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Normal Ergodic Actions

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The von Neumann–Halmos theory of ergodic transformations with discrete spectrum makes use of the duality theory of locally compact abelian groups to characterize those transformations preserving a probability measure, which are defined by a rotation on a compact abelian group. We use the recently developed duality between general locally compact groups and Hopf–von Neumann algebras to characterize those actions of a locally compact group, preserving a σ-finite measure, which are defined by a dense embedding in another group. They are characterized by the property of normality, previously introduced by the author, and motivated by Mackey's theory of virtual groups. The discrete spectrum theory is readily seen to come out as the special case in which the invariant measure is finite.

1. Introduction

The notion of a normal ergodic action of a locally compact group G, with an invariant probability measure, was introduced by the author in [11, 13] in order to study a generalization in the case in which G is not abelian of the von Neumann–Halmos uniqueness-existence theorems for ergodic transformations with discrete spectrum. More generally, we defined the notion of a normal ergodic extension of a given action to study the uniqueness-existence theorems in the context of relatively discrete spectrum [11]. For extensions, these considerations are necessary even if G is abelian. The motivation for our definition came from Mackey's theory of virtual groups. Normal actions and extensions are the virtual subgroup analogues of normal subgroups in group theory.

It follows from [5, Theorem 1; 11, Theorem 5.7] that for a normal G-space X with discrete spectrum, we have up to isomorphism modulo null sets, $X = K$, a compact group, and the action $k \cdot g$ is given by $k\phi(g)$ where $\phi: G \to K$ is a homomorphism with dense range. A generalization of this result to normal extensions with relatively discrete spectrum is also contained in [11]. Although it was not explicitly stated, the results of [11, 12] in fact imply that every normal action (or extension) with an invariant probability measure has discrete (or

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relatively discrete) spectrum, and hence is of the above form. A proof of this appears in Section 2.

The main point of this paper, however, is to remove the condition that the action of G on X preserves a finite measure, an assumption which plays a crucial role in the proof of the above theorem. Thus, we suppose only that X has a σ-finite measure invariant and ergodic under G. The notion of normality applies equally well to this more general situation (see Section 2), and the main result of this paper is the following.

THEOREM. *If X has a σ-finite invariant measure μ under G, and the action is normal, then there is a locally compact (second countable) group H and a homomorphism $\phi: G \to H$ with dense range such that up to isomorphism modulo null sets, $X = H$, μ is equivalent to the Haar measure, and the action is given by $h \cdot g = h\phi(g)$.*

This result can be looked at in two ways: as describing the structure of normal actions, or alternatively, as providing a new criterion for an action to be defined by a homomorphism into a locally compact group.

The proof of the theorem is itself of some interest. One of the first results in this direction, when G is the group of integers or the real line, goes back to Halmos and von Neumann [2], where the duality theory of locally compact abelian groups was exploited. In this paper, we employ the more recently developed duality between general locally compact groups and Hopf–von Neumann algebras. In particular, the main theorem of Takesaki's paper [10] will be the device we use to construct the group H.

The outline of this paper is as follows. Section 2 contains some preliminary definitions and the result about normal actions with finite invariant measure, based on the techniques of [11, 12]. Sections 3–5 establish various properties of normal actions, which are then applied in Section 6 to the construction of a suitable Hopf–von Neumann algebra which yields the main theorem. Most of the difficulties however, appear in Sections 3–5, which are quite technical and measure theoretic in nature. Finally, Section 7 applies the main theorem in an alternate approach to the case of finite invariant measure.

2. Normal Actions and Extensions with Finite Invariant Measure

We begin by recalling some basic definitions. (See [8, 11] for details and discussions of these ideas. We suppose that (S, μ) is a standard measure space, that there is a right Borel action of G and S, and that μ is σ-finite, invariant, and ergodic under G. Here, G is a second countable locally compact group. A Borel function $\alpha: S \times G \to M$ is a standard Borel group, is called a cocycle if for each $g, h \in G$, $\alpha(s, gh) = \alpha(s, g)\, \alpha(sg, h)$ for almost all s. If α and β are cocyles into M, then α and β are called equivalent, or cohomologous, if there is a Borel function $a: S \to M$ such that for each g, $a(s)\,\alpha(s, g)\,a(sg)^{-1} = \beta(s, g)$ for almost all s. If π is a unitary representation of G on a Hilbert space H, then the restriction of π to $S \times G$ is the cocycle defined by $\alpha(s, g) = \pi(g)$. Thus, $\alpha: S \times G \to U(H)$. The term restriction is used because it is the virtual subgroup analog of the restriction of a group representation to a subgroup (see [11, Example 2.7]).

Since G acts on S, there is a naturally induced unitary representation of G on $L^2(S)$ defined by $(U(g)f)(s) = f(sg)$.

DEFINITION 2.1. (S, μ) is a normal G-space if the restriction of the representation U to $S \times G$ is equivalent to the identity cocycle.

PROPOSITION 2.2. *Suppose $H \subset G$ is a closed subgroup. Then G/H is a normal G-space if and only if H is a normal subgroup.*

Proof. This is [11, Proposition 5.3] (where the implicit assumption that G/H had a finite invariant measure was not needed.)

When μ is finite and invariant, the structure of a normal action can be completely described using results of [5, 11, 12].

PROPOSITION 2.3. *Suppose (S, μ) is a normal G-space where μ is an invariant probability measure. Then S has discrete spectrum; i.e., the natural induced representation U of G on $L^2(S)$ is the direct sum of finite-dimensional representations.*

Proof. Let $H \subset L^2(S)$ be the orthogonal complement of the subspace generated by the finite dimensional G-invariant subspaces, and suppose $H \neq \{0\}$. Let π be $U \mid H$. Let $A: S \to U(L^2(S))$ such that for each g, $A(s)U(g)A(sg)^{-1} = I$ for almost all s. Let $P: L^2(S) \to H$ be the orthogonal projection. Choose an element $f \in L^2(S)$ with $PA(s)^{-1}f \neq 0$ for $s \in S_0$, where S_0 is some set with positive measure. Let $h \in L^2(S \times S)$ be defined by $h(s, t) = (A(s)^{-1}f)(t)$. We claim that h is essentially invariant under the product action of G on $S \times S$. For any $g \in G$, and almost all $(s, t) \in S \times S$, we have

$$\begin{aligned} h(sg, tg) &= (A(sg)^{-1}f)(tg) \\ &= (U(g)^{-1}A(s)^{-1}f)(tg) \\ &= (A(s)^{-1}f)(t) = h(s, t). \end{aligned}$$

Now let $\tilde{P}: L^2(S; L^2(S)) \to L^2(S; H)$ be the orthogonal projection, so that $\tilde{P} = \int^{\oplus} P_s \, d\mu$, where P_s is a copy of P. Identifying $L^2(S; L^2(S))$ with $L^2(S \times S)$, we see that $H \subset L^2(S)$ G-invariant implies that $L^2(S; H)$ is G-invariant. Thus, $\tilde{P}h$ is also G-invariant and by the choice of f, $\tilde{P}h \neq 0$. We can identify $L^2(S; H)$ with $L^2(S) \otimes H$, and under this identification, the product G-action on $L^2(S; H) \subset L^2(S \times S)$ corresponds to the representation $U \otimes \pi$. Since $\tilde{P}h$ is G-invariant, $U \otimes \pi$ contains the identity representation, and it follows [4, Lemma 8.1] that π contains a nonzero finite-dimensional subrepresentation. This contradicts the choice of H, and so S must have discrete spectrum.

COROLLARY 2.4. *If S is a normal G-space with a finite invariant measure, then there is a compact group K and a homomorphism $\phi: G \to K$ with dense range such that S is essentially isomorphic to K, with the action of $g \in G$ on K given by right translation by $\phi(g)$.*

Proof. This follows from Proposition 2.3, [5, Theorem 1], and [11, Theorem 5.7].

