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Self-energy of one electron in non-relativistic QED

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Abstract

We investigate the self-energy of one electron coupled to a quantized radiation field by extending the ideas developed in Hainzl (Ann. H. Poincaré, in press). We fix an arbitrary cut-off parameter Λ and recover the α^2-term of the self-energy, where α is the coupling parameter representing the fine structure constant. Thereby we develop a method which allows to expand the self-energy up to *any power* of α. This implies that perturbation theory in α is correct if Λ is fix. As an immediate consequence we obtain enhanced binding for electrons.

Keywords: QED; Self-energy; Enhanced binding

1. Introduction and main results

In recent times the self-energy of an electron coupled to a photon field was studied in several articles. In [10], Lieb and Loss showed that in the limit of large cut-off parameter Λ, perturbation theory is conceptually wrong.

A different method of investigating the self-energy was developed in Hainzl [5]. Therein the cut-off parameter Λ was fixed and the self-energy in the case of small coupling parameter α was studied. It turned out that one photon is enough to recover the first order in α.

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By similar methods Hainzl and Seiringer evaluated in [6] the mass renormalization via the dispersion relation and proved that after renormalizing the mass the binding energy of an electron in the field of a nucleus, to leading order in α, has a finite limit as Λ goes to infinity.

As our main result in the present paper we recover the next to leading order, the α^2-term, of the self-energy of an electron.

As a byproduct of the proof we develop a method which allows to expand the self-energy, step by step, up to *any power* of α, and implies at the same time that perturbation theory, in α, is *correct* if Λ is kept fix, in a case where there is no spectral gap.

As an immediate consequence of our main result we obtain enhanced binding for electrons. This means that a dressed electron in the field of an external potential V can have a bound state even if the corresponding Schrödinger operator $p^2 + V$ has only essential spectrum. Enhanced binding for charged particles without spin was previously proven in [7].

1.1. Self-energy

The self-energy of an electron is defined as the bottom of the spectrum of the so-called Pauli–Fierz operator

$$T = (p + \sqrt{\alpha}A(x))^2 + \sqrt{\alpha}\sigma \cdot B(x) + H_{\mathrm{f}}, \tag{1.1}$$

acting on the Hilbert space

$$\mathscr{H} = \mathscr{L}^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathscr{F},$$

where $\mathscr{F} = \oplus_{n=0}^{+\infty} \mathscr{L}_b^2(\mathbb{R}^{3n}; \mathbb{C}^{2n})$ is the Fock space for the photon field and $\mathscr{L}_b^2(\mathbb{R}^{3n})$ is the space of symmetric functions in $\mathscr{L}^2(\mathbb{R}^{3n})$ representing n-photons states. For $n = 0$, this space is simply $\mathbb{C}|0\rangle$, where $|0\rangle (\epsilon \mathbb{C}^2)$ is the vacuum vector. This operator is essentially self-adjoint on $\mathscr{D}(\Delta) \cap \mathscr{D}(H_{\mathrm{f}})$ (see [8]), where $\mathscr{D}$ denotes the operator domain. Its spectrum is of the form $[\Sigma_\alpha, +\infty[$, where the self-energy Σ_α is a complicated function of α (and of a cutoff parameter Λ to be introduced below). Without radiation field, the Hamiltonian is, of course, simply the Laplace operator $-\Delta$ with respective spectrum $[0, +\infty[$.

We fix units such that the Planck constant $\hbar = 1$, the speed of light $c = 1$, and the electron mass $m = \frac{1}{2}$. The electron charge is then given by $e = \sqrt{\alpha}$ with $\alpha \approx 1/137$, the fine structure constant. In the present paper, α plays the role of a small dimensionless number, which measures the coupling to the radiation field. Our results hold for *sufficiently small values* of α. σ is the vector of Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$. Recall that the σ_i's are hermitian 2×2 complex matrices and fulfill the anti-commutation relations $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\mathbb{1}_{\mathbb{C}^2}\delta_{i,j}$. The operator $p = -i\nabla$ is the electron momentum while A is the magnetic vector potential. The magnetic field is $B = \operatorname{curl} A$.

The vector potential is

$$A(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\chi(|k|)}{2\pi|k|^{1/2}} \epsilon^{\lambda}(k)[a_{\lambda}(k)e^{ik\cdot x} + a_{\lambda}^*(k)e^{-ik\cdot x}]\,\mathrm{d}k,$$

and the corresponding magnetic field reads

$$B(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\chi(|k|)}{2\pi|k|^{1/2}} (k \times i\epsilon^{\lambda}(k))[a_{\lambda}(k)e^{ik\cdot x} - a_{\lambda}^*(k)e^{-ik\cdot x}]\,\mathrm{d}k,$$

where the annihilation and creation operators a_{λ} and a_{λ}^*, respectively, satisfy the usual commutation relations

$$[a_{\nu}(k), a_{\lambda}^*(q)] = \delta(k-q)\delta_{\lambda,\nu}$$

and

$$[a_{\lambda}(k), a_{\nu}(q)] = 0, \quad [a_{\lambda}^*(k), a_{\nu}^*(q)] = 0.$$

The vectors $\epsilon^{\lambda}(k) \in \mathbb{R}^3$ are orthonormal polarization vectors perpendicular to k, and they are chosen in a such a way that

$$\epsilon^2(k) = \frac{k}{|k|} \wedge \epsilon^1(k). \tag{1.2}$$

The function $\chi(|k|)$ describes the ultraviolet cutoff for the interaction at large wave numbers k. We choose for χ the Heaviside function $\Theta(\Lambda - |k|)$. (More general cut-off functions would work but let us nevertheless emphasize the fact that we shall sometimes use the radial symmetry of χ in the proofs.) Throughout the paper we assume Λ to be an arbitrary but fixed positive number.

The photon field energy H_{f} is given by

$$H_{\mathrm{f}} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| a_{\lambda}^*(k) a_{\lambda}(k)\,\mathrm{d}k \tag{1.3}$$

and the field momentum reads

$$P_{\mathrm{f}} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} k a_{\lambda}^*(k) a_{\lambda}(k)\,\mathrm{d}k. \tag{1.4}$$

In the following, we use the notation

$$A(x) = D(x) + D^*(x), \quad B(x) = E(x) + E^*(x) \tag{1.5}$$

for the vector potential, respectively, the magnetic field.

The operators D^* and E^* create a photon wave function $G(k)e^{-ik\cdot x}$ and $H(k)e^{-ik\cdot x}$, respectively, where $G(k) = (G^1(k), G^2(k))$ and $H(k) = (H^1(k), H^2(k))$ are vectors of one-photon states, given by

$$G^{\lambda}(k) = \frac{\chi(|k|)}{2\pi|k|^{1/2}}\epsilon^{\lambda}(k) \tag{1.6}$$

and

$$H^{\lambda}(k) = \frac{-i\chi(|k|)}{2\pi|k|^{1/2}} k \wedge \epsilon^{\lambda}(k) = -ik \wedge G^{\lambda}(k). \tag{1.7}$$

It turns out to be convenient to denote a general vector $\Psi \in \mathscr{H}$ as a direct sum

$$\Psi = \bigoplus_{n \geqslant 0} \psi_n, \tag{1.8}$$

where $\psi_n = \psi_n(x, k_1, \ldots, k_n)$ is a n-photons state. For simplicity, we do not include the variables corresponding to the polarization of the photons and the spin of the electron.

From [5] we know that the first-order term in α of the self-energy

$$\Sigma_\alpha = \inf \operatorname{spec} T \tag{1.9}$$

is given by

$$\alpha\pi^{-1}\Lambda^2 - \alpha\langle 0|E\mathscr{A}^{-1}E^*|0\rangle = 2\alpha\pi^{-1}[\Lambda - \ln(1+\Lambda)], \tag{1.10}$$

where $\mathscr{A} = P_{\mathrm{f}}^2 + H_{\mathrm{f}}$ and $|0\rangle$ is the vacuum in the Fock space $\mathscr{F}$. Recall that the vacuum polarization, $\alpha\langle 0|A^2|0\rangle = \alpha\pi^{-1}\Lambda^2$, enters somehow ab initio the game, whereas the second term on the r.h.s. of (1.10) stems from the magnetic field B. But now, for the next to leading order α^2 all terms contribute.

Theorem 1 (Expansion of the self-energy up to second-order). *Let Λ be fixed. Then, for α small enough,*

$$\begin{aligned}\Sigma_\alpha =\ & \alpha[\pi^{-1}\Lambda^2 - \langle 0|E\mathscr{A}^{-1}E^*|0\rangle] - \alpha^2[\langle 0|DD\mathscr{A}^{-1}D^*D^*|0\rangle \\ & + \langle 0|E\mathscr{A}^{-1}E\mathscr{A}^{-1}E^*\mathscr{A}^{-1}E^*|0\rangle + 4\langle 0|E\mathscr{A}^{-1}P_{\mathrm{f}}\cdot D\mathscr{A}^{-1}P_{\mathrm{f}}\cdot D^*\mathscr{A}^{-1}E^*|0\rangle \\ & - 2\langle 0|E\mathscr{A}^{-1}E\mathscr{A}^{-1}D^*D^*|0\rangle - \langle 0|E\mathscr{A}^{-1}E^*|0\rangle\,||\mathscr{A}^{-1}E^*|0\rangle||^2] \\ & + \mathscr{O}(\alpha^{5/2}\ln(1/\alpha)).\end{aligned} \tag{1.11}$$

Remark 1. Throughout the paper the notation $\mathscr{O}(f(\alpha))$ means that there is a positive constant C such that $|\mathscr{O}(f(\alpha))| \leqslant Cf(\alpha)$.

Remark 2. In Remark 3 in the following section, we explain how (1.11) can be guessed from formal perturbation theory.

1.2. Enhanced binding

As an immediate consequence of Theorem 1 we are able to prove enhanced binding for electrons, which was already shown in [7] for charged bosons. Namely, if we take a negative radial potential $V = V(|x|)$ with compact support such that $p^2 + V$ has purely continuous spectrum, thus no bound state, but a so-called zero-resonance which satisfies the equation

$$\psi(x) = -\frac{1}{4\pi}\int \frac{V(y)\psi(y)}{|x-y|}\,\mathrm{d}y \tag{1.12}$$

then after turning on the radiation field, even for infinitely small coupling α, the Hamiltonian

$$H_\alpha = T + V \tag{1.13}$$

has a ground state. To this end, we use a result of Griesemer et al. [4] stating that the inequality

$$\inf \operatorname{spec} H_\alpha < \Sigma_\alpha \tag{1.14}$$

guarantees the existence of a ground state. Earlier the existence of a ground state, for small coupling, has been proven in [1].

Theorem 2 (Enhanced binding). *Let V be a negative continuous function, which is radially symmetric and with compact support. Assume that the corresponding Schrödinger operator $p^2 + V$ has no eigenvalue, but that there exists a non-trivial radial solution of* (1.12). *Then at least for small values of α the operator H_α has a ground state.*

In the dipole approximation enhanced binding in the limit of large coupling α was shown in [9].

2. Proof of Theorem 1

We will follow the methods developed in [5] and extend the ideas therein. For sake of a simplified notation, we introduce the unitary transform

$$U = e^{iP_{\mathrm{f}}\cdot x} \tag{2.1}$$

acting on $\mathscr{H}$. Notice that

$$U(E^*(x)\psi(x)) = H(k)\psi(x)$$

and

$$U(D^*(x)\psi(x)) = G(k)\psi(x).$$

More generally, for a n-photons component, we have

$$U(E^*(x)\psi_n(x, k_1, \dots, k_n)) = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} H(k_i)\psi_n(x, k_1, \dots, \check{k_i}, \dots, k_{n+1})$$

and

$$U(D^*(x)\psi_n(x, k_1, \dots, k_n)) = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} G(k_i)\psi_n(x, k_1, \dots, \check{k_i}, \dots, k_{n+1}),$$

where the notation $\check{}$ means that the corresponding variable has been omitted. Since

$$UpU^* = p - P_{\mathrm{f}}, \tag{2.2}$$

we obtain

$$UTU^* = (p - P_{\mathrm{f}} + \sqrt{\alpha}A)^2 + \sqrt{\alpha}\sigma \cdot B + H_{\mathrm{f}}, \tag{2.3}$$

where $A = A(0)$ and $B = B(0)$.

Obviously,

$$\inf \operatorname{spec}[UTU^*] = \inf \operatorname{spec} T. \tag{2.4}$$

Therefore in the following, we will rather work with UTU^* which we still denote by T.

We also introduce the notation

$$L = (p - P_{\mathrm{f}})^2 + H_{\mathrm{f}}, \tag{2.5}$$

$$\mathscr{P} = p - P_{\mathrm{f}}, \tag{2.6}$$

$$F_{\mathrm{f}}^* = 2P_{\mathrm{f}} \cdot D^* + \sigma \cdot E^* \tag{2.7}$$

and

$$F^* = 2\mathscr{P} \cdot D^* + \sigma \cdot E^*. \tag{2.8}$$

Let us also recall that

$$A^2 = \Lambda^2\pi^{-1} + 2D^* \cdot D + D^* \cdot D^* + D \cdot D, \tag{2.9}$$

since $D \cdot D^* - D^* \cdot D = \Lambda^2\pi^{-1}$. (In the following, we shall often denote DD instead of $D \cdot D$, and similarly for E, E^* or D^*, for simplicity.)

Before turning to the proof of the theorem per se, we give an heuristic argument based on formal perturbation theory with respect to α to derive (1.11).

Remark 3. Let us consider again the unitary transform of T given by (2.1) and (2.3). In (2.3), p $(\in \mathbb{R}^3)$ appears as a parameter, and for fixed p, the operator UTU^* acts on $\mathscr{F}$.

