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Higher Order Bifurcations of Limit Cycles

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This paper shows that asymmetrically perturbed, symmetric Hamiltonian systems of the form

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \pm(x \pm x^3) + \lambda_1 y + \lambda_2 x^2 + \lambda_3 xy + \lambda_4 x^2 y,\end{aligned}$$

with analytic $\lambda_j(\varepsilon) = O(\varepsilon)$, have at most two limit cycles that bifurcate for small $\varepsilon \neq 0$ from any period annulus of the unperturbed system. This fact agrees with previous results of Petrov, Dangelmayr and Guckenheimer, and Chicone and Iliev, but shows that the result of three limit cycles for the asymmetrically perturbed, exterior Duffing oscillator, recently obtained by Jebrane and Żołądek, is incorrect. The proofs follow by deriving an explicit formula for the kth-order Melnikov function, $M_k(h)$, and using a Picard–Fuchs analysis to show that, in each case, $M_k(h)$ has at most two zeros. Moreover, the method developed in this paper for determining the higher-order Melnikov functions also applies to more general perturbations of these systems.

1. INTRODUCTION

In this paper, we study the asymmetrically perturbed *global center*

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - x^3 + \lambda_1 y + \lambda_2 x^2 + \lambda_3 xy + \lambda_4 x^2 y,\end{aligned} \tag{1}$$

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the asymmetrically perturbed *truncated pendulum*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + x^3 + \lambda_1 y + \lambda_2 x^2 + \lambda_3 xy + \lambda_4 x^2 y, \end{aligned} \tag{2}$$

and the asymmetrically perturbed *Duffing oscillator*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + \lambda_1 y + \lambda_2 x^2 + \lambda_3 xy + \lambda_4 x^2 y, \end{aligned} \tag{3}$$

with analytic $\lambda_j(\varepsilon) = \sum_{k=1}^{\infty} \lambda_{jk} \varepsilon^k$. For $\varepsilon = 0$, systems (1)–(3) are Hamiltonian, with phase portraits shown in Figs. 1, 2, and 3 respectively. In each case we prove that at most two limit cycles bifurcate from any period annulus of the unperturbed system for small $\varepsilon \neq 0$.

These systems occur in oscillating chemical reactor models and they have been studied by several authors; see the references at the end of the paper.

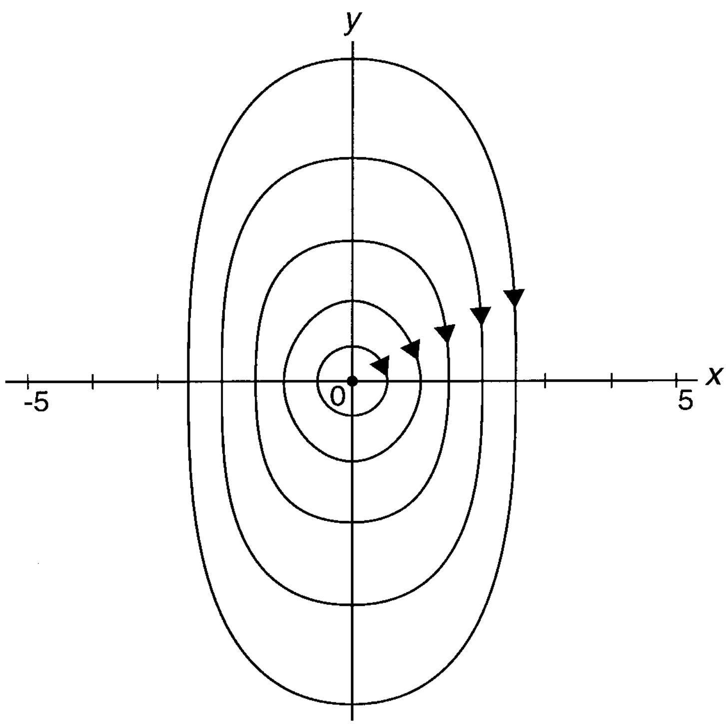

FIGURE 1

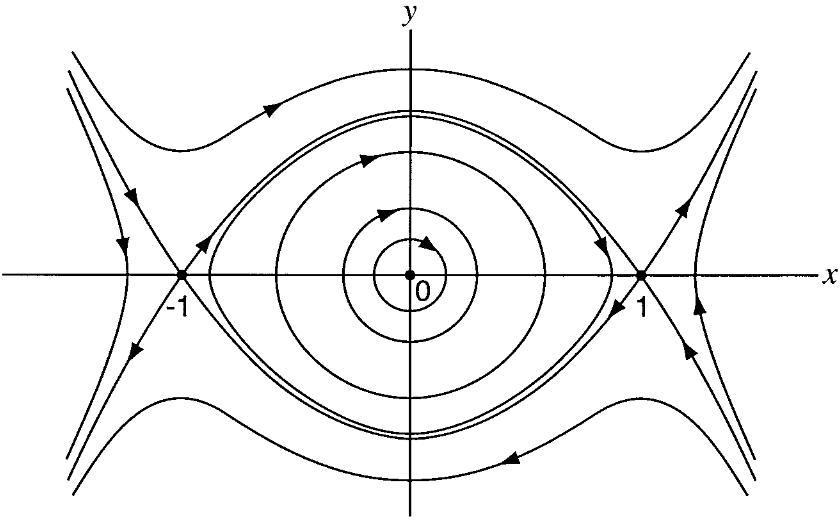

FIGURE 2

In particular, it was recently proved in [3] and [9] that the asymmetrically perturbed global center (1) has at most two limit cycles and this agrees with the earlier work of Dangelmayr and Guckenheimer [5], who used a computer to obtain that fact, and with the results of Petrov [14] for the system (1). As expected, the first-order analysis in ε does not suffice to determine the limit cycles in (1)–(3). A natural approach, deriving

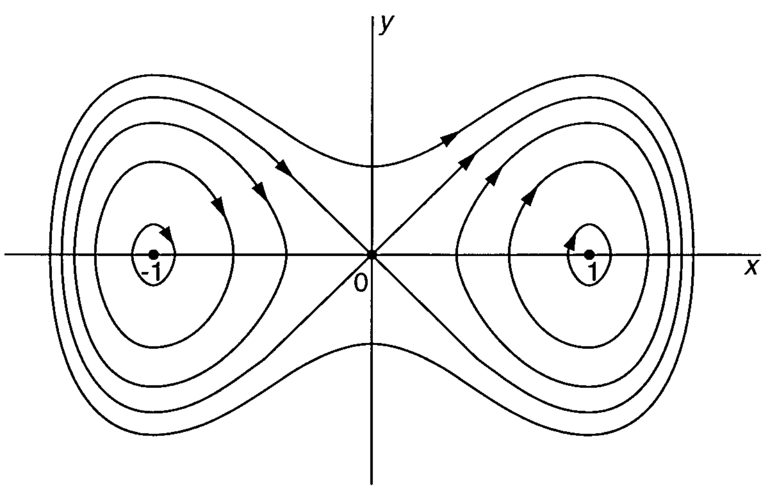

FIGURE 3

formulas for the higher order Melnikov functions and analyzing the related Picard–Fuchs and Riccati equations, is used to prove the results in the present paper concerning the number of limit cycles in the perturbed systems; however, a new method is proposed to derive the higher order Melnikov functions for (1)–(3). It seems it is shorter than the method used by Chicone and Jacobs in [3], [4] and simpler than using Françoise's procedure from [7] for the systems (1)–(3). Indeed, when applied to (1), (2) and the exterior Duffing oscillator in (3), Françoise's method requires a calculation of growing complexity at each successive step. Thus, in practice only the first few Melnikov functions could be derived for these cases by using the procedure from [7]. See Remark 2.3 in [9] and Remark 2.5 in Section 2 below regarding this latter procedure. Furthermore, it would be considerably more difficult to derive a structural result of Bautin type for the truncated pendulum or for the Duffing oscillator, as was done in [3] for the global center. (This was confirmed in private communications with Carmen Chicone.) In any case, the method used in this paper determines the higher order Melnikov functions for all of the period annuli that occur in the systems (1)–(3), including the exterior annulus for the Duffing oscillator (3). Our construction does not require deriving a structural result of Bautin type as in [3] and it also applies to more general perturbations of these systems. In particular, a formula for the higher-order Melnikov functions for the perturbed Duffing oscillator with an arbitrary cubic perturbation in the second equation is derived in the appendix.