We note that there are analogs of Proposition 2.3 and Corollary 2.4 for extensions. We include them, modulo a few details, for completeness. Thus, let (X, μ), (Y, v) be ergodic Lebesgue G-spaces, μ and v invariant probability measures, and $p: X \to Y$ a measure-preserving G-map. See [11, Definition 5.4] for the definition of X being a normal extension of Y.

PROPOSITION 2.5. *Suppose X is a normal extension of Y. Then X has relatively discrete spectrum over Y* [11, Definition 5.1].

Proof. Let $F_y = p^{-1}(y)$, $\mu = \int^{\oplus} \mu_y \, dy$, $H_y = L^2(F_y, \mu_y)$ and $\alpha(y, g)$: $H_{yg} \to H_y$ the naturally induced cocycle [11; Example 2.3]. Let $\beta(x, g) = \alpha(p(x), g)$ and $H_x = H_{p(x)}$. Normality means that β is equivalent to the identity. Thus, there is a Hilbert space H_0 and a Borel field $A_x: H_x \to H_0$, unitary for almost all x, and such that for each g and almost all x, $A(x)\,\beta(x, g)\,A(xg)^{-1} = I$.

Suppose X does not have relatively discrete spectrum over Y. Let $V \subset L^2(X)$, $V \neq \{0\}$, be the orthogonal complement of the space spanned by the G-invariant finite-dimensional subbundles of $L^2(X) = \int^{\oplus} L^2(F_y)\, dv(y)$. Then $V = \int^{\oplus} V_y$. Let $V_x = V_{p(x)} \subset H_x$, and $\gamma = \alpha \mid V$; i.e., $\gamma(y, g) = \alpha(y, g) \mid V_{xg}$. By the choice of V, γ has no nontrivial finite-dimensional subcocycles. Let $P_x: H_x \to V_x$ be orthogonal projection, and $\tilde{P}: \int^{\oplus} H_x \, d\mu \to \int^{\oplus} V_x$ be orthogonal projection; i.e., $\tilde{P} = \int^{\oplus} P_x \, d\mu(x)$. Now choose an element $z \in H_0$ such that $P_x A(x)^{-1} z \neq 0$ for x in a nonnull set. Define $h \in \int^{\oplus} H_x \, d\mu(x)$ by $h = \int^{\oplus} h_x$, where $h_x = A(x)^{-1} z$. Let σ be the representation of G on $\int^{\oplus} H_x \, d\mu$ induced by β[11, Example 2.4]. Then

$$\begin{aligned} \sigma(h)_x &= \beta(x, g)\, h_{xg} \\ &= \beta(x, g)\, A(xg)^{-1} z \\ &= A(x)^{-1} z = h_x \end{aligned}$$

for almost all x. Thus, h is invariant under σ.

Since $V = \int^{\oplus} V_y$ is invariant under the representation of G induced by α, $\int^{\oplus} V_x \subset \int^{\oplus} H_x$ will be invariant under the representation of G induced by β. Thus $\tilde{P}h \in \int^{\oplus} V_x$ will also be G-invariant, and by the choice of z, $\tilde{P}h \neq 0$. Now $\int^{\oplus} V_x$ can be indentified with $\int^{\oplus} (V_y \otimes L^2(F_y))\, dv(y)$, and the representation σ becomes the representation induced by the $Y \times G$ cocycle $\gamma \otimes \alpha$. Since

$\tilde{P}h$ is G-invariant, $\gamma \otimes \alpha$ contains the identity cocycle representation, and it follows from [11, Lemma 2.13] that γ must have a finite-dimensional subcocycle. This contradicts the choice of V, which shows that X must have relatively discrete spectrum over Y.

COROLLARY 2.6. *If X is a normal extension of Y, where X and Y have finite invariant measures, then there is a compact group K and a minimal cocycle* [11, *Definition* 3.7] *$c: Y \times G \to K$ such that up to isomorphism modulo null sets, $X = Y \times_c K$* [11, Sect. 3].

It is Corollary 2.4 that we will generalize to the case of a σ-finite measure. The proof given below provides a new proof of Corollary 2.4. It seems likely that Corollary 2.6 can be generalized as well, perhaps using similar methods. We will not attempt that here, but rather leave it to a future paper.

3. THE INDUCED POINT TTANSFORMATIONS

In this section, we begin the proof of the theorem stated in the introduction. Thus, suppose S is a normal G-space, with m a σ-finite invariant ergodic measure on S. It will often be convenient to work with an equivalent finite quasi-invariant measure μ on S. We remark that there is a natural isomorphism of $L^2(S, m)$ with $L^2(S, \mu)$, and we usually use the same symbol to denote naturally identified unitary operators. Let T denote the natural induced representation of G on $L^2(S)$ described in Section 2. Normality implies that there is a Borel field of unitary operators $U(s): L^2(S) \to L^2(S)$ such that for each g and almost all s, $U(s)\, T(g)\, U(sg)^{-1} = I$. The idea of the proof is to use the operators $U(s)$ to prove that $L^\infty(S)$ has the structure of a commutative Hopf–von Neumann algebra with an involution and invariant measure [10]. We will then apply Takesaki's theorem [10, Theorem 2] to obtain our result.

We begin by showing that we can modify $U(s)$ so that these maps induce point transformations of S. Let B be the Boolean σ-algebra of projection operators in $L^2(S)$ defined by multiplication by the characteristic functions of subsets of S. Then it is well known that $T(g)$ leaves B invariant, i.e., $T(g)^{-1}BT(g) = B$ for all $g \in G$. Let B_s be the Boolean σ-algebra defined by $B_s = U(s)\, BU(s)^{-1}$. The space $\mathscr{B}$ of Boolean σ-algebras of projection operators on H has a standard Borel structure [1; 11, Lemma 4.5], and since $U(s)$ is a Borel field, $s \to B_s$ is a Borel function. We claim that this function is G-invariant. For $g \in G$, we have, for almost all s,

$$\begin{aligned} B_{sg} &= U(sg)\, BU(sg)^{-1} \\ &= U(sg)\, T(g)^{-1}BT(g)\, U(sg)^{-1} \\ &= U(s)\, BU(s)^{-1} = B_s . \end{aligned}$$

Since $\mathscr{B}$ is standard, it follows by the ergodicity of G on S that B_s is constant on a conull set. Choosing a given unitary operator A such that $AB_sA^{-1} = B$ for almost all s, replacing $U(s)$ by $AU(s)$, then changing on a Borel null set, we can assume $U(s)^{-1}BU(s) = B$ for all s.

Now let $U: L^2(S \times S) \to L^2(S \times S)$ be defined by $U = \int^{\oplus} U(s)\, d\mu$, where we have identified $L^2(S \times S)$ with $L^2(S; L^2(S) = \int^{\oplus} L^2(S)\, d\mu$. Then U preserves the Boolean algebra $B(S \times S) = \int^{\oplus} B\, d\mu(s)$. Thus, there is a Borel map Ψ on $S \times S$ such that (i) Ψ preserves the measure class of $\mu \times \mu$, and (ii) Ψ^*: $B(S \times S) \to B(S \times S)$, $\Psi^* = \tilde{U}$ where $\tilde{U}$ is the automorphism of $B(S \times S)$ induced by U (i.e., $\tilde{U}(P) = UPU^{-1}$) [8, Theorem 2.1].

Let p_1: $S \times S \to S$ be projection onto the first factor. Then $p_1{}^*$: $B(S) \to B(S \times S)$ and we have $\tilde{U}p_1{}^* = p_1{}^*$; that is $\Psi^* p_1{}^* = p_1{}^*$, and hence $p_1 \circ \Psi = p_1$ almost everywhere [8, Theorem 2.1]. Now $\{(s, t) \mid p_1\Psi(s, t) = s\}$ is conull Borel, so changing $\Psi(s, t)$ on a Borel null set we can assume Ψ is Borel, measure-class preserving, and $p_1 \circ \Psi = p_1$ for all $(s, t) \in S \times S$.

