JOURNAL OF NUMBER THEORY 13, 337–354 (1981)

A Geometric Approach to Equal Sums of Fifth Powers

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Communicated by J. W. S. Cassels

Received October 30, 1979

Most of the known results about the Diophantine equation $x^5 + y^5 + z^5 = u^5 + v^5 + w^5$ are shown to be particular instances of a simple geometrical construction. By studying a K3 surface contained in the fourfold, we show that there are an infinity of parametric solutions also satisfying $x + y + z = u + v + w$, $x - y = u - v$; and we show that these may be effectively determined.

1. INTRODUCTION

1.1. We are concerned in this study with the Diophantine equation

$$x^5 + y^5 + z^5 = u^5 + v^5 + w^5. \tag{1}$$

The first non-trivial examples of two equal sums of three fifth powers are apparently due to Sastry and Chowla [7], who exhibit the one-parameter curve of solutions

$$(t^5 + 75)^5 + (t^5 - 75)^5 + (-50t)^5 = (t^5 + 25)^5 + (t^5 - 25)^5 + (10t^3)^5$$

and Subba Rao [8] who gives

$$3^5 + (-4)^5 + 29^5 = 10^5 + 20^5 + 28^5.$$

Moessner [5] found an example where each fifth power is positive:

$$49^5 + 75^5 + 107^5 = 39^5 + 92^5 + 100^5$$

and gave [6] two one-parameter solutions of degrees 7 and 36. Swinnerton–Dyer [9] shows how to construct two-parameter solutions of degrees 30 and 64; and Lander [3] produces a two-parameter solution, not given explicitly, of which he gives a special case as a one-parameter solution of degree 9. Lander, Parkin, and Selfridge [4] list many numerical solutions.

0022-314X/81/030337-18$02.00/0

1.2. We show in Section 2 how most of the known results concerning parametric solutions of (1) may be put into the context of a simple geometric construction. This section is essentially a survey of the known results, although we do produce an explicit two-parameter solution of degree 9, and some further one-parameter solutions.

In Section 3, we adopt an approach similar to that taken for equal sixth powers in Bremner [1] (henceforth denoted ESP). The fourfold (1) contains a two-parameter family of K3 surfaces, for putting

$$X = x + u, \qquad Y = y + v, \qquad Z = z + w,$$
$$\lambda = x - u, \qquad \mu = y - v, \qquad \nu = z - w,$$

Eq. (1) transforms to the generically non-singular quartic surface in $\mathbb{P}^3$ given by

$$\lambda(5X^4 + 10\lambda^2X^2 + \lambda^4) + \mu(5Y^4 + 10\mu^2Y^2 + \mu^4) + \nu(5Z^4 + 10\nu^2Z^2 + \nu^4) = 0.$$

Certain members of this family contain rational straight lines and so we correspondingly expect there to be an interesting Néron–Severi group defined over $\mathbb{Q}$. Such is the case for the surface given by $\lambda : \mu : \nu = 1 : 1 : -2$ which contains eight rational lines. We study this surface in detail; in particular, pencils of elliptic curves lie upon it, and we can obtain non-trivial automorphisms as in ESP, Section 5. We then show that a result similar to that conjectured in ESP, Section 7, is true; that is, we exhibit two explicit automorphisms of the surface such that any parametrizable curve defined over $\mathbb{Q}$ of arithmetic genus 0 arises as the image of one of the straight lines by repeated application of the two automorphisms and symmetries.

Thus all such curves on the surface can be effectively determined.

2. A Simple Construction

We regard the equation

$$x^5 + y^5 + z^5 = u^5 + v^5 + w^5$$

as defining a quintic fourfold V in five-dimensional projective space. Suppose given a rational point P of V; if we can construct through P a straight line l which also passes through three further rational points of V, then the remaining point of intersection of l with V is necessarily rational, and in this way we can define a rational automorphism of the fourfold. There are several obvious ways in which to define the line l. To this end we note that V contains the 15 distinct rational planes given in Table I.

TABLE I

π_1 : $x=u$, $y=v$, $z=w$	π_2 : $x=u$, $y=w$, $z=v$	π_3 : $x=u$, $y=-z$, $v=-w$	π_4 : $x=v$, $y=u$, $z=w$	π_5 : $x=v$, $y=w$, $z=u$
π_6 : $x=v$, $y=-z$, $u=-w$	π_7 : $x=w$, $y=u$, $z=v$	π_8 : $x=w$, $y=v$, $z=u$	π_9 : $x=w$, $y=-z$, $u=-v$	π_{10} : $x=-y$, $z=u$, $v=-w$
π_{11} : $x=-y$, $z=v$, $u=-w$	π_{12} : $x=-y$, $z=w$, $u=-v$	π_{13} : $x=-z$, $y=u$, $v=-w$	π_{14} : $x=-z$, $y=v$, $u=-w$	π_{15} : $x=-z$, $y=w$, $u=-v$

First, we can fix three such planes, and then take l to be a common transversal passing through P (though to be well defined it is necessary that no two of the planes meet in a line). Second, we can fix two planes, and take l to be a common transversal with a double point at P in its intersection with V. Third, we can fix one plane, insisting that l have a triple point at P; and fourth, we can demand l to be a line with P a point of order 4.

In general we wish to examine the images of the 15 rational planes under the automorphisms constructed above, for these will necessarily be rational surfaces given in terms of two parameters. Notice that any line joining a point of one rational plane to a point of any other rational plane lies inside the linear fourfold

$$\Pi: x + y + z = u + v + w;$$

and so in all but the fourth construction above, it is necessary that P lie on Π, and then automatically the surfaces resulting as the images of the rational planes lie inside Π.

We now examine the individual cases, in each instance denoting the coordinates of P by $(a_1 a_2 a_3 b_1 b_2 b_3)$.

2.1. *P simple point*

The symmetric group S_6 acts on the rational planes in an obvious manner, and so acts on triples of the π_i. It is a simple exercise to count the conjugacy classes, and there are five such, corresponding to the five possibilities for the pattern of intersection of three such planes: three lines; two lines and one point; one line and two points; or three points, either coincident or distinct. Representative triples may be taken respectively to be $\{\pi_1\pi_2\pi_3\}$, $\{\pi_1\pi_2\pi_4\}$, $\{\pi_1\pi_2\pi_6\}$, $\{\pi_1\pi_5\pi_7\}$, $\{\pi_1\pi_5\pi_9\}$.