The main results of this paper are summarized in the following theorems for the systems (1)–(3).

THEOREM 1.1 (The Global Center). *The system* (1) *with* $\lambda_j(\varepsilon) = \sum_{k=1}^{\infty} \lambda_{jk}\varepsilon^k$:

(i) *has a center at* $(0, 0)$ *if and only if* $\lambda_1^2 + \lambda_4^2 = \lambda_2\lambda_3 = 0$,

(ii) *has at most one limit cycle that bifurcates from the period annulus of* (1) *for small* $\varepsilon \neq 0$, *if* $\lambda_2\lambda_3 = 0$, *and*

(iii) *has at most two limit cycles that bifurcate from the period annulus of* (1) *for small* $\varepsilon \neq 0$.

THEOREM 1.2 (The Truncated Pendulum). *The system* (2) *with* $\lambda_j(\varepsilon) = \sum_{k=1}^{\infty} \lambda_{jk}\varepsilon^k$:

(i) *has a center at* $(0, 0)$ *if and only if* $\lambda_1^2 + \lambda_4^2 = \lambda_2\lambda_3 = 0$,

(ii) *has at most one limit cycle that bifurcates from the period annulus of* (2) *for small* $\varepsilon \neq 0$, *if* $\lambda_2\lambda_3 = 0$, *and*

(iii) *has at most two limit cycles that bifurcate from the period annulus of* (2) *for small* $\varepsilon \neq 0$.

THEOREM 1.3 (The Interior Duffing Oscillator). *The system* (3) *with* $\lambda_j(\varepsilon) = \sum_{k=1}^{\infty} \lambda_{jk} \varepsilon^k$:

(i) *has a center at both of the critical points* $(\frac{1}{2}(\lambda_2 \pm \sqrt{\lambda_2^2 + 4}), 0) = (\pm 1 + O(\varepsilon), 0)$ *if and only if* $\lambda_1^2 + \lambda_3^2 + \lambda_4^2 = 0$,

(ii) *has at most one limit cycle that bifurcates from either one of the interior period annuli of* (3) *for small* $\varepsilon \neq 0$, *if* $\lambda_1 \lambda_3 \lambda_4 = 0$, *and*

(iii) *has at most two limit cycles that bifurcate from either one of the interior period annuli of* (3) *for small* $\varepsilon \neq 0$.

THEOREM 1.4 (The Exterior Duffing Oscillator). *The system* (3) *with* $\lambda_j(\varepsilon) = \sum_{k=1}^{\infty} \lambda_{jk} \varepsilon^k$:

(i) *has a continuous band of periodic orbits on the exterior of a compound separatrix cycle if and only if* $\lambda_1^2 + \lambda_4^2 = \lambda_2 \lambda_3 = 0$, *and*

(ii) *has at most two limit cycles that bifurcate from the exterior period annulus of* (3) *for small* $\varepsilon \neq 0$.

Remark. The following distributions of limit cycles, which bifurcate out of the three period annuli (that is, the left, right, and exterior period annuli) of (3), are possible: $(l, r, e) = (0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$, $(1, 0, 1)$, $(0, 1, 1)$, $(1, 0, 2)$, $(0, 1, 2)$, $(1, 1, 1)$. This statement follows easily from [1], [11], [13] and the results formulated above.

We note that Theorem 4(ii) agrees with the earlier results of Dangelmayr and Guckenheimer [5] and Petrov [14] for the asymmetrically perturbed exterior Duffing oscillator (3), but that it does not agree with Jebrane and Żołądek's Theorem 2 on p. 2 in [11] which claims that three limit cycles can bifurcate from the exterior period annulus of the asymmetrically perturbed Duffing oscillator (see equation (1) in [11], which can be written in the form of Eq. (3) above). It is our contention that Theorem 4(ii) above corrects Theorem 2 in [11]. In particular, we note that the inequality preceding Lemma 11 on p. 11 of [11] is incorrect.

The proofs in the sections below establish the main result of this paper that at most two limit cycles bifurcate from any of the period annuli in the systems (1)–(3) for small $\varepsilon \neq 0$. The kth-order Melnikov functions for these systems, $M_k(h)$, are derived in the next Section 2 and the number of zeros of $M_k(h)$, for any of the cases, is determined in Section 3 by analyzing the corresponding Picard–Fuchs and Riccati equations. We recall that $M_k(h)$ appear as coefficients in the power series in ε,

$$d_E(h, \varepsilon) = M_1(h)\,\varepsilon + M_2(h)\,\varepsilon^2 + M_3(h)\varepsilon^3 + \cdots$$

of the corresponding *displacement function* $d_E(h, \varepsilon)$ (equal to the increment of the energy along a section to the Hamiltonian flow). Therefore the limit cycles correspond to isolated zeros of the first nonvanishing Melnikov function. Thus, the exact upper bound for the number of zeros of $M_k(h)$, counted with multiplicities, determines the cyclicity, i.e., the maximum of limit cycles that emerge from the related period annulus. The results concerning the cyclicity of the annuli around centers of (1)–(3) and the continuous band of cycles on the exterior of a separatrix cycle of (3) follow from the structure of the Melnikov functions for (1)–(3) and from symmetries of these systems. The remaining results in Theorems 1–4 are of a simpler nature and appear in the literature. In particular, Theorem 1(ii) is established in [3]; Theorem 2(ii) is established in [1]; and Theorems 3(ii) and 4(i) are established in [1] and [13].

2. COMPUTATION OF $M_k(h)$

We are going to derive in detail a formula for the kth-order Melnikov function only in the case of perturbed Duffing oscillator (3). The remaining systems can be considered similarly. Let H be the Hamiltonian function of the unperturbed system (that is (3), with $\varepsilon = 0$). For $k \in \mathbb{Z}$ consider the integral

$$J_k(h) = \oint_{\delta(h)} x^k y\, dx, \qquad h \in \Sigma, \quad \text{with derivative} \quad J'_k(h) = \oint_{\delta(h)} \frac{x^k}{y}\, dx, \tag{2.1}$$

where Σ is the open interval for which the algebraic curve $H = h$ contains an oval $\delta(h)$ within a continuous set of ovals $\mathscr{A}$ (a period annulus).

2.1. *The interior Duffing oscillator*

In this case, $\Sigma = (-\frac{1}{4}, 0)$.

THEOREM 2.1. *The Melnikov functions for the perturbed interior Duffing oscillator* (3) *have the form*

$$M_1(h) = \lambda_{11} J_0(h) + \lambda_{31} J_1(h) + \lambda_{41} J_2(h), \qquad h \in \Sigma,$$

and if $M_1(h) \equiv \cdots \equiv M_{k-1}(h) \equiv 0$ *in* Σ *for some* $k \geqslant 2$, *then*

$$M_k(h) = \lambda_{1k} J_0(h) + \lambda_{3k} J_1(h) + \lambda_{4k} J_2(h), \qquad h \in \Sigma.$$

Proof. Denote

$$H(\lambda_2) = H(x, y, \lambda_2) = \tfrac{1}{2}(y^2 - x^2) + \tfrac{1}{4}x^4 - \tfrac{1}{3}\lambda_2 x^3. \tag{2.2}$$

For definiteness, let us consider the family of ovals inside the right branch of the "eight loop". It is well known that for $h \in \Sigma = (-\frac{1}{4}, 0)$,

$$\begin{aligned} M_1(h) &= \oint_{H(0)=h} (\lambda_{11} y + \lambda_{31} xy + \lambda_{41} x^2 y)\, dx \\ &= \lambda_{11} J_0(h) + \lambda_{31} J_1(h) + \lambda_{41} J_2(h). \end{aligned}$$