For each s, define $\phi(s)$: $S \to S$ by $\phi(s)(t) = p_2\Psi(s, t)$.

LEMMA 3.1. *For s in a conull Borel set, $\phi(s)$ preserves the measure class of μ.*

Proof. We first note that $s \to \phi(s)_*\mu$ is a Borel function from S into $M(S)$, the space of finite measures on S. (See [3], e.g., for a discussion of this space.) To see this, let $E \subset S$ be Borel. Then $\mu(\phi(s)^{-1}(E)) = \mu(\Psi^{-1}(S \times E) \cap p_1^{-1}(s))$. This is a Borel function of s by virtue of Fubini's theorem and the fact that Ψ is Borel. Now $\mu \times \mu = \int^{\oplus} \mu\, d\mu(s)$, so $\Psi_*(\mu \times \mu) = \int^{\oplus} \phi(s)_*\mu\, d\mu(s)$. Since $\Psi_*(\mu \times \mu) \sim \mu \times \mu$, we must have, for almost all s, $\phi(s)_*\mu \sim \mu$. Furthermore, it follows from [9; Lemma 1.1] and the fact that $s \to \phi(s)_*\mu$ is Borel, that $\{s \mid \phi(s)_*\mu \sim \mu\}$ is a conull Borel set.

We also note that this implies that for s in a conull Borel set

$$\phi(s)^* = \tilde{U}(s): B(S) \to B(S).$$

Ultimately, we use the maps $\phi(s)$ to define a Hopf–von Neumann algebra. In order to do this, the maps must satisfy several conditions. This and the following two sections are devoted to demonstrating that we can modify $\phi(s)$ in such a way that the required conditions hold. We begin this task with the following.

LEMMA 3.2. *Let $\lambda(g)$: $S \to S$ be $\lambda(g)(s) = sg$. Then for each g and almost all s, $\lambda(g)\,\phi(s) = \phi(sg)$ almost everywhere.*

Proof. For each g, $U(s)\, T(g) = U(sg)$ for almost all s, so $\tilde{U}(s)\, \tilde{T}(g) = \tilde{U}(sg)$ on $B(S)$, i.e., $\phi(s)^*\lambda(g)^* = \phi(sg)^*$, from which the result follows.

Since Ψ^* is a Boolean isomorphism, Ψ is injective on a conull Borel set [8, Theorem 2.1]. Hence, for almost all s, $\phi(s)$ is a Borel isomorphism from a conull Borel set onto its image, which is also conull Borel. Hence $\phi(s)^{-1}$ can be defined almost everywhere for almost all s, and we can extend it to the remainder of S by letting it equal some fixed non-atom $t_0 \in S$.

Although $\phi(s)$ (and $\phi(s)^{-1}$) will not necessarily preserve the G-invariant measure m, we now show that we can modify ϕ so that it will. The map $s \to m_s = \phi(s)_*^{-1}m$ is Borel, as can be seen as in the proof of Lemma 3.1. By Lemma 3.2, for each g and almost all s,

$$m_{sg} = \phi(sg)_*^{-1}m = \phi(s)_*^{-1}\lambda(g)_*^{-1}m = m_s.$$

Since the space of σ-finite measures is countably separated, ergodicity of G on S implies m_s is constant on a conull set, say $m_s = v$. Choose an arbitrary s_0 with $\phi(s_0)_*^{-1}m = v$. Let $\bar{\phi}(s) = \phi(s)\,\phi(s_0)^{-1}$. Then $\bar{\phi}(s)_*^{-1}m = \phi(s_0)_* m_s = m$ for almost all s. Thus, replacing ϕ by $\bar{\phi}$ and U by $U(s_0)^{-1}U(s)$, we can acutally assume that $\phi(s)_* m = m$ for almost all s.

Let $W(s) = \phi(s)^*: L^2(S, m) \to L^2(S, m)$, i.e., $(W(s)f)(t) = f(\phi(s)t)$. Since $\phi(s)$ is a measure preserving isomorphism, $W(s)$ is unitary. $U(s)$ is not necessarily equal to $W(s)$, but they induce the same automorphism of $B(S)$. Further, from Lemma 3.2, we have $W(s)\,T(g)\,W(sg)^{-1} = I$ for each g and almost all s. The result of these remarks is that we can replace U by W, and hence assume that (almost everywhere) $U(s)$ is actually the unitary induced by the measure preserving point transformation $\phi(s)$.

We now consider relations between the various maps $U(s)$.

LEMMA 3.3. *There is a Borel field of unitary operators $A(t)$, $t \in S$, such that for almost all (s, t),*

$$U(\phi(s)t) = A(t)\,U(t)\,U(s) \tag{3.3}$$

Proof. Let $W_t(s) = U(\phi(s)t)\,U(s)^{-1}U(t)^{-1}$. Then for each g and almost all (s, t),

$$\begin{aligned} W_t(sg) &= U(\phi(sg)t)\,U(sg)^{-1}U(t)^{-1}. \\ &= U(\phi(sg)t)\,T(g)^{-1}U(s)^{-1}U(t)^{-1} \\ &= U([\phi(sg)t]\,g^{-1})U(s)^{-1}U(t)^{-1} \\ &= U(\lambda(g)^{-1}\phi(sg)t)U(s)^{-1}U(t)^{-1} \\ &= U(\phi(s)t)\,U(s)^{-1}U(t)^{-1} = W_t(g), \end{aligned}$$

by Lemma 3.2. Since $W_t: S \to U(L^2(S))$ is Borel, and $U(L^2(S))$ is a standard Borel space, for almost all t, W_t is constant on a conull set. Letting $A(t)$ be this constant, the result follows.

We will need a result similar to Lemma 3.3 that gives an expression for $U(\phi(s)^{-1}t)$. To do this, let G act on $S \times S$ by $(s, t)g = (sg, tg)$ and let E be the space of ergodic components of the action for some (essentially unique) ergodic decomposition [7, p. 192]. Let $\delta: S \times S \to E$ take each point to its ergodic component, and let $n = \delta_*(\mu \times \mu)$.

LEMMA 3.4. *There is a Borel field of unitary operators* $B: E \to U(L^2(S))$ *such that for almost all* (s, t),

$$U(\phi(s)^{-1}t) = B(\delta(s, t))\, U(t)\, U(s)^{-1}. \tag{3.4}$$

Proof. Let $W(s, t) = U(\phi(s)^{-1}t)\, U(s)\, U(t)^{-1}$. Then for each g and almost all (s, t),

$$\begin{aligned} W(sg, tg) &= U(\phi(sg)^{-1}tg)\, U(sg)\, U(tg)^{-1} \\ &= U(\phi(s)^{-1}\lambda(g)^{-1}tg)\, U(s)\, T(g)\, T(g)^{-1}U(t)^{-1} \\ &= U(\phi(s)^{-1}t)\, U(s)\, U(t)^{-1}. \end{aligned}$$

Thus, $W: S \times S \to U(L^2(S))$ is Borel and G-invariant, and since $U(L^2(S))$ is standard, there is a Borel function $B: E \to U(L^2(S))$ such that $B(\delta(s, t)) = W(s, t)$ for almost all (s, t). The result now follows.

We often need points that behave well with respect to various almost everywhere conditions. Thus let $S_0 \subset S$ be a conull Borel set such that $s \in S_0$ implies the following:

(i) $\phi(s)$ and $\phi(s)^{-1}$ preserve the measure m, and $\phi(s)^* = U(s)$ (c1)

(ii) $\{t \in S \mid (3.3), \text{ and } 3.4 \text{ hold}\}$ is conull (c2)

(iii) (3.2) holds. (c3)

We make further restrictions on S_0 throughout the paper. A technical condition we shall need presently is the following.

