace we obtain the desired conclusion. ∎

Now we are in the position to obtain an algebraic description of the pairs admitting scattering. Given a contraction S commuting with $\mathring{U}$ we consider an auxiliary nonnegative operator

$$\Gamma=\begin{pmatrix} I & S^* \\ S & I \end{pmatrix}$$

on $H\oplus H$ and equip $H\oplus H$ with the Γ-norm $(h,h)_\Gamma \stackrel{\text{def}}{=} (\Gamma h, h)_{H\oplus H}$. We denote by $\mathscr{H}_S$ the completion of the quotient space of $H\oplus H$ with respect to the Γ-norm. Since the operator

$$\mathring{\mathscr{U}}=\begin{pmatrix} \mathring{U} & 0 \\ 0 & \mathring{U} \end{pmatrix}$$

commutes with Γ we see that it extends to the unitary operator on $\mathscr{H}_S$.

The formula (3) establishes a one-to-one correspondence between the factorizations of the form $S=W_+^*W_-$, $W_\pm$ being isometries, and the factorizations $\Gamma=W^*W$, W being a bounded operator from $H\oplus H$ to H. We now fix any factorization $\Gamma=W^*W$, W: $H\oplus H\to H$. Then there exists a unitary operator U on H (see Corollary 3.1) such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H}_S & \xrightarrow{W} & H \\ {\scriptstyle \mathring{\mathcal{U}}}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{H}_S & \xrightarrow{W} & H \end{array} \tag{4}$$

The operator U is uniquely determined on $\operatorname{ran} W = \operatorname{clos}(\operatorname{ran} W_- + \operatorname{ran} W_+)$. Since $\Gamma = W^* W$, the operator W embeds $\mathcal{H}_S$ isometrically into H. It is clear that the diagram (4) describes pairs (W_-, W_+), $(U, \mathring{U})$ satisfying (1).

Suppose now that $(U, \mathring{U}) \in \Omega$ and the wave operators are complete, i.e., $\operatorname{ran} W = H$. Using Lemma 3.1 and the diagram (4), we can easily describe all pairs $(U, \mathring{U}) \in \Omega$ with the scattering matrix $S = W_+^* W_-$. The pairs $(U, \mathring{U})$ are parametrized by the isometries $\mathcal{U}$: $H \to H$ satisfying

$$\operatorname*{s-lim}_{|n| \to +\infty} (\mathcal{U} - I) \mathring{U}^n = 0. \tag{5}$$

The isometry $\mathcal{U}$ determines the wave operators $V_\pm = \mathcal{U} W_\pm$ and the diagram (4), where $W = (V_-, V_+)$, determines a unitary operator U. To obtain (5) one should write $V_\pm - I = \mathcal{U}(W_\pm - I) + \mathcal{U} - I$ and apply (2) to $W_\pm$ and $V_\pm$.

The fact that S commutes with $\mathring{U}$ (see Corollary 3.1) restricts the class of scattering operators. Indeed, S commutes with the spectral measure of $\mathring{U}$ [6] and hence must have a nontrivial reducing subspace if the spectrum of $\mathring{U}$ consists of more than one point. The case $\mathring{U} = \lambda I$, $|\lambda| = 1$, is not interesting since then $U = \mathring{U}$ by (1) and $W_- = W_+ = I$ by the definition of the wave operators. It is well known that the unilateral shift $\mathcal{S}$ on the Hardy class H^2 has no non trivial reducing subspaces [28]. Thus not every contraction can be a scattering operator.

The condition (2) imposes further restrictions on the spectral measure of $\mathring{U}$. It is well known that $W_\pm = I$ if the spectrum of $\mathring{U}$ is discrete [19]. We show that the same conclusion holds for some types of continuous spectra.

Let $M(\mathbb{T})$ be the Banach space of finite complex Borel measures on $\mathbb{T}$ and $M_0(\mathbb{T})$ be its closed subspace consisting of measures μ such that the Fourier transform

$$\hat{\mu}(n) = \int_{\mathbb{T}} \bar{z}^n \, d\mu$$

vanishes as $n \to \pm\infty$. The subspace $M_0(\mathbb{T})$ is an (L)-ideal of $M(\mathbb{T})$, i.e., it contains an arbitrary measure ν which is absolutely continuous with respect to μ, $\mu \in M_0(\mathbb{T})$. Since $M(\mathbb{T})$ is a complete complex lattice [35] we can apply the Radon–Nikodym theorem to show that every μ in $M(\mathbb{T})$ can be uniquely decomposed as $\mu = \mu_1 + \mu_0$, where $\mu_0 \in M_0(\mathbb{T})$ and μ_1 is singular to any measure in $M_0(\mathbb{T})$ ($\mu_1 \perp M_0(\mathbb{T})$). Let $\mathring{\mathscr{E}}$ be the spectral measure of $\mathring{U}$:

$$\mathring{U} = \int_{\mathbb{T}} z \, d\mathring{\mathscr{E}}. \tag{6}$$

Using (6) and the above decomposition of $M(\mathbb{T})$ we can split H into the orthogonal sum $H = H_1 \oplus H_0$ of reducing subspaces of $\mathring{U}$. We put $\mu_{h,g}$ to be the measure on $\mathbb{T}$ such that $\mu_{h,g}(\Delta) = (\mathring{\mathscr{E}}(\Delta)h, g)$, $\Delta \subset \mathbb{T}$. Then $H_0 = \{h \in H\colon \mu_{h,h} \in M_0(\mathbb{T})\}$ and $H_1 = \{h \in H\colon \mu_{h,h} \perp M_0(\mathbb{T})\}$. The polarization formula for complex quadratic forms shows that H_0 and H_1 are linear subsets of H. It turns out that only trivial scattering can occur on H_1.