Assume first that $\lambda_{1j} = \lambda_{3j} = \lambda_{4j} = 0$ for $j = 1, 2, ..., k-1$, with some $k \geqslant 2$. Under this assumption, we proceed to prove that $M_k(h) = \lambda_{1k} J_0(h) + \lambda_{3k} J_1(h) + \lambda_{4k} J_2(h)$, $h \in \Sigma$. We fix $h \in (-\frac{1}{4}, 0)$ and denote by ξ the greatest positive solution of $H(\xi, 0, 0) = h$. Take $P(\xi, 0)$ and for small ε consider the Poincaré first return map for (3), $\xi \to \xi_1 = \mathscr{P}(\xi, \varepsilon)$, defining a point $P_1(\xi_1, 0)$. Let $d(\xi, \varepsilon) = \xi_1 - \xi$ be the related displacement function and $d_E(h, \varepsilon) = H(\xi_1, 0, 0) - h$ be the corresponding increment of energy. Obviously, $d_E(h, \varepsilon) = d(\xi, \varepsilon)[\xi^3 - \xi + O(\varepsilon)]$. We write system (3) in a Pfaffian form

$$dH(x, y, \lambda_2) - (\lambda_1 y + \lambda_3 xy + \lambda_4 x^2 y)\, dx = 0 \tag{2.3}$$

and integrate (2.3) along the phase curve γ of (3) connecting P and P_1. One obtains

$$\int_\gamma dH(x, y, \lambda_2) = \int_\gamma (\lambda_1 y + \lambda_3 xy + \lambda_4 x^2 y)\, dx. \tag{2.4}$$

On the left, we have, for some ξ_* between ξ and ξ_1,

$$H(\xi_1, 0, \lambda_2) - H(\xi, 0, \lambda_2) = d(\xi, \varepsilon)\, \partial_x H(\xi_*, 0, \lambda_2) = d(\xi, \varepsilon)[\xi^3 - \xi + O(\varepsilon)].$$

On the right, we have

$$\begin{aligned} &\int_\gamma (\lambda_1 y + \lambda_3 xy + \lambda_4 x^2 y)\, dx \\ &\qquad = \varepsilon^k \oint_{H(0)=h} (\lambda_{1k} y + \lambda_{3k} xy + \lambda_{4k} x^2 y)\, dx + O(\varepsilon^{k+1}). \end{aligned}$$

In combination, this yields

$$d(\xi, \varepsilon) = \sum_{j=1}^{\infty} \varepsilon^j d_j(\xi) = \frac{\varepsilon^k}{\xi^3 - \xi} \oint_{H(0)=h} (\lambda_{1k} y + \lambda_{3k} xy + \lambda_{4k} x^2 y)\, dx + O(\varepsilon^{k+1}).$$

Hence $d_j(\xi) = 0$, $j < k$ and

$$d_k(\xi) = \frac{1}{\xi^3 - \xi} M_k(h) = \frac{1}{\xi^3 - \xi} [\lambda_{1k} J_0(h) + \lambda_{3k} J_1(h) + \lambda_{4k} J_2(h)].$$

Thus, $M_j(h) = 0$, $j < k$, and $M_k(h) = \lambda_{1k} J_0(h) + \lambda_{3k} J_1(h) + \lambda_{4k} J_2(h)$ for $h \in \Sigma$, which realizes the main step in the proof. To complete the proof of Theorem 2.1, it remains to use the linear independence of integrals appearing in $M_k(h)$.

LEMMA 2.2. *The functions $J_0(h)$, $J_1(h)$ and $J_2(h)$, related to the interior Duffing oscillator* (3), *are linearly-independent on $\Sigma = (-\frac{1}{4}, 0)$.*

The proof of the above lemma is obtained as a by-product from the considerations in the next section and admits the following geometric interpretation. If the functions $J_0(h)$, $J_1(h)$, and $J_2(h)$ were dependent in Σ, then the plane curve $Q(h) = J_1(h)/J_0(h)$, $R(h) = J_2(h)/J_0(h)$, $h \in \Sigma$, would be flat (that is, a line segment). As we will establish later, this curve always is strictly convex which implies the linear independence of $J_k(h)$.

Using Lemma 2.2, we see that $M_1(h) \equiv 0$ is equivalent to $\lambda_{11} = \lambda_{31} = \lambda_{41} = 0$ which implies that $M_2(h) = \lambda_{12} J_0(h) + \lambda_{32} J_1(h) + \lambda_{42} J_2(h)$, according to the first step in the proof. Now if $M_j(h) \equiv 0$ for $j \leqslant 2$, then $\lambda_{1j} = \lambda_{3j} = \lambda_{4j} = 0$ for $j \leqslant 2$ which implies that $M_3(h) = \lambda_{13} J_0(h) + \lambda_{33} J_1(h) + \lambda_{43} J_2(h)$, by the first step, and so on. The proof is then completed by induction.

2.2. *The exterior Duffing oscillator*

For this case, $\Sigma = (0, \infty)$.

THEOREM 2.3. *The Melnikov functions for the perturbed exterior Duffing oscillator* (3) *have the form*

$$M_1(h) = \lambda_{11} J_0(h) + \lambda_{41} J_2(h), \qquad h > 0, \tag{2.5}$$

and if $M_1(h) \equiv \cdots \equiv M_{k-1}(h) \equiv 0$ in Σ for some $k \geqslant 2$, then

$$M_k(h) = \lambda_{1k} J_0(h) + \lambda_{4k} J_2(h) + \tfrac{1}{3} \sum_{i+j=k} \lambda_{2i} \lambda_{3j} J_4'(h), \qquad h > 0. \tag{2.6}$$

Proof. We proceed as above. As is well known,

$$M_1(h) = \lambda_{11} J_0(h) + \lambda_{41} J_2(h), \qquad h \in \Sigma.$$

Define

$$\mu = \lambda_2 \lambda_3 = \sum_{k=2}^{\infty} \mu_k \varepsilon^k, \qquad \mu_k = \sum_{i+j=k} \lambda_{2i} \lambda_{3j} \quad (\mu_1 = 0)$$

and assume for the moment that $\lambda_{1j} = \lambda_{4j} = \mu_j = 0$ for $j = 1, 2, ..., k-1$, with some $k \geqslant 2$. We next fix $h > 0$ and denote by ξ the positive solution of $H(\xi, 0, 0) = h$. As above take $P(\xi, 0)$ and for small ε consider the Poincaré first return map for (3). The left-hand side of (2.4) again equals $d(\xi, \varepsilon)$ $[\xi^3 - \xi + O(\varepsilon)]$. On the right, we have

$$\int_\gamma (\lambda_1 y + \lambda_3 xy + \lambda_4 x^2 y)\, dx$$

$$= \varepsilon^k \oint_{H(0)=h} (\lambda_{1k} y + \lambda_{4k} x^2 y)\, dx + \lambda_3 \int_\gamma xy\, dx + O(\varepsilon^{k+1}).$$

We recall that $\gamma = \gamma(\lambda)$ is the phase curve of (3), which for each $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ begins at the same point $(\xi, 0)$ and ends at $(\xi_1, 0)$. Below we will find the first approximation of the integral

$$I(\lambda) = \int_{\gamma(\lambda)} xy\, dx$$

for small λ, using the analyticity of $\gamma(\lambda)$. As

$$I(0) = \int_{\gamma(0)} xy\, dx = \oint_{H(0)=h} xy\, dx = 0,$$

we have

$$I(\lambda) = I(\lambda_1, 0, 0, 0) + I(0, \lambda_2, 0, 0) + I(0, 0, \lambda_3, 0)$$

$$+ I(0, 0, 0, \lambda_4) + O\left(\sum_{i \neq j} |\lambda_i \lambda_j|\right)$$

$$= I(0, \lambda_2, 0, 0) + I(0, 0, \lambda_3, 0) + O(\varepsilon^k).$$

Let us denote by $K(x, y, \lambda_3)$ the first integral of the system

$$\dot{x} = y,$$

$$\dot{y} = x - x^3 + \lambda_3 xy.$$

Obviously, K is an even function of x. Therefore

$$I(0, 0, \lambda_3, 0) = \oint_{K(x, y, \lambda_3) = K(\xi, 0, \lambda_3)} xy\, dx = 0.$$