Given a triple of rational planes, and supposing P to be a point of Π, then each plane with P generates a unique threefold lying in Π, which has to contain any common transversal l of the three planes passing through P, so l occurs in the intersection of the three threefolds. This intersection is of dimension one, and so precisely l, if and only if no two of the three planes meet in a line and the three planes have no point in common. So for the construction to be well defined we can restrict attention to the triple $\{\pi_1\pi_5\pi_9\}$.

Using the above characterization of the transversal, it is easy to write the equations of l:

$$\begin{aligned}
x:y:z:u:v:w = {} & \lambda[(a_1-b_1)(a_1-b_3)-(a_2-b_2)(b_1+b_2)] \\
& +\mu[(a_1-b_2)(a_1-b_3)-(a_3-b_1)(b_1+b_2)] \\
& :\lambda[-(a_1-b_3)(a_3-b_3)-(a_2+a_3)(a_2-b_2)] \\
& +\mu[(a_1-b_3)(a_2-b_3)-(a_2+a_3)(a_3-b_1)] \\
& :\lambda[-(a_2-b_2)(b_1+b_2)-(a_1-b_1)(a_1-b_3)] \\
& +\mu[-(a_1-b_3)(a_2-b_3)-(a_2+a_3)(a_3-b_1)] \\
& :\lambda[-(a_2-b_2)(b_1+b_2)-(a_1-b_1)(a_1-b_3)] \\
& +\mu[(a_1-b_2)(a_1-b_3)-(a_3-b_1)(b_1+b_2)] \\
& :\lambda[(a_1-b_1)(a_1-b_3)-(a_2-b_2)(b_1+b_2)] \\
& +\mu[(a_1-b_3)(a_2-b_3)-(a_2+a_3)(a_3-b_1)] \\
& :\lambda[-(a_1-b_3)(a_3-b_3)-(a_2+a_3)(a_2-b_2)] \\
& +\mu[-(a_1-b_3)(a_2-b_3)-(a_2+a_3)(a_3-b_1)].
\end{aligned}$$

P is given by $\lambda:\mu=1:1$, and the points of intersection with π_1, π_5, π_9 by $\lambda:\mu=0:1$, $1:0$, $(b_1-a_3):(a_2-b_2)$, respectively.

If now P is to describe π_i, then we must have $i=11$ or 13. For if π_i intersects π_1, π_5, or π_9 in a line, it is readily checked that the transversal l is actually contained in π_i, and so a fifth point of intersection is not well defined. But the image of π_{11} is immediately verified to be π_{13} and vice versa, so the rational planes do not actually give rise to new surfaces. However one could, for instance, let P describe the surface discovered in (2) below, thereby creating further new surfaces. For example, starting with $P=(3, -24, -28, 67, -62, -54)$ we deduce the fifth point of intersection (40163, −6166, −37254, −33245, 37420, −7432).

2.2. *P double point*

In a disguised form this is the construction of Lander [3].

We fix two rational planes, and again it is necessary to assume that they do not intersect in a line. From symmetry we can restrict attention to $\{\pi_1\pi_5\}$.

Given P, the transversal l is realised as the intersection of the two threefolds defined by π_1, P and π_5, P, together with the tangent fourfold T to V at P.

The equation of T is $a_1^4 x + a_2^4 y + a_3^4 z = b_1^4 u + b_2^4 v + b_3^4 w$, and so the equations of l are

$$\begin{aligned}
x:y:z:u:v:w = {} & \lambda[(a_1-b_1)(a_3^4-b_1^4)-(a_2-b_2)(a_2^4-b_3^4)]\\
& +\mu[(a_1-b_2)(a_2^4-b_2^4)-(a_3-b_1)(a_3^4-b_3^4)]\\
& :\lambda[(a_2-b_2)(a_1^4-b_2^4)-(a_3-b_3)(a_3^4-b_1^4)]\\
& +\mu[(a_2-b_3)(a_3^4-b_3^4)-(a_1-b_2)(a_1^4-b_1^4)]\\
& :\lambda[(a_3-b_3)(a_2^4-b_3^4)-(a_1-b_1)(a_1^4-b_2^4)]\\
& +\mu[(a_3-b_1)(a_1^4-b_1^4)-(a_2-b_3)(a_2^4-b_2^4)]\\
& :\lambda[(a_3-b_3)(a_2^4-b_3^4)-(a_1-b_1)(a_1^4-b_2^4)]\\
& +\mu[(a_1-b_2)(a_2^4-b_2^4)-(a_3-b_1)(a_3^4-b_3^4)]\\
& :\lambda[(a_1-b_1)(a_3^4-b_1^4)-(a_2-b_2)(a_2^4-b_3^4)]\\
& +\mu[(a_2-b_3)(a_3^4-b_3^4)-(a_1-b_2)(a_1^4-b_1^4)]\\
& :\lambda[(a_2-b_2)(a_1^4-b_2^4)-(a_3-b_3)(a_3^4-b_1^4)]\\
& +\mu[(a_3-b_1)(a_1^4-b_1^4)-(a_2-b_3)(a_2^4-b_2^4)].
\end{aligned}$$

P is given by $\lambda:\mu = 1:1$, and the points of intersection with π_1, π_5 by $\lambda:\mu = 0:1$, $1:0$. If P describes the plane π_i, then for a well-defined image π_i must intersect π_1, π_5 in distinct points, and so it suffices by symmetry to consider the image of π_9.