Let $d(s, t) = A(\phi(s)^{-1}t)\, B(\delta(s, t))$, where $s \in S_0$. Then d is essentially G-invariant, and hence $d(s, t) = B_0(\delta(s, t))$ almost everywhere, where B_0 is defined on E. A further condition on S_0 we require is

(iv) $s \in S_0$ implies $B_0(\delta(s, t)) = d(s, t)$ for almost all t. (c4)

Another condition we will require stems from the following.

LEMMA 3.5. *Let E be an ergodic decomposition of $S \times S$. For each $s \in S$, let $\delta_s = \delta \mid \{(s, t) \mid t \in S\}$, so $\delta_s: S \to E$. Then for almost all s, $(\delta_s)_*\mu \sim n$.*

Proof. Define a map $F: S \times S \to S \times S$ by $F(s, t) = (s, \phi(s)^{-1}t)$. Then for almost all (s, t), $F(sg, tg) = F(s, t) \circ g$, where $(\circ g)$ is the operation $(y, z) \circ g = (yg, z)$. Thus, F sets up an isomorphism of the Boolean G-spaces $(B(S \times S), \cdot)$ and $(B(S \times S), \circ)$ which preserves by projection on the first factor. Since the conclusion of the theorem holds for $(B(S \times S), \circ)$ it is not hard to see that it holds for $(B(S \times S), \cdot)$ as well.

We now state the condition on s we will need:

(v) $s \in S_0$ implies (3.5) is true. (c5)

We now modify $\phi(s)$ and $U(s)$ so as to make the equality in Lemma 3.3 more manageable. Choose a point $s_0 \in S$; and let $\bar{\phi}(s) = \phi(s) \circ \phi(s_0)^{-1}$ and

$$\bar{U}(s) = U(s_0)^{-1} U(s)$$

LEMMA 3.6. *There is a Borel field $\bar{A}(t)$ such that if $s \in S_0$, then for almost all t,*

$$\bar{U}(\bar{\phi}(s)t) = \bar{A}(t)\, \bar{U}(t)\, \bar{U}(s).$$

Proof. Let $s \in S_0$. Then for almost all t,

$$\begin{aligned} \bar{U}(\bar{\phi}(s)t) &= U(s_0)^{-1} U(\phi(s)\, \phi(s_0)^{-1} t) \\ &= U(s_0)^{-1} A(\phi(s_0)^{-1} t)\, U(\phi(s_0)^{-1} t)\, U(s) \\ &= U(s_0)^{-1} A(\phi(s_0)^{-1} t)\, B(\delta(s_0, t))\, U(t)\, U(s_0)^{-1} U(s) \\ &= \bar{A}(t)\, \bar{U}(t)\, \bar{U}(s), \end{aligned}$$

where

$$\bar{A}(t) = U(s_0)^{-1} A(\phi(s_0)^{-1} t)\, B(\delta(s_0, t))\, U(s_0).$$

The point of Lemma 3.6 is the following.

COROLLARY 3.7. *$\bar{A}(t) = I$ for almost all t.*

Proof. Let $s = s_0$ in Lemma 3.6. Since $\bar{\phi}(s_0)t = t$ for almost all t and $\bar{U}(s_0) = I$, we obtain for almost all t, $\bar{U}(t) = \bar{A}(t)\, \bar{U}(t)$.

We now wish to establish a similar result for the corresponding modification of $B(\delta(s, t))$.

LEMMA 3.8. *For all $s \in S_0$,*

$$\bar{U}(\bar{\phi}(s)^{-1} t) = \bar{U}(t)\, \bar{U}(s)^{-1} \textit{ for almost all } t.$$

Proof. For $s \in S_0$, we have for almost all t,

$$\begin{aligned} \bar{U}(\bar{\phi}(s)^{-1} t) &= U(s_0)^{-1} U(\phi(s_0)\, \phi(s)^{-1} t) \\ &= U(s_0)^{-1} A(\phi(s)^{-1} t)\, U(\phi(s)^{-1} t)\, U(s_0) \\ &= U(s_0)^{-1} A(\phi(s)^{-1} t)\, B(\delta(s, t))\, U(t)\, U(s)^{-1} U(s_0) \\ &= c(s, t)\, \bar{U}(t)\, \bar{U}(s)^{-1}, \end{aligned}$$

where $c(s, t) = U(s_0)^{-1} A(\phi(s)^{-1} t)\, B(\delta(s, t))\, U(s_0)$. By Corollary 3.7, and the expression for $\bar{A}(t)$ in the proof of Lemma 3.6, $c(s_0, t) = I$ for almost all t. This implies $d(s_0, t) = I$ for almost all t (see condition (iv) on S_0 for the definition of d), and by condition (iv) that $B_0(\delta(s_0, t)) = I$ for almost all t. Now condition (v) on S_0 implies that $B_0(e) = I$ for almost all $e \in E$. But for $s \in S_0$,

$$c(s, t) = U(s_0)^{-1} B_0(\delta(s, t))\, U(s_0) \text{ for almost all } t.$$

Condition (v) on S_0 then implies $c(s, t) = I$ for almost all t. The result follows.

For notational ease, we hereafter call $\bar{U}$ and $\bar{\phi}$ by U and ϕ, respectively. We summarize some of our results in the following theorem.

THEOREM 3.9. *We can choose the operators $U(s)$ and the maps $\phi(s)$ in such a way that for $s \in S_0$, a conull Borel set in S,* (i) *$U(\phi(s)t) = U(t)\, U(s)$ for almost all all t;* (ii) *$U(\phi(s)^{-1}t) = U(t)\, U(s)^{-1}$ for almost all t. Furthermore, there is an element $s_0 \in S$ such that $\phi(s_0)t = t$* a.e., *and $U(s_0) = I$.*

It will be necessary for us to modify U and ϕ again later on. It is easy to check that if $t_0 \in S_0$, and we make the modification $\bar{U}(s) = U(t_0)^{-1}U(s)$, $\bar{\phi}(s) = \phi(s)\,\phi(t_0)^{-1}$, that Theorem 3.9 remains true, with the role of s_0 now played by t_0, and S_0 unchanged.

4. The Involution

We now construct a measure class preserving map on S which will enable us to define an involution on $L^\infty(S)$. In order to do this, it will be necessary to examine more closely the ergodic decomposition of $S \times S$ under the product action. Let $F\colon S \times S \to S \times S$ be $F(s, t) = (s, \phi(s)^{-1}t)$. Then, for almost all (s, t), $F(sg, tg) = F(s, t) \circ g$, where, as above $(\circ g)$ is the operation $(y, z) \circ g = (yg, z)$. Then F^* is an isomorphism of the Boolean G-spaces $B(S \times S, \cdot)$ and $B(S \times S, \circ)$. Hence, there are invariant conull Borel sets $T_1, T_2 \subset S \times S$, and a bijective G-map $A\colon T_1 \to T_2$ such that $A = F$ a.e. Let $p\colon S \times S \to S$ be projection on the second factor and let $q = p \circ A$. Then $q\colon T_1 \to S$ preserves measure class and it is clear that decomposing $\mu \times \mu$ with respect to μ over the (G-invariant) fibers of q defines an ergodic decomposition of $\mu \times \mu$. We will call $\{q^{-1}(s) \cap (S \times S - T_1)\}$ the ergodic pieces of $S \times S$.