THEOREM 3.1. *If $(U, \mathring{U}) \in \Omega$ then $U = \mathring{U}$ on H_1. If $\mathring{U}$ is an arbitrary unitary operator with $H_0 \neq \{\mathbb{O}\}$ then there exists a unitary U with $(U, \mathring{U}) \in \Omega$ such that $W_+ = W_-$ and $W_+ h \neq h$ for $h \in H_0$, $h \neq 0$.*

The first statement of Theorem 3.1 is an easy consequence of the following lemma.

LEMMA 3.2. *Let V be a bounded operator satisfying* $\lim_{n \to +\infty} \|V\mathring{U}^n h\| = 0$ *for $h \in H$. Then $V | H_1 = 0$.*

Proof. Since V is weakly continuous, we see that V vanishes on the closed linear span G of the weak-limit points of $\{\mathring{U}^n h\}_{n \geqslant 0}$, $h \in H$. We show that $H_1 \subset G$. Let $g \perp G$. Then

$$\lim_{n \to +\infty} (\mathring{U}^n h, g) = 0 \tag{7}$$

for every h in H. Since $\mu_{g,g} \geqslant 0$ we conclude (put $h = g$ in (7)) that $\mu_{g,g} \in M_0(\mathbb{T})$, i.e., $g \in H_0$. It follows that $g \perp H_1$. ∎

Proof of Theorem 3.1. Suppose that $(U, \mathring{U}) \in \Omega$ and put $V_\pm = W_\pm - I$. Applying Lemma 3.2 to V_+, $\mathring{U}$ and V_-, $\mathring{U}^*$ we obtain that $V_\pm | H_1 \equiv 0$. Now (1) implies that $U = \mathring{U}$ on H_1.

We turn to the second statement. Since $H_0 \neq \{\mathbb{O}\}$, it follows that H_0 contains a subspace isomorphic to $L^2(d\mu)$, $\mu \in M_0(\mathbb{T})$ and therefore $\dim H_0 = \aleph_0$. Let $\{e_n\}_{n \geqslant 0}$ be any orthonormal basis in H_0 and $\{\lambda_n\}_{n \geqslant 0}$ be any sequence of points of $\mathbb{T}$ satisfying $\lambda_n \neq 1$, $\lim_{n \to +\infty} \lambda_n = 1$. Then W, defined by $We_n = \lambda_n e_n$, is a unitary operator on H. Clearly, $Wh = h$ iff $h = 0$. Since $\lim_n \lambda_n = 1$ the operator $W - I$ is compact. Since $\lim_{|n| \to +\infty} \mathring{U}^n h = 0$ in the weak topology for $h \in H_0$ ($\mu_{h,g} \in M_0(\mathbb{T})$, $g \in H$), it follows that $\lim_{|n| \to +\infty} \|(W-I)\mathring{U}^n h\| = 0$. Thus (W, W) satisfies (2). But $I = W^*W$ commutes with $\mathring{U}$. By Corollary 3.1 there exists a unitary operator U satisfying (1). Obviously $(U, \mathring{U}) \in \Omega$. ▮

Closed subsets of $\mathbb{T}$ which cannot support a nonzero element of $M_0(\mathbb{T})$ are called the uniqueness sets in the narrow sense, or briefly the U_0-sets. This class of sets has been studied in detail [5, 18, 22]. In particular, there exist perfect U_0-sets. If the spectrum $\sigma(\mathring{U})$ of a unitary operator $\mathring{U}$ is a U_0-set then $(U, \mathring{U}) \in \Omega \Rightarrow U = \mathring{U}$ by Theorem 3.1. Thus any scattering operator must commute with a "large" spectral measure.

The following lemma shows that there are also restrictions on the spectral properties of wave operators.

LEMMA 3.3. *Let W be a wave operator. Then $V = W - I$ is not essentially left invertible.*

Proof. Suppose that $W = \lim_{n \to +\infty} U^{-n}\mathring{U}^n$. By Lemma 3.2, $V \mid H_1 \equiv 0$. If $\dim H_1 = \aleph_0$ then V is an operator with large kernel and therefore V is not essentially left invertible. Suppose therefore that $\dim H_1 < \aleph_0$. Then $\dim H_0 = \aleph_0$. Suppose further that there exists a bounded operator B such that $BV = I + K$, where K is compact. For every $h \in H_0$, $\lim_{n \to \infty} \mathring{U}^n h = 0$ in the weak topology and hence $\lim_{n \to \infty} \|K\mathring{U}^n h\| = 0$. Since $\lim_{n \to \infty} \|BV\mathring{U}^n h\| = 0$ by (2), we obtain a contradiction. ▮

The property of wave operators stated in the lemma is responsible for the relation of approximate orthonormal systems to the scattering theory.