Taking $P=(r, p, -p, -q, q, r)$, the fifth point of intersection is given by

$$\begin{aligned}
\lambda+\mu:\lambda-\mu = {} & r^8 - 2(p^2+q^2)\,r^6 - 2(p^4+p^2q^2+q^4)\,r^4\\
& + (p^2+q^2)(p^4+q^4)\,r^2 + (p^8+p^6q^2+p^4q^4+p^2q^6+q^8):\\
& 2r(p+q)(p^2+q^2)^2(r^2-pq).
\end{aligned}$$

So abbreviating the polynomial $c_0 p^n + c_1 p^{n-1} q + \cdots + c_n q^n$ by $(c_0, c_1, \ldots, c_n)$, we find the fifth point of intersection has coordinates given by

$$\begin{aligned}
& x-w:y+z:v+u:v-u:y-z:x+w\\
& = 2(p^2+q^2)(q^4-p^4)\,r(r^2-pq):\\
& \quad 2(p^2+q^2)(r^4-q^4)\,r(r^2-pq):\\
& \quad 2(p^2+q^2)(r^4-p^4)\,r(r^2-pq):\\
& \quad qr^8 - 2q(1,0,1)\,r^6 + 2p(1,0,2,1,1)\,r^4\\
& \quad - q(1,2,3,4,1,2,-1)\,r^2 + q(1,0,1,0,1,0,1,0,1):
\end{aligned}$$

$$pr^8 - 2p(1, 0, 1)\, r^6 + 2q(1, 1, 2, 0, 1)\, r^4$$
$$- p(-1, 2, 1, 4, 3, 2, 1)\, r^2 + p(1, 0, 1, 0, 1, 0, 1, 0, 1):$$
$$r^9 - 2r^5(1, 1, 1, 1, 1) + r^3(1, 1, 1, 1)^2$$
$$+ r(1, 0, -1, -2, -3, -2, -1, 0, 1).$$

There are curves of degree 8 lying on this surface; for example, the image of the curve on π_9 given by $r = (\alpha + 1)p + \alpha q$, $\alpha \in \mathbb{Q}$, is in general of degree 8. The particular case $\alpha = 1$ gives the parametrization:

$$x = (389, 1111, 1599, 1435, 897, 417, 137, 29, 2),$$
$$y = (123, 41, 129, 421, 543, 319, 87, 3, -2),$$
$$z = (442, 1647, 2603, 2363, 1275, 409, 77, 13, 3),$$
$$u = (272, 729, 1425, 1685, 1221, 527, 135, 19, 3),$$
$$v = (474, 1655, 2647, 2363, 1263, 385, 49, -3, -1),$$
$$w = (208, 415, 259, 171, 231, 233, 117, 29, 1).$$

2.3. *P triple point*

This occurs in Swinnerton–Dyer [9] who fixes π_7 and lets P describe π_1. The image is a surface of degree at most 30.

2.4. *P point of order* 4

This again occurs in Swinnerton–Dyer [9] who argues as follows. Take P to be a point of π_1, and let l have equations:

$$x : y : z : u : v : w$$
$$= a_1\lambda + (b_1 + c_1)\mu : a_2\lambda + (b_2 + c_2)\mu : a_3\lambda + (b_3 + c_3)\mu :$$
$$a_1\lambda + (b_1 - c_1)\mu : a_2\lambda + (b_2 - c_2)\mu : a_3\lambda + (b_3 - c_3)\mu.$$

Then P is of order 4 provided

$$\sum a_1^4 c_1 = \sum a_1^3 b_1 c_1 = \sum a_1^2 c_1(c_1^2 + 3b_1^2) = 0,$$

where the summation is over the cyclic permutations of the suffices 1, 2, 3. Eliminating b_1 between the last two equations gives the necessary and sufficient condition for P to have order 4:

$$\sum a_1^4 c_1 = 0; \qquad 3c_1c_2c_3 \sum a_1^2 c_1^3 = \text{square}. \qquad (2)$$

Eliminating c_3 gives the sextic curve over $\mathbb{Q}(a_1 a_2 a_3)$:

$$3c_1c_2(a_1^4c_1 + a_2^4c_2)[3a_1^4a_2^4c_1c_2(a_1^4c_1 + a_2^4c_2) - a_1^2(a_3^{10} - a_1^{10})c_1^3 - a_2^3(a_3^{10} - a_2^{10})c_2^3] = \text{square}. \qquad (3)$$

The problem of finding points on (3) is difficult. The discriminant of the cubic factor is $-27a_1^4a_2^4a_3^{20}(a_1^5 + a_2^5 + a_3^5)(a_1^5 + a_2^5 - a_3^5)(a_1^5 - a_2^5 + a_3^5)(a_1^5 - a_2^5 - a_3^5)$ and so there is no non-trivial specialization of a_1, a_2, a_3 to rational polynomials in one variable which will reduce the genus of the curve (3). All points that are known are furnished by Swinnerton–Dyer, who satisfies (2) by taking $a_i = d_i^3$, $i = 1, 2, 3$, and $\sum d_1^2 c_1 = 0$. This gives rise to a two-parameter solution of degree at most 64; specialising to $d_1 = a^2$, $d_2 = a$, $d_3 = 1$ results in the parametric solution of degree 36 given by Moessner [6].

3. A Particular Surface

3.1. As mentioned in the Introduction, we will now restrict attention to the surface lying inside V given by $x - u : y - v : z - w = 1 : 1 : -2$. This amounts to studying the surface W given by

$$W: \quad \begin{aligned} x^5 + y^5 + z^5 &= u^5 + v^5 + w^5 \\ x + y + z &= u + v + w \\ x - y &= u - v. \end{aligned} \qquad (4)$$

We make the transformation

$$\begin{aligned} x &= r + s + t, \qquad & u &= -r + s + t, \\ y &= r - s + t, & v &= -r - s + t, \\ z &= p - 2r, & w &= p + 2r \end{aligned} \qquad (5)$$

under which the equation of W becomes

$$(p^2 - t^2)(p^2 + t^2 + 8r^2) = (s^2 - r^2)(s^2 + 3r^2 + 6t^2). \qquad (6)$$

W contains the eight rational lines

$$\begin{array}{llll} L_1\colon r = s & L_2\colon r = s & L_3\colon r = -s & L_4\colon r = -s \\ \quad p = t, & \quad p = -t, & \quad p = t, & \quad p = -t, \\ L_5\colon r = t & L_6\colon r = t & L_7\colon r = -t & L_8\colon r = -t \\ \quad p = s, & \quad p = -s, & \quad p = s, & \quad p = -s, \end{array}$$

and the intersection with W of the quadric $p^2 - t^2 = s^2 - r^2$ is precisely these eight lines, each with multiplicity one; so a hyperplane section Π on W satisfies

$$\Pi \sim \tfrac{1}{2}(L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8). \tag{7}$$