Let $r(s, t) = (t, s)$. Then r is a G-map, and hence, modulo null sets, permutes the ergodic pieces of $S \times S$. More precisely, $q \circ r$ is G-invariant, and hence there is a Borel map $a\colon S \to S$ and an invariant conull Borel set $T_3 \subset T_1$ such that $q(r(z)) = a(q(z))$ for all $z \in T_3$. Replacing T_3 by $T_3 \cap r(T_3)$, we can assume $T_3 = r(T_3)$. The above remarks imply that if $H \subset T_3$ is a union of ergodic pieces, then $r(H)$ is also a union of ergodic pieces. (Ergodic pieces in T_3 are ergodic pieces of $S \times S$ intersected with T_3.)

Let S_1 be the conull Borel set of elements $t \in S$ that satisfy the following conditions:

(i) $\phi(t)^{-1}$ is measure preserving

(ii) $F(s, t) = A(s, t)$ and $F(t, s) = A(t, s)$ for almost all s.

(iii) $\{s \mid (s, t) \in T_3\}$ is conull

(iv) $\{s \mid (t, s) \in A(T_3)\}$ is conull.

The main result of this section is the following:

PROPOSITION 4.1. *Let* $j_t(s) = \phi(s)^{-1}t$. *Then if* $t \in S_1$, j_t *is measure class preserving.*

Proof. Let $E \subset S$, and suppose $\mu(E) > 0$. Then $s \in j_t^{-1}(E)$ if and only if $(s, t) \in F^{-1}(S \times E)$. Then for fixed $t \in S_1$, the following sets have the same measure:

$$\{s \mid (s, t) \in F^{-1}(S \times E)\},$$
$$\{s \mid (s, t) \in A^{-1}(S \times E)\} \qquad \text{by (ii)},$$
$$\{s \mid (t, s) \in r(A^{-1}(S \times E) \cap T_3)\} \qquad \text{by (iii)},$$
$$\{s \mid (t, \phi(t)^{-1}s) \in A(r(A^{-1}(S \times E) \cap T_3))\} \qquad \text{by (ii)}.$$

Now $\{y \mid (t, y) \in A(rA^{-1}(S \times E) \cap T_3)\}$ has positive measure since $r(A^{-1}(S \times E)) \cap T_3$ is a union of ergodic pieces and has positive measure, and by condition (iv). Since $\phi(t)^{-1}$ is measure preserving, we obtain $\mu(j_t^{-1}(E)) > 0$. A similar argument shows $\mu(E) = 0 \Rightarrow \mu(j_t^{-1}(E)) = 0$.

We conclude this section with some further technical results we will need, which are consequences of Proposition 4.1.

We require that elements $s \in S_0$ now satisfy additional conditions.

(vi) $s \in S_0$ implies $\{y \mid U(\phi(y)^{-1}s) = U(s)\, U(y)^{-1}\}$ is conull. (c6)

(vii) $s \in S_1$, so that j_s is measure class preserving. (c7)

The only difficulty with these conditions is that there is no guarantee that s_0 satisfies them. We thus modify U and ϕ as above. Choose t_0 so that conditions (c1-c7) are satisfied. Let

$$\bar{\phi}(s) = \phi(s) \circ \phi(t_0)^{-1},$$
$$\bar{U}(s) = U(t_0)^{-1}U(s).$$

LEMMA 4.2. *For any* $s \in S_0$ (*i.e., satisfying* c1–c7), $\bar{U}(\bar{\phi}(y)^{-1}s) = \bar{U}(s)\, \bar{U}(y)^{-1}$ *for almost all* y.

Proof. $\bar{U}(\bar{\phi}(y)^{-1}s) = U(t_0)^{-1}U(\phi(t_0)\,\phi(y)^{-1}s)$. Since j_s preserves measure class, for almost all y, this becomes

$$\begin{aligned} U(t_0)^{-1}U(\phi(t_0)\,\phi(y)^{-1}s) &= U(t_0)^{-1}U(\phi(y)^{-1}s)\, U(t_0) \\ &= U(t_0)^{-1}U(s)\, U(y)^{-1}U(t_0) \\ &= \bar{U}(s)\, \bar{U}(y)^{-1}. \end{aligned}$$

Thus, relabeling $\bar{U}$, $\bar{\phi}$, t_0 by U, ϕ, s_0, respectively, we can assume, without losing our previous results, that conditions c6 and c7 hold for S_0.

COROLLARY 4.3. *$\phi(\phi(y)^{-1}s_0) = \phi(y)^{-1}$ almost everywhere, for almost all y.*

Proof. By condition c6,

$$U(\phi(y)^{-1}s_0) = U(s_0)\, U(y)^{-1} = U(y)^{-1} \text{ a.e.}$$

5. ESSENTIAL INJECTIVITY OF U

We show in this section that it suffices to consider the case in which U: $S \to U(L^2(S))$ is injective on a conull set. A variation of our notation will be helpful. Let q: $S \to U(L^2(S))$ be $q(s) = U(s)$ and $Q(s)$: $U(L^2(S)) \to U(L^2(S))$ be defined by $Q(s)x = xq(s)$.

LEMMA 5.1. *If $s \in S_0$, then*

(i) *$(q \circ \phi(s))t = (Q(s) \circ q)(t)$ for almost all t.*

(ii) *$Q(s)$ and $Q(s)^{-1}$ preserve the measure class of $v = q_*(\mu)$.*

Proof. (i) is clear. Let $F \subset q(S)$, and $D = \{t \mid U(\phi(s)t) = U(t)\, U(s)\}$. Then $q^{-1}(Q(s)^{-1}F) \cap D = \phi(s)^{-1}(q^{-1}F) \cap D$ and since D is conull and $\phi(s)$ measure class preserving, F is null if and only if $Q(s)^{-1}F$ is null. Similarly,

$$q^{-1}(Q(s)F) \cap D' = \phi(s)(q^{-1}F) \cap D',$$

where

$$D' = \{t \mid U(\phi(s)^{-1}t) = U(t)\, U(s)^{-1}\},$$

and (ii) follows readily.

Via the map q: $S \to X = q(S)$, we can identify $L^2(X, v)$ as a subspace of $L^2(S, \mu)$. Lemma 5.1 implies that for $s \in S_0$, $U(s)(L^2(X)) = L^2(X)$. $U(\cdot)$ can be considered as defined on X, and we let $W(x) = U(x) \mid L^2(X)$ for $x \in q(S_0)$, and an arbitrary unitary on $L^2(X)$ otherwise. Lemma 5.1 also implies that for $s \in S_0$, $Q(s)^*$: $B(X) \to B(X)$, we have $Q(s)^* = \tilde{W}(q(s))$. Since $U(s)$ leaves $L^2(X)$ invariant for almost all s, so will $T(g)$. Thus, there is an induced factor action of G on X [11, Proposition 2.1]. Let $T_0(g)$ be $T(g) \mid L^2(X)$. The relation $U(s)\, T(g)\, U(sg)^{-1} = I$ implies $W(x)\, T_0(g)\, W(xg)^{-1} = I$, and hence X is a normal G-space. We now show that W is essentially injective.

LEMMA 5.2. *If $x, y \in q(S_0)$, and $W(x) = W(y)$, then $U(x) = U(y)$. (So in fact, $x = y$).*

Proof. Let $q(s_1) = x$, $q(s_2) = y$, where $s_i \in S_0$. Then for almost all $z \in X$,

$$Q(s_1)z = S(s_2)z.$$

Hence for almost all t,

$$Q(s_1)\, q(t) = Q(s_2)\, q(t),$$

which, from Lemma 5.1, implies

$$U(\phi(s_1)t) = U(\phi(s_2)t).$$

But this implies

$$U(t)\, U(s_1) = U(t)\, U(s_2) \qquad \text{for almost all } t,$$

which shows $U(x) = U(y)$.