Denote

$$E(p) = \inf \operatorname{spec} UTU^*,$$

where p is kept fixed. Let us assume that $E(p) \geqslant E(0)$, which is known to be true in the case without B-field [3] but still open for the full hamiltonian. Under this assumption,

$$E(0) = \inf \operatorname{spec} T = \inf \operatorname{spec} \tilde{T}_\alpha,$$

where

$$\tilde{T}_\alpha = \mathscr{A} + \sqrt{\alpha}(F_{\mathrm{f}} + F_{\mathrm{f}}^*) + \alpha H^1 + \alpha\pi^{-1}\Lambda^2,$$

with F_{f} being defined by (2.7) above and $H^1 = D \cdot D + D^* \cdot D^* + 2D^* \cdot D$.

Because $\mathscr{A}|0\rangle = 0$, the vacuum vector $|0\rangle$ is an eigenvector of $\tilde{T}_0$. Since we are interested in the small α case, we can apply "formal" perturbation theory as found in classical textbooks (see, for example, [12]). Up to second order, we then get an approximate "ground" state

$$\begin{aligned}\Psi_\alpha &= |0\rangle - \sqrt{\alpha}\mathscr{A}^{-1}(F_{\mathrm{f}} + F_{\mathrm{f}}^*)|0\rangle + \alpha\mathscr{A}^{-1}(F_{\mathrm{f}} + F_{\mathrm{f}}^*)\mathscr{A}^{-1}(F_{\mathrm{f}} + F_{\mathrm{f}}^*)|0\rangle \\ &\quad - \alpha\mathscr{A}^{-1}H^1|0\rangle - \alpha||\mathscr{A}^{-1}\sigma \cdot E^*|0\rangle||^2|0\rangle \\ &= |0\rangle - \sqrt{\alpha}\mathscr{A}^{-1}\sigma \cdot E^*|0\rangle + \alpha\mathscr{A}^{-1}F^*\mathscr{A}^{-1}\sigma \cdot E^*|0\rangle \\ &\quad - \alpha\mathscr{A}^{-1}D^* \cdot D^*|0\rangle,\end{aligned} \tag{2.10}$$

since $P_{\mathrm{f}}|0\rangle = F_{\mathrm{f}}|0\rangle = 0$, $D\mathscr{A}^{-1}\sigma \cdot E^*|0\rangle = 0$ (see the proof of (2.24)) and

$$\mathscr{A}^{-1}\sigma \cdot E\mathscr{A}^{-1}\sigma \cdot E^*|0\rangle = ||\mathscr{A}^{-1}\sigma \cdot E^*|0\rangle||^2|0\rangle.$$

Following [12], this leads to an approximate energy

$$E(0) \sim \frac{\langle 0|\tilde{T}_\alpha|\Psi_\alpha\rangle}{\langle 0|\Psi_\alpha\rangle},$$

which yields exactly the right-hand side in (1.11). Note that the expression (2.10) will be used in Section 2.1 to build a trial function for the upper bound of Σ_α. Let us emphasize that this perturbation argument is only formal since 0 is not an isolated eigenvalue of $\mathscr{A}$ and Kato's perturbation method can therefore not be applied directly.

Let us now turn to the proof of Theorem 1.

For any general $\Psi \in \mathscr{H}$, we have

$$(\Psi, T\Psi) = \Lambda^2 \alpha \pi^{-1} ||\Psi||^2 + ||p\psi_0||^2 + 2\alpha \sum_{n \geqslant 1} (\psi_n, D^* D \psi_n) + \mathscr{E}_0[\psi_0, \psi_1] + \sum_{n \geqslant 0} \mathscr{E}[\psi_n, \psi_{n+1}, \psi_{n+2}], \tag{2.11}$$

where, as in [5],

$$\mathscr{E}_0[\psi_0, \psi_1] = (\psi_1, L\psi_1) + 2\sqrt{\alpha}\, \mathrm{Re}(F^* \psi_0, \psi_1) \tag{2.12}$$

and

$$\mathscr{E}[\psi_n, \psi_{n+1}, \psi_{n+2}] = (\psi_{n+2}, L\psi_{n+2}) + 2\, \mathrm{Re}(\sqrt{\alpha} F^* \psi_{n+1} + \alpha D^* D^* \psi_n, \psi_{n+2}). \tag{2.13}$$

For simplicity, in this section, we shall actually work in the momentum representation of the electron space. A n-photons function ψ_n will then be looked at as $\psi_n(l,k)$ with $k = (k_1, \dots, k_n)$, where l stands for the momentum variable of the electron and is obtained from the position variable x by Fourier transform. In that case $\mathscr{P}$ is simply a multiplication operator, and for short we use

$$\mathscr{P}\psi_n(l, k_1, \dots, k_n) = \left(l - \sum_{i=1}^{n} k_i \right) \psi_n =: \mathscr{P}_n \psi_n \tag{2.14}$$

and similarly

$$H_{\mathrm{f}} \psi_n(l, k_1, \dots, k_n) = \sum_{i=1}^{n} |k_i| \psi_n =: H_{\mathrm{f}}^n \psi_n. \tag{2.15}$$

2.1. Upper bound for Σ_α

As usual, the trick is to exhibit a cleverly chosen trial function. In [5], the leading order term in α is obtained by a trial function $\overline{\Psi}^{(n)}$ with only one photon. The idea to get the second order term is to add a two-photon component whose $\mathscr{L}^2$ norm is of the order of α. The choice of our trial function draws its inspiration from the formal perturbation method which is explained in Remark 3 above. More precisely, using (2.10), we define the sequence of trial wave functions

$$\begin{aligned} \Psi^{(n)} &= f_n \uparrow \otimes \Psi_\alpha \\ &= \overline{\Psi}^{(n)} + \alpha f_n \uparrow \otimes \mathscr{A}^{-1} [\sigma \cdot E^* + 2P_{\mathrm{f}} \cdot D^*] \mathscr{A}^{-1} \sigma \cdot E^* |0\rangle \\ &\quad - \alpha f_n \uparrow \otimes \mathscr{A}^{-1} D^* D^* |0\rangle \end{aligned} \tag{2.16}$$

with $\uparrow$ denoting the spin-up vector $(1,0)$ in $\mathbb{C}^2$, $f_n \in H^1(\mathbb{R}^3;\mathbb{R})$, $||f_n|| = 1$ and $||pf_n|| \to 0$ when n goes to infinity, and where

$$\overline{\Psi}^{(n)} = f_n \uparrow \otimes |0\rangle - \sqrt{\alpha} f_n \uparrow \otimes \mathscr{A}^{-1} \sigma \cdot E^* |0\rangle. \tag{2.17}$$

Let us already observe that the choice for the trial function will also appear more natural after the proof of the lower bound (see below the expected decomposition (2.32) and (2.34) — with $n = 0$—of a two-photon state close to the ground state).

We are going to check that

$$\lim_{n \to +\infty} \frac{(\Psi^{(n)}, T\Psi^{(n)})}{||\Psi^{(n)}||^2} = \mathscr{E}_1 \alpha + \mathscr{E}_2 \alpha^2 + \mathcal{O}(\alpha^3), \tag{2.18}$$

where

$$\mathscr{E}_1 = \pi^{-1} \Lambda^2 - \langle 0 | E \mathscr{A}^{-1} E^* | 0 \rangle \tag{2.19}$$

and

$$\begin{aligned}\mathscr{E}_2 = &- \langle 0 | DD\mathscr{A}^{-1} D^* D^* | 0 \rangle - \langle 0 | E\mathscr{A}^{-1} E \mathscr{A}^{-1} E^* \mathscr{A}^{-1} E^* | 0 \rangle \\ &- 4 \langle 0 | E \mathscr{A}^{-1} P_{\mathrm{f}} \cdot D \mathscr{A}^{-1} P_{\mathrm{f}} \cdot D^* \mathscr{A}^{-1} E^* | 0 \rangle \\ &- 2 \langle 0 | E \mathscr{A}^{-1} E \mathscr{A}^{-1} D^* D^* | 0 \rangle + \langle 0 | E \mathscr{A}^{-1} E^* | 0 \rangle || \mathscr{A}^{-1} E^* | 0 \rangle ||^2, \end{aligned} \tag{2.20}$$

respectively, denote the coefficient of α and α^2 in (1.11).

We first point out that, for any N-photon wave function φ_N, we have

$$L(f_n \otimes \mathscr{A}^{-1} \varphi_N) - f_n \otimes \varphi_N \to 0 \quad \text{in } H^{-1}(\mathbb{R}^3;\mathbb{R}) \otimes \mathscr{L}^2(\mathbb{R}^3, \mathbb{C}^2)^N\text{-weak}, \tag{2.21}$$

as n goes to infinity in virtue of the fact that $\lim_{n \to +\infty} ||pf_n|| = 0$, and since, by definition of L and $\mathscr{A}$,

$$L(f_n \otimes \mathscr{A}^{-1} \varphi_N) = f_n \otimes \varphi_N - 2pf_n \otimes P_{\mathrm{f}} \mathscr{A}^{-1} \varphi_N + p^2 f_n \otimes \mathscr{A}^{-1} \varphi_N. \tag{2.22}$$

Then, with the help of (2.11) and the fact that $||f_n|| = 1$, easy calculations yield

$$\begin{aligned}&(\Psi^{(n)}, T\Psi^{(n)}) \\ &= \alpha \pi^{-1} \Lambda^2 ||\Psi^{(n)}||^2 + ||pf_n||^2 + 2\alpha ||D\psi_1^{(n)}||^2 + 2\alpha ||D\psi_2^{(n)}||^2 \\ &\quad + (\psi_1^{(n)}, L\psi_1^{(n)}) + 2\sqrt{\alpha}\, \mathrm{Re}(F^* f_n \uparrow, \psi_1^{(n)}) \\ &\quad + (\psi_2^{(n)}, L\psi_2^{(n)}) + 2\sqrt{\alpha}\, \mathrm{Re}(F^* \psi_1^{(n)}, \psi_2^{(n)}) + 2\alpha\, \mathrm{Re}(D^* D^* f_n \uparrow, \psi_2^{(n)}) \\ &= \alpha \pi^{-1} \Lambda^2 ||\Psi^{(n)}||^2 - \alpha \langle 0 | E \mathscr{A}^{-1} E^* | 0 \rangle + o_n(1) + \mathcal{O}(\alpha^3) \end{aligned} \tag{2.23a}$$

$$+2\alpha^2||D\mathscr{A}^{-1}\sigma\uparrow\cdot E^*|0\rangle||^2-\alpha^2\langle 0|DD\mathscr{A}^{-1}D^*D^*|0\rangle \tag{2.23b}$$

$$-\alpha^2\langle 0|\sigma\uparrow\cdot E\mathscr{A}^{-1}F_{\rm f}\mathscr{A}^{-1}F_{\rm f}^*\mathscr{A}^{-1}\sigma\uparrow\cdot E^*|0\rangle \tag{2.23c}$$

$$+2\alpha^2\,\mathrm{Re}(L^{-1}F^*\mathscr{A}^{-1}\sigma\uparrow\cdot E^*f_n,D^*D^*f_n\uparrow), \tag{2.23d}$$

where $o_n(1)$ refers to a quantity that goes to 0 as n goes to infinity and is some error term coming from the fact that $\lim_{n\to+\infty}||pf_n||=0$, while $\mathcal{O}(\alpha^3)$ comes from the $\alpha||D\psi_2^{(n)}||^2$ term. The proof of the fact that

$$(2.23\mathrm{a})=-\alpha\langle 0|E\mathscr{A}^{-1}E^*|0\rangle+o_n(1)$$

is detailed in [5]. We first check that $D\psi_1^{(n)}=0$, or, equivalently,

$$D\mathscr{A}^{-1}\sigma\uparrow\cdot E^*|0\rangle=0. \tag{2.24}$$

This simply follows from the relation

$$\sum_{\lambda=1,2}\epsilon_i^\lambda\epsilon_j^\lambda=\delta_{i,j}-\frac{k_ik_j}{|k|^2}, \tag{2.25}$$

and the obvious observation that, for every $i\in\{1,2,3\}$,

$$D_i\mathscr{A}^{-1}\sigma\uparrow\cdot E^*|0\rangle=\sum_{j=1}^3\sigma_j\uparrow\sum_{\lambda=1,2}\int_{\mathbb{R}^3}\frac{G_i^\lambda(k)H_j^\lambda(k)}{|k|^2+|k|}\,\mathrm{d}k,$$

with the three vectors $\sigma_j\uparrow$, $j=1,2,3$, being linearly independent. Then, if $\boldsymbol{\varepsilon}^{\mathbf{j\,l\,n}}$ denotes the totally antisymmetric epsilon-tensor, we obtain, for every $i,j\in\{1,2,3\}$,

$$\sum_{\lambda=1,2}\int_{\mathbb{R}^3}\frac{G_i^\lambda(k)H_j^\lambda(k)}{|k|^2+|k|}\,\mathrm{d}k=\sum_{\lambda=1,2}\sum_{l,n=1}^3 i\int_{\mathbb{R}^3}\frac{\chi(|k|)\epsilon_i^\lambda(k)[\boldsymbol{\varepsilon}^{\mathbf{j\,l\,n}}\epsilon_l^\lambda(k)k_n]}{|k|^3+|k|^2}\,\mathrm{d}k$$
$$=\sum_{l,n=1}^3 i\int_{\mathbb{R}^3}\frac{\chi(|k|)[\delta_{i,l}-\frac{k_ik_l}{|k|^2}]\boldsymbol{\varepsilon}^{\mathbf{j\,l\,n}}k_n}{|k|^3+|k|^2}\,\mathrm{d}k=0. \tag{2.26}$$

Concerning (2.23d), we use the anti-commutation relations of the σ_j's and the fact that the functions $H^\lambda(k)$ belong to $(i\mathbb{R})^3$ while $G^\lambda(k)$ belong to $\mathbb{R}^3$ to check that

$$\mathrm{Re}(L^{-1}\mathscr{P}\cdot D^*\mathscr{A}^{-1}\sigma\cdot\uparrow\cdot E^*f_n,D^*D^*f_n\uparrow)=o_n(1),$$

and to deduce that

$$(2.23\mathrm{d})=2\alpha^2||f_n||^2\langle 0|E\mathscr{A}^{-1}E\mathscr{A}^{-1}D^*D^*|0\rangle+o_n(1).$$

We now turn to (2.23c) and check that

$$(2.23c) = -\alpha^2 \langle 0|E\mathscr{A}^{-1}E\mathscr{A}^{-1}E^*\mathscr{A}^{-1}E^*|0\rangle - 4\alpha^2 \langle 0|E\mathscr{A}^{-1}P_f \cdot D\mathscr{A}^{-1}P_f \cdot D^*\mathscr{A}^{-1}E^*|0\rangle, \tag{2.27}$$

since the cross term $\mathrm{Re}\langle 0|E\mathscr{A}^{-1}P_f \cdot D\mathscr{A}^{-1}E^*\mathscr{A}^{-1}E^*|0\rangle$ vanishes thanks again to the fact that G is real valued while H is purely imaginary.