The intersection matrix of the L_i is as follows:

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
L_1	−2	1	1	0	1	0	0	1
L_2	1	−2	0	1	0	1	1	0
L_3	1	0	−2	1	0	1	1	0
L_4	0	1	1	−2	1	0	0	1
L_5	1	0	0	1	−2	1	1	0
L_6	0	1	1	0	1	−2	0	1
L_7	0	1	1	0	1	0	−2	1
L_8	1	0	0	1	0	1	1	−2

Symmetries of W, with their action on the lines L_i, are shown in the following table (L_m is abbreviated to m),

TABLE II

Symmetry	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
$r \to -r$	3	4	1	2	7	8	5	6
$s \to -s$	3	4	1	2	6	5	8	7
$r, s \to -r, -s$	1	2	3	4	8	7	6	5
$t \to -t$	2	1	4	3	7	8	5	6
$r, t \to -r, -t$	4	3	2	1	5	6	7	8
$s, t \to -s, -t$	4	3	2	1	8	7	6	5
$p \to -p$	2	1	4	3	6	5	8	7

together with the eight further symmetries obtained by interchanging s and t, which has the effect of interchanging (L_1, L_2, L_3, L_4) and (L_5, L_6, L_7, L_8), respectively.

W has 16 conics defined over $\mathbb{Q}$ typified by the divisor $\Pi - L_1 - L_2$, which is realised by the equations $\{p = t,\ s^2 + 3r^2 + 6t^2 = 0\}$, and so does not possess any rational points. Further, W is fibred by pencils of elliptic curves, for example, $\Pi - L_i$ of degree 3. We consider in detail the curve of intersection E_λ lying on W:

$$E_\lambda\colon \quad \begin{aligned} &(4\lambda + 1)(p^2 - t^2) = (2\lambda + 3)(s^2 - r^2) \\ &(2\lambda + 3)(p^2 + t^2 + 8r^2) = (4\lambda + 1)(s^2 + 3r^2 + 6t^2). \end{aligned}$$

E_λ possesses the points $(r, s, t, p) = (\pm\lambda^{1/2}, \pm\lambda^{1/2}, \pm 1, 1)$ defined over $\mathbb{Q}(\lambda^{1/2})$ and so is an abelian variety (of dimension 1) over $\mathbb{Q}(\lambda^{1/2})$. We shall show in

Appendix A that with $P_0 = (\lambda^{1/2}, -\lambda^{1/2}, -1, 1)$ as the zero for the group law, then the group of $\mathbb{Q}(\lambda^{1/2})$ points on E_λ is generated by $P_1 = (\lambda^{1/2}, \lambda^{1/2}, 1, 1)$, $P_2 = (-\lambda^{1/2}, -\lambda^{1/2}, 1, 1)$, points of order 2, and $P_3 = (\lambda^{1/2}, \lambda^{1/2}, -1, 1)$, a point of infinite order. Now let Γ be a curve on W which is defined over $\mathbb{Q}$. As in ESP, Section 3.3, Γ meets E_λ in a divisor of points defined over $\mathbb{Q}(\lambda)$, and so by subtracting an appropriate multiple of P_0, we can associate to Γ, via the Jacobian, a point of E_λ defined over $\mathbb{Q}(\lambda^{1/2})$. That is, $\Gamma \cdot E_\lambda - rP_0$ is an element of G_a/G_t for a suitable integer r, and so $\Gamma \cdot E_\lambda \sim r_0 P_0 + r_1 P_1 + r_2 P_2 + r_3 P_3$ for integers r_i, $i = 0, \ldots, 3$. Since $\Gamma \cdot E_\lambda$ is defined over $\mathbb{Q}(\lambda)$, we also have $\Gamma \cdot E_\lambda \sim r_0 \bar{P}_0 + r_1 \bar{P}_1 + r_2 \bar{P}_2 + r_3 \bar{P}_3$, where $\bar{P}_i$ denotes the conjugate point to P_i under $\lambda^{1/2} \to -\lambda^{1/2}$, and thus $2\Gamma \cdot E_\lambda \sim \sum_{i=0}^{3} r_i(P_i + \bar{P}_i) = (r_0 L_4 + (r_1 + r_2) L_1 + r_3 L_2) \cdot E_\lambda$.

Consequently there is a function f_λ on E_λ defined over $\mathbb{Q}(\lambda)$ with divisor $(2\Gamma - r_0 L_4 - (r_1 + r_2) L_1 - r_3 L_2) \cdot E_\lambda$ and so (N.B. ESP, 3.3)

$$2\Gamma \sim r_0 L_4 + (r_1 + r_2) L_1 + r_3 L_2 + (\text{components of } E_\lambda \text{ which split}). \tag{8}$$

To find the E_λ which are singular is straightforward. There is one decomposition as the sum of four lines:

$$\lambda = 1 : E \sim L_5 + L_6 + L_7 + L_8 \tag{9}$$

and no less than ten decompositions as the sum of two conics:

$\lambda = -\frac{1}{4}$: $\quad E \sim (\Pi - L_1 - L_2) + (\Pi - L_3 - L_4)$

$\lambda = -\frac{3}{2}$: $\quad E \sim (\Pi - L_1 - L_3) + (\Pi - L_2 - L_4)$

$\lambda = 0$: $\quad E \sim (s = r\theta_0) + (s = -r\theta_0) \qquad \theta_0^2 = -\frac{3}{2}$

$\lambda = -\frac{2}{3}$: $\quad E \sim (t = r\theta_0) + (t = -r\theta_0)$

$\lambda = -\frac{3}{7}$: $\quad E \sim (p = s\theta_1) + (p = -s\theta_1) \qquad \theta_1^2 = -\frac{7}{3}$

$\lambda = \infty$: $\quad E \sim (p = t\theta_1) + (p = -t\theta_1)$

$\lambda = \dfrac{-8 + 5\sqrt{2}}{7}$: $\quad E \sim (p = r\theta_2) + (p = -r\theta_2) \qquad \theta_2^2 = \dfrac{-8 - 5\sqrt{2}}{2}$