On the other hand, using the definition of $H(\lambda_2)$ in (2.2), we obtain

$$\begin{aligned} I(0, \lambda_2, 0, 0) &= \oint_{H(\lambda_2) + (1/3)\lambda_2 \xi^3 = h} xy\, dx \\ &= \lambda_2 \oint_{H(0) = h} x(dy/d\lambda_2)\, dx + O(\lambda_2^2) \\ &= \lambda_2 \oint_{H(0) = h} \frac{x^4 - \xi^3 x}{3y}\, dx + O(\lambda_2^2). \end{aligned}$$

Indeed, from $H(x, y, \lambda_2) + \frac{1}{3}\lambda_2 \xi^3 = h$ it follows $(dy/d\lambda_2)\, \partial_y H = -\partial_{\lambda_2} H - \frac{1}{3}\xi^3$ and $dy/d\lambda_2 = (x^3 - \xi^3)/3y$. In combination, all this yields

$$\begin{aligned} \lambda_3 \int_\gamma xy\, dx &= \lambda_2 \lambda_3 \oint_{H(0) = h} \frac{x^4 - \xi^3 x}{3y}\, dx + O(\varepsilon^{k+1}) \\ &= \varepsilon^k \mu_k \oint_{H(0) = h} \frac{x^4}{3y}\, dx + O(\varepsilon^{k+1}). \end{aligned}$$

Putting together all these estimates, we get

$$d(\xi, \varepsilon) = \sum_{j=1}^{\infty} \varepsilon^j d_j(\xi) = \frac{\varepsilon^k}{\xi^3 - \xi} \oint_{H(0) = h} \left(\lambda_{1k} y + \lambda_{4k} x^2 y + \mu_k \frac{x^4}{3y} \right) dx + O(\varepsilon^{k+1}).$$

Hence $d_j(\xi) = 0$, $j < k$ and

$$d_k(\xi) = \frac{1}{\xi^3 - \xi} M_k(h) = \frac{1}{\xi^3 - \xi} \left[\lambda_{1k} J_0(h) + \lambda_{4k} J_2(h) + \frac{1}{3} \mu_k J_4'(h) \right].$$

Thus we have established, under the additional assumption above, that $M_j(h)=0$ for $j<k$ and $M_k(h)$ is given by (2.6), which realizes the main step in the proof.

For completeness, we formulate the result for linear independence that we will use to finish the proof.

LEMMA 2.4. *The functions* $J_0(h)$, $J_2(h)$ *and* $J_4'(h)$ *related to the exterior Duffing oscillator* (3), *are linearly-independent on* $\Sigma=(0,\infty)$.

The proof is similar to the proof of Lemma 2.2. By the linear independence, $M_1(h)\equiv 0$ is equivalent to $\lambda_{11}=\lambda_{41}=\mu_1=0$ which implies that (2.6) holds with $k=2$, by the first part of the proof. Now if, in addition, $M_2(h)\equiv 0$, then we obtain that $\lambda_{12}=\lambda_{42}=\mu_2=0$. Using again the first step, we conclude that (2.6) holds for $k=3$, etc. The proof is completed by induction.

Remark 2.5. We are aware that the result in Theorem 2.1 could be easily derived by applying the procedure from [7], but the result concerning the exterior Duffing oscillator in Theorem 2.3 is much more difficult to obtain. Recall that, applied to (3), Françoise's (recursion) formula would give (see [9, Remark 2.3]) $M_k(h)=\oint_{H=h}\Omega_k$, where

$$\Omega_1=\omega_1, \qquad \Omega_j=\omega_j+\sum_{i+s=j} q_i\omega_s, \quad 2\leqslant j\leqslant k,$$

with $\omega_j=(\lambda_{1j}y+\lambda_{2j}x^2+\lambda_{3j}xy+\lambda_{4j}x^2y)\,dx$ and q_j, $1\leqslant j\leqslant k-1$ determined successively from representations $\Omega_j=q_j dH+dQ_j$; H is the Hamiltonian function of the unperturbed system. In attempting to apply the above algorithm, we observe that in the internal case $h<0$ one obtains $q_k=0\ \forall k\geqslant 1$ and the formula from Theorem 2.1 easily follows. At the same time, in the external case $h>0$ even for q_1 we have

$$q_1=\frac{\lambda_{31}}{\sqrt{2}}\left(\frac{\pi}{2}\operatorname{sign} y-\arctan\frac{x^2-1}{y\sqrt{2}}\right).$$

The next q_2 is given by the formula $q_2=(\lambda_{32}/\lambda_{31})\,q_1+\lambda_{31}r_1$ where r_1 is determined from the requirement that the form $q_1xy\,dx-r_1\,dH$ be exact. Clearly, the formulas for q_2, q_3, etc. will be much more complicated.

Remark 2.6. Repeating the proof of Theorem 2.3, we obtain that the Melnikov functions for the perturbed global center (1) (or the truncated pendulum (2)) are given by (2.5) and (2.6), where $h>0$ (respectively, $h\in(0,\frac{1}{4})$).

3. ANALYSIS OF THE PICARD–FUCHS AND RICCATI EQUATIONS

In this section we give a detailed proof of Theorem 3(iii) and of Theorem 4(ii) and outline the proofs of Theorems 1 and 2. Taking into account the structure of the Melnikov functions established in the previous section, it suffices to consider the following particular perturbations,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + \varepsilon(\lambda_1 y + \lambda_3 xy + \lambda_4 x^2 y), \end{aligned} \tag{3.1i}$$

for the interior Duffing oscillator, and

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + \varepsilon(\varepsilon\lambda_1 y + \lambda_2 x^2 + \lambda_3 xy + \varepsilon\lambda_4 x^2 y), \end{aligned} \tag{3.1e}$$

for the exterior Duffing oscillator. In (3.1), the parameters λ_j are assumed to be independent of ε.

Denote $R(h) = J_2(h)/J_0(h)$, where $J_k(h)$ is given by (2.1). The statements in the following two lemmas are well known, see, e.g., [11, 12, 15, 16].

LEMMA 3.1. (i) *The integrals J_k, $k = 0, 1, 2$ satisfy the Picard–Fuchs system*

$$\begin{aligned} 4hJ_0' + J_2' &= 3J_0, \\ (4h+1)\, J_1' &= 4J_1, \\ 4hJ_0' + (12h+4)\, J_2' &= 15J_2. \end{aligned} \tag{3.2}$$

(ii) *The ratio R satisfies the Riccati equation*

$$4h(4h+1)\, R' = 5R^2 + (8h-4)R - 4h \quad \text{or} \quad \begin{aligned} \dot{h} &= 4h(4h+1), \\ \dot{R} &= 5R^2 + (8h-4)\, R - 4h. \end{aligned} \tag{3.3}$$

The phase portrait of (3.3) is shown in Fig. 4. In the figure, we have denoted by Γ_i and by Γ_e the graph of the ratio $R(h)$ for $h \in (-\frac{1}{4}, 0)$ and $h > 0$ respectively. Let $\Gamma = S_1 \cup \Gamma_p \cup \Gamma_c$ where Γ_p and Γ_c form the unstable manifold of the saddle $S_1(0, 0)$.