The last second-order term which appears in (1.11) is easily recovered, once we have observed from (2.16) and (2.17) that

$$||\Psi^{(n)}||^2 = 1 + \alpha||\mathscr{A}^{-1}E^*|0\rangle||^2 + \mathcal{O}(\alpha^2).$$

Hence (2.18), by dividing the l.h.s. of (2.23) by $||\Psi^{(n)}||^2$.

2.2. *Lower bound for Σ_α*

The proof will be divided into two steps. First, in Section 2.2.1, we deduce a priori estimates for any state which is "close enough" to the ground state energy. Next in Section 2.2.2 we use these estimates to recover the α^2-term of the self-energy.

2.2.1. *A priori estimates*

Our first step will consist in improving a bit further the estimates in [5]. Indeed, we may choose a state Ψ in $\mathscr{H}$, close enough to the ground state, such that $||\Psi|| = 1$ and

$$\Sigma_\alpha \leqslant (\Psi, T\Psi) \leqslant \Sigma_\alpha + C\alpha^2 \leqslant \alpha\pi^{-1}\Lambda^2 - \alpha\langle 0|E\mathscr{A}^{-1}E^*|0\rangle + C\alpha^2, \tag{2.28}$$

where, here and below, C denotes a positive constant that is independent of α (but that might possibly dependent on Λ). We thus have as in [5]

$$\sum_{n\geqslant 0}(\psi_n, L\psi_n) \leqslant C\alpha, \tag{2.29}$$

hence

$$\sum_{n\geqslant 0}(\psi_n, (D^*D + E^*E)\psi_n) \leqslant C\alpha, \tag{2.30}$$

in virtue of Griesemer et al. [4, Lemma A.4]. We now observe that

$$\mathscr{E}_0[\psi_0, \psi_1] = -\alpha||L^{-1/2}F^*\psi_0||^2 + (h_1, Lh_1), \tag{2.31}$$

where

$$\psi_1 = -\sqrt{\alpha}L^{-1}F^*\psi_0 + h_1, \tag{2.32}$$

and that, for every $n \geqslant 0$,

$$\mathscr{E}[\psi_n, \psi_{n+1}, \psi_{n+2}] = -\alpha||L^{-1/2}F^*\psi_{n+1} + \sqrt{\alpha}L^{-1/2}D^*D^*\psi_n||^2 + (h_{n+2}, Lh_{n+2}), \tag{2.33}$$

where

$$\psi_{n+2} = -\sqrt{\alpha}L^{-1}F^*\psi_{n+1} - \alpha L^{-1}D^*D^*\psi_n + h_{n+2}. \tag{2.34}$$

Comparing with (2.11), we thus rewrite

$$(\Psi, T\Psi) = \alpha\Lambda^2\pi^{-1}||\Psi||^2 - \alpha||L^{-1/2}F^*\psi_0||^2 \tag{2.35a}$$

$$-\alpha\sum_{n\geqslant 0}||L^{-1/2}F^*\psi_{n+1} + \sqrt{\alpha}L^{-1/2}D^*D^*\psi_n||^2 \tag{2.35b}$$

$$+||p\psi_0||^2 + 2\alpha\sum_{n\geqslant 1}(\psi_n, D^*D\psi_n) + \sum_{n\geqslant 1}(h_n, Lh_n). \tag{2.35c}$$

Our first step will consist in observing that the estimates in [5] yield

$$\sum_{n\geqslant 1}(h_n, Lh_n) \leqslant C\alpha^2 \tag{2.36}$$

and

$$||p\psi_0||^2 \leqslant C\alpha^2, \tag{2.37}$$

thereby improving the estimate on the zeroth-order term in (2.29). These bounds will follow from the fact that only the terms in the first two lines of (2.35) contribute to recover the first to leading order term up to $\mathcal{O}(\alpha^2)$. Hence, all the (positive) terms in (2.35c) are at most of the order of α^2.

Indeed, on the one hand, we recall from [5] that

$$|\alpha(\sigma\cdot E^*\psi_0, L^{-1}\sigma\cdot E^*\psi_0) - \alpha||\psi_0||^2\langle 0|E\mathscr{A}^{-1}E^*|0\rangle| \leqslant C\alpha||p\psi_0||^2,$$

$$\mathrm{Re}(\sigma\cdot E^*\psi_0, L^{-1}\mathscr{P}\cdot D^*\psi_0) = 0$$

and

$$\alpha(\mathscr{P}\cdot D^*\psi_0, L^{-1}\mathscr{P}\cdot D^*\psi_0) \leqslant C\alpha||p\psi_0||^2.$$

Hence

$$|\alpha||L^{-1/2}F^*\psi_0||^2 - \alpha||\psi_0||^2\langle 0|E\mathscr{A}^{-1}E^*|0\rangle| \leqslant C\alpha||p\psi_0||^2. \tag{2.38}$$

Therefore, concerning the last term in (2.35a), we have

$$-\alpha||L^{-1/2}F^*\psi_0||^2 = -\alpha||\psi_0||^2\langle 0|E\mathscr{A}^{-1}E^*|0\rangle + \mathcal{O}(\alpha^2), \tag{2.39}$$

thanks to (2.29).

On the other hand, we now estimate the different terms in (2.35b), for every $n\geqslant 0$. More precisely,

$$\begin{aligned}(2.35\text{b}) &= -\alpha||L^{-1/2}F^*\psi_{n+1} + \sqrt{\alpha}L^{-1/2}D^*D^*\psi_n||^2\\ &= -\alpha||L^{-1/2}F^*\psi_{n+1}||^2 - \alpha^2(\psi_n, DDL^{-1}D^*D^*\psi_n) \qquad (2.40\text{a})\end{aligned}$$

$$-2\alpha^{3/2}\,\mathrm{Re}(F^*\psi_{n+1}, L^{-1}D^*D^*\psi_n). \tag{2.40b}$$

It is shown in [5], that

$$|||L^{-1/2}F^*\psi_{n+1}||^2 - ||\psi_{n+1}||^2\langle 0|E\mathscr{A}^{-1}E^*|0\rangle| \leqslant C(\psi_{n+1}, L\psi_{n+1}). \tag{2.41}$$

This follows from the three bounds

$$\begin{aligned}&|(\sigma\cdot E^*\psi_{n+1}, L^{-1}\sigma\cdot E^*\psi_{n+1}) - ||\psi_{n+1}||^2\langle 0|E\mathscr{A}^{-1}E^*|0\rangle|\\ &\quad\leqslant C(\psi_{n+1}, L\psi_{n+1}),\end{aligned}$$

$$(\mathscr{P}\cdot D^*\psi_{n+1}, L^{-1}\mathscr{P}\cdot D^*\psi_{n+1}) \leqslant C(\psi_{n+1}, L\psi_{n+1}) \tag{2.42}$$

and

$$|\mathrm{Re}(\mathscr{P}\cdot D^*\psi_{n+1}, L^{-1}\sigma\cdot E^*\psi_{n+1})| \leqslant C(\psi_{n+1}, H_{\mathrm{f}}\psi_{n+1}),$$

whose proofs are detailed in [5]. (See also the proof of Lemma B.1 in Appendix B, which follows the same patterns.) Moreover, from Lemma 2 in the appendix of [5],

$$\begin{aligned}&|\alpha^2(\psi_n, DDL^{-1}D^*D^*\psi_n) - \alpha^2||\psi_n||^2\langle 0|DD\mathscr{A}^{-1}D^*D^*|0\rangle|\\ &\quad\leqslant C\alpha^2(\psi_{n+1}, L\psi_{n+1}).\end{aligned} \tag{2.43}$$

Actually, only the upper bounds of (2.42) and (2.43) are proven in [5] which indeed suffices for the first-order term, but following the methods described in Appendix B estimates (2.42) and (2.43) are easily derived.

For (2.40b), we get from the proof of Lemma C.2 in Appendix C

$$\begin{aligned}&\alpha^{3/2}|(F^*\psi_{n+1}, L^{-1}D^*D^*\psi_n)|\\ &\quad\leqslant C\alpha^2||\psi_n||^2 + C\alpha(\psi_{n+1}, L\psi_{n+1}) + C\alpha(\psi_n, L\psi_n).\end{aligned} \tag{2.44}$$

Summing up (2.41), (2.43) and (2.44) over $n \geqslant 0$ and using (2.39) and (2.29), we first deduce from (2.35) that

$$\begin{aligned} &\alpha\pi^{-1}\Lambda^2 - \alpha\|\Psi\|^2\langle 0|E\mathscr{A}^{-1}E^*|0\rangle + \mathcal{O}(\alpha^2) \\ &\geqslant \Sigma_\alpha \geqslant (\Psi, T\Psi) + \mathcal{O}(\alpha^2) \\ &= \alpha\pi^{-1}\Lambda^2\|\Psi\|^2 - \alpha\|\Psi\|^2\langle 0|E\mathscr{A}^{-1}E^*|0\rangle + \mathcal{O}(\alpha^2) \\ &\quad + \|p\psi_0\|^2 + 2\alpha\sum_{n\geqslant 1}(\psi_n, D^*D\psi_n) + \sum_{n\geqslant 1}(h_n, Lh_n). \end{aligned}$$

Whence (2.36) and (2.37).

We now make use of these bounds to derive the second-order terms in (1.11).

2.2.2. *Recovering the α^2-terms*

As a first consequence of (2.37), we deduce from (2.38) that

$$-\alpha\|L^{-1/2}F^*\psi_0\|^2 = -\alpha\|\psi_0\|^2\langle 0|E\mathscr{A}^{-1}E^*|0\rangle + \mathcal{O}(\alpha^3). \tag{2.45}$$

It turns out that, although it was not necessary hitherto, we now have to introduce an infrared regularization as in [6] to deal with the terms in (2.35b) (or equivalently in (2.40a) and (2.40b)). Therefore, in definition (2.13) of $\mathscr{E}$ we replace the operator L by

$$L_\alpha \equiv L + \alpha^3,$$

and the extra term $\alpha^3\sum_{n\geqslant 2}\|\psi_n\|^2$ contributes as an additional $\mathcal{O}(\alpha^3)$ in (2.11). The definition of h_{n+1} has, of course, to be modified accordingly by replacing L^{-1} by L_α^{-1} in (2.34). We shall nevertheless keep the same notation for h_{n+1}, and we also emphasize the fact that bound (2.36) obviously remains true.

Keeping this minor modification in mind, we now go back to (2.35) and we shall now use decompositions (2.32) and (2.34) of ψ_{n+1}, $n\geqslant 1$, in terms of ψ_n, ψ_{n-1} and h_{n+1} to exhibit the remaining second-order terms, as guessed from the upper bound.

More precisely, the following quantity is now to be estimated:

$$\begin{aligned} &-\alpha\|L_\alpha^{-1/2}F^*\psi_{n+1} + \sqrt{\alpha}L_\alpha^{-1/2}D^*D^*\psi_n\|^2 \\ &\quad = -\alpha\|L_\alpha^{-1/2}F^*h_{n+1}\|^2 - \alpha^2\|L_\alpha^{-1/2}F^*L_\alpha^{-1}F^*\psi_n\|^2 \end{aligned} \tag{2.46a}$$

$$-\alpha^2\|L_\alpha^{-1/2}D^*D^*\psi_n\|^2 - \alpha^3\|L_\alpha^{-1/2}F^*L_\alpha^{-1}D^*D^*\psi_{n-1}\|^2 \tag{2.46b}$$

$$+2\alpha^2\,\mathrm{Re}(L_\alpha^{-1}F^*L_\alpha^{-1}F^*\psi_n, D^*D^*\psi_n) \tag{2.46c}$$

$$+2\alpha^{3/2}\,\mathrm{Re}(L_\alpha^{-1}F^*L_\alpha^{-1}F^*\psi_n, F^*h_{n+1}) \tag{2.46d}$$

$$-2\alpha^{3/2}\,\mathrm{Re}(L_\alpha^{-1}D^*D^*\psi_n, F^*h_{n+1}) \tag{2.46e}$$

$$-2\alpha^{5/2}\,\mathrm{Re}(L_\alpha^{-1}F^*L_\alpha^{-1}F^*\psi_n, F^*L_\alpha^{-1}D^*D^*\psi_{n-1}) \tag{2.46f}$$

$$-2\alpha^{5/2}\,\mathrm{Re}(L_\alpha^{-1}F^*L_\alpha^{-1}D^*D^*\psi_{n-1}, D^*D^*\psi_n) \tag{2.46g}$$

$$+2\alpha^2\,\mathrm{Re}(L_\alpha^{-1}F^*L_\alpha^{-1}D^*D^*\psi_{n-1}, F^*h_{n+1}), \tag{2.46h}$$

with here and below the convention that the terms containing ψ_{n-1} vanish for $n = 0$.

In order to lighten the presentation, the sequence of the proof has been organized as follows. The contributing terms in (2.46a) and (2.46c) are investigated in Appendix B and the terms in (2.46d)–(2.46h) are shown to be of higher order in Appendix C.