THEOREM 3.2. *Let W be any isometry such that the operator $W - I$ is not essentially left invertible. Then there exists a pair $(U, \mathring{U})$ in Ω such that $W = W_+ = W_-$.*

Proof. By Theorem 2.1 we can find a complete orthonormal system $\{e_n\}_{n \in \mathbb{Z}}$ such that $\lim_{|n| \to +\infty} \|(W-I)e_n\| = 0$. We define $\mathring{U}$ by $\mathring{U}e_n = e_{n+1}$. To define U we consider the orthonormal system $\{f_n\}_{n \in \mathbb{Z}}$, $f_n =^{\text{def}} We_n$, and put $Uf_n = f_{n+1}$, $n \in \mathbb{Z}$. If $\dim(H \ominus WH) > 0$ we define U to be an arbitrary unitary operator on $H \ominus WH$. It is clear that $UW = W\mathring{U}$. Since $\{e_n\}$ is a basis in H it suffices to prove that $\lim_{|n| \to +\infty} \|(W-I)\mathring{U}^n e_m\| = 0$. Then (2) holds automatically. But $\|(W-I)\mathring{U}^n e_m\| = \|(W-I)e_{n+m}\| \to 0$ as $|n| \to +\infty$ by the construction. ▮

4. The Inverse Scattering Problem. The Case of the Homogeneous Lebesgue Spectrum

A unitary operator $\mathring{U}$ is characterized (up to unitary equivalence) by the spectral type $[\mathring{\mathscr{E}}]$ of its spectral measure $\mathring{\mathscr{E}}$ and the function $\mathscr{M}(\mathring{U})$ of its spectral multiplicity [6]. In this section we suppose that $\mathscr{M}(\mathring{U}) \equiv \text{const}$ and that $[\mathring{\mathscr{E}}]$ is the type of the Lebesgue measure m on $\mathbb{T}$. Let E be any Hilbert space with $\dim E = \mathscr{M}(\mathring{U})$. We consider the Hilbert space $L^2(E)$ of E-valued square-summable function f on $\mathbb{T}$ satisfying

$$\|f\|^2_{L^2(E)} = \int_{\mathbb{T}} \|f(t)\|^2_E \, dm(t) < +\infty.$$

The spectral theorem applied to $\mathring{U}$ says that there exists a unitary map $\mathscr{F}$: $H \to L^2(E)$ such that $\mathscr{F}\mathring{U}\mathscr{F}^*$ coincides with the shift $\mathscr{S}: f \mapsto zf$. If S is the scattering operator of a pair $(U, \mathring{U})$ then $\mathscr{F}S\mathscr{F}^*$ commutes with $\mathscr{S}$ and therefore equals a contractive operator-valued function $S_{\mathscr{F}}(t)$, $t \in \mathbb{T}$, which is called the suboperator of the scattering operator S [1]. To simplify the notation we often drop the index $\mathscr{F}$ in $S_{\mathscr{F}}$: $S_{\mathscr{F}}(t) = S(t)$.

The main result of this section solves the inverse scattering problem in case of homogeneous Lebesgue spectrum of $\mathring{U}$.

Theorem 4.1. *Let S be a contraction on H which is unitarily equivalent to an operator-valued function $S(t)$ on $L^2(E)$. Then S is the scattering operator of a pair $(U, \mathring{U})$.*

Unitary Coupling and the Function Model. Here we state basic facts of the Adamjan–Arov theory of unitary couplings. An isometry V on a Hilbert space $\mathscr{D}$ is called a simple semiunitary operator if $\bigcap_{n\geq 0} V^n\mathscr{D} = \{0\}$. If $\mathfrak{N} = \mathscr{D} \ominus V\mathscr{D}$ then

$$\mathscr{D} = \bigoplus_{k=0}^{\infty} V^k\mathfrak{N}.$$

Definition. A unitary operator $\mathscr{U}$ on $\mathscr{H}$ is called a unitary coupling of simple semiunitary operators V_-, V_+ on $\mathscr{D}_-$, $\mathscr{D}_+$ if

$$\begin{aligned} &(1)\quad \mathscr{D}_- \subset \mathscr{H},\ \mathscr{D}_+ \subset \mathscr{H}; \\ &(2)\quad \mathscr{U}^{-1}|\mathscr{D}_- = V_-,\ \mathscr{U}|\mathscr{D}_+ = V_+. \end{aligned} \tag{1}$$

The subspace $\mathscr{D}_-$ is called the incoming subspace for the unitary group $\{\mathscr{U}^n\}_{n\in\mathbb{Z}}$ and $\mathscr{D}_+$ is called the outgoing subspace for $\{\mathscr{U}^n\}_{n\in\mathbb{Z}}$.