We now want to show that assuming the theorem true for the normal G-space X implies that it is also true for S. To do this, we will need a rather delicate statement of the theorem. What we will assume, and this will be shown in Section 6, is that we can modify the maps $W(x)$, in a way to be spelled out below, to obtain maps $\overline{W}(x)$, so that there is (i) a locally compact group H, (ii) a homomorphism of G into H with dense range, (iii) a conull G-invariant set $X_1 \subset X$, and (iv) a Borel measure class preserving G-isomorphism $\theta\colon X_1 \to H$ with the following further property. Let $\Phi = \theta^*\colon L^\infty(H) \to L^\infty(X_1)$. Then for almost all x,

$$\Phi^{-1}\overline{W}(x)\Phi = \sigma(\theta(x)), \tag{$*$}$$

where σ is the right regular representation of H. The restriction we need on $\overline{W}(x)$ is that

$$\overline{W}(x) = W(x_n)^{-1} \cdots W(x_1)^{-1} W(x), \tag{$**$}$$

where the x_i can be chosen (not arbitrarily) within a given conull set. The point of this last statement is that we want to ensure that $x_i \in q(S_0)$. Conditions $(*)$ and $(**)$ do not appear specifically in the statement of the theorem. However, condition $(*)$ is Lemma 6.7 and $(**)$ can be seen by examining the proof. Finally, we now proceed to show how this implies the theorem for S.

Let $\overline{U}(s) = U(x_n)^{-1} \cdots U(x_1)^{-1} U(s)$, and $\bar{\phi}(s) = \phi(s)\,\phi(s_1)^{-1} \cdots \phi(s_n)^{-1}$ where $s_i \in S_0$ and $q(s_i) = x_i$. Then Lemma 5.2 clearly holds for $\overline{U}$ and $\overline{W}$ as well as U and W.

LEMMA 5.3. *For almost all x, y, z, $\overline{W}(x)\, \overline{W}(y) = \overline{W}(z)$ implies $\overline{U}(x)\, \overline{U}(y) = \overline{U}(z)$.*

Proof. Let $y \in q(S_0)$, so $y = q(t_0)$, $t_0 \in S_0$. Then for almost all s, $\overline{U}(\bar{\phi}(t_0)s) = \overline{U}(s)\, \overline{U}(t_0)$ since $s_1, \ldots, s_n \in S_0$. Thus, for almost all s,

$$\overline{W}(q(s))\, \overline{W}(y) = \overline{W}(q(\bar{\phi}(t_0)s)).$$

Hence, by Lemma 5.2 for almost all s, z, we have $\overline{W}(q(s))\,\overline{W}(y) = \overline{W}(z)$ implies $\overline{U}(q(\bar{\phi}(t_0)s)) = \overline{U}(z)$. From the above, this means $\overline{U}(s)\,\overline{U}(t_0) = \overline{U}(z)$. This completes the proof.

THEOREM 5.4. *Under the above assumptions, U is actually essentially injective. Thus S is essentially isomorphic to X, and so the conclusion of the theorem holds for S.*

Proof. Let $\theta\colon X_1 \to H$ as described above. Then for almost all h,

$$\overline{W}(\theta^{-1}h) = \Phi\sigma(h)\,\Phi^{-1}.$$

Hence,

$$\overline{W}(\theta^{-1}k)\overline{W}(\theta^{-1}h) = \overline{W}(\theta^{-1}(hk))$$

for almost all h, k. Let $W_0(h) = \overline{U}(\theta^{-1}h)$. It follows from Lemma 5.3 that $W_0(hk) = W_0(h)\,W_0(k)$. Thus, we get an induced action of H on S which has as a factor action the action on X. Since X is essentially isomorphic to H as an H-space, $S \to X$ must be essentially injective if S is ergodic [14, Lemma 8.23]. Hence, it remains only to show this latter condition. But a set invariant under almost all h for the Boolean action defined by $W_0(h)$ will be invariant under the action defined by $\overline{U}(s)$ for almost all s. Since $\overline{U}(s)\,T(g) = \overline{U}(sg)$ a.e., this implies that there would be an element invariant under $T(g)$ for all g, which contradicts the ergodicity of G on S. This completes the proof.

We may thus assume from this point onward, that U is injective on a conull set. Making a modification similar to those above, it is easy to see that we can suppose s_0 is in this set, and we require the further condition on S_0:

(viii) U is injective on S_0. (C8)

We draw some important corollaries.

COROLLARY 5.5. *Let $j = j_{s_0}$ (see Proposition 4.1). Then $j^2 = id$ almost everywhere.*

Proof. $j(s) = \phi(s)^{-1}s_0$, so $j^2(s) = \phi(\phi(s)^{-1}s_0)^{-1}s_0$. Now $s \to \phi(s)^{-1}s_0$ is measure class preserving, so by condition C6 (see also Lemma 4.2), for almost all s,

$$\begin{aligned} U(j^2(s)) &= U(s_0)\,U(\phi(s)^{-1}s_0)^{-1} \\ &= U(s_0)\,U(s)\,U(s_0)^{-1} = U(s). \end{aligned}$$

The result follows since U is essentially injective.

COROLLARY 5.6. *$t \to \phi(t)s$ is measure class preserving for almost all s.*

Proof. For almost all (s, t).

$$U(\phi(t)s) = U(\phi(\phi(t)^{-1}s_0)^{-1}s),$$

and by essential injectivity,

$$\phi(t)s = \phi(\phi(t)^{-1}s_0)^{-1}s = i_s(j_{s_0}(t))$$

almost everywhere. The results follows from Proposition 4.1.

6. Proof of the Main Theorem

We now describe the Hopf–von Neumann algebra structure on $L^\infty(S)$. The various measure theoretic results of the preceding sections will make verification of the Hopf–von Neumann algebra axioms fairly straightforward. The reader is referred to [10] for definitions and results pertaining to Hopf–von Neumann algebras.

Definition 6.1. *Let* $\delta: L^\infty(S) \to L^\infty(S) \otimes L^\infty(S) = L^\infty(S \times S)$ be given by $\delta(f)(s, t) = f(\phi(t)s)$, where $f \in L^\infty(S)$.

Proposition 6.2. *δ is an isomorphism of $L^\infty(S)$ into $L^\infty(S \times S)$.*

Proof. δ is the map induced by the function $S \times S \to S$, $(s, t) \to \phi(t)s$. Since $\psi(s, t) = (s, \phi(s)t)$ is measure class preserving, so is $(s, t) \to \phi(s)t$.

We now show that δ is a comultiplication.

Proposition 6.3. $(\delta \otimes i) \circ \delta = (i \otimes \delta) \circ \delta$ *(these are maps* $L^\infty(S) \to L^\infty(S) \otimes L^\infty(S) \otimes L^\infty(S) = L^\infty(S \times S \times S)$*).*

Proof. Consider $((\delta \otimes i)g)(s, t, u)$ where $g \in L^\infty(S \times S)$. For almost all u, we have $g_u \in L^\infty(S)$ and then

$$\begin{aligned}(\delta \otimes i)(g)(s, t, u) &= \delta(g_u)(s, t)\\ &= g_u(\phi(t)s).\end{aligned}$$

Now if $g = \delta f$, then

$$(\delta f)_u(y) = \delta f(y, u) = f(\phi(u)y).$$

So

$$\begin{aligned}(\delta \otimes i)(\delta f)(s, t, u) &= (\delta f)_u(\phi(t)s)\\ &= f(\phi(u)\,\phi(t)s).\end{aligned}$$

On the other hand, consider $((i \otimes \delta)g)(s, t, u)$ where $g \in L^\infty(S \times S)$. Then for each s, we have $g_s \in L^\infty(S)$, and $((i \otimes \delta)g)(s, t, u) = \delta(g_s)(t, u) = g_s(\phi(u)t)$. If $g = \delta f$ for $f \in L^\infty(S)$, then

$$g_s(y) = (\delta f)(s, y) = f(\phi(y)s).$$

Hence $(\delta f)_s(\phi(u)t) = f(\phi(\phi(u)t)s)$. But for almost all (s, t, u), $\phi(u)\phi(t)s = \phi(\phi(u)t)s$, and therefore $(i \otimes \delta)\delta = (\delta \otimes i)\delta$.