Admitting these lemmas for a while, we thus have from Lemmas B.2 and B.3 in Appendix B and (2.29) and (2.36),

$$\begin{aligned}(2.46a) = {} & -\alpha(1-\|\psi_0\|^2)\langle 0|E\mathscr{A}^{-1}E^*|0\rangle \\ & +\alpha^2\langle 0|E\mathscr{A}^{-1}E^*|0\rangle\|\mathscr{A}^{-1}E^*|0\rangle\|^2 - \alpha^2\langle 0|E\mathscr{A}^{-1}E\mathscr{A}^{-1}E^*\mathscr{A}^{-1}E^*|0\rangle \\ & -4\alpha^2\langle 0|E\mathscr{A}^{-1}\mathscr{P}_{\mathrm{f}}\cdot D\mathscr{A}^{-1}\mathscr{P}_{\mathrm{f}}\cdot D^*\mathscr{A}^{-1}E^*|0\rangle + \mathcal{O}(\alpha^{5/2}\ln(1/\alpha)).\end{aligned} \tag{2.47}$$

From (2.43) and (2.29) again, we identify the second order term in (2.46b); namely,

$$(2.46b) = -\alpha^2\langle 0|DD\mathscr{A}^{-1}D^*D^*|0\rangle + \mathcal{O}(\alpha^3), \tag{2.48}$$

since the second term in (2.46b) is easily checked to be $\mathcal{O}(\alpha^3)$. (Note that (2.43) remains true when L is replaced by L_α.)

The last contributing terms follows from Lemma B.4 and (2.29)

$$(2.46c) = 2\alpha^2\langle 0|E\mathscr{A}^{-1}E\mathscr{A}^{-1}D^*D^*|0\rangle + \mathcal{O}(\alpha^{5/2}\ln(1/\alpha)). \tag{2.49}$$

Finally, using the a priori estimates (2.29) and (2.36), and with the help of Lemmas C.1–C.5, we deduce that

$$(2.46d) + (2.46e) + (2.46f) + (2.46g) + (2.46h) = \mathcal{O}(\alpha^{5/2}\ln(1/\alpha)). \tag{2.50}$$

To deduce (1.11) we go back to (2.35). We simply bound from below the terms in (2.35c) by zero, and identify (2.35a) and (2.35b), by using (2.45) and by inserting (2.47)–(2.50) in (2.46).

Remark 4. It would be possible to improve the error estimates to $\mathcal{O}(\alpha^3)$, but we do not want to overburden the paper with too many estimates. We just mention as an example that, from the proof of the upper bound, we know that we may choose a state Ψ in $\mathscr{H}$, close enough to the ground state, such that $||\Psi|| = 1$ and

$$\Sigma_\alpha \leqslant (\Psi, T\Psi) \leqslant \Sigma_\alpha + C\alpha^3 \leqslant \alpha\pi^{-1}\Lambda^2 + \alpha\mathscr{E}_1 + \alpha^2\mathscr{E}_2 + \mathcal{O}(\alpha^3).$$

Then, arguing as in Section 2.2.1, we infer from (2.35) that actually

$$\sum_{n\geqslant 0} (h_{n+1}, Lh_{n+1}) + ||p\psi_0||^2 \leqslant C\alpha^{5/2}\ln(1/\alpha). \tag{2.51}$$

This new and better bound now helps to improve all error estimates on quantities which involve h_{n+1} and $||p\psi_0||^2$ (like (2.38), for example), and so on by a kind of bootstrap argument.

Remark 5. By means of the methods developed throughout the proof it is now possible to expand the self-energy up to any power of α, but unfortunately the number of estimates rapidly increase. We know from perturbation theory that to gain the α^3-term we just need to add the term

$$-\sqrt{\alpha}\mathscr{A}^{-1}(F + F^*)\psi_2 - \alpha\mathscr{A}^{-1}D^*D^*\psi_1 \tag{2.52}$$

and normalize the corresponding state. The one- and two-photon parts ψ_1 and ψ_2 are defined in the upper bound (see (2.16)). Notice that (2.52) also includes the one-photon term $\alpha^{3/2}\mathscr{A}^{-1}F(\mathscr{A}^{-1}F^*\mathscr{A}^{-1}E^* + \mathscr{A}^{-1}D^*D^*)|0\rangle$.

3. Proof of Theorem 2

To prove the theorem we will proceed similarly to [7] and check the binding condition of Griesemer et al. [4] for H_α. Namely, we will show that

$$\inf \operatorname{spec} H_\alpha < \Sigma_\alpha - \delta\alpha^2 + \mathcal{O}(\alpha^{5/2}\ln(1/\alpha)), \tag{3.1}$$

for some positive constant δ. To this end, we define a one and a two-photon state similar to the previous section to recover the self-energy, and we add an extra appropriately chosen one-photon component which involves the gradient of an electron function which is close to a zero-resonance state; that is, a radial solution of the equation

$$\psi(x) = -\frac{1}{4\pi}\int \frac{V(y)\psi(y)}{|x-y|}\,\mathrm{d}y. \tag{3.2}$$

Let r_0 denote the radius of the support of V, then, due to Newton's theorem,

$$\psi(x) = \frac{C}{|x|} \tag{3.3}$$

for $|x| \geqslant r_0$ and an appropriate constant C. Notice that ψ satisfies

$$-\Delta\psi + V(x)\psi = 0. \tag{3.4}$$

Due to elliptic regularity properties (see e.g. [11]), we infer that $\psi \in C^2(\mathbb{R}^3)$.

To make ψ an $\mathscr{L}^2$-function we are going to truncate it. It turns out to be reasonable to do so at distance $|x| \sim 1/\alpha$ from the origin. To this end, we take two functions $u(t)$ and $v(t)$ in $C^2(\mathbb{R})$ with $u^2 + v^2 = 1$, $u = 1$ for $t \in [0, 1]$ and $u = 0$ for $t \geqslant 2$, and we define

$$\psi_\epsilon(x) = \psi(x)u(\epsilon\alpha|x|). \tag{3.5}$$

Assume $1/(\epsilon\alpha) \geqslant 2r_0$, so

$$\psi_\epsilon(x) = \frac{C}{|x|}u(\epsilon\alpha|x|) \tag{3.6}$$

for $|x| \geqslant r_0$. Therefore, we may find positive constants C_1 and C_2, depending on r_0, such that

$$||p^2\psi_\epsilon||^2 \leqslant C_1||p\psi_\epsilon||^2 \leqslant \alpha\epsilon C_2||\psi_\epsilon||^2. \tag{3.7}$$

(Notice that $||\psi_\epsilon||^2 = C(\alpha\epsilon)^{-1}$.)

Throughout the previous section we have worked with the operator $A(0)$. Here, the Hamiltonian also depends on the electron variable x. In order to adapt the method developed in the previous section, we introduce again the unitary transform

$$U = e^{iP_{\mathrm{f}}\cdot x} \tag{3.8}$$

acting on the Hilbert space $\mathscr{H}$. When applied to a n-photons function φ_n we obtain $U\varphi_n = e^{i(\sum_{i=1}^n k_i)\cdot x}\varphi_n(x, k_1, \ldots, k_n)$.

Since $UpU^* = p - P_{\mathrm{f}}$ we infer the corresponding transform for the Hamiltonian H_α

$$UH_\alpha U^* = (p - P_{\mathrm{f}} + \sqrt{\alpha}A)^2 + \sqrt{\alpha}\sigma \cdot B + H_{\mathrm{f}} + V(x), \tag{3.9}$$

which we denote again by H_α. Notice that in the above equation $A = A(0)$ and $B = B(0)$.

We now define the trial function

$$\begin{aligned}\Psi_\epsilon = {} & \psi_\epsilon\uparrow - \sqrt{\alpha}\mathscr{A}^{-1}(\sigma\uparrow)E^*\psi_\epsilon - d\sqrt{\alpha}\mathscr{A}^{-1}\mathscr{P}\cdot D^*\psi_\epsilon - \alpha\mathscr{A}^{-1}D^*\cdot D^*\psi_\epsilon \\ & + \alpha\mathscr{A}^{-1}(\sigma\uparrow)E^*\mathscr{A}^{-1}(\sigma\uparrow)E^*\psi_\epsilon + 2\alpha\mathscr{A}^{-1}\mathscr{P}D^*\mathscr{A}^{-1}(\sigma\uparrow)E^*\psi_\epsilon,\end{aligned} \tag{3.10}$$

with $\mathscr{A} = P_{\mathrm{f}}^2 + H_{\mathrm{f}}$.

Comparing with the minimizing sequence for Σ_α in (2.16) and (2.17) we have replaced in (3.10) the mere electron function f_n by ψ_ϵ and have added an extra one-photon component $-d\sqrt{\alpha}\mathscr{A}^{-1}\mathscr{P}\cdot D^*\psi_\epsilon$, which will be responsible for lowering the energy, whereas the other one- and two-photon parts will help to recover Σ_α.

For short, we denote the one- and two-photon terms in Ψ_ϵ by ψ_1 and ψ_2, respectively. Obviously, the terms $(\psi_1, P_{\mathrm{f}}\cdot p\psi_1)$ and $(\psi_2, P_{\mathrm{f}}\cdot p\psi_2)$ vanish, which can be immediately seen by integrating over the field variables, having in mind (1.2) and the fact that $\mathscr{A}$ commutes with the reflection $k\to -k$.

By means of Schwarz' inequality and (3.7) we infer

$$|([2\sqrt{\alpha}p\cdot D^* + \sqrt{\alpha}\sigma\cdot E^*]\sqrt{\alpha}\mathscr{A}^{-1}p\cdot D^*\psi_\epsilon,\psi_2)| \\ + |(\psi_2, p_x^2\psi_2)| \leqslant ||\Psi_\epsilon||^2\mathcal{O}(\alpha^{5/2}). \tag{3.11}$$

Taking into account the negativity of V and the estimates in the proof of the upper bound in Section 2 we arrive at

$$(\Psi_\epsilon, H_\alpha\Psi_\epsilon) \leqslant (\psi_\epsilon, [p^2+V]\psi_\epsilon) - d\alpha(\psi_\epsilon, p\cdot D\mathscr{A}^{-1}p\cdot D^*\psi_\epsilon) \\ + \alpha d^2[(\psi_\epsilon, p\cdot D\mathscr{A}^{-1}p\cdot D^*\psi_\epsilon) + (\psi_\epsilon, p\cdot D\mathscr{A}^{-1}p^2\mathscr{A}^{-1}p\cdot D^*\psi_\epsilon)] \\ + [\Sigma_\alpha + \mathcal{O}(\alpha^{5/2}\ln(1/\alpha))]\,||\Psi_\epsilon||^2. \tag{3.12}$$

Using the Fourier transform we are able to evaluate explicitly

$$(\psi_\epsilon, p\cdot D\mathscr{A}^{-1}p\cdot D^*\psi_\epsilon) = \sum_{\lambda=1,2}\int |\hat{\psi}_\epsilon(l)|^2\frac{[G^\lambda(p)\cdot l]^2}{|p|^2+|p|}\,\mathrm{d}p\,\mathrm{d}l \\ = ||p\psi_\epsilon||^2\pi^{-1}\int_0^\Lambda\int_{-1}^1\frac{\chi(|p|)x^2}{1+|p|}\,\mathrm{d}x\,d|p| \\ = \frac{2}{3\pi}\ln(1+\Lambda)||p\psi_\epsilon||^2 \tag{3.13}$$

and analogously

$$(\psi_\epsilon, p\cdot D\mathscr{A}^{-1}p^2\mathscr{A}^{-1}p\cdot D^*\psi_\epsilon) = \frac{2}{3\pi}\ln(1+\Lambda)||p^2\psi_\epsilon||^2 \\ \leqslant C_1\frac{2}{3\pi}\ln(1+\Lambda)||p\psi_\epsilon||^2. \tag{3.14}$$

Minimizing the corresponding terms in (3.12) with respect to d, leads to the requirement $d=\frac{1}{2(C_1+1)}$.

Finally, it remains to choose an appropriate ϵ to guarantee that

$$(\psi_\epsilon, [p^2 + V]\psi_\epsilon) - \alpha \frac{\ln(1+\Lambda)}{6\pi(C_1+1)} ||p\psi_\epsilon||^2 < -\alpha\nu||p\psi||^2, \tag{3.15}$$

for some $\nu(\epsilon) > 0$. By IMS localization formula (see e.g. [2, Theorem 3.2])

$$(\psi_\epsilon, [p^2 + V]\psi_\epsilon) = (\psi, [p^2 + V]\psi) - (\psi v, [p^2 + V]\psi v) + (\psi, [|\nabla v|^2 + |\nabla u|^2]\psi). \tag{3.16}$$

The first term on the r.h.s. vanishes by assumption, the second one is positive, and the third one is bounded by

$$(\psi, [|\nabla v|^2 + |\nabla u|^2]\psi) \leqslant C(\epsilon\alpha)^2 \int_{2(\epsilon\alpha)^{-1} \geqslant |x| \geqslant (\epsilon\alpha)^{-1}} \frac{1}{|x|^2}\,\mathrm{d}x \leqslant C\alpha\epsilon,$$

the constant depending on $\max\{|v'(t)| + |u'(t)| \mid t \in [1,2]\}$. Since

$$||p\psi_\epsilon||^2 \geqslant ||p\psi||^2 - C\epsilon\alpha, \tag{3.17}$$

we obtain (3.15) for ϵ small enough. Consequently,

$$(\Psi_\epsilon, H_\alpha\Psi_\epsilon)/(\Psi_\epsilon, \Psi_\epsilon) \leqslant -\delta(\epsilon)\alpha^2 + \Sigma_\alpha + \mathcal{O}(\alpha^{5/2}\ln(1/\alpha)), \tag{3.18}$$

which implies our claim.

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Appendix A. Auxiliary operators

For convenience we introduce the operators

$$|D| = \sum_{\lambda=1,2} \int \frac{\chi(|k|)}{2\pi|k|^{1/2}} a_\lambda(k)\,\mathrm{d}k, \tag{A.1}$$

$$|E| = \sum_{\lambda=1,2} \int \frac{\chi(|k|)|k|^{1/2}}{2\pi} a_\lambda(k)\,\mathrm{d}k, \tag{A.2}$$

$$|X| = \sum_{\lambda=1,2} \int \frac{\chi(|k|)}{2\pi |k|^{1/2} [|k| + \alpha^3]^{1/2}} a_\lambda(k)\,\mathrm{d}k. \tag{A.3}$$

It is easily proved, using the commutation relations between the annihilation and creation operators, that

$$|X|\,|X|^* = |X|^*|X| + 2\pi^{-1}(\Lambda + 3\alpha^3 \ln(1/\alpha) - \alpha^3 \ln(\Lambda + \alpha^3)). \tag{A.4}$$

Moreover, analogously to Griesemer et al. [4, Lemma A.4] we obtain the following.