An important example of a unitary coupling is provided by the scattering theory. Let $\mathring{\mathscr{U}}$ be a unitary operator with homogeneous Lebesgue spectrum and $(\mathscr{U}, \mathring{\mathscr{U}}) \in \Omega$. Then $\mathscr{F}\mathring{\mathscr{U}}\mathscr{F}^* = \mathscr{S}$ on $L^2(E)$. Every function f in $L^2(E)$ expands into the orthogonal Fourier series

$$f = \sum_{k \in \mathbb{Z}} \hat{f}(k) z^k, \qquad \hat{f}(k) = \int_{\mathbb{T}} f \bar{z}^k \, dm.$$

We consider two orthogonal subspaces

$$H^2_+(E) = \{f \in L^2(E) \colon \hat{f}(n) = 0, n = -1, -2, ...\},$$
$$H^2_-(E) = \{f \in L^2(E) \colon \hat{f}(n) = 0, n = 0, 1, ...\}$$

of $L^2(E)$. If we put $\mathscr{D}_- = W_- \mathscr{F}^* H^2_-(E)$ and $\mathscr{D}_+ = W_+ \mathscr{F}^* H^2_+(E)$ then $\mathscr{U}^{-1}\mathscr{D}_- \subset \mathscr{D}_-$ and $\mathscr{U}\mathscr{D}_+ \subset \mathscr{D}_+$. So $\mathscr{U}$ is a unitary coupling of $\mathscr{U}^{-1} | \mathscr{D}_-$ and $\mathscr{U} | \mathscr{D}_+$.

The above construction can be reversed. We denote by $\mathscr{P}(\mathfrak{N}_\pm)$ the orthogonal projections in $\mathscr{H}$ onto the deficiency subpsaces $\mathfrak{N}_+ = \mathscr{D}_\pm \ominus V\mathscr{D}_\pm$. Then every element h of $\mathscr{H}^+_{\mathscr{U}} \stackrel{\text{def}}{=} \operatorname{span}\{\mathscr{U}^{-n}\mathscr{D}_+ \colon n = 0, 1, ...\}$ can be expanded into the orthogonal series

$$h = \sum_{k \in \mathbb{Z}} \mathscr{U}^k \mathscr{P}(\mathfrak{N}_+) \mathscr{U}^{-k} h.$$

Hence the map π_+,

$$\pi_+ f = \sum_{k \in \mathbb{Z}} \mathscr{U}^k \hat{f}(k),$$

is an isometry of $L^2(\mathfrak{N}_+)$ onto $\mathscr{H}^+_{\mathscr{U}}$. Similarly,

$$\pi_- f = \sum_{k \in \mathbb{Z}} \mathscr{U}^{k+1} \hat{f}(k)$$

is an isometry of $L^2(\mathfrak{N}_-)$ onto $\mathscr{H}^-_{\mathscr{U}} \stackrel{\text{def}}{=} \operatorname{span}\{\mathscr{U}^n \mathscr{D}_- \colon n = 0, 1, ...\}$. Obviously, we have $\mathscr{D}_- = \pi_- H^2_-(\mathfrak{N}_-)$ and $\mathscr{D}_+ = \pi_+ H^2_+(\mathfrak{N}_+)$. In addition

$$\mathscr{U}\pi_\pm = \pi_\pm \mathscr{S} \tag{1}$$

and consequently $\pi^*_+ \pi_-$ commutes with $\mathscr{S}$. The operator $\pi^*_+ \pi_-$ is called the scattering suboperator of the coupling $\mathscr{U}$.

The following theorem is an important consequence of the Adamjan–Arov theory.

THEOREM (Adamjan and Arov) [1]. *Let S be an arbitrary essentially bounded function on $\mathbb{T}$ whose values are contractions from $\mathfrak{N}_-$ to $\mathfrak{N}_+$. Then S is the scattering suboperator of a unitary coupling $\mathscr{U}$.*

For the completeness of the exposition we present a proof based on the Pavlov form of the function model [30, 27].

Proof. We equip the space $\mathscr{H} = L^2(\mathfrak{N}_-) \oplus L^2(\mathfrak{N}_+)$ with the seminorm

$$\|f \oplus g\|^2 = \int_{\mathbb{T}} \left\langle \begin{pmatrix} I & S^*(\xi) \\ S(\xi) & I \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathfrak{N}_- \oplus \mathfrak{N}_+} dm(\xi) \tag{3}$$

and denote the completion of the quotient space obtained by the same symbol $\mathscr{H}$. Clearly, $\mathscr{U} = \mathscr{S} \oplus \mathscr{S}: f \oplus g \mapsto \mathscr{S}f \oplus \mathscr{S}g$ is a unitary operator on $\mathscr{H}$. We define the isometries $\pi_\pm : L^2(E) \to \mathscr{H}$ by

$$\pi_- f = f \oplus 0, \qquad \pi_+ g = 0 \oplus g. \tag{4}$$

It is easy to check that $\pi_+^*(f \oplus g) = Sf + g$. It follows that $\pi_+^* \pi_- = S$. Now $\mathscr{U}\pi_\pm = \pi_\pm \mathscr{S}$ and we obtain that $\mathscr{U}$ is a unitary coupling of $\mathscr{U}^{-1} | \mathscr{D}_-$ and $\mathscr{U} | \mathscr{D}_+$, where $\mathscr{D}_- = \pi_- H^2_-(\mathfrak{N}_-)$, $\mathscr{D}_+ = \pi_+ H^2_+(\mathfrak{N}_+)$. ∎