$\lambda = \dfrac{-8 - 5\sqrt{2}}{7}$: $\quad E \sim (p = r\bar{\theta}_2) + (p = -r\bar{\theta}_2) \qquad \bar{\theta}_2^2 = \dfrac{-8 + 5\sqrt{2}}{2}$

$$\lambda = \frac{-19+5\sqrt{13}}{6} : E \sim (t = s\theta_3) + (t = -s\theta_3) \qquad \theta_3^2 = \frac{-19-5\sqrt{13}}{6}$$

$$\lambda = \frac{-19-5\sqrt{13}}{6} : E \sim (t = s\bar{\theta}_3) + (t = -s\bar{\theta}_3) \qquad \bar{\theta}_3^2 = \frac{-19+5\sqrt{13}}{6}$$

It follows that the field of definition for the full Néron–Severi group of W over $\mathbb{C}$, is rather large: indeed, it has to be a (finite) extension of $\mathbb{Q}(\sqrt{-6}, \sqrt{-21}, \sqrt{(-8-5\sqrt{2})/2}, \sqrt{(-19-5\sqrt{13})/6})$. But fortunately we are interested here only with the subgroup of the Néron–Severi group which is defined over $\mathbb{Q}$; and the components defined over $\mathbb{Q}$ are just L_5, L_6, L_7, L_8, $\Pi - L_1 - L_2$, $\Pi - L_3 - L_4$, $\Pi - L_1 - L_3$, $\Pi - L_2 - L_4$, and $2\Pi - L_1 - L_2 - L_3 - L_4$.

From (8) and (9) we deduce that 2Γ is linearly equivalent to an integer combination of $L_1, \ldots, L_8$ and Π, so that 4Γ is linearly equivalent to an integer combination of $L_1, \ldots, L_8$.

Demanding integer intersection number with each L_i and conic of type $\Pi - L_i - L_j$ shows that Γ has to be an integer combination of the following:

$$D_1 = L_1; \qquad D_2 = \tfrac{1}{2}(L_2 + L_8); \qquad D_3 = \tfrac{1}{2}(L_3 + L_8);$$
$$D_4 = \tfrac{1}{2}(L_4 + L_7); \qquad D_5 = \tfrac{1}{2}(L_5 + L_8); \qquad D_6 = \tfrac{1}{2}(L_6 + L_7);$$
$$D_7 = L_7; \qquad D_8 = \tfrac{1}{2}(\Pi - L_1 - L_8).$$

The intersection matrix of the D_i is as follows:

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
D_1	-2	1	1	0	1	0	0	1
D_2	1	-1	$-\frac{1}{2}$	1	$-\frac{1}{2}$	1	1	$\frac{1}{2}$
D_3	1	$-\frac{1}{2}$	-1	1	$-\frac{1}{2}$	1	1	$\frac{1}{2}$
D_4	0	1	1	-1	1	$-\frac{1}{2}$	-1	0
D_5	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	-1	1	1	$\frac{1}{2}$
D_6	0	1	1	$-\frac{1}{2}$	1	-1	-1	0
D_7	0	1	1	-1	1	-1	-2	0
D_8	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$

Let $\Gamma \sim \sum_{i=1}^{8} d_i D_i$, $d_i \in \mathbb{Z}$. Then $(\Gamma \cdot \Gamma) \equiv 0 \bmod 2$ automatically implies that $d_8 \equiv 0$ (2) and $-d_2^2 - d_3^2 - d_4^2 - d_5^2 - d_6^2 - d_2 d_3 - d_2 d_5 - d_3 d_5 - d_4 d_6 \equiv 0$ (2) i.e.,

$$d_2 + d_3 + d_4 + d_5 + d_6 + d_2 d_3 + d_2 d_5 + d_3 d_5 + d_4 d_6 \equiv 0 \ (2). \tag{10}$$

Now D_5, D_6 are not both elements of $NS(W, \mathbb{Q})$, for otherwise there exists a divisor D in $NS(W, \mathbb{Q})$ with $2D \sim L_5 + L_6 + L_7 + L_8$ and we have seen in (9) that this is not the case. So

$$d_5 d_6 \equiv 0 \ (2). \tag{11}$$

This is more generally true, in that there is no divisor D in $NS(W, \mathbb{Q})$ with $2D$ linearly equivalent to a divisor of degree 4 and genus 1. For otherwise $(2D \cdot 2D) = 0$, so $(D \cdot D) = 0$, and D would be an effective divisor of degree 2 and genus 1, impossible. Thus considering the elliptic pencils $L_2 + L_4 + L_7 + L_8$; $L_2 + L_6 + L_7 + L_8$; $L_3 + L_4 + L_7 + L_8$; $L_3 + L_6 + L_7 + L_8$; $L_4 + L_5 + L_7 + L_8$, we obtain, respectively,

$$\begin{gathered} d_2 d_4 \equiv 0; \qquad d_2 d_6 \equiv 0; \qquad d_3 d_4 \equiv 0; \\ d_3 d_6 \equiv 0; \qquad d_4 d_5 \equiv 0 \bmod 2. \end{gathered} \tag{12}$$

Multiplying (10) by d_4 and using (11) and (12) we deduce $d_4 \equiv 0$ (2); and in exactly the same way, $d_6 \equiv 0$ (2). Multiplying (10) by d_2, d_3, d_5, we deduce $d_2 \equiv d_3 \equiv d_5$ (2).

Accordingly, Γ is an integer combination of $L_1, \ldots, L_8$, Π and $D = \frac{1}{2}(L_2 + L_3 + L_5 + L_8)$. Now if D is a genuine element of $NS(W, \mathbb{Q})$, then D is effective and $D + L_1$ having square 0 is a decomposition of an elliptic pencil F of degree 3 having L_1 as one of its components. But $F \sim D + L_1 \sim 2\Pi - L_4 - L_6 - L_7 - D$ and so arises as the residual intersection on W of a conic passing through L_4, L_6, L_7 and D. But the only conic to contain L_4, L_6, L_7 is easily verified to be $p^2 - t^2 = s^2 - r^2$, which as we have already seen intersects W in $L_1, \ldots, L_8$. Thus D is not an element of $NS(W, \mathbb{Q})$, so $NS(W, \mathbb{Q})$ is generated as a $\mathbb{Z}$-module by $L_1, \ldots, L_8$ and Π.