Lemma A.1. *For* (A.1)–(A.3) *we have*

$$|D|^*|D| \leqslant \frac{2}{\pi} \Lambda H_\mathrm{f}, \tag{A.5}$$

$$|E|^*|E| \leqslant \frac{2\pi}{3} \Lambda H_\mathrm{f}, \tag{A.6}$$

$$|X|^*|X| \leqslant C[|\ln(1/\alpha)| + |\ln(1+\Lambda)|] H_\mathrm{f}. \tag{A.7}$$

Remark 6. These newly defined operators now act on real functions. Nevertheless, to simplify the notation, we shall often write $|X|\psi$ instead of $|X|\,|\psi|$ for the $\mathbb{C}^2$-valued functions we are considering.

Proof. We only prove inequality (A.7). The proof for the other terms work similarly and is given in [4, Lemma A.4].

Take an arbitrary $\Psi \in \mathscr{H}$ and fix the photons number n. Then by means of Schwarz' inequality

$$\begin{aligned}(\psi_n, |X|^*|X|\psi_n) &\leqslant 2\left(\int \sqrt{\rho_{\psi_n}(k)|k|^{1/2}} \frac{\chi(|k|)}{|k|[|k| + \alpha^3]^{1/2}} \mathrm{d}k \right)^2 \\ &\leqslant C[|\ln(1/\alpha)| + |\ln(1+\Lambda)|] \int \rho_{\psi_n}(k)|k|\,\mathrm{d}k, \end{aligned} \tag{A.8}$$

since with the usual definition

$$\rho_{\psi_n}(k) = n \int |\psi_n(l, k, k_2, \dots, k_n)|^2\,\mathrm{d}l\,\mathrm{d}k_2 \dots \mathrm{d}k_n \tag{A.9}$$

for the one-photon density, we have

$$\int_{\mathbb{R}^3} \rho_{\psi_n}(k)|k|\,\mathrm{d}k = (\psi_n, H_\mathrm{f}\psi_n) \tag{A.10}$$

while

$$\int \frac{\chi(|k_{n+1}|)^2}{|k_{n+1}|^2(|k_{n+1}|+\alpha^3)}\,\mathrm{d}k_{n+1} \sim \ln(1/\alpha) \tag{A.11}$$

for α small enough. □

From now on, in order to lighten the notation, $\mathrm{d}^n k$ stands for $\mathrm{d}k_1 \dots \mathrm{d}k_n$.

Appendix B. Evaluation of the contributing terms in (2.46)

Recall our notation

$$\mathscr{P} = p - P_{\mathrm{f}}, \quad F = 2\mathscr{P} \cdot D + \sigma \cdot E. \tag{B.1}$$

In the momentum representation of the electron space, $\mathscr{P}$ is simply a multiplication operator and for short we use

$$\mathscr{P}\psi_n(l, k_1, \dots, k_n) = \left(l - \sum_{i=1}^{n} k_i\right)\psi_n =: \mathscr{P}_n \psi_n, \tag{B.2}$$

and similarly

$$H_{\mathrm{f}}\psi_n(l, k_1, \dots, k_n) = \sum_{i=1}^{n} |k_i|\psi_n =: H_{\mathrm{f}}^n \psi_n. \tag{B.3}$$

We shall also denote

$$L_\alpha^n = |\mathscr{P}_n|^2 + H_{\mathrm{f}}^n + \alpha^3.$$

For the sake of simplicity, we will use in the following the convention:

$$|H|^2 := \sum_{\lambda=1,2} |H^\lambda|^2, \quad |G|^2 := \sum_{\lambda=1,2} |G^\lambda|^2,$$

and additionally for all $a \in \mathbb{R}^3$

$$|a \cdot G|^2 := \sum_{\lambda=1,2} |a \cdot G^\lambda|^2.$$

These conventions are suggested by our definition of H and G.

Before evaluating in Lemma B.2 the first term in (2.46a), we need the following preliminary lemma.

Lemma B.1. *For every* $n \geqslant 0$,

$$\Big| \, ||L_\alpha^{-1} F^* \psi_n||^2 - ||\psi_n||^2 ||\mathscr{A}^{-1} E^* |0\rangle ||^2 \Big| \leqslant C\Big[\sqrt{\alpha}||\psi_n||^2 + \alpha^{-1/2}||\mathscr{P}\psi_n||^2 + \ln(1/\alpha)\,(\psi_n, H_{\mathrm{f}}\psi_n)\Big]. \tag{B.4}$$

Proof. The l.h.s. of (B.4) is the sum of three terms:

$$||L_\alpha^{-1} F^* \psi_n||^2 = ||L_\alpha^{-1} \sigma \cdot E^* \psi_n||^2 + 4||L_\alpha^{-1} \mathscr{P} \cdot D^* \psi_n||^2 + 4\,\mathrm{Re}(L_\alpha^{-1}\sigma \cdot E^*\psi_n, L_\alpha^{-1}\mathscr{P} \cdot D^*\psi_n). \tag{B.5}$$

Each term is separately investigated in the three steps below.

Step 1: The first term $||L_\alpha^{-1}\sigma \cdot E^*\psi_n||^2$ is the one which contributes, and we show that

$$| \, ||L_\alpha^{-1}\sigma \cdot E^*\psi_n||^2 - ||\psi_n||^2 \; ||\mathscr{A}^{-1}E^*|0\rangle||^2| \leqslant C[\sqrt{\alpha}||\psi_n||^2 + \alpha^{-1/2}||\mathscr{P}\psi_n||^2 + (\psi_n, H_{\mathrm{f}}\psi_n)].$$

This term is decomposed into a sum of two terms I_n and II_n, depending whether the same photon is created on both sides or not. Thanks to permutational symmetry and the anti-commutation relations of the Pauli matrices, they are, respectively, given by

$$\mathrm{I}_n = \int \frac{|H(k_{n+1})|^2 \, |\psi_n(l, k_1, \dots, k_n)|^2}{(|\mathscr{P}_{n+1}|^2 + H_{\mathrm{f}}^{n+1} + \alpha^3)^2} \, \mathrm{d}l \, \mathrm{d}k_1 \dots \mathrm{d}k_{n+1} \tag{B.6}$$

and

$$\mathrm{II}_n = n \sum_{i,j=1}^{3} \int \frac{(\sigma_j\psi_n(l, k_1, \dots, k_n), \sigma_i\psi_n(l, k_2, \dots, k_{n+1}))}{(|\mathscr{P}_{n+1}|^2 + H_{\mathrm{f}}^{n+1} + \alpha^3)^2} \times H_j(k_{n+1})\overline{H}_i(k_1) \, \mathrm{d}l \, \mathrm{d}k_1 \dots \mathrm{d}k_{n+1}, \tag{B.7}$$

where the ¯ in the second line above refers to the complex conjugate. We first evaluate II_n, for which it is simply checked that

$$\mathrm{II}_n \leqslant Cn \int \frac{|H(k_1)| \, |H(k_{n+1})|}{|k_{n+1}| \, |k_1|} \times |\psi_n(l, k_1, \dots, k_n)| \, |\psi_n(l, k_2, \dots, k_{n+1})| \, \mathrm{d}l \, \mathrm{d}^{n+1}k$$

$$\leqslant C \int \frac{\chi(|k|)}{|k|^2} \mathrm{d}k (\psi_n, H_{\mathrm{f}}\psi_n),$$

thanks to (A.9) and (A.10). We now examine $\mathrm{I}_n - ||\psi_n||^2 ||\mathscr{A}^{-1} E^* |0\rangle||^2$ and observe that

$$||\mathscr{A}^{-1} E^* |0\rangle|| = \int_{\mathbb{R}^3} \frac{|H(k)|^2}{(|k|^2 + |k|)^2} \, \mathrm{d}k.$$

We first write $L_\alpha^{n+1} = Q_{n+1} + |\mathscr{P}_n|^2 + H_\mathrm{f}^n + \alpha^3 - 2\mathscr{P}_n \cdot k_{n+1}$, with $Q_{n+1} = |k_{n+1}|^2 + |k_{n+1}|$. The following quantity is then to be evaluated:

$$\begin{aligned} &\mathrm{I}_n - ||\psi_n||^2 ||\mathscr{A}^{-1} E^* |0\rangle||^2 \\ &\quad = \int |H(k_{n+1})|^2 \, |\psi_n(l, k_1, \dots, k_n)|^2 \left[\frac{1}{(L_\alpha^{n+1})^2} - \frac{1}{Q_{n+1^2}} \right] \mathrm{d}l \, \mathrm{d}^{n+1} k. \end{aligned}$$

We now point out that

$$\frac{1}{(Q+b)^2} = \frac{1}{Q^2} - \frac{2b}{Q(Q+b)^2} - \frac{b^2}{Q^2 (Q+b)^2}, \tag{B.8}$$

apply this expression with $Q = Q_{n+1} + |\mathscr{P}_n|^2$ and $b = H_\mathrm{f}^n + \alpha^3 - 2\mathscr{P}_n \cdot k_{n+1}$, and insert the corresponding expression into (B.6). I_n then appears as a sum of three contributions

$$\begin{aligned} A_n &= \int \frac{|H(k_{n+1})|^2}{(|\mathscr{P}_n|^2 + Q_{n+1})^2} |\psi_n(l, k_1, \dots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+1} k, \\ B_n &= 2 \int \frac{|H(k_{n+1})|^2 (2\mathscr{P}_n \cdot k_{n+1} - H_\mathrm{f}^n - \alpha^3)}{(|\mathscr{P}_n|^2 + Q_{n+1})(L_\alpha^{n+1})^2} \, |\psi_n(l, k_1, \dots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+1} k \end{aligned}$$

and

$$C_n = \int \frac{|H(k_{n+1})|^2 (H_\mathrm{f}^n + \alpha^3 - 2\mathscr{P}_n \cdot k_{n+1})^2}{(|\mathscr{P}_n|^2 + Q_{n+1})^2 \, (L_\alpha^{n+1})^2} |\psi_n(l, k_1, \dots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+1} k.$$

First, applying again (B.8) with $Q = Q_{n+1}$ and $b = |\mathscr{P}_n|^2$, it is easily seen that

$$\left| A_n - ||\psi_n||^2 ||\mathscr{A}^{-1} E^* |0\rangle||^2 \right| \leqslant C \int \frac{\chi(|k_{n+1}|)}{|k_{n+1}|^2} \, \mathrm{d}k_{n+1} \, ||\mathscr{P}\psi_n||^2,$$

by using $\frac{|\mathscr{P}_n|^2}{|\mathscr{P}_n|^2+Q_{n+1}} \leqslant 1$. Concerning B_n, we get on the one hand

$$\int \frac{|H(k_{n+1})|^2 (H_{\mathrm{f}}^n + \alpha^3)}{(|\mathscr{P}_n|^2 + Q_{n+1})\,(L_\alpha^{n+1})^2} |\psi_n(l, k_1, \dots, k_n)|^2 \,\mathrm{d}l\, \mathrm{d}^{n+1}k$$
$$\leqslant C \int \frac{\chi(|k_{n+1}|)}{|k_{n+1}|^2} \mathrm{d}k_{n+1} \, [(\psi_n, H_{\mathrm{f}}\psi_n) + \alpha^3 ||\psi_n||^2],$$

while, on the other hand, and with the help of Schwarz' inequality,

$$\left| \int \frac{|H(k_{n+1})|^2 \, (\mathscr{P}_n \cdot k_{n+1})}{(|\mathscr{P}_n|^2 + Q_{n+1})\,(L_\alpha^{n+1})^2} |\psi_n(l, k_1, \dots, k_n)|^2 \,\mathrm{d}l\, \mathrm{d}^{n+1}k \right|$$
$$\leqslant C \int \frac{\chi(|k_{n+1}|)}{|k_{n+1}|} \mathrm{d}k_{n+1} \, ||\psi_n|| \, ||\mathscr{P}\psi_n||.$$

For C_n, using Young's inequality to deal with the cross term, we easily get

$$|C_n| \leqslant C \int \frac{\chi(|k_{n+1}|)}{|k_{n+1}|^2} \mathrm{d}k_{n+1} [(\psi_n, H_{\mathrm{f}}\psi_n) + \alpha^3 ||\psi_n||^2]$$
$$+ \; C \int \frac{\chi(|k_{n+1}|)}{|k_{n+1}|} \mathrm{d}k_{n+1} \, ||\mathscr{P}\psi_n||^2,$$

since $\frac{H_{\mathrm{f}}^n + \alpha^3}{L_\alpha^{n+1}} \leqslant 1$.

Step 2: We now show the following bound on the second diagonal term:

$$(L_\alpha^{-1} \mathscr{P} \cdot D^* \psi_n, L_\alpha^{-1} \mathscr{P} \cdot D^* \psi_n) \leqslant C \ln(1/\alpha) \, (\psi_n, L\psi_n). \tag{B.9}$$

This quantity is again the sum of two terms $\mathrm{I}_n + \mathrm{II}_n$. We first consider the "diagonal" term I_n for which the same photon is created in both sides. It is worth observing that, thanks to our choice of gauge for the potential vector A, $G^\lambda(k) \cdot k = 0$. Then, the first term is bounded from above by

$$\mathrm{I}_n \leqslant \int \frac{|G(k_{n+1})|^2 \, |\mathscr{P}_n|^2 \, |\psi_n(l, k_1, \dots, k_n)|^2}{(|\mathscr{P}_{n+1}|^2 + H_{\mathrm{f}}^{n+1} + \alpha^3)^2} \mathrm{d}l \, \mathrm{d}k_1 \dots \mathrm{d}k_{n+1}$$
$$\leqslant C \left(\int \frac{|G(k_{n+1})|^2}{|k_{n+1}| \, (|k_{n+1}| + \alpha^3)} \mathrm{d}k_{n+1} \right) \|\mathscr{P}\psi_n\|^2$$
$$\leqslant C \ln(1/\alpha) \, \|\mathscr{P}\psi_n\|^2,$$

in virtue of (A.11).