Proof of Theorem 4.1. By the Adamjan–Arov theorem there exists a unitary coupling $\mathscr{U}$ on a Hilbert space $\mathscr{H}$ such that a given contractive operator-valued function $S(t)$ is the scattering suboperator of $\mathscr{U}$, i.e.,

$$S(t) = \pi_+^* \pi_-. \tag{5}$$

We define $\mathscr{D}_+$ and $\mathscr{D}_-$ by

$$\mathscr{D}_+ = \pi_+ H^2_+(E), \qquad \mathscr{D}_- = \pi_- H^2_-(E).$$

Now we fix a complete orthonormal system $\{e_j : 0 \leqslant j < \dim E\}$ in E and consider the family

$$e_{nj} = \begin{cases} \pi_+(z^n e_j), & n \geqslant 0 \\ \pi_-(z^n e_j), & n < 0. \end{cases} \tag{6}$$

It is clear that $\{e_{nj} : n \geqslant 0, j \geqslant 0\}$ is a complete orthonormal system in $\mathscr{D}_+$ while $\{e_{nj} : n < 0, j \geqslant 0\}$ is a complete orthonormal system in $\mathscr{D}_-$. The following lemma claims that $\mathscr{D}_+$ and $\mathscr{D}_-$ are almost orthogonal. ∎

LEMMA 4.1. *Let $e \in E$ then*

$$\lim_{n \to +\infty} \|\mathbf{P}_-(S^* z^n e)\| = 0.$$

Proof. Since $S^*e \in L^2(E)$ we have

$$S^*e = \sum_{k \in \mathbb{Z}} c_k z^k, \qquad c_k \in E, k \in \mathbb{Z},$$

$\sum_{k \in \mathbb{Z}} \|c_k\|^2 < +\infty$. The result follows from the identity

$$\|\mathbf{P}_-(S^*z^n e)\|^2 = \sum_{k < -n} \|c_k\|^2. \quad \blacksquare$$

It can easily be checked that the orthogonal projection $\mathscr{P}_-$ in $\mathscr{H}$ onto $\mathscr{D}_- = \pi_- H^2_-(E)$ is given by

$$\mathscr{P}_- = \pi_- \mathbf{P}_- \pi_-^*.$$

Let $F = \mathscr{H} \ominus \mathscr{D}_-$. Then

$$\operatorname{dist}(h, F) = \|\mathscr{P}_- h\| = \|\mathbf{P}_- \pi_-^* h\|. \tag{7}$$

We fix j and using (6) and (7) obtain for $n \geqslant 0$ that

$$\operatorname{dist}(e_{nj}, F) = \|\mathbf{P}_-(\pi_-^* \pi_+ z^n e_j)\|.$$

Observing that $\pi_-^* \pi_+ = S^*$ by (5) and applying Lemma 4.1, we obtain that $\lim_{n \to +\infty} \operatorname{dist}(e_{nj}, F) = 0$ for every j. Now Corollary 2.1 guarantees the existence of a complete orthonormal system $\{f_{nj}\}_{n \geqslant 0}$ in F such that

$$\lim_{n \to +\infty} \|e_{nj} - f_{nj}\| = 0, \qquad j = 1, 2, ...$$

It is clear that the family

$$f_{nj} = \begin{cases} f_{nj}, & n \geqslant 0 \\ e_{nj}, & n < 0 \end{cases}$$

is a complete orthonormal system in $\mathscr{H}$ since $\mathscr{H} = F \oplus \mathscr{D}_-$. We consider an auxiliary unitary map V: $L^2(E) \to \mathscr{H}$ defined by $V(z^n e_j) = f_{nj}$, $n \in \mathbb{Z}$, $0 \leqslant j < \dim E$. Let $\mathring{U} = \mathscr{S}$ and $U = V^* \mathscr{U} V$. To prove the existence of the wave operator W_+ for $(U, \mathring{U})$ we fix m, $m \in \mathbb{Z}$, and j. Suppose that $n \in \mathbb{Z}$ and $n + m \geqslant 0$. Then

$$\begin{aligned} U^{-n} \mathring{U}^n (z^m e_j) &= V^* \mathscr{U}^{-n} V(z^{n+m} e_j) = V^* \mathscr{U}^{-n} f_{n+m,j} \\ &= V^* \mathscr{U}^{-n} e_{n+m,j} + V^* \mathscr{U}^{-n}(f_{n+m,j} - e_{n+m,j}). \end{aligned}$$