3.2. We now proceed to construct two explicit automorphisms of W, defined over $\mathbb{Q}$; we already have the symmetries listed in Table II.

Consider first the curve E_λ with, in the notation of ESP, Section 5, the transversal C equal to L_2. The subgroup of $NS(W, \mathbb{Q})$ which is orthogonal to the components of E_λ and L_2 is generated by the divisor $-L_1 + L_2 + L_3 - L_4 + L_5 + 2L_6 + 2L_7 + L_8$, which is accordingly reversed by the involution ϕ_1 arising from E_λ and L_2 (ESP, Theorem 5.4).

Now ϕ_1 interchanges L_6 and L_7 and fixes L_5, L_8, $\Pi - L_1 - L_2$, $\Pi - L_1 - L_3$, L_2. Solving for the images of L_i gives the representation of ϕ_1 as an 8×8 matrix with respect to the L_i as basis:

$$\phi_1: \begin{bmatrix} . & 1 & 1 & -1 & 1 & 2 & 2 & 1 \\ . & 1 & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ -1 & 1 & 1 & . & 1 & 2 & 2 & 1 \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & 1 \end{bmatrix}. \tag{13}$$

Thus ϕ_1 reduces the degree of the divisor $\Gamma \sim \sum n_i L_i$ provided $n_1 + n_4 < 0$ (of course, the n_i may all be half odd integers). If we first apply to Γ a symmetry, and then apply ϕ_1 we can obtain a divisor of reduced degree if any of the following inequalities hold:

$$\begin{aligned} &n_1 + n_4 < 0, \\ &n_2 + n_3 < 0, \\ &n_5 + n_8 < 0, \\ &n_6 + n_7 < 0. \end{aligned} \tag{14}$$

Second, consider the pencil F linearly equivalent to $L_2 + L_4 + L_7 + L_8$, with transversal L_3. The rational decompositions of F are

$$\begin{aligned} &F \sim L_2 + L_4 + L_7 + L_8, \\ &F \sim (\Pi - L_1 - L_3) + (\Pi - L_5 - L_6), \\ &F \sim (\Pi - L_1 - L_5) + (\Pi - L_3 - L_6); \end{aligned}$$

the subgroup of $NS(W, \mathbb{Q})$ orthogonal to L_3 and the components of F is generated by $-L_1 + L_2 + L_3 + 2L_4 + L_5 - L_6 + 2L_7 + L_8$, which is reversed by the involution ϕ_2 arising from F and L_3. Now ϕ_2 interchanges L_4 and L_7 and fixes L_2, L_8, $\Pi - L_1 - L_3$, $\Pi - L_1 - L_5$, L_3; thus ϕ_2 is represented by the matrix:

$$\phi_2: \begin{bmatrix} . & 1 & 1 & 2 & 1 & -1 & 2 & 1 \\ . & 1 & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & 1 & . & . & . \\ -1 & 1 & 1 & 2 & 1 & . & 2 & 1 \\ . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & 1 \end{bmatrix}, \tag{15}$$

and ϕ_2 with symmetries will achieve a reduction in degree of $\sum n_i L_i$ if any of the following inequalities hold:

$$\begin{aligned} n_1 + n_6 &< 0, \\ n_1 + n_7 &< 0, \\ n_2 + n_5 &< 0, \\ n_2 + n_8 &< 0, \\ n_3 + n_5 &< 0, \\ n_3 + n_8 &< 0, \\ n_4 + n_6 &< 0, \\ n_4 + n_7 &< 0. \end{aligned} \tag{16}$$

We can now state the principal result.

THEOREM. *Let Γ be an irreducible curve on W defined over $\mathbb{Q}$ and of arithmetic genus 0. Then Γ can be generated from either the line L_1 or the conic $\Pi - L_1 - L_2$ by repeated applications of the involutions ϕ_1, ϕ_2 and the symmetries of W, in such a way that each application of ϕ_1 or ϕ_2 strictly increases the degree of the curve under consideration.*

Proof. Take $\Gamma \sim \sum n_i L_i$ to have minimal degree among those irreducible curves on W, defined over $\mathbb{Q}$, which cannot be generated in the manner described in the theorem. Then the n_i cannot satisfy inequality (14) or (16), so we must have $n_1 + n_4 \geqslant 0, \ldots, n_4 + n_7 \geqslant 0$. Also, since Γ is irreducible, we have 24 inequalities which result from demanding non-negative intersection number with each rational line and conic. This total set of inequalities defines a convex cone in eight-dimensional affine space, and it suffices to show that any point of this cone corresponds to a divisor of arithmetic genus strictly greater than zero; that is, from (9) of ESP, that $(\Gamma \cdot \Gamma) \geqslant 0$ on this cone.

Elementary rearrangement gives

$$\begin{aligned} 4(\Gamma \cdot \Gamma) = (\deg \Gamma)^2 &- 4(n_1 - n_4)^2 - 4(n_2 - n_8)^2 \\ &- 4(n_3 - n_5)^2 - 4(n_6 - n_7)^2 \\ &- 2(n_2 - n_3 - n_5 + n_8)^2 - 2(n_1 + n_4 - n_6 - n_7)^2 \\ &- 3(n_1 - n_2 - n_3 + n_4 - n_5 + n_6 + n_7 - n_8)^2, \end{aligned} \tag{17}$$

and so on the hyperplane $\deg \Gamma = \text{const}$, the form $(\Gamma \cdot \Gamma)$ takes its minimum value at a vertex of the resulting convex polyhedron; so in turn it suffices to show that $(\Gamma \cdot \Gamma) \geqslant 0$ on the extremal rays of the cone. This is now a finite computation, readily performed by computer; and indeed $(\Gamma \cdot \Gamma) \geqslant 0$ on the extremal rays. An outline of the computation is given in Appendix B.