For the second term, we use $\frac{|\mathscr{P}|^2}{(|\mathscr{P}|^2+H_\mathrm{f}+\alpha^3)^2}\leqslant\frac{1}{2}(H_\mathrm{f}+\alpha^3)^{-1}$ and proceed as follows:

$$\mathrm{II}_n \leqslant n \sum_{\lambda=1,2} \int \frac{|G^\lambda(k_{n+1})|\,|\mathscr{P}_{n+1}|^2\,|G^\lambda(k_1)|}{(|\mathscr{P}_{n+1}|^2+H_\mathrm{f}^{n+1}+\alpha^3)^2} \times |\psi_n(l,k_1,\dots,k_n)|\,|\psi_n(l,k_2,\dots,k_{n+1})|\,\mathrm{d}l\,\mathrm{d}^{n+1}k$$

$$\leqslant C\,(\psi_n,|X|^*|X|\psi_n)\leqslant C\ln(1/\alpha)\,(\psi_n,H_\mathrm{f}\psi_n),$$

where the operator $|X|$ has been defined by (A.3) in Appendix A. (B.9) follows.

Step 3. Finally, we deal with the cross term in (B.5) and show that

$$|\mathrm{Re}(L_\alpha^{-1}\sigma\cdot E^*\psi_n, L_\alpha^{-1}\mathscr{P}\cdot D^*\psi_n)|\leqslant C\,(\psi_n,H_\mathrm{f}\psi_n).$$

Indeed, the term which corresponds to the case when one photon interacts with itself vanishes thanks to the fact that G is real-valued while H has purely imaginary components. Observe now that, thanks to

$$\frac{|\mathscr{P}|}{|\mathscr{P}|^2+H_\mathrm{f}+\alpha^3}\leqslant\frac{1}{2}(H_\mathrm{f}+\alpha^3)^{-1/2}\leqslant\frac{1}{2}H_\mathrm{f}^{-1/2}, \tag{B.10}$$

$$\frac{|\mathscr{P}|}{(|\mathscr{P}|^2+H_\mathrm{f}+\alpha^3)^2}\leqslant\frac{1}{2}(H_\mathrm{f}+\alpha^3)^{-3/2}\leqslant\frac{1}{2}H_\mathrm{f}^{-3/2},$$

and $(H_\mathrm{f}^{n+1})^{3/2}\geqslant|k_{n+1}|^{5/4}\,|k_1|^{1/4}$. Then the remaining part gives

$$|\mathrm{Re}(L_\alpha^{-1}\sigma\cdot E^*\psi_n, L_\alpha^{-1}\mathscr{P}\cdot D^*\psi_n)|\leqslant n\sum_{\lambda=1,2}\int\frac{|H^\lambda(k_{n+1})|\,|\mathscr{P}_{n+1}|\,|G^\lambda(k_1)|}{(L_\alpha^{n+1})^2} \times|\psi_n(l,k_1,\dots,k_n)|\,|\psi_n(l,k_2,\dots,k_{n+1})|\,\mathrm{d}l\,\mathrm{d}^{n+1}k$$

$$\leqslant C\int\frac{\chi(|k|)}{|k|^{5/2}}\,\mathrm{d}k(\psi_n,H_\mathrm{f}\psi_n).$$

Lemma B.1 follows collecting all above estimates. □

Let us now turn to the following.

Lemma B.2 (Evaluating the first term in (2.46a)).

$$-\alpha\sum_{n\geqslant0}||L_\alpha^{-1/2}F^*h_{n+1}||^2 = -\alpha(1-||\psi_0||^2)\,\langle 0|E\mathscr{A}^{-1}E^*|0\rangle + \alpha^2\langle 0|E\mathscr{A}^{-1}E^*|0\rangle||\mathscr{A}^{-1}E^*|0\rangle||^2 + \mathscr{O}(\alpha^{5/2}\ln(1/\alpha)). \tag{B.11}$$

Proof. As a direct consequence of (2.41) and (2.36), we first get

$$- \alpha \sum_{n\geqslant 0} ||L_\alpha^{-1/2} F^* h_{n+1}||^2 = -\alpha \left(\sum_{n\geqslant 0} ||h_{n+1}||^2 \right) \langle 0|E\mathscr{A}^{-1}E^*|0\rangle + \mathcal{O}(\alpha^3). \tag{B.12}$$

(Note that (2.41) remains true with L replaced with L_α.) Next, we show that

$$\sum_{n\geqslant 0} ||h_{n+1}||^2 = 1 - ||\psi_0||^2 - \alpha||\mathscr{A}^{-1}E^*|0\rangle||^2 + \mathcal{O}(\alpha^{3/2}\ln(1/\alpha)). \tag{B.13}$$

To this extent, using definitions (2.32) and (2.34) of h_{n+1}, we get

$$\begin{aligned}
&\sum_{n\geqslant 0} ||\psi_{n+1}||^2 \\
&= 1 - ||\psi_0||^2 \\
&= \sum_{n\geqslant 0} ||h_{n+1} - \sqrt{\alpha}\, L_\alpha^{-1} F^* \psi_n - \alpha\, L_\alpha^{-1} D^* D^* \psi_{n-1}||^2 \\
&= \sum_{n\geqslant 0} ||h_{n+1}||^2 + \alpha \sum_{n\geqslant 0} ||L_\alpha^{-1} F^* \psi_n||^2 - 2\sqrt{\alpha} \sum_{n\geqslant 0} \operatorname{Re}(h_{n+1}, L_\alpha^{-1} F^* \psi_n) \\
&\quad - 2\alpha \sum_{n\geqslant 0} \operatorname{Re}(h_{n+1}, L_\alpha^{-1} D^* D^* \psi_{n-1}) + \mathcal{O}(\alpha^{3/2}),
\end{aligned}$$

where $\mathcal{O}(\alpha^{3/2})$ comes both from the term $\alpha^2 \sum_{n\geqslant 0} ||L_\alpha^{-1} D^* D^* \psi_{n-1}||^2$, and from the term $\alpha^{3/2} \sum_{n\geqslant 0} \operatorname{Re}(L_\alpha^{-1} F^* \psi_n, L_\alpha^{-1} D^* D^* \psi_{n-1})$, which is of the order of $\alpha^{3/2}$, thanks to Schwarz' inequality and Lemma B.1 and the fact that

$$||L_\alpha^{-1} D^* D^* \psi_{n-1}||^2 \leqslant C(||\psi_{n-1}||^2 + \ln(1/\alpha)(\psi_{n-1}, H_{\mathrm{f}} \psi_{n-1})). \tag{B.14}$$

Indeed, the diagonal part is obviously bounded by

$$||\psi_{n-1}||^2 \int \frac{|G(k_{n+1})|^2 |G(k_{n+1})|^2}{(|k_{n+1}| + |k_{n+2}|)^2} \, \mathrm{d}k_{n+1} \, \mathrm{d}k_{n+2},$$

whereas the off-diagonal part is estimated by $(\psi_{n-1}, |X|^* |X| \psi_{n-1})$.

With the help of Lemma B.1 in Appendix B, we have

$$\alpha \sum_{n\geqslant 0} ||L_\alpha^{-1} F^* \psi_n||^2 = \alpha||\mathscr{A}^{-1}E^*|0\rangle||^2 + \mathcal{O}(\alpha^{3/2}).$$

Next, we prove that

$$\sqrt{\alpha} \sum_{n\geqslant 0} |(h_{n+1}, L_\alpha^{-1} F^* \psi_n)| \leqslant C\, \alpha^{3/2} \ln(1/\alpha). \tag{B.15}$$

Let us indicate the main lines of the proof (B.15). Thanks to the permutational symmetry of the photons variable, we have

$$
\begin{aligned}
&|(h_{n+1}, L_\alpha^{-1} F^* \psi_n)| \\
&\quad \leqslant \sqrt{n+1} \sum_{\lambda=1,2} \int \frac{[2|G^\lambda(k_{n+1}) \cdot \mathscr{P}_{n+1}| + |H^\lambda(k_{n+1})|]}{|\mathscr{P}_{n+1}|^2 + H_\mathrm{f}^{n+1} + \alpha^3} \\
&\qquad \times |h_{n+1}(l, k_1, \ldots, k_{n+1})| |\psi_n(l, k_1, \ldots, k_n)| \, \mathrm{d}l \, \mathrm{d}k_1 \ldots \mathrm{d}k_{n+1}.
\end{aligned}
$$

We begin with analyzing the term involving H which appears to be easier to deal with than the term involving G. This is due to the two facts that

$$
\frac{|H^\lambda(k_{n+1})|}{L_\alpha^{n+1}} \leqslant C \frac{\chi(|k_{n+1}|)}{|k_{n+1}|^{1/2}}, \tag{B.16}
$$

whereas

$$
\frac{|\mathscr{P}_{n+1} \cdot G^\lambda(k_{n+1})|}{L_\alpha^{n+1}} \leqslant C \frac{\chi(|k_{n+1}|)}{|k_{n+1}|^{1/2} (|k_{n+1}| + \alpha^3)^{1/2}} \tag{B.17}
$$

in virtue of (B.10).

On the one hand, using the fact that $|\mathscr{P}_{n+1}|^2 + H_\mathrm{f}^{n+1} + \alpha^3 \geqslant |k_{n+1}|$, the H-term may be bounded by

$$
\begin{aligned}
&\sqrt{n+1} \sum_{\lambda=1,2} \int \frac{|h_{n+1}(l, k_1, \ldots, k_{n+1})| \, |H^\lambda(k_{n+1})|}{|\mathscr{P}_{n+1}|^2 + H_\mathrm{f}^{n+1}} \\
&\quad \times |\psi_n(l, k_1, \ldots, k_n)| \, \mathrm{d}l \, \mathrm{d}^{n+1}k \\
&\qquad \leqslant C \sqrt{n+1} \int |h_{n+1}(l, k_1, \ldots, k_{n+1})| \, |k_{n+1}|^{1/2} \\
&\qquad \times |\psi_n(l, k_1, \ldots, k_n)| \frac{\chi(|k_{n+1}|)}{|k_{n+1}|} \, \mathrm{d}l \, \mathrm{d}^{n+1}k \\
&\qquad \leqslant C \, (h_{n+1}, H_\mathrm{f} h_{n+1})^{1/2} \, \|\psi_n\|,
\end{aligned} \tag{B.18}
$$

thanks to Schwarz' inequality. On the other hand, for the G-term, we shall make use of (B.10) to deduce the bound

$$
\begin{aligned}
&\sqrt{n+1} \sum_{\lambda=1,2} \int \frac{|h_{n+1}(l, k_1, \ldots, k_{n+1})| \, |G^\lambda(k_{n+1}) \cdot \mathscr{P}_{n+1}|}{|\mathscr{P}_{n+1}|^2 + H_\mathrm{f}^{n+1} + \alpha^3} \\
&\quad \times |\psi_n(l, k_1, \ldots, k_n)| \, \mathrm{d}l \, \mathrm{d}^{n+1}k
\end{aligned}
$$

$$\leqslant C\sqrt{n+1}\int \frac{|h_{n+1}(l,k_1,\ldots,k_{n+1})|\,|k_{n+1}|^{1/2}\,\chi(|k_{n+1}|)}{(|k_{n+1}|+\alpha^3)^{1/2}\,|k_{n+1}|}$$

$$\times |\psi_n(l,k_1,\ldots,k_n)|\,\mathrm{d}l\,\mathrm{d}^{n+1}k$$

$$\leqslant C\,(h_{n+1},H_{\mathrm{f}}h_{n+1})^{1/2}\left(\int \frac{\chi(|k_{n+1}|)}{|k_{n+1}|^2\,(|k_{n+1}|+\alpha^3)}\,\mathrm{d}k_{n+1}\right)^{1/2}\|\psi_n\|$$

$$\leqslant C\ln(1/\alpha)^{1/2}(h_{n+1},H_{\mathrm{f}}h_{n+1})^{1/2}\,\|\psi_n\|, \tag{B.19}$$

thanks to (A.11). Gathering together (B.18) and (B.19), we deduce that

$$|(h_{n+1},L_\alpha^{-1}F^*\psi_n)|\leqslant C\ln(1/\alpha)^{1/2}(h_{n+1},H_{\mathrm{f}}h_{n+1})^{1/2}\,\|\psi_n\|$$
$$\leqslant C\,\alpha\,\|\psi_n\|^2+C\ln(1/\alpha)\alpha^{-1}\,(h_{n+1},H_{\mathrm{f}}h_{n+1});$$

hence, (B.15) thanks to (2.36).