Since $n+m \geqslant 0$ we obtain from (6) that $e_{n+m,j} = \pi_+(z^{n+m}e_j)$. It follows from (2) that $\mathscr{U}^{-n}\pi_+ = \pi_+ S^{-n}$, which yields $\mathscr{U}^{-n}e_{n+m,j} = \mathscr{U}^{-n}\pi_+(z^{n+m}e_j) = \pi_+(\mathscr{S}^{-n}z^{n+m}e_j) = \pi_+(z^m e_j)$. Next,

$$\lim_{n \to +\infty} \|V^*\mathscr{U}^{-n}(f_{n+m,j} - e_{n+m,j})\|$$

$$= \lim_{n \to +\infty} \|f_{nj} - e_{nj}\| = 0$$

and we obtain that

$$\operatorname*{s-lim}_{n \to +\infty} U^{-n}\mathring{U}^n(z^m e_j)$$

$$= V^*\pi_+(z^m e_j), \qquad m \in \mathbb{Z},\ j \geqslant 0.$$

Similarly,

$$\operatorname*{s-lim}_{n \to -\infty} U^{-n}\mathring{U}^n(z^m e_j)$$

$$= V^*\pi_-(z^m e_j), \qquad m \in \mathbb{Z},\ j \geqslant 0.$$

It follows that $(U, \mathring{U}) \in \Omega$ and the wave operators $W_\pm$ satisfy $W_\pm = V^*\pi_\pm$. Hence

$$W_+^* W_- = \pi_+^* V V^* \pi_- = \pi_+^* \pi_- = S(t)$$

according to (5). If now $\mathscr{F} S \mathscr{F}^* = S(t)$, where $\mathscr{F}$ is the unitary operator $\mathscr{F}\colon H \to L^2(E)$ such that $\mathscr{F}\mathring{U}\mathscr{F}^* = \mathscr{S}$, then S is obviously the scattering operator of the pair $(\mathscr{F}^* U \mathscr{F}, \mathscr{F}^* \mathscr{S} \mathscr{F})$.

The construction used in the proof of Theorem 4.1 gives us a pair $(U, \mathring{U})$ in Ω such that $S = W_+^* W_-$ and the wave operators W_+, W_- are complete. The algebraic description presented in Section 3 and Lemma 3.1 yield a description of all pairs in Ω with the scattering operator S. These pairs are parametrized by the isometries $\mathscr{U}\colon H \to H$ satisfying (5) of Section 3.

THEOREM 4.2. *Let S be a contraction on H which is unitarily equivalent to an operator-valued function on $L^2(E)$. Then S is the scattering operator of a pair $(U, \mathring{U})$ with incomplete wave operators.*

Proof. We recall that $\{z^n e_j\colon n \in \mathbb{Z},\ 0 \leqslant j < \dim E\}$ is a complete orthonormal sequence in $L^2(E)$. Applying Lemma 2.3 to $\{z^n e_j\}_{n \in \mathbb{Z}}$, $0 \leqslant j < \dim E$, and $F = L^2(E)$ as $|n| \to +\infty$, we obtain an infinite-dimensional subspace G in $L^2(E)$ such that

$$\lim_{|n| \to +\infty} \|\mathscr{P}_G(z^n e_j)\| = 0, \qquad 0 \leqslant j < \dim E.$$

Let now $F = L^2(E) \ominus G$. Then by Lemma 2.2 and Corollary 2.1 there exists an isometry $\mathscr{U}\colon L^2(E) \to F$ such that

$$\lim_{|n| \to +\infty} \|\mathscr{U}(z^n e_j) - z^n e_j\| = 0, \quad 0 \leqslant j < \dim E.$$

It follows that $\mathscr{U}$ satisfies (5) of Section 3. We now put $V_\pm = \mathscr{U} W_\pm$, where $W_\pm$ are the wave operators obtained in Theorem 4.1. Since $\mathscr{U}^*\mathscr{U} = I_{L^2(E)}$ we have $V_+^* V_- = W_+^* W_- = S$. Next,

$$\begin{aligned} \operatorname*{s-lim}_{n \to \pm\infty} (V_\pm - I)\mathring{U}^n = {} & \operatorname*{s-lim}_{n \to \pm\infty} \mathscr{U}(W_\pm - I)\mathring{U}^n \\ & + \operatorname*{s-lim}_{n \to \pm\infty} (\mathscr{U} - I)\mathring{U}^n = 0. \end{aligned}$$

It follows that there exists a unitary operator $\tilde{U}$ on $L^2(E)$ such that $(\tilde{U}, \mathring{U}) \in \Omega$ and $V_\pm$ are the wave operators of $(\tilde{U}, \mathring{U})$. Since $\operatorname{ran} V_\pm \subset \operatorname{ran} \mathscr{U} \subset F$ we see that V_+ and V_- are not complete. ▮