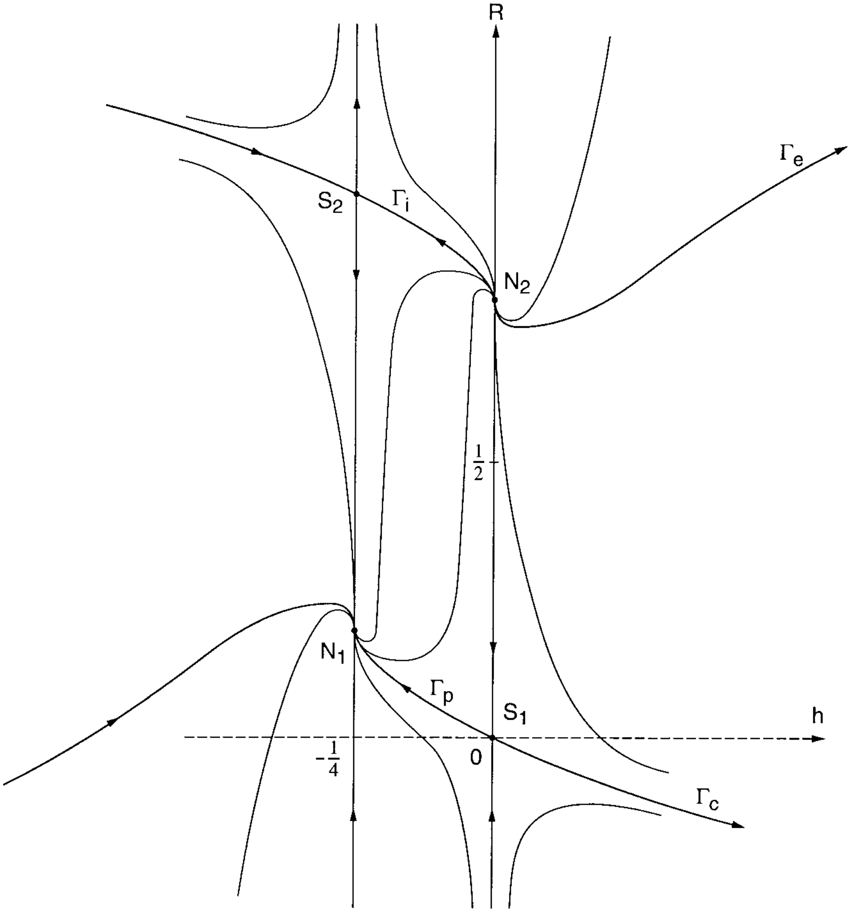

FIGURE 4

LEMMA 3.2. (i) *The phase portrait of* (3.3) *is symmetric with respect to the point* $(-\frac{1}{8}, \frac{1}{2})$.

(ii) Γ *is strictly decreasing and strictly convex.*

(iii) Γ_e *has unique non-degenerate minimum at* (h_1, R_1) *and unique inflection point at* (h_2, R_2) *where* $0 < h_1 < h_2$ *and* $\frac{3}{4} < R_1 < \frac{4}{5}$.

(iv) *As* $h \to +\infty$, *the points on* Γ_e, Γ *satisfy respectively* $R \sim \pm C\sqrt{h}$ *with* $C \in (0, 2)$.

(v) *Near the improper node* $N_2(0, \frac{4}{5})$, *the points* (h, R) *on* $\Gamma_i \cup \Gamma_e$ *satisfy*

$$R = \tfrac{4}{5} + \tfrac{3}{5} h \ln |h| + mh + O((h \ln |h|)^2).$$

Proof of Theorem 3(iii). We need to investigate the function

$$M(h)=\lambda_1 J_0(h)+\lambda_3 J_1(h)+\lambda_4 J_2(h), \qquad h\in(-\tfrac{1}{4},0)$$

for $\lambda_1\lambda_3\lambda_4\neq 0$. Our goal is to prove that $M(h)$ has at most two zeros in $(-\frac{1}{4},0)$. Making use of (3.2), we obtain (following [10]) that M satisfies the equation

$$(h+\tfrac{1}{4})\,M'-M=[(4\lambda_4 h-5\lambda_1)\,J_2+(4\lambda_1-4\lambda_1 h-4\lambda_4 h)\,J_0]/16h.$$

Solving this equation, we get the integral representation

$$\begin{aligned}M(h)&=\left(h+\frac{1}{4}\right)\left(M'\left(-\frac{1}{4}\right)\right.\\&\qquad\left.+\int_{-1/4}^{h}\frac{(4\lambda_4 s-5\lambda_1)\,J_2(s)+[4\lambda_1-4(\lambda_1+\lambda_4)s]\,J_0(s)}{s(4s+1)^2}\,ds\right)\\&=\left(h+\frac{1}{4}\right)U(h).\end{aligned}$$

Note that the integral is convergent near $-\frac{1}{4}$ since the numerator $N(s)$ of the integrand has a double zero at $-\frac{1}{4}$ (this is because J_0, J_2 and $R-1$ vanish for $h=-\frac{1}{4}$).

LEMMA 3.3. *The function $N(s)$ has in $(-\frac{1}{4},0)$ at most one zero.*

Assuming Lemma 3.3 was already proved, this yields that the above integral function has at most one critical value in $(-\frac{1}{4},0)$, hence $U(h)$ and together with it $M(h)$ have no more than two zeros there.

To prove Lemma 3.3, we consider two cases. The first one is when the two linear functions in N have a common zero (this occurs for $\lambda_4+5\lambda_1=0$). Then N can be expressed as $N=-5\lambda_1 J_0(R-\frac{4}{5})(4s+1)$. As $R>\frac{4}{5}$ in the considered interval, this yields no zero for N. The second case is when the coefficients have no common zero. Then $N(h)$ has the same zeros as the difference $R-r(h)$ where

$$r(h)=\frac{4(\lambda_1+\lambda_4)\,h-4\lambda_1}{4\lambda_4 h-5\lambda_1}.$$

We see that this hyperbola goes through the critical points N_2 and $S_2(-\frac{1}{4},1)$. Calculating the equation of the contact points of r with the flow in (3.3), we therefore have

$$(d/dt)(R-r(h))\,|_{R=r}=(\dot R-\dot h r'(h))\,|_{R=r}=4h(4h+1)\,r'(h)\,P_1(h)=0$$

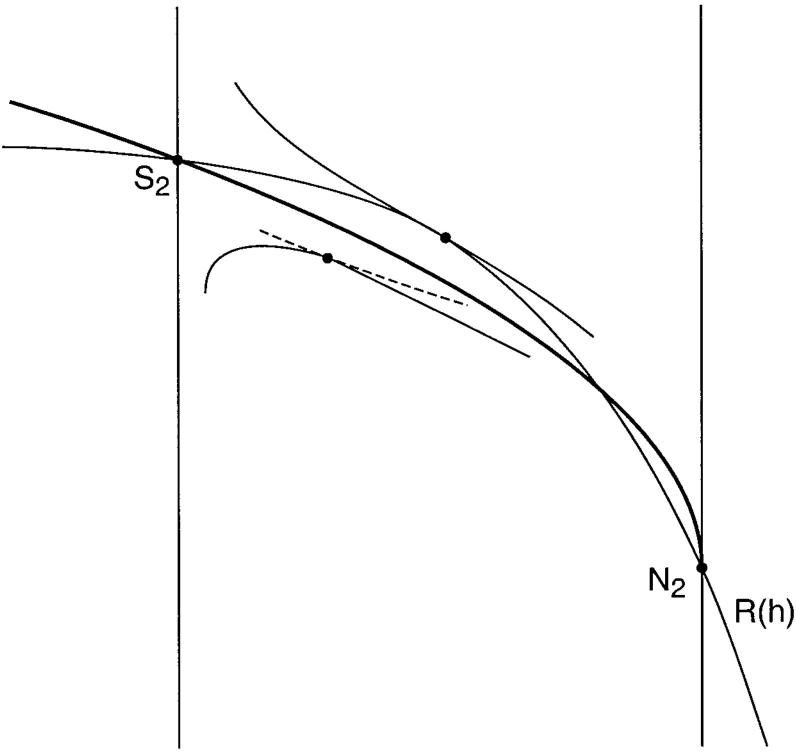

FIGURE 5

where $P_1(h)$ is a first-order polynomial. This means that the hyperbola intersects the curve Γ_i at most once, otherwise there would be at least two contact points on $r(h)$ for $h \in (-\frac{1}{4}, 0)$, see Fig. 5. This proves the lemma and together Theorem 3(iii).