Finally, we bound the last term in a similar way by

$$\alpha\sum_{n\geqslant 0}|(h_{n+1},L_\alpha^{-1}D^*D^*\psi_{n-1})|\leqslant C\alpha^2\ln(1/\alpha). \tag{B.20}$$

Indeed, we recall that

$$D^*\cdot D^*\psi_{n-1}(l,k_1,\ldots,k_{n+1})=\frac{2}{\sqrt{n(n+1)}}\sum_{\lambda,\mu=1,2}\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}G^\lambda(k_i)\cdot G^\mu(k_j)$$
$$\times\,\psi_{n-1}(l,k_1,\ldots,\check{k}_i,\ldots,\check{k}_j,\ldots,k_{n+1}).$$

Thus, thanks to permutational symmetry and since $|\mathscr{P}_{n+1}|^2+H_{\mathrm{f}}^{n+1}+\alpha^3\geqslant 2\,(|k_n|+\alpha^3/2)^{1/2}\,(|k_{n+1}|+\alpha^3/2)^{1/2}$, we may bound this term as follows

$$|(h_{n+1},L_\alpha^{-1}D^*D^*\psi_{n-1})|$$
$$\leqslant\sum_{\lambda,\mu=1,2}\sqrt{n(n+1)}\int\frac{|G^\lambda(k_n)|\,|G^\mu(k_{n+1})|}{(|k_n|+\alpha^3/2)^{1/2}\,(|k_{n+1}|+\alpha^3/2)^{1/2}}$$
$$\times|\psi_{n-1}(l,k_1,\ldots,k_{n-1})|\,|h_{n+1}(l,k_1,\ldots,k_{n+1})|\,\mathrm{d}l\,\mathrm{d}k_1\ldots\mathrm{d}k_{n+1}$$
$$\leqslant C\,(|X|\,|h_{n+1}|,|X|^*\,|\psi_{n-1}|)$$
$$\leqslant C\big[\alpha^{-1}\ln(1/\alpha)\,(h_{n+1},H_{\mathrm{f}}h_{n+1})+\alpha\|\psi_{n-1}\|^2$$
$$+\,\alpha\ln(1/\alpha)\,(\psi_{n-1},H_{\mathrm{f}}\psi_{n-1})\big],$$

where the operator $|X|$ has been defined by (A.3) in Appendix A and where the last inequality follows from Schwarz' inequality, (A.4) and (A.7). Hence (B.20) thanks to (2.36).

Hence (B.13). Finally (B.11) follows by inserting (B.13) into (B.12). □

We now prove the following

Lemma B.3 (Evaluating the second term in (2.46a)). *For every* $n \geqslant 0$,

$$\Big| \, \|L_\alpha^{-1/2} F^* L_\alpha^{-1} F^* \psi_n\|^2 - \|\psi_n\|^2 \langle 0 | E \mathscr{A}^{-1} E \mathscr{A}^{-1} E^* \mathscr{A}^{-1} E^* | 0 \rangle$$

$$- \; 4 \, \|\psi_n\|^2 \langle 0 | E \mathscr{A}^{-1} \mathscr{P}_{\mathrm{f}} \cdot D \mathscr{A}^{-1} \mathscr{P}_{\mathrm{f}} \cdot D^* \mathscr{A}^{-1} E^* | 0 \rangle \Big|$$

$$\leqslant C \Big[\sqrt{\alpha} \|\psi_n\|^2 + \alpha^{-1/2} \|\mathscr{P} \psi_n\|^2 + \ln(1/\alpha)(\psi_n, L\psi_n) \Big]. \tag{B.21}$$

Proof. Thanks to the permutational symmetry, we have

$$\|L_\alpha^{-1/2} F^* L_\alpha^{-1} F^* \psi_n\|^2$$

$$= \sum_{\lambda,\mu=1,2} \sum_{i=1}^{n+1} \sum_{j=i+1}^{n+2} \sum_{\gamma,\gamma',v,v'=1}^{3} \int$$

$$\Bigg(\frac{(\overline{H}_\gamma^\lambda(k_{n+2})\sigma_\gamma + 2\mathscr{P}_{n+2} \cdot G^\lambda(k_{n+2}))(\overline{H}_{\gamma'}^\mu(k_{n+1})\sigma_{\gamma'} + 2\mathscr{P}_{n+1} \cdot G^\mu(k_{n+1}))}{L_\alpha^{n+2}(L_\alpha^{n+1})^2}$$

$$+ \frac{(\overline{H}_\gamma^\lambda(k_{n+1})\sigma_\gamma + 2\mathscr{P}_{n+2} \cdot G^\lambda(k_{n+1}))(\overline{H}_{\gamma'}^\mu(k_{n+2})\sigma_{\gamma'} + 2\overline{\mathscr{P}}_{n+1} \cdot G^\mu(k_{n+2}))}{L_\alpha^{n+2} \, L_\alpha^{n+1} \, \overline{L}_\alpha^{n+1}}$$

$$\times \psi_n(l, k_1, \ldots, k_n), (H_v^\lambda(k_i)\sigma_v + \mathscr{P}_{n+1} \cdot G^\lambda(k_i)) \, (H_{v'}^\mu(k_j)\sigma_{v'} + 2\mathscr{P}_{n+2} \cdot G^\mu(k_j))$$

$$\times \, \psi_n(l, k_1, \ldots, \check{k}_i, \ldots, \check{k}_j, \ldots, k_{n+2}) \Bigg) \, \mathrm{d}l \, \mathrm{d}^{n+2}k, \tag{B.22}$$

where $\overline{\mathscr{P}}_{n+1} = l - \sum_{i=1, \neq n+1}^{n+2} k_i$ and $\overline{L}_\alpha^{n+1} = \overline{\mathscr{P}}_{n+1}^2 + \sum_{i=1, \neq n+1}^{n+2} |k_i| + \alpha^3$. To avoid confusion corresponding to our notation we restrict our attention to the first term in (B.22). The proof of the second part works analogously. The first quantity in (B.22) is decomposed in a sum of three terms I_n, II_n and III_n, which correspond, respectively, to the cases $i = n+1$ and $j = n+2$, $i \neq n+1$ and $j = n+2$ and $i, j \notin \{n+1, n+2\}$. The terms will be respectively examined in the three steps below.

Step 1: We first consider the diagonal term I_n. We use the fact that H is complex valued while G is real valued to cancel all terms which involve an odd number of H's terms. In virtue of the anti-commutation properties of the Pauli matrices, we may write

$$\mathrm{I}_n = \int \frac{|H(k_{n+2})|^2 \, |H(k_{n+1})|^2}{L_\alpha^{n+2} \, (L_\alpha^{n+1})^2} |\psi_n(l, k_1, \ldots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k \tag{B.23}$$

$$+ \; 4 \int \frac{|H(k_{n+1})|^2 \, |\mathscr{P}_{n+2} \cdot G(k_{n+2})|^2}{L_\alpha^{n+2} \, (L_\alpha^{n+1})^2} |\psi_n(l, k_1, \ldots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k$$

$$+ \; 16 \int \frac{|\mathscr{P}_{n+1} \cdot G(k_{n+1})|^2 \, |\mathscr{P}_{n+2} \cdot G(k_{n+2})|^2}{L_\alpha^{n+2} \, (L_\alpha^{n+1})^2} |\psi_n(l, k_1, \ldots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k$$

$$+ \; 4 \int \frac{|\mathscr{P}_{n+1} \cdot G(k_{n+1})|^2 \, |H(k_{n+2})|^2}{L_\alpha^{n+2} \, (L_\alpha^{n+1})^2} |\psi_n(l, k_1, \ldots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k$$

$$+ \; 4 \sum_{\lambda,\mu=1,2} \int \frac{\mathscr{P}_{n+1} \cdot G^\lambda(k_{n+1}) \, \mathscr{P}_{n+2} \cdot G^\mu(k_{n+2}) \, H^\mu(k_{n+2}) \cdot H^\lambda(k_{n+1})}{L_\alpha^{n+2} \, (L_\alpha^{n+1})^2}$$

$$\times \, |\psi_n(l, k_1, \ldots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k. \tag{B.24}$$

The first two terms will be the contributing ones and we leave them temporarily apart. The three others are bounded as follows:

$$\int \frac{|G(k_{n+1})|^2 \, |\mathscr{P}_{n+2}|^2 |G(k_{n+2})|^2}{L_\alpha^{n+2} \, (L_\alpha^{n+1})^2} |\mathscr{P}_n|^2 \, |\psi_n(l, k_1, \ldots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k$$

$$\leqslant C \left(\int \frac{\chi(|k|)}{|k|^2} \mathrm{d}k \right)^2 ||\mathscr{P}\psi_n||^2,$$

by using that $\mathscr{P}_{n+1} \cdot G^\lambda(k_{n+1}) = \mathscr{P}_n \cdot G^\lambda(k_{n+1})$, similarly

$$\int \frac{|\mathscr{P}_n|^2 \, |G(k_{n+1})|^2 \, |H(k_{n+2})|^2}{L_\alpha^{n+2} (L_\alpha^{n+1})^2} |\psi_n(l, k_1, \ldots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k$$

$$\leqslant C \int \chi(|k_{n+2}|) \, \mathrm{d}k_{n+2} \int \frac{\chi(|k_{n+1}|)}{|k_{n+1}|^2 \, (|k_{n+1}| + \alpha^3)} \mathrm{d}k_{n+1} \, ||\mathscr{P}\psi_n||^2$$

$$\leqslant C \ln(1/\alpha) ||\mathscr{P}\psi_n||^2,$$

thanks to (A.11), and

$$\sum_{\lambda,\mu=1,2} \int \frac{|H^\lambda(k_{n+2})|\,|H^\mu(k_{n+1})|\,|\mathscr{P}_n|\,|G^\mu(k_{n+1})|\,|\mathscr{P}_{n+1}|\,|G^\lambda(k_{n+2})|}{L_\alpha^{n+2}(L_\alpha^{n+1})^2} \times |\psi_n(l,k_1,\dots,k_n)|^2 \mathrm{d}l\,\mathrm{d}^{n+2}k$$

$$\leqslant C \int \frac{\chi(|k_{n+2}|)}{|k_{n+2}|}\,\mathrm{d}k_{n+2} \int \frac{\chi(|k_{n+1}|)}{|k_{n+1}|^{3/2}}\,\mathrm{d}k_{n+1}\,||\psi_n||\,||\mathscr{P}\psi_n||$$

$$\leqslant C[\sqrt{\alpha}\,||\psi_n||^2 + \alpha^{-1/2}\,||\mathscr{P}\psi_n||^2],$$

with the help of (B.10). We now turn to (B.23) and check that

$$\left| \int \frac{|H(k_{n+2})|^2\,|H(k_{n+1})|^2}{L_\alpha^{n+2}(L_\alpha^{n+1})^2}\,|\psi_n(l,k_1,\dots,k_n)|^2\,\mathrm{d}l\,\mathrm{d}^{n+2}k - ||\psi_n||^2 \int \frac{|H(k_{n+2})|^2|H(k_{n+1})|^2}{Q_{n+2}(Q_{n+1})^2}\,\mathrm{d}k_{n+1}\,\mathrm{d}k_{n+2} \right|$$

$$\leqslant C[\sqrt{\alpha}\,||\psi_n||^2 + \alpha^{-1/2}\,||\mathscr{P}\psi_n||^2 + \ln(1/\alpha)(\psi_n, H_\mathrm{f}\psi_n)], \tag{B.25}$$

with $Q_{n+2} = |k_{n+2} + k_{n+1}|^2 + |k_{n+2}| + |k_{n+1}|$ and $Q_{n+1} = |k_{n+1}|^2 + |k_{n+1}|$. Observe that

$$\langle 0|E\mathscr{A}^{-1}E\mathscr{A}^{-1}E^*\mathscr{A}^{-1}E^*|0\rangle = \int \frac{|H(k_{n+2})|^2\,|H(k_{n+1})|^2}{Q_{n+2}(Q_{n+1})^2}\,\mathrm{d}k_{n+1}\mathrm{d}k_{n+2} + \int \frac{|H(k_{n+2})|^2\,|H(k_{n+1})|^2}{Q_{n+2}Q_{n+1}(|k_{n+2}|^2 + |k_{n+2}|)}\,\mathrm{d}k_{n+1}\,\mathrm{d}k_{n+2}.$$

We first apply (B.8) to $(L_\alpha^{n+1})^2$ with $Q = Q_{n+1} + |\mathscr{P}_n|^2$ and $b = -2k_{n+1}\cdot\mathscr{P}_n + H_\mathrm{f}^n + \alpha^3$. By simple arguments which are very similar to those used in the course of the proof of Lemma B.1 above (that we skip to reduce the length of the calculations), we check that

$$\left| \int \frac{|H(k_{n+2})|^2\,|H(k_{n+1})|^2}{L_\alpha^{n+2}(L_\alpha^{n+1})^2}\,|\psi_n(l,k_1,\dots,k_n)|^2\,\mathrm{d}l\,\mathrm{d}^{n+2}k - \int \frac{|H(k_{n+2})|^2\,|H(k_{n+1})|^2}{L_\alpha^{n+2}(Q_{n+1} + |\mathscr{P}_n|^2)^2}\,|\psi_n(l,k_1,\dots,k_n)|^2\,\mathrm{d}l\,\mathrm{d}^{n+2}k \right|$$

$$\leqslant C[\sqrt{\alpha}\,||\psi_n||^2 + \alpha^{-1/2}\,||\mathscr{P}\psi_n||^2 + (\psi_n, H_\mathrm{f}\psi_n)].$$

Next, we apply

$$\frac{1}{Q+b} = \frac{1}{Q} - \frac{b}{Q(Q+b)} \tag{B.26}$$

to L_α^{n+2} with $Q = Q_{n+2}$ and $b = -2(k_{n+2}+k_{n+1})\cdot\mathscr{P}_n + |\mathscr{P}_n|^2 + H_\mathrm{f}^n + \alpha^3$ and obtain that

$$\left|\int \frac{|H(k_{n+2})|^2\,|H(k_{n+1})|^2}{L_\alpha^{n+2}(Q_{n+1}+|\mathscr{P}_n|^2)^2}\,|\psi_n(l,k_1,\dots,k_n)|^2\,\mathrm{d}l\,\mathrm{d}^{n+2}k\right.$$
$$\left. - \int \frac{|H(k_{n+2})|^2\,|H(k_{n+1})|^2}{Q_{n+2}(Q_{n+1}+|\mathscr{P}_n|^2)^2}\,|\psi_n(l,k_1,\dots,k_n)|^2\,\mathrm{d}l\,\mathrm{d}^{n+2}k\right|$$
$$\leqslant C[\sqrt{\alpha}\,||\psi_n||^2 + \alpha^{-1/2}\,||\mathscr{P}\psi_n||^2 + (\psi_n, H_\mathrm{f}\psi_n)].$$