Remarks. Since none of the rational conics actually possess rational points, it follows that a curve of arithmetic genus 0, defined over $\mathbb{Q}$, which is rationally parametrizable, arises by repeated application to L_1 of the symmetries of W, and ϕ_1, ϕ_2.

In virtue of the last clause in the statement of the theorem, all such parametrizable curves on W can be constructively found. Indeed, using the matrix representations (13) and (15) of ϕ_1, ϕ_2, it is easy to generate a complete list of divisors up to any preassigned degree, corresponding to rationally parametrizable curves.

Up to degree 20, there are up to symmetry just six divisors of the required type of degree greater than 1:

$$\begin{aligned}
\text{degree } 7: \quad & L_2 + L_3 - L_4 + L_5 + 2L_6 + 2L_7 + L_8 \\
& L_2 + L_3 + 2L_4 + L_5 - L_6 + 2L_7 + L_8 \\
\text{degree } 13: \quad & -2L_1 + 2L_2 + 2L_3 + L_4 + 2L_5 + 4L_6 + 2L_7 + 2L_8 \\
& -2L_1 + 2L_2 + 2L_3 + 4L_4 + 2L_5 + L_6 + 2L_7 + 2L_8 \\
\text{degree } 19: \quad & -2L_1 + 3L_2 + 3L_3 + 3L_5 + 6L_6 + 3L_7 + 3L_8 \\
& -2L_1 + 3L_2 + 3L_3 + 6L_4 + 3L_5 + 3L_7 + 3L_8.
\end{aligned}$$

Because of the explicit definition of ϕ_1 and ϕ_2 it is a straightforward calculation to realize such divisors in terms of a rational parametrization. The first two divisors listed above, respectively, give rise to the following solutions of (6).

$$\begin{aligned}
r &= 3\mu^6 + 29\mu^4 + 48\mu^2 + 20, \\
s &= 7\mu^6 + 61\mu^4 + 112\mu^2 + 20, \\
t &= \mu(5\mu^6 + 39\mu^4 + 52\mu^2 + 4), \\
p &= \mu(5\mu^6 + 51\mu^4 + 108\mu^2 + 36)
\end{aligned}$$

and

$$\begin{aligned}
r &= (50, -115, 270, -27, -106, 135, 10, -25), \\
s &= (75, -80, -133, 228, -95, -272, 185, -100), \\
t &= (-50, -35, -250, 309, -206, 199, -70, -25), \\
p &= (-75, -60, 149, -84, 239, -332, 135, -100)
\end{aligned}$$

using abbreviated notation. We then immediately have solutions of

$$x^5 + y^5 + z^5 = u^5 + v^5 + w^5$$

via transformation (5).

APPENDIX A: NORMAL FORM OF THE ELLIPTIC PENCIL E_λ

We first of all give maps which put E_λ into normal form. Let $\nu^2 = \lambda$, and put

$$m : n = -(3\lambda^2 + 19\lambda + 3)r + 5\lambda(2\lambda + 3)s - (7\lambda^2 - 4\lambda - 3)\nu t : s - \nu t,$$

$$\begin{aligned} j : n^2 = {} & (3\lambda^2 + 19\lambda + 3)r - (3\lambda^2 - \lambda - 2)s^2 - 5\lambda(4\lambda + 1)t^2 \\ & + 2p[5\lambda(4\lambda + 1)t - \nu((3\lambda^2 + 19\lambda + 3)r \\ & - (3\lambda^2 - \lambda - 2)s)] : (s - \nu t)^2. \end{aligned}$$

Then it can be verified that

$$\begin{aligned} \tau = {} & -\tfrac{1}{2}(3\lambda^2 + 19\lambda + 3)j + \tfrac{1}{2}m^2 - \tfrac{1}{2}(49\lambda^4 \\ & - 116\lambda^3 - 421\lambda^2 - 131\lambda - 6)n^2, \end{aligned}$$

$$\sigma = m\tau - 5\lambda(2\lambda + 3)(7\lambda^2 + 16\lambda + 2)[m - (7\lambda^2 - 4\lambda - 3)n]$$

give a birational map from E_λ to the curve

$$\begin{aligned} G : \sigma^2 = {} & \tau[\tau - 25\lambda(2\lambda + 3)(4\lambda + 1)n^2] \\ & \times [\tau - (7\lambda^2 + 16\lambda + 2)(3\lambda^2 + 19\lambda + 3)n^2]. \end{aligned}$$

G has the two-isogenous curve

$$\begin{aligned} G' : S^2 = {} & T[T^2 + 2(21\lambda^4 + 381\lambda^3 + 681\lambda^2 + 161\lambda + 6)TN^2 \\ & + (\lambda - 1)^4(3\lambda + 2)^2(7\lambda + 3)^2 N^4]. \end{aligned}$$

We list for reference the images on G of the following eight $\mathbb{Q}(\nu)$-points of E_λ:

r	s	t	p	τ/n^2	σ/n^3
ν	ν	1	1	$25\lambda(2\lambda + 3)(4\lambda + 1)$	0
ν	ν	-1	1	$5(4\lambda + 1)(3\lambda^2 + 19\lambda + 3)$	$(-7\lambda^2 + 4\lambda + 3)\tau/n^2$
ν	$-\nu$	1	1	$5\lambda(2\lambda + 3)(3\lambda^2 + 19\lambda + 3)$	$(3\lambda^2 - \lambda - 2)\tau/n^2$
ν	$-\nu$	-1	1	∞	∞
$-\nu$	$-\nu$	1	1	0	0
$-\nu$	$-\nu$	-1	1	$5(4\lambda + 1)(7\lambda^2 + 16\lambda + 2)$	$(-3\lambda^2 + \lambda + 2)\tau/n^2$
$-\nu$	ν	1	1	$5\lambda(2\lambda + 3)(7\lambda^2 + 16\lambda + 2)$	$(7\lambda^2 - 4\lambda - 3)\tau/n^2$
$-\nu$	ν	-1	1	$(7\lambda^2 + 16\lambda + 2)(3\lambda^2 + 19\lambda + 3)$	0