Remark 3.4. It is perhaps interesting to mention the following fact concerning the cyclicity of the period annulus around the quadratic isochronous center $\mathscr{S}_4$, see e.g. [4]. Chicone and Jacobs in their study [4] found (after rather complicated argument) that the period annulus around $\mathscr{S}_4$ is of cyclicity two. It turns out that the proof of our Theorem 3(iii) could also be used as a short proof for that fact. Indeed, one can take coordinates in which the first integral and integrating factor of the corresponding isochronous system are $H = x^{-4}(\frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}) = h$ and $F = x^{-5}$; that is,

$$\dot{x} = xy,$$
$$\dot{y} = 2y^2 - x^2 + 1.$$

To study limit cycles in any quadratic perturbation of this system, one needs to consider (see [10]) the function (in fact, the first Melnikov function),

$$M(h) = \mu_1 I_0(h) + \mu_2 I_1(h) + \mu_3 I_{-1}(h)$$

where $I_k(h) = \int_{H=h} x^{k-5} y\, dx$. If in the proof of our Theorem 3(iii) one replaces the integrals (J_0, J_1, J_2) with the above defined integrals (I_1, I_0, I_{-1}) respectively, and $(\lambda_1, \lambda_3, \lambda_4)$ by (μ_2, μ_1, μ_3) respectively, then the proof goes without any further changes.

Proof of Theorem 4 (ii). Below we estimate the number of zeros in $(0, \infty)$ of the function

$$M(h) = \lambda_1 J_0(h) + \lambda_4 J_2(h) + \tfrac{1}{3} \lambda_2 \lambda_3 J'_4(h).$$

By Lemma 3.2(iii), if $\lambda_2 \lambda_3 = 0$, then $M(h)$ has at most two positive zeros. Therefore we will assume below that $\lambda_2 \lambda_3 \neq 0$. We divide the proof of the fact that M has no more than two positive zeros into three steps. The first step is to prove the following weaker estimate.

LEMMA 3.5. *The function $M(h)$ has in $(0, \infty)$ at most three zeros (counting the multiplicity).*

Proof. We use the relations

$$(k+6) J_{k+3} = (2k+6) J_{k+1} + 4khJ_{k-1}, \qquad J'_{k+3} - J'_{k+1} = kJ_{k-1}$$

to obtain that

$$J'_4 = \frac{5J_2 + 4hJ_0}{4h+1}. \tag{3.4}$$

We therefore can express M as

$$M = \frac{J_0}{3(4h+1)} \{[(12\lambda_1 + 4\lambda_2\lambda_3) h + 3\lambda_1] + [12\lambda_4 h + (3\lambda_4 + 5\lambda_2\lambda_3)] R\}.$$

As above, if the two coefficients have a common zero, then M has the representation

$$M = \frac{J_0 (R - \frac{1}{5})}{3(4h+1)} [12\lambda_4 h + (3\lambda_4 + 5\lambda_2\lambda_3)]$$

and hence, by Lemma 3.2(iii), it will have at most one zero. In the general case of independent coefficients, the zeros of M coincide with the zeros of the function $R-r(h)$ where

$$r(h)=-\frac{(12\lambda_1+4\lambda_2\lambda_3)\,h+3\lambda_1}{12\lambda_4 h+(3\lambda_4+5\lambda_2\lambda_3)}.$$

Since $r(-\frac{1}{4})=\frac{1}{5}$, the hyperbola $r(h)$ goes through the node $N_1(-\frac{1}{4},\frac{1}{5})$. Thus the equation of the contact points, which the hyperbola has with the flow generated by (3.3), takes the form $(d/dt)(R-r(h))|_{R=r}=(4h+1)\,r'(h)\,P_2(h)=0$ where P_2 is a second degree polynomial. This means that for $h>0$ the graph of $r(h)$ has at most two contact points with the flow in (3.3). Taking into account that N_1 lies on the hyperbola and the position of the curve Γ_e (see Fig. 4), we conclude that only one of the branches of the hyperbola could intersect Γ_e. As r contains at most two contact points, this yields that r can intersect Γ_e in at most three points. Hence, the zeros of M are at most three. Lemma 3.5 is proved.

Let us consider the plane curve $\mathscr{S}:(Q(h),R(h)),\ h>0$ where $Q(h)=J_4'(h)/J_0(h)$. Provided $\mathscr{S}$ is regular (that is $|Q'(h)|+|R'(h)|\neq 0$), the number of zeros of M coincides with the number of the intersection points of $\mathscr{S}$ with the straight line $\lambda_1+\frac{1}{3}\lambda_2\lambda_3 q+\lambda_4 r=0$ in the $(q,r)-$plane. Theorem 4(ii) will be proved if we establish that the curve $\mathscr{S}$ is regular, properly posed and strictly convex. The first step consists in proving that $\mathscr{S}$ is convex near its ends.

LEMMA 3.6. (i) *We have $Q'(h)<0$, hence $\mathscr{S}$ is regular and properly posed.*

(ii) *As $h\to +0$, the points on $\mathscr{S}$ satisfy*

$$Q\to 4-0,\qquad R\sim\tfrac{1}{5}Q,\qquad R-\tfrac{1}{5}Q\sim\tfrac{12}{5}h(Q)>0$$

where $h(Q)$ is the solution of $Q-4=3h\ln|h|$ and $h''(Q)>0$ near $Q=4$.

(iii) *As $h\to +\infty$, the points on $\mathscr{S}$ satisfy*

$$Q\to 1+0,\qquad R\sim C_1(Q-1)^{-1},\quad \text{where}\quad C_1\in(0,5).$$

Proof. (i) Using (3.4) we see that Q can be expressed as

$$Q(h)=\frac{5R(h)+4h}{4h+1}. \tag{3.5}$$

Then, by (3.3) and (3.5), $Q'(h)<0$ for $h>0$ is equivalent to $5R(5R-8h-4)-4h<0$. As the line $r=\frac{8}{5}h+\frac{4}{5}$ is a line without contact, we get $5R-8h-4<0$ which implies $Q'<0$. (ii). By Lemma 3.2(v), we have

$$Q=4+3h\ln|h|+(5m-12)\,h+O((h\ln|h|)^2).$$

These asymptotic expansions of Q and R yield (ii) immediately.

(iii) Replacing $R\sim C\sqrt{h}$ in (3.5), we get the last statement in Lemma 3.6.

LEMMA 3.7. *The curve $\mathcal{S}$ has no inflection point.*

Proof. If $\mathcal{S}$ has an inflection point, then Lemma 3.6 yields that $\mathcal{S}$ has either a degenerate inflection point or at least two different inflection points. In both cases, there exist values of λ_k, for which the straight line $\lambda_1+\frac{1}{3}\lambda_2\lambda_3 q+\lambda_4 r=0$ will have at least four intersections with $\mathcal{S}$, counting multiplicity (see Fig. 6). By Lemma 3.5, this is however impossible.