The results obtained in this section permit us to construct a function model for general scattering with homogeneous Lebesgue spectrum. We consider the Pavlov model for unitary coupling, i.e., the Hilbert space $\mathscr{H} = L^2(E) \oplus L^2(E)$ equipped with the seminorm (3), and define the unitary operator U by $U = \mathscr{S} \oplus \mathscr{S}$. Next we fix any complete orthonormal system $\{f_{nj}\colon n \in \mathbb{Z},\ 0 \leqslant j < \dim E\}$ in $\mathscr{H}$ such that

$$\begin{aligned} & \lim_{n \to -\infty} \|f_{nj} - \pi_-(z^n e_j)\| \\ & \quad = \lim_{n \to +\infty} \|f_{nj} - \pi_+(z^n e_j)\| = 0, \qquad j \geqslant 0, \end{aligned} \tag{8}$$

where π_-, π_+ are defined by (4) and $\{e_j\colon 0 \leqslant j < \dim E\}$ is an arbitrary orthonormal basis in E. We put $\mathring{U} f_{nj} = f_{n+1,j}$, $n \in \mathbb{Z}$, $0 \leqslant j < \dim E$. Then $(U, \mathring{U}) \in \Omega$ and the scattering operator of $(U, \mathring{U})$ is unitarily equivalent to the operator-valued function $S(\zeta)$, $\zeta \in \mathbb{T}$. Thus we obtain function models of scattering parametrized by complete orthonormal systems satisfying (8). In case S^* is analytic in $\mathbb{D}$ the system $\{\pi_-(z^{-(n+1)}e_j),\ \pi_+(z^n e_j)\colon n = 0, 1, 2, \ldots,\ 0 \leqslant j < \dim E\}$ is orthogonal. Hence we obtain a direct relation of complete orthogonal systems approximating incomplete orthogonal families to pairs of unitary operators admitting scattering.

Theorem 4.1 can easily be extended to strongly continuous unitary groups. We denote by $L^2(\mathbb{R}, E)$ the Hilbert space of E-valued square-summable functions on the real line $\mathbb{R}$.

COROLLARY 4.1. *Let S be a contraction which is unitarily equivalent to an operator-valued function on $L^2(\mathbb{R}, E)$. Then there exists a pair of strongly continuous unitary groups such that S is their scattering operator.*

Proof. The unitary map $\mathscr{F}: L(\mathbb{R}, E) \to L^2(\bigoplus_{n \in \mathbb{Z}} E)$ defined by

$$(\mathscr{F} f)(e^{ix}) = \{f(x + 2\pi n)\}_{n \in \mathbb{Z}}, \qquad 0 \leqslant x < 2\pi$$

sends the unitary operator $\mathring{U}$, $\mathring{U}f(x) = e^{ix}f(x)$, $f \in L^2(\mathbb{R}, E)$, to the shift $\mathscr{S}$ on $L^2(\bigoplus_{n \in \mathbb{Z}} E)$. It follows from Theorem 4.1 that for every contractive operator-valued function $S(x)$ on $\mathbb{R}$ there exists a unitary operator U such that S is the scattering operator of the pair $(U, \mathring{U})$ and the wave operators W_-, W_+ are complete.

We consider a strongly continuous unitary group $\{\mathring{U}(t)\}_{t \in \mathbb{R}}$ defined by

$$\mathring{U}(t)\, f(x) = e^{itx} f(x), \quad f \in L^2(\mathbb{R}, E).$$

Clearly $\mathring{U}(n) = \mathring{U}^n$, $n \in \mathbb{Z}$. We have

$$\operatorname*{s-lim}_{t \to \pm\infty} (W_\pm - I)\, \mathring{U}(t) = \operatorname*{s-lim}_{t \to \pm\infty} (W_\pm - I)\, \mathring{U}^{[t]} \mathring{U}(t - [t]) = 0 \tag{9}$$

since $\{\mathring{U}(s) f\}_{0 \leqslant s \leqslant 1}$ is compact, $f \in L^2(\mathbb{R}, E)$. Since $S(x)$ commutes with $\mathring{U}(t)$ for every $t \in \mathbb{R}$, it follows from Corollary 3.1 that there exists a unitary operator $U(t)$ satisfying

$$U(t)\, W_\pm = W_\pm\, \mathring{U}(t).$$

Since the wave operators are complete the operator $U(t)$ is uniquely determined. It is also clear that $\{U(t)\}_{t \in \mathbb{R}}$ is strongly continuous. Now (9) yields

$$W_\pm = \operatorname*{s-lim}_{t \to \pm\infty} U(-t)\, \mathring{U}(t). \quad \blacksquare$$

REFERENCES

1. V. M. ADAMJAN AND D. Z. AROV, On unitary couplings of semi-unitary operators, *Amer. Math. Soc. Transl.* (2) **95** (1970), 75–129.
2. N. K. BARI, On stability of some properties of orthogonal systems, *Dokl. Akad. Nauk SSSR* **33** (1941), 342–345. [Russian]
3. N. K. BARI. Sur les systèmes complets de fonctions orthogonales, *Mat. Sb. Akad. Nauk SSSR* **14** (1944), 51–106. [French]
4. N. K. BARI, Biorthogonal systems and bases in Hilbert space, *Uchen. Zap. Moskov. Gos. Univ. Mat.* (4) **148** (1951), 69–107. [Russian]
5. N. K. BARI, "A Treatise on Trigonometric Series," Macmillan Co., New York, 1964.
6. M. S. BIRMAN AND M. Z. SOLOMJAK, "Spectral Theory of Self-adjoint Operators on a Hilbert Space," Reidel, Dordrecht/Tokyo, 1987.
7. L. N. DOVBYSH, N. K. NIKOL'SKII, AND V. N. SUDAKOV, How "good" can a non-hereditarily complete family be? *Zap. Nauch. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **73** (1977), 52–69. [Russian]