We claim that G' has only the trivial cover (N.B. ESP, 3.2). For we certainly have $T = \Delta A^2$, $\Delta = \Delta(\nu)$ being square-free, and where we can assume without loss of generality that Δ, A, $N \in \mathbb{Z}[\nu]$ and $(A, N) = 1$. Then $\Delta \mid (\lambda-1)^4(3\lambda+2)^2(7\lambda+3)^2$, and with $K = K(\lambda) = 21\lambda^4 + 381\lambda^3 + 681\lambda^2 + 161\lambda + 6$, we have

$$\Delta A^4 + 2KA^2N^2 + \frac{(\lambda-1)^4(3\lambda+2)^2(7\lambda+3)^2}{\Delta} N^4 = \text{square}.$$

Now both $(3\lambda+2)$ and $(7\lambda+3)$ are prime ideals of $\mathbb{Z}[\nu]$; so if $3\lambda+2 \mid \Delta$, then modulo $(3\lambda+2)$, we have $384A^2N^2 \equiv$ square, which would imply that 6 is a square in $\mathbb{Q}(\sqrt{-2/3})$, a contradiction. Similarly, if $7\lambda+3 \mid \Delta$, then modulo $(7\lambda+3)$, we have $4032A^2N^2 \equiv$ square, implying 7 is a square in $\mathbb{Q}(\sqrt{-3/7})$, a contradiction. So Δ can only be divisible by $\nu \pm 1$; but modulo $(4\lambda+1)$,

$$\left(\Delta A^2 + \frac{625}{256} N^2\right)^2 \equiv \Delta \cdot \text{square}$$

whence $\Delta(i/2)$ is a square in $\mathbb{Q}(i)$, and Δ can only be ± 1. Finally, mod(ν)

$$(\Delta A^2 + 6N^2)^2 \equiv \Delta \cdot \text{square}$$

and so $\Delta = -1$ is impossible.

On G, let $\tau = \delta a^2$, δ square-free, with $\delta, a, n \in \mathbb{Z}[\nu]$, $(a, n) = 1$. Then $\delta \mid M$, $M = 25\lambda(4\lambda+1)(2\lambda+3)(7\lambda^2+16\lambda+2)(3\lambda^2+19\lambda+3)$ (where all the factors are prime in $\mathbb{Z}[\nu]$, except $\lambda = \nu^2$) and

$$\delta a^4 - Ka^2n^2 + \frac{M}{\delta} n^4 = \text{square}. \tag{18}$$

Equivalently,

$$(2\delta a^2 - Kn^2)^2 - (\lambda-1)^4(3\lambda+2)^2(7\lambda+3)^2 n^4 = \delta \cdot \text{square}. \tag{19}$$

Suppose $\nu \mid \delta$. Then $-6a^2n^2 \equiv$ square, mod(ν), whence $\nu \mid a$, $\nu \nmid n$, and (18) gives an impossible congruence mod(ν^2). Thus $\nu \nmid \delta$, so $\delta = \delta(\lambda)$ is a genuine element of $\mathbb{Z}[\lambda]$.

Now $K(1) = 2.25^2$, so (19) clearly implies $\delta(1)$ is a square in $\mathbb{Q}$. Further, modulo $(3\lambda+2)$, we have from (19)

$$(2\delta a^2 - 192n^2)^2 \equiv \delta \cdot \text{square}$$

so that if $\delta' = \delta(-2/3)$ is not a square of $\mathbb{Q}(\sqrt{-2/3})$, then necessarily $96\delta'$ is a square, that is, $-\delta'$ is a square, in $\mathbb{Q}(\sqrt{-2/3})$.

This set of conditions immediately reduces the number of possibilities for δ to eight, corresponding to the eight points we have already listed. Thus from the formulae of ESP, 3.1, we deduce that the group $\mathfrak{G}$ of $\mathbb{Q}(\nu)$-points on G has precisely one generator of infinite order.

We claim that this generator may be taken as the point

$$P = (5(4\lambda + 1)(3\lambda^2 + 19\lambda + 3), 5(4\lambda + 1) \times (3\lambda^2 + 19\lambda + 3)(-7\lambda^2 + 4\lambda + 3)).$$

For it is straightforward to show, as in ESP, 3.2, that the only points of $\mathfrak{G}$ of finite order are the two-division points.

Let g be a generator of infinite order in $\mathfrak{G}$; then since P is clearly not divisible by 2 in $\mathfrak{G}$, we may assume that P is an odd multiple of g (by adding to g, if necessary, a point of order 2). So it suffices to show that P is not divisible in $\mathfrak{G}$, and this is straightforward when modelled on a similar demonstration in Section 4 of Cassels, Ellison, and Pfister [2].

It now follows that $\mathfrak{G} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$, with generators for the $\mathbb{Q}(\nu)$-points on E_λ being given by $(\nu, \nu, 1, 1)$, $(-\nu, -\nu, 1, 1)$ of order 2, and $(\nu, \nu, -1, 1)$ of infinite order.

APPENDIX B: The Computer Program

We give here a brief outline of the computer program used to calculate the vertices of the convex polyhedron defined in Section 4; the algorithm is due to Swinnerton–Dyer.

We start with a basic simplex S_8 determined by eight linearly independent planes. Inductively, suppose there is given a simplex S_n determined by n planes; that is, suppose there is given a set V_n of "vertices," and a set E_n of pairs of vertices, which identify the "edges" of the polyhedral simplex S_n. Now introduce a further plane π. Vertices which are "cut off" by π can be immediately rejected, and a new vertex can only be created where π cuts an old edge. So it is simple to construct the vertex set V_{n+1} of S_{n+1}.

To determine E_{n+1} we first run through E_n, discarding, retaining, or amending each edge as appropriate. All that remains is then to augment E_n by adding in the new edges which lie entirely within π. Such edges join two vertices lying in π; and to test whether two such vertices are joined by an edge, it is sufficient to check that both vertices lie on five (other than π) linearly independent planes. This is the method we use in our program. As pointed out by Swinnerton-Dyer, it is more efficient to argue that if PP' is an old edge cut by π at Q, where we retain P and discard P', then PQ is one new edge through Q and others arise from the 2-manifolds through PP' intersected with π (slight modifications are necessary if $Q = P$).

When the final simplex has been constructed, it is then simple to run through the vertices, checking that the quadratic form of intersection there is non-negative.

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