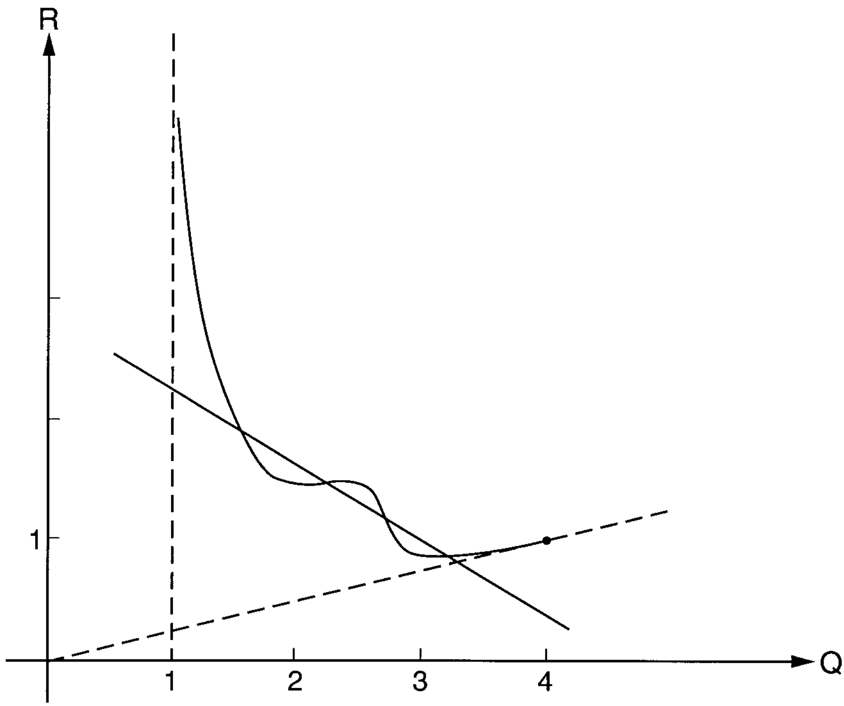

FIGURE 6

The results proved above establish that the curve $\mathscr{S}$ is regular and strictly convex. Therefore, each line can intersect $\mathscr{S}$ in at most two points, counting multiplicity. This implies Theorem 4(ii).

Remark 3.8. The following particular integrals (3.6) and (3.7) (we have to mention that $J'_2 = J'_4 - J_0$)

$$I_i(h) = \alpha J_0(h) + \beta J_1(h) + J_2(h), \qquad h \in (-\tfrac{1}{4}, 0) \tag{3.6}$$

$$I_e(h) = \alpha J_0(h) + \lambda\beta J'_2(h) + J_2(h), \qquad h \in (0, \infty) \tag{3.7}$$

appear in an investigation of the two-parameter system [11]

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \lambda + x - x^3 + \varepsilon y(\alpha + \beta x + x^2), \end{aligned}$$

where λ and ε are small (see also [5]). It seems to us that the treatment of (3.6) and (3.7) presented in [11] is much more complicated than ours above. Moreover, Jebrane and Żołądek found in [11] that (3.7) can have three zeros, which is not the case. Therefore the bifurcation diagram presented in [11, Theorem 2] is not correct.

Proof of Theorems 1, 2(ii), (iii). We have to count the zeros of

$$M(h) = \lambda_1 J_0(h) + \lambda_4 J_2(h) + \tfrac{1}{3}\lambda_2\lambda_3 J'_4(h)$$

respectively for $h \in (0, \infty)$ and $h \in (0, \frac{1}{4})$. We recall that the contour of integration in $J_k(h)$ is $\delta(h) \subset \{H = h\}$ where the Hamiltonian function is given respectively by $H = \frac{1}{2}(y^2 + x^2) \pm \frac{1}{4}x^4$. For these cases we obtain instead of (3.2), (3.3) the equations

$$\begin{aligned} 4hJ'_0 - J'_2 &= 3J_0, \\ \mp 4hJ'_0 + (12h \pm 4)\, J'_2 &= 15J_2. \end{aligned} \qquad \text{and} \qquad \begin{aligned} \dot{h} &= 4h(4h \pm 1), \\ \dot{R} &= -5R^2 + (8h \mp 4)\, R \pm 4h. \end{aligned}$$

Using these equations, the proof of (ii) is easily reduced to showing that the curves Γ_c and Γ_p we have defined before Lemma 3.2 are strictly decreasing. The proof of (iii) reduces to showing that each hyperbola passing through the critical point $N_1(-\frac{1}{4}, \frac{1}{5})$ of (3.3) can intersect the curve Γ in at most two points (counting multiplicity). The details are as in [9].

APPENDIX

The approach from Section 2 can be applied to other cases. To compare with how the method works in more general situations, let us consider, for example, the Duffing oscillator with an arbitrary cubic perturbation in the second equation. Performing an affine change, we get

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + \lambda_1 y + \lambda_2 x^2 + \lambda_3 xy + \lambda_4 x^2 y + \lambda_5 y^2 + \lambda_6 xy^2 + \lambda_7 y^3. \end{aligned} \tag{A1}$$

For $\lambda_1 = \lambda_3 = \lambda_4 = \lambda_7 = 0$, the system is integrable and has a first integral

$$\begin{aligned} &H(x, y, \lambda_2, \lambda_5, \lambda_6) \\ &\quad = e^{-(2\lambda_5 x + \lambda_6 x^2)} \left[\frac{y^2}{2} - \int_0^x e^{2\lambda_5(x-s) + \lambda_6(x^2 - s^2)} (s - s^3 + \lambda_2 s^2)\, ds \right] \end{aligned}$$

with an integrating factor $M(x, \lambda) = e^{-(2\lambda_5 x + \lambda_6 x^2)}$. Using this fact, we can rewrite (A1) in a Pfaffian form

$$dH(x, y, \lambda_2, \lambda_5, \lambda_6) - M(x, \lambda)(\lambda_1 y + \lambda_3 xy + \lambda_4 x^2 y + \lambda_7 y^3)\, dx = 0.$$

As above, we consider two cases.

(i) *Case* $h < 0$. For definiteness, we take the family of ovals inside the right branch of the “eight loop”. We know that

$$M_1(h) = \oint_{H(0) = h} (\lambda_{11} y + \lambda_{31} xy + \lambda_{41} x^2 y + \lambda_{71} y^3)\, dx \equiv 0$$

is equivalent to $\lambda_{11} = \lambda_{31} = \lambda_{41} = \lambda_{71} = 0$. Assume that $\lambda_{1j} = \lambda_{3j} = \lambda_{4j} = \lambda_{7j} = 0$ for $j = 1, 2, ..., k-1$. Using the same construction as before, we get

$$\int_\gamma dH(x, y, \lambda_2, \lambda_5, \lambda_6) = \int_\gamma M(x, \lambda)(\lambda_1 y + \lambda_3 xy + \lambda_4 x^2 y + \lambda_7 y^3)\, dx. \tag{A2}$$

On the left, we have

$$\begin{aligned} H(\xi_1, 0, \lambda_2, \lambda_5, \lambda_6) - H(\xi, 0, \lambda_2, \lambda_5, \lambda_6) &= d(\xi, \varepsilon)\, \partial_x H(\xi_*, 0, \lambda_2, \lambda_5, \lambda_6) \\ &= d(\xi, \varepsilon)[\xi^3 - \xi + O(\varepsilon)]. \end{aligned}$$

The integral on the right equals

$$\varepsilon^k \oint_{H(0) = h} (\lambda_{1k} y + \lambda_{3k} xy + \lambda_{4k} x^2 y + \lambda_{7k} y^3)\, dx + O(\varepsilon^{k+1}).$$

In combination, this yields

$$d(\xi, \varepsilon) = \sum_{j=1}^{\infty} \varepsilon^j d_j(\xi)$$
$$= \frac{\varepsilon^k}{\xi^3 - \xi} \oint_{H(0)=h} (\lambda_{1k} y + \lambda_{3k} xy + \lambda_{4k} x^2 y + \lambda_{7k} y^3)\, dx + O(\varepsilon^{k+1}).$$

Hence $d_j(\xi) = 0$, $j < k$ and

$$d_k(\xi) = \frac{1}{\xi^3 - \xi} M_k(h)$$
$$= \frac{1}{\xi^3 - \xi} \left[\lambda_{1k} J_0(h) + \lambda_{3k} J_1(h) + \lambda_{4k} J_2(h) + \frac{3}{7} \lambda_{7k}(4hJ_0(h) + J_2(h)) \right].$$

Thus, $M_j(h) = 0$, $j < k$ and we have proved:

LEMMA A.1. *For any k, the following formula holds for the Melnikov functions related to any of the small annuli in* (A1):

$$M_k(h) = \lambda_{1k} J_0(h) + \lambda_{3k} J_1(h) + \lambda_{4k} J_2(h) + \tfrac{3}{7} \lambda_{7k}(4hJ_0(h) + J_2(h)),$$
$$h \in (-\tfrac{1}{4}, 0).$$