Proposition 6.3 shows that $(L^\infty(S), \delta)$ is a Hopf–von Neumann algebra, which is clearly commutative. We now defined an involution of this algebra.

Let $j\colon S \to S$ be $j = j_{s_0}$ (see Section 4). Then j preserves measure class and hence induces a map $J\colon L^\infty(S) \to L^\infty(S)$, $(Jf)s = f(j(s))$. Since $(J^2f)(s) = J(f \circ j)(s) = f(j^2s) = f(s)$ for almost all s, by Corollary 5.5, J is an involution. Let $\sigma\colon L^\infty(S) \otimes L^\infty(S) \to L^\infty(S) \otimes L^\infty(S)$ be defined by $\sigma(f \otimes g) = g \otimes f$. Under the isomorphism of $L^\infty(S \times S)$ with $L^\infty(S) \otimes L^\infty(S)$, this map is identified with the map of $L^\infty(S \times S)$ defined by $(\sigma f)(x, y) = f(y, x)$. We claim:

PROPOSITION 6.4. $\sigma \circ \delta \circ J = (J \otimes J) \circ \delta$. *These are maps* $L^\infty(S) \to L^\infty(S) \otimes L^\infty(S) = L^\infty(S \times S))$.

Proof. Since σ, δ, J are all induced by point maps (say σ_*, δ_*, j, respectively) it suffices to show that $j \circ \delta_* \circ \sigma_* = \delta_* \circ (j \times j)$ almost everywhere. For $(t, s) \in S \times S$, the left side is $(j \circ \delta_*)(s, t) = j(\phi(t)s) = \phi(\phi(t)s)^{-1}s_0$. Now $U(\phi(\phi(t)s)^{-1}s_0) = U(\phi(t)s)^{-1}$ a.e. (by condition c6 on S_0) $= U(t)^{-1}U(s)$. But we also have $U(\phi(s)\phi(t)^{-1}s_0) = U(t)^{-1}U(s)$ a.e. since j is measure class preserving. By the essential injectivity of U, we conclude $(j \circ \delta_*)(s, t) = \phi(s)\phi(t)^{-1}s_0$ a.e.

On the other hand,

$$\begin{aligned}(\delta_* \circ (j \times j))(t, s) &= \delta_*(j(t), j(s))\\ &= \phi(j(s))(j(t))\\ &= \phi(\phi(s)^{-1}s_0)\,\phi(t)^{-1}s_0 .\end{aligned}$$

Since j preserves measure class, this $= \phi(s)^{-1}\phi(t)^{-1}s_0$ a.e. This proves the lemma.

Finally, we verify that the invariant measure m is also an invariant measure in the sense of Hopf–von Neumann algebras. Since $L^\infty(S)$ is a von Neumann algebra, the automorphism $J\colon L^\infty(S) \to L^\infty(S)$ induces a map J_* on the predual space, i.e., $J_*\colon L^1(S) \to L^1(S)$. If $\lambda \in L^1(S)$, this map is given by $J_*(\lambda) = (\lambda \circ j)\Delta$, where $\Delta = dj_*m/dm$.

LEMMA 6.5. *If* $f, g, h \in L^\infty(S, m) \cap L^1(S, m)$. *Then*

$$m \otimes m((f \otimes g)\,\delta h) = m \otimes m((h \otimes J_*g)\,\delta f).$$

Proof. Identifying $L^\infty(S) \otimes L^\infty(S)$ with $L^\infty(S \times S)$, the left side becomes

$$\iint_{S\times S} f(s)\,g(t)\,h(\phi(t)s)\,ds\,dt \tag{*}$$

and the right side

$$\iint_{S\times S} h(s)\,g(jt)\,f(\phi(t)s)\,\Delta(t)\,ds\,dt. \tag{**}$$

Replacing t by $j(t)$ in (*), we obtain

$$\iint_{S\times S} f(s)\,g(jt)\,h(\phi(jt)s)\,\Delta(t)\,ds\,dt = \iint_{S\times S} f(s)\,g(jt)\,h(\phi(t)^{-1}s)\,\Delta(t)\,ds\,dt.$$

Now replace s by $\phi(t)s$. Since $\phi(t)$ preserves measure for almost all t, we obtain $(*) = (**)$.

We summarize our results.

THEOREM 6.6. *$(L^\infty(S), \delta, J)$ is an involutive Hopf–von Neumann algebra with invariant measure m.*

As a consequence of Takesaki's duality theorem [10, Theorem 2], there is an isomorphism Φ of $(L^\infty(S), \delta, J, m)$ with $(L^\infty(H), \delta_H, j_H, \mu_H)$ where H is a locally compact group. We note that this implies $\delta_H \circ \Phi = (\Phi \otimes \Phi) \circ \delta$. Since $\Phi: L^\infty(S) \to L^\infty(H)$ is an algebra isomorphism, Φ is induced by a Borel map $\theta: H \to S$ which is injective and measure preserving [8, Theorem 2.1]. Let σ be the right regular representation of H.

LEMMA 6.7. *For almost all s,*

$$\sigma(\theta^{-1}(s)) = \Phi U(s)\,\Phi^{-1}.$$

Proof. Let $\lambda \in L^2(H)$. We want $(\sigma(\theta^{-1}s))\lambda)h = (\Phi U(s)\,\Phi^{-1}\lambda)h$ for almost all $(s, h) \in S \times H$. For this it suffices to show

$$(\sigma(\theta^{-1}s)\lambda)(\theta^{-1}t) = (\Phi U(s)\,\Phi^{-1}\lambda)(\theta^{-1}t) \qquad (*)$$

for almost all $(s, t) \in S \times S$. Now the left side of (*) is $(\delta_H\lambda)(\theta^{-1}t, \theta^{-1}s)$ by the definition of δ_H. Since θ induces Φ, θ^{-1} induces Φ^{-1}, and this $= (\Phi \otimes \Phi^{-1})$ $(\delta_H\lambda)(t, s)$. Since Φ is a Hopf algebra isomorphism, this

$$\begin{aligned} &= (\delta \circ \Phi^{-1})\,\lambda(t, s) = \delta(\Phi^{-1}\lambda)(t, s) \\ &= \Phi^{-1}\lambda(\Phi(s)t) = U(s)\,\Phi^{-1}\lambda(t) \\ &= \Phi U(s)\,\Phi^{-1}\lambda(\theta^{-1}t), \end{aligned}$$

completing the proof.

COROLLARY 6.8. *For each $g \in G$, $\Phi T(g)\,\Phi^{-1} = \sigma(h)$ for some $h \in H$.*

Proof. $T(g) = U(s)^{-1}U(sg)$ for almost all s. We can choose s so that this equality holds and $\Phi U(s)\,\Phi^{-1} = \sigma(\theta^{-1}s)$,

$$\Phi U(sg)\,\Phi^{-1} = \sigma(\theta^{-1}(sg)).$$

But then

$$T(g) = \Phi^{-1}\sigma(\theta^{-1}s)^{-1}\phi(\theta^{-1}(sg))\Phi.$$

We are finally ready to prove the main theorem.

THEOREM 6.9. *If S is a normal G-space with a σ-finite invariant measure, then there is a locally compact group H such that S is esesntially isomorphic to the G-space H, where the action of G on H is defined by a homomorphism of G into H with dense range.*

Proof. Define a homomorphism $A: G \to H$ by $A(g) = \sigma^{-1}(\Phi T(g) \Phi^{-1})$ (recall that $\sigma: H \to U(L^2(H))$ is a Borel isomorphism onto its image). Consider the corresponding action of G on H. The induced unitary on $L^2(H)$, say $W(g)$, is just $\Phi T(g) \Phi^{-1}$. But since $\Phi: L^\infty(S) \to L^\infty(H)$ is an algebra isomorphism, Φ is also a Boolean isomorphism of $B(S) \to B(H)$. Thus Φ is an isomorphism of the pairs $(T(g), B(S))$, $(W(g), B(H))$. It follows that the actions of G on S and H are essentially isomorphic. Finally, $A(G)$ must be dense in H, since the action is ergodic.