8. P. ERDÖS, H. S. SHAPIRO, AND A. L. SHIELDS, Large and small subspaces of Hilbert space, *Michigan Math. J.* **12** (1965), 169–178.
9. L. D. FADDEEV, On the Friedrichs model in the theory of perturbation of continuous spectrum, *Trudy Mat. Inst. Steklov.* **73** (1964), 292–313. [Russian]
10. J. B. GARNETT, "Bounded Analytic Functions," Academic Press, New York/San Francisco, 1981.
11. I. C. GOHBERG AND M. G. KREIN, "Introduction in the Theory of Linear Non-selfadjoint Operators," Nauka, Moscow, 1965. [Russian]
12. I. C. GOHBERG AND A. S. MARKUS, On some inequalities for eigenvalues and matrix entries of linear operators, *Izv. Akad. Nauk. Moldav. SSR Ser. Fiz.-Teckhn. Mat. Nauk.* **5** (1962), 103–108. [Russian]
13. I. C. GOHBERG AND A. S. MARKUS, Some relations between eigen-values and matrix entries of linear operators, *Mat. Sb.* **64** (1964), 481–496. [Russian]
14. V. I. GURARII, The existence of a non-hereditarily complete family in an arbitrary separable Banach space, *Zap. Nauchn. Leningrad Otdel. Mat. Inst. Steklov. (LOMI)* **92** (1979), 274–277.
15. P. R. HALMOS, "A Hilbert Space Problem Book," Springer-Verlag, New York/Berlin, 1982.
16. P. R. HALMOS AND V. S. SUNDER, "Bounded Integral Operators on L^2 Spaces," Springer-Verlag, Berlin/New York, 1978.
17. J. W. HELTON, Discrete time systems, operator models, and scattering Theory, *J. Funct. Anal.* **16** (1974), 15–38.
18. J.-P. KAHANE, "Séries de Fourier absolument convergentes," Springer-Verlag, Berlin/New York, 1970. [French]
19. T. KATO, "Perturbation Theory for Linear Operators," Springer-Verlag, Berlin/New York, 1966.
20. P. D. LAX AND R. S. PHILLIPS, Scattering theory, *Bull. Amer. Math. Soc.* **70** (1964), 130–142.
21. P. D. LAX AND R. S. PHILLIPS, "Scattering Theory," Academic Press, New York, 1967.
22. R. LYONS, "A Characterization of Measures Whose Fourier–Stieltjes Transforms Vanish at Infinity," Thesis, University of Michigan, 1983.
23. A. S. MARKUS, A problem of spectral analysis for operators with point spectrum, *Izv. Akad. Nauk SSSR Ser. Mat.* **34** (1970), 662–688.
24. B. SZ.-NAGY AND C. FOIAŞ, "Harmonic Analysis of Operators on Hilbert Space," North-Holland, Amsterdam, 1970.
25. D. J. NEWMAN AND H. S. SHAPIRO, The Taylor coefficients of inner functions, *Michigan Math. J.* **9** (1962), 249–255.
26. N. K. NIKOL'SKII AND V. I. VASYUNING, Notes on two functions models, *in* "The Bieberbach Conjecture," Mathematical Surveys, Vol. 21, pp. 113–141, Amer. Math. Soc., Providence, RI, 1986.
27. N. K. NIKOL'SKII AND V. I. VASYUNING, A unified approach to function models, and the transcription problem, LOMI preprints, E-5-86.
28. N. K. NIKOL'SKII, "Treatise on the Shift Operator," Springer-Verlag, Berlin/New York, 1985.
29. A. M. OLEVSKII, "Fourier Series with Respect to General Orthogonal Systems," Springer-Verlag, Berlin/New York, 1975.
30. B. S. PAVLOV, Dilation theory and spectral analysis of nonselfadjoint differential operators, *Amer. Math. Soc. Transl.* (2) **115** (1980), 103–142.
31. B. S. PAVLOV AND S. V. PETRAS, On the singular spectrum of a weak perturbation of the multiplication by an independent variable, *Funktsional. Anal. i Prilozhen.* (2) **4** (1970), 54–61. [Russian]

32. D. B. PEARSON, An example in potential scattering illustrating the breakdown of asymptotic completeness, *Comm. Math. Phys.* **40** (1975), 125–146.
33. S. C. POWER, "Hankel Operators on Hilbert Space," Pitman, Boston/Melbourne, 1982.
34. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics. III. Scattering Theory," Academic Press, New York/London, 1979.
35. H. H. SCHAEFER, "Topological Vector Spaces," Macmillan Co., New York, 1966.