(ii) *Case $h > 0$.* For the exterior Duffing oscillator, the higher order Melnikov functions appear to be of very complicated nature and we are not able to calculate effectively $M_k(h)$, $k \geqslant 3$ for the whole perturbation as given in (A1). We know that

$$M_1(h) = \oint_{H(0)=h} (\lambda_{11} y + \lambda_{41} x^2 y + \lambda_{71} y^3)\, dx \equiv 0$$

is equivalent to $\lambda_{11} = \lambda_{41} = \lambda_{71} = 0$. The left-hand side of (A2) again equals $d(\xi, \varepsilon)[\xi^3 - \xi + O(\varepsilon)]$. To calculate $M_2(h)$, we see that in the right-hand side the integral has asymptotics

$$\varepsilon^2 \oint_{H(0)=h} (\lambda_{12} y + \lambda_{42} x^2 y + \lambda_{72} y^3)\, dx + \lambda_3 \int_\gamma M(x, \lambda)\, xy\, dx + O(\varepsilon^3).$$

To find the first order approximation of the integral

$$I(\lambda) = \int_{\gamma(\lambda)} M(x, \lambda)\, xy\, dx$$

for small λ, we again use that

$$I(0)=\int_{\gamma(0)} xy\,dx=\oint_{H(0)=h} xy\,dx=0,$$

which yields

$$\begin{aligned}I(\lambda)&=I(0,\lambda_2,\lambda_3,0,\lambda_5,\lambda_6,0)+O(\varepsilon^2)\\&=I(\lambda_2)+I(\lambda_3)+I(\lambda_5)+I(\lambda_6)+O(\varepsilon^2).\end{aligned}$$

Just as above, we obtain $I(\lambda_3)=I(\lambda_6)=0$ (because of symmetry) and $I(\lambda_2)=\frac{1}{3}\lambda_2 J_4'(h)+O(\lambda_2^2)$. Similarly,

$$\begin{aligned}I(\lambda_5)&=\oint_{H(x,\,y,\,\lambda_5)=H(\xi,\,0,\,\lambda_5)} e^{-2\lambda_5 x}xy\,dx\\&=\lambda_5\oint_{H(0)=h}[-2x^2y+x(dy/d\lambda_5)]\,dx+O(\lambda_5^2)\\&=\lambda_5\oint_{H(0)=h}[-x^2y-(2x^4)/(3y)+(2x^6)/(5y)]\,dx+O(\lambda_5^2).\end{aligned}$$

Indeed, differentiating with respect to λ_5 the equality $H(x,y,\lambda_5)=H(\xi,0,\lambda_5)$ yields $(dy/d\lambda_5)\,\partial_y H=\partial_{\lambda_5}H(\xi,0,\lambda_5)-\partial_{\lambda_5}H(x,y,\lambda_5)$ and after we calculate the derivatives one obtains $dy/d\lambda_5=xy-2(x^3-\xi^3)/3y+2(x^5-\xi^5)/5y$. In combination, all this yields

$$\begin{aligned}&\lambda_3\int_\gamma M(x,\lambda)\,xy\,dx\\&\quad=\varepsilon^2\{\tfrac{1}{3}\lambda_{21}\lambda_{31}J_4'(h)+\lambda_{31}\lambda_{51}[-J_2(h)-\tfrac{2}{3}J_4'(h)+\tfrac{2}{5}J_6'(h)]\}+O(\varepsilon^3)\end{aligned}$$

Putting together all these estimates, we get

$$\begin{aligned}M_2(h)&=\lambda_{12}J_0+\lambda_{42}J_2+\tfrac{3}{7}\lambda_{72}(4hJ_0+J_2)\\&\quad+\tfrac{1}{3}\lambda_{21}\lambda_{31}J_4'+\tfrac{1}{15}\lambda_{31}\lambda_{51}(3J_2-4J_4')\\&=(\lambda_{12}+\tfrac{12}{7}\lambda_{72}h)\,J_0+(\lambda_{42}+\tfrac{3}{7}\lambda_{72}+\tfrac{1}{3}\lambda_{31}\lambda_{51})\,J_2\\&\quad+(\tfrac{1}{3}\lambda_{21}\lambda_{31}-\tfrac{4}{15}\lambda_{31}\lambda_{51})\,J_4',\end{aligned}$$

which unfortunately does not suffice to determine the general form of $M_k(h)$ because the condition for $M_2(h)=0$ is

$$\lambda_{12}=\lambda_{72}=\lambda_{42}+\tfrac{1}{5}\lambda_{31}\lambda_{51}=\tfrac{1}{3}\lambda_{21}\lambda_{31}-\tfrac{4}{15}\lambda_{31}\lambda_{51}=0.$$

Provided $M_1(h) = \cdots = M_{k-1}(h) \equiv 0$, the kth order Melnikov function is

$$M_k(h) = \sum_{j=1}^{k} \frac{1}{(k-j)!} \times \frac{d^{k-j}}{d\varepsilon^{k-j}} \int_{\gamma(\lambda)} M(x, \lambda)(\lambda_{1j} y + \lambda_{3j} xy + \lambda_{4j} x^2 y + \lambda_{7j} y^3)\, dx \bigg|_{\varepsilon=0}.$$

The problem is how to calculate effectively the second, third etc. derivatives with respect to ε. To obtain a definitive result, we will consider the particular case when $\lambda_2 \lambda_3 \lambda_4 \lambda_5 = 0$. Then one can derive an explicit formula for the higher order Melnikov functions. Take for example the case $\lambda_2 = 0$. Then repeating the above calculation (with arbitrary k instead of $k = 2$) we get

LEMMA A2. *For any $k \geqslant 2$, the following formula holds for the Melnikov functions of $(A1)$ with $\lambda_2 = 0$, corresponding to the exterior annulus*:

$$M_k(h) = \lambda_{1k} J_0 + \lambda_{4k} J_2 + \tfrac{3}{7} \lambda_{7k}(4hJ_0 + J_2) + \tfrac{1}{15} \sum_{i+j=k} \lambda_{3i} \lambda_{5j}(3J_2 - 4J'_4).$$

Indeed, taking $\mu = \lambda_3 \lambda_5 = \sum_{k=2}^{\infty} \mu_k \varepsilon^k$, $\mu_k = \sum_{i+j=k} \lambda_{3i} \lambda_{5j}$ $(\mu_1 = 0)$ and assuming that $\lambda_{1j} = \lambda_{4j} = \lambda_{7j} = \mu_j = 0$ for $j < k$, we have on the right of (A2)

$$\varepsilon^k \oint_{H(0)=h} (\lambda_{1k} y + \lambda_{4k} x^2 y + \lambda_{7k} y^3)\, dx + \lambda_3 \int_\gamma M(x, \lambda)\, xy\, dx + O(\varepsilon^{k+1})$$

and

$$\int_{\gamma(\lambda)} M(x, \lambda)\, xy\, dx = I(\lambda) = I(\lambda_3, \lambda_5, \lambda_6) + O(\varepsilon^k)$$
$$= I(\lambda_3, \lambda_6) + I(\lambda_5) + o(\lambda_5) + O(\varepsilon^k).$$

Further, the system

$$\dot{x} = y,$$
$$\dot{y} = x - x^3 + \lambda_3 xy + \lambda_6 xy^2$$

has a first integral (it is not elementary) $K(x, y, \lambda_3, \lambda_6)$ which is an even function of x. Therefore, by symmetry,

$$I(\lambda_3, \lambda_6) = \oint_{K(x,\, y,\, \lambda_3,\, \lambda_6) = K(\xi,\, 0,\, \lambda_3,\, \lambda_6)} e^{-\lambda_6 x^2} xy \, dx = 0.$$

The proof then goes as above.

We note finally that the problem for setting up the *sharp* upper bound to the number of zeros of the Melnikov functions $M_k(h)$ derived in Lemmas A1 and A2 is far from being trivial and it remains beyond the scope of the present paper.

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