COROLLARY 6.10. *If S has a finite invariant measure, the group H is compact, and S has discrete spectrum.*

Proof. H is compact if and only if the Haar measure is finite.

Remark. This provides a different proof of Corollary 2.4.

Remark. It seems likely that Theorem 6.9 remains true under the weaker assumption that S has a probability measure quasi-invariant and ergodic under G. With still further attention to measure theoretic detail, one may be able to show that a normal G-space with a quasi-invariant probability measure actually has a σ-finite invariant measure, by a method similar to that of [14, Lemma 8.33].

7. Finite Invariant Measures

In this section, we give a direct proof that G-spaces with finite invariant measure and discrete spectrum (where the spectrum satisfies a certain additional condition, which always holds if G is abelian) are normal, and then use Corollary 6.10 to describe their structure.

PROPOSITION 7.1. *Suppose G acts ergodically on S, with a finite invariant measure, and that $L^2(S)$ has an orthonormal basis of eigenvectors of the action. Then S is a normal G-space.*

Proof. Let $\{f_i\}$ be an orthonormal basis of eigenvectors, so that $T(g)f_i = \lambda(g)f_i$, $\lambda(g) \in \mathbb{C}$, and let $H_i = \mathbb{C}f_i$. We claim that for each i, the cocycle $\alpha(s, g) = T(g) \mid H_i$ is equivalent to the identity. For this, it suffices to show that there is a G-invariant function in $L^2(S; H_i)$ for the induced representation U^α of G. But $\theta: S \to H_i$, $\theta(s) = \bar{f}_i(s) f_i$ clearly satisfies the required conditions.

As a consequence, we obtain the von Neumann–Halmos theorem.

COROLLARY 7.2. *If $T: S \to S$ is ergodic with pure point spectrum, then T is equivalent to a rotation on a compact abelian group.*

Proof. Proposition 7.1 and Corollary 6.10.

More generally, we obtain a special case of Mackey's theorem [5] by these methods.

PROPOSITION 7.3. *Suppose S is an ergodic G-space with finite invariant measure. Let $U(g)$ be the natural representation of G on $L^2(S)$ suppose $U = \sum_{\pi \in L}^{\oplus} (\dim \pi)\pi$, where L is a collection of equivalence classes of finite-dimensional irreducible representations. Then S is a normal G-space and hence the conclusion of Corollary 6.10. holds.*

Proof. Fix $\pi \in L$, and let $n = \dim \pi$. Choose orthonormal elements $f_i ,..., f_n \in L^2(S)$ such that $H = \operatorname{span}\{f_i\}$ is invariant and irreducible under G, and $U \mid H$ is equivalent to π. Let $a_{ij}(g)$ be functions on G such that $U(g)f_j = \sum a_{ij}(g)f_i$. Then the hypothesis of the theorem implies there is an orthonormal set of functions h_{ik} in $L^2(S)$, $1 \leqslant i, k \leqslant n$, such that (i) $h_{i1} = f_i$ and (ii) for each k, $U(g)\, h_{jk} = \sum_i a_{ij}(g)\, h_{ik}$.

To show the action is normal, it suffices to show that the cocycle $\alpha(s, g) = U(g) \mid H$ is equivalent to the identity, since π is arbitrary. To do this, it suffices to produce nonzero G-invariant functions $\theta_k: S \to H$, $k = 1,..., n$, such that $\{\theta_k(s)\}$ are orthogonal for almost all s.

Define $\theta_k(s, t) = \sum_j \bar{h}_{jk}(s) f_j(t)$, which we can consider as a function $S \to H$. Then

$$\begin{aligned}\theta_k(sg, tg) &= \sum_j \bar{h}_{jk}(sg) f_j(tg)\\ &= \sum_j \left(\sum_i \bar{a}_{ij}(g)\, \bar{h}_{ik}(s)\right)\left(\sum_p a_{pj}(g) f_p(t)\right)\\ &= \sum_{i,p} \sum_j \bar{a}_{ij}(g)\, a_{pj}(g)\, \bar{h}_{ik}(s) f_p(t).\end{aligned}$$

Since $a_{ij}(g)$ is unitary for each g,

$$\begin{aligned}\sum_{i,p}\left(\sum_j \bar{a}_{ij}(g)\, a_{pj}(g)\, \bar{h}_{ik}(s) f_p(t)\right) &= \sum_{i,p} \delta_{ip} \bar{h}_{ik}(s) f_p(t)\\ &= \theta_k(s, t).\end{aligned}$$

Thus, θ_k are invariant, and it suffices to show that for each fixed i, k, $i \neq k$, and almost all s, that

$$A(s) = \int_S \theta_k(s, t)\, \bar{\theta}_i(s, t)\, dt = 0.$$

But this is just

$$A(s) = \int \left(\sum_j \bar{h}_{jk}(s) f_j(t) \right) \left(\sum_p h_{pi}(s) \bar{f}_p(t) \right) dt$$

$$= \sum_{j,p} \bar{h}_{jk}(s)\, h_{pi}(s) \int_S f_j(t) \bar{f}_p(t)\, dt$$

$$= \sum_j \bar{h}_{jk}(s)\, h_{ji}(s).$$

Now an argument similar to the one above, showing that θ_k is invariant, will show that $A(s)$ is G-invariant. By ergodicity, $A(s)$ is essentially constant. But $\int A(s)\, ds = \sum_j \langle h_{ji} \mid h_{jk} \rangle = 0$. Thus, $A(s) = 0$ almost everywhere, completing the proof.

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References

1. E. G. Effros, The borel space of von Neumann algebras on a separable Hilbert space, *Pac. J. Math.* **15** (1964), 1153–1164.
2. P. R. Halmos and J. von Neumann, Operator methods in classical mechanics, II, *Ann. Math.* **43** (1942), 332–350.
3. K. Lange, Borel sets of probability measures, *Pac. J. Math.* **48** (1973), 141–161.
4. G. W. Mackey, Induced representations of locally compact groups, I, *Ann. Math.* **55** (1952), 101–139.
5. G. W. Mackey, Ergodic transformation groups with a pure point spectrum, *Ill. J. Math.* **8** (1964), 593–600.
6. G. W. Mackey, Ergodic theory and virtual groups, *Math. Ann.* **166** (1966), 187–207.
7. G. W. Mackey, Ergodic theory and its significance for statistical mechanics and probability theory, *Adv. Math.* **12** (1974), 178–268.
8. A. Ramsay, Virtual groups and group actions, *Adv. Math.* **6** (1971), 253–322.
9. A. Ramsay, Boolean duals of virtual groups, *J. Funct. Anal.* **15** (1974), 56–101.
10. M. Takesaki, A characterization of group algebras as a converse of Tannaka–Stinespring–Tatsuuma duality theorem, *Amer. J. Math.* **91** (1969), 529–564.
11. R. J. Zimmer, Extensions of ergodic group actions, *Ill. J. Math.* **20** (1976), 373–409.
12. R. J. Zimmer, Ergodic actions with generalized discrete spectrum, *Ill. J. Math.* **20** (1976), 555–588.
13. R. J. Zimmer, Extensions of ergodic actions and generalized discrete spectrum, *Bull. A.M.S.* **81** (1975), 633–636.
14. V. S. Varadarajan, "Geometry of Quantum Theory," Vol. II, Van Nostrand, Princeton, N.J., 1970.