Finally, applying again (B.8) with $Q = Q_{n+1}$ and $b = |\mathscr{P}_n|^2$, we get

$$\left|\int \frac{|H(k_{n+2})|^2\,|H(k_{n+1})|^2}{Q_{n+2}(Q_{n+1}+|\mathscr{P}_n|^2)^2}\,|\psi_n(l,k_1,\dots,k_n)|^2\,\mathrm{d}l\,\mathrm{d}^{n+2}k\right.$$
$$\left. - \int \frac{|H(k_{n+2})|^2\,|H(k_{n+1})|^2}{Q_{n+2}(Q_{n+1})^2}\,|\psi_n(l,k_1,\dots,k_n)|^2\,\mathrm{d}l\,\mathrm{d}^{n+2}k\right|$$
$$\leqslant C[\sqrt{\alpha}\,||\psi_n||^2 + \alpha^{-1/2}\,||\mathscr{P}\psi_n||^2 + (\psi_n, H_\mathrm{f}\psi_n)].$$

The proof of (B.25) is then over and we now regard the term in (B.24) and show that

$$\left|\int \frac{|H(k_{n+1})|^2\,|\mathscr{P}_{n+2}\cdot G(k_{n+2})|^2}{L_\alpha^{n+2}(L_\alpha^{n+1})^2}\,|\psi_n(l,k_1,\dots,k_n)|^2\,\mathrm{d}l\,\mathrm{d}^{n+2}k\right.$$
$$\left. - \,||\psi_n||^2 \int \frac{|(k_{n+2}+k_{n+1})\cdot G(k_{n+2})|^2|H(k_{n+1})|^2}{Q_{n+2}(Q_{n+1})^2}\,\mathrm{d}k_{n+1}\mathrm{d}k_{n+2}\right|$$
$$\leqslant C[\sqrt{\alpha}\,||\psi_n||^2 + \alpha^{-1/2}\,||\mathscr{P}\psi_n||^2 + (\psi_n, H_\mathrm{f}\psi_n)], \tag{B.27}$$

where

$$\langle 0|E\mathscr{A}^{-1}\mathscr{P}_\mathrm{f}\cdot D\mathscr{A}^{-1}\mathscr{P}_\mathrm{f}\cdot D^*\mathscr{A}^{-1}E^*|0\rangle$$
$$= \int \frac{|(k_{n+2}+k_{n+1})\cdot G(k_{n+2})|^2|H(k_{n+1})|^2}{Q_{n+2}(Q_{n+1})^2}\,\mathrm{d}k_{n+1}\,\mathrm{d}k_{n+2}$$
$$+ \int \frac{|(k_{n+2}+k_{n+1})\cdot G(k_{n+2})|^2|H(k_{n+1})|^2}{Q_{n+2}\,Q_{n+1}(|k_{n+2}|^2+|k_{n+2}|)}\,\mathrm{d}k_{n+1}\,\mathrm{d}k_{n+2}.$$

The proof is exactly the same as for (B.25), therefore we only sketch the main lines. Applying (B.8) to $(L_\alpha^{n+1})^2$ with $Q = |\mathscr{P}_n|^2 + Q_{n+1}$ and $b = -2\mathscr{P}_n \cdot k_{n+1} + H_f^n + \alpha^3$, we first arrive at

$$\left| \int \frac{|H(k_{n+1})|^2\, |\mathscr{P}_{n+2} \cdot G(k_{n+2})|^2}{L_\alpha^{n+2}(L_\alpha^{n+1})^2} |\psi_n(l, k_1, \dots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k \right.$$
$$\left. - \int \frac{|H(k_{n+1})|^2\, |\mathscr{P}_{n+2} \cdot G(k_{n+2})|^2}{L_\alpha^{n+2}(Q_{n+1} + |\mathscr{P}_n|^2)^2} |\psi_n(l, k_1, \dots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k \right|$$
$$\leqslant C[\sqrt{\alpha}\, ||\psi_n||^2 + \alpha^{-1/2}\, ||\mathscr{P}\psi_n||^2 + (\psi_n, H_f \psi_n)].$$

Next, again from (B.26), with $Q = Q_{n+1}$ and $b = |\mathscr{P}_n|^2$, we obtain

$$\left| \int \frac{|H(k_{n+1})|^2\, |\mathscr{P}_{n+2} \cdot G(k_{n+2})|^2}{L_\alpha^{n+2}(Q_{n+1} + |\mathscr{P}_n|^2)^2} |\psi_n(l, k_1, \dots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k \right.$$
$$\left. - \int \frac{|H(k_{n+1})|^2\, |\mathscr{P}_{n+2} \cdot G(k_{n+2})|^2}{L_\alpha^{n+2}(Q_{n+1})^2} |\psi_n(l, k_1, \dots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k \right|$$
$$\leqslant C||\mathscr{P}\psi_n||^2,$$

and we use (B.26) with $Q = Q_{n+2}$ and $b = -2\mathscr{P}_n \cdot (k_{n+1} + k_{n+2}) + |\mathscr{P}_n|^2 + H_f^n + \alpha^3$ to get

$$\left| \int \frac{|H(k_{n+1})|^2\, |\mathscr{P}_{n+2} \cdot G(k_{n+2})|^2}{L_\alpha^{n+2}(Q_{n+1})^2} |\psi_n(l, k_1, \dots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k \right.$$
$$\left. - \int \frac{|H(k_{n+1})|^2\, |\mathscr{P}_{n+2} \cdot G(k_{n+2})|^2}{Q_{n+2}(Q_{n+1})^2} |\psi_n(l, k_1, \dots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k \right|$$
$$\leqslant C[\sqrt{\alpha}\, ||\psi_n||^2 + \alpha^{-1/2}\, ||\mathscr{P}\psi_n||^2 + (\psi_n, H_f \psi_n)].$$

Finally, since $\mathscr{P}_{n+2} = \mathscr{P}_n - (k_{n+1} + k_{n+2})$ and $G^\lambda(k_{n+2}) \cdot k_{n+2} = 0$, we obtain

$$\int \frac{|H(k_{n+1})|^2\, |\mathscr{P}_{n+2} \cdot G(k_{n+2})|^2}{Q_{n+2}(Q_{n+1})^2} |\psi_n(l, k_1, \dots, k_n)|^2 \, \mathrm{d}l \, \mathrm{d}^{n+2}k$$
$$= ||\psi_n||^2 \int \frac{|H(k_{n+1})|^2\, |(k_{n+1} + k_{n+2}) \cdot G(k_{n+2})|^2}{Q_{n+2}(Q_{n+1})^2} \mathrm{d}k_{n+1} \mathrm{d}k_{n+2}$$

$$+\ 2\sum_{\lambda=1,2}\int \frac{|H(k_{n+1})|^2(k_{n+1}\cdot G^\lambda(k_{n+2}))(\mathscr{P}_n\cdot G^\lambda(k_{n+2}))}{Q_{n+2}(Q_{n+1})^2}$$
$$\times|\psi_n(l,k_1,\dots,k_n)|^2\mathrm{d}l\,\mathrm{d}^{n+2}k$$
$$+\int\frac{|H(k_{n+1})|^2\,|\mathscr{P}_n\cdot G(k_{n+2})|^2}{Q_{n+2}(Q_{n+1})^2}|\psi_n(l,k_1,\dots,k_n)|^2\,\mathrm{d}l\,\mathrm{d}^{n+2}k.$$

The second term in the r.h.s. vanishes when integrated first with respect to k_{n+1} since H and Q_{n+1} are radially symmetric functions, whereas the second term is easily bounded by

$$C\int\frac{\chi(|k_{n+2}|)}{|k_{n+2}|^2}\mathrm{d}k_{n+2}\int\frac{\chi(|k_{n+1}|)}{|k_{n+1}|}\mathrm{d}k_{n+1}||\mathscr{P}\psi_n||^2.$$

This concludes the proof of (B.27).

Step 2: We now regard the term II_n which, thanks to permutational symmetry, can be bounded by

$$|\mathrm{II}_n|\leqslant Cn\sum_{\lambda,\mu=1,2}\int\frac{(|H^\mu(k_{n+2})|+2|\mathscr{P}_{n+2}\cdot G^\mu(k_{n+2})|)^2}{L_\alpha^{n+2}(L_\alpha^{n+1})^2}$$
$$\times(|H^\lambda(k_{n+1})|+2|\mathscr{P}_{n+1}\cdot G^\lambda(k_{n+1})|)(|H^\lambda(k_1)|+2|\mathscr{P}_{n+1}\cdot G^\lambda(k_1)|)$$
$$\times|\psi_n(l,k_1,\dots,k_n)|\,|\psi_n(l,k_2,\dots,k_{n+1})|\,\mathrm{d}l\,\mathrm{d}^{n+2}k.$$

We are going to show that

$$|\mathrm{II}_n|\leqslant C\ln(1/\alpha)(\psi_n,H_\mathrm{f}\psi_n).$$

First observe that it is enough to study the case of $|H(k_{n+2})|^2+4|\mathscr{P}_{n+2}\cdot G(k_{n+2})|^2$. Since

$$\frac{|H(k_{n+2})|^2}{L_\alpha^{n+2}(L_\alpha^{n+1})^2}\leqslant C\frac{\chi(|k_{n+2}|)}{(L_\alpha^{n+1})^2},$$

whereas, using $\mathscr{P}_{n+2}\cdot G^\lambda(k_{n+2})=\mathscr{P}_{n+1}\cdot G^\lambda(k_{n+2})$,

$$\frac{|\mathscr{P}_{n+2}\cdot G(k_{n+2})|^2}{L_\alpha^{n+2}(L_\alpha^{n+1})^2}\leqslant C\frac{\chi(|k_{n+2}|)}{|k_{n+2}|^2\,L_\alpha^{n+1}},$$

in virtue of (B.10), it is easily seen that the $|H|^2$ contribution is the most delicate to handle since it involves a higher power of $|k_1|+|k_{n+1}|$ at the denominator. We thus concentrate on this term. Moreover, comparing (B.16) and (B.17) it is easily seen that the "worse" term may be bounded as follows:

$$
\begin{aligned}
& n \sum_{\lambda=1,2} \int \frac{|\mathscr{P}_{n+1} \cdot G^{\lambda}(k_{n+1})|\, |\mathscr{P}_{n+1} \cdot G^{\lambda}(k_1)|}{(L_{\alpha}^{n+1})^2} \\
& \quad \times |\psi_n(l, k_1, \dots, k_n)|\, |\psi_n(l, k_2, \dots, k_{n+1})\, \mathrm{d}l\, \mathrm{d}^{n+1}k \\
& \quad \leqslant Cn \sum_{\lambda=1,2} \int \frac{|G^{\lambda}(k_{n+1})|\, |G^{\lambda}(k_1)|}{|k_{n+1}|^{1/2}\, (|k_{n+1}| + \alpha^3)^{1/2} |k_1|^{1/2}\, (|k_1| + \alpha^3)^{1/2}} \\
& \qquad \times |\psi_n(l, k_1, \dots, k_n)|\, |\psi_n(l, k_2, \dots, k_{n+1})|\, \mathrm{d}l\, \mathrm{d}^{n+1}k \\
& \quad \leqslant C \ln(1/\alpha)(\psi_n, H_{\mathrm{f}} \psi_n),
\end{aligned}
$$

thanks to Schwarz' inequality and (A.11).

Step 3: We finally consider the full off-diagonal term that we first roughly bound by

$$
\begin{aligned}
|\mathrm{III}_n| \leqslant & \; C\, n(n-1) \sum_{\lambda,\mu=1,2} \\
& \int \frac{(|H^{\lambda}(k_{n+2})| + |\mathscr{P}_{n+2} \cdot G^{\lambda}(k_{n+2})|)\, (|H^{\mu}(k_{n+1})| + |\mathscr{P}_{n+1} \cdot G^{\mu}(k_{n+1})|)}{L_{\alpha}^{n+2}\, (L_{\alpha}^{n+1})^2} \\
& \quad \times (|H^{\lambda}(k_1)| + |\mathscr{P}_{n+2} \cdot G^{\lambda}(k_1)|)\, (|H^{\mu}(k_2)| + |\mathscr{P}_{n+1} \cdot G^{\mu}(k_2)|) \\
& \quad \times |\psi_n(l, k_1, \dots, k_n)|\, |\psi_n(l, k_3, \dots, k_{n+2})|\, \mathrm{d}l\, \mathrm{d}^{n+2}k.
\end{aligned}
$$

The term only involving the H's is bounded by

$$
\begin{aligned}
|\mathrm{III}_n| \leqslant & \; Cn(n-1) \sum_{\lambda,\mu=1,2} \int \frac{|H^{\lambda}(k_{n+2})|\, |H^{\mu}(k_{n+1})|\, |H^{\lambda}(k_1)| |H^{\mu}(k_2)|}{H_{\mathrm{f}}^{n+1} |k_2| |k_{n+1}|} \\
& \quad \times |\psi_n(l, k_1, \dots, k_n)|\, |\psi_n(l, k_3, \dots, k_{n+2})|\, \mathrm{d}l\, \mathrm{d}^{n+2}k \\
\leqslant & \; C\, \| |E| H_{\mathrm{f}}^{-1/2} |D| |\psi_n| \|^2 \leqslant C(\psi_n, H_{\mathrm{f}} \psi_n),
\end{aligned}
$$

and the corresponding term with the G's reads

$$
\begin{aligned}
|\mathrm{III}_n| \leqslant & \; C\, n(n-1) \sum_{\lambda,\mu=1,2} \int \frac{|G^{\lambda}(k_{n+2})|\, |G^{\mu}(k_{n+1})|\, |G^{\lambda}(k_1)|\, |G^{\mu}(k_2)|}{L_{\alpha}^{n+2} (L_{\alpha}^{n+1})^2} \\
& \quad \times |\mathscr{P}_{n+1}|^2\, |\mathscr{P}_{n+1}|^2\, |\psi_n(l, k_1, \dots, k_n)|\, |\psi_n(l, k_3, \dots, k_{n+2})|\, \mathrm{d}l\, \mathrm{d}^{n+2}k
\end{aligned}
$$

$$\leqslant Cn(n-1)\sum_{\lambda,\mu=1,2}\int\frac{|G^{\lambda}(k_{n+2})|\,|G^{\mu}(k_{n+1})|\,|G^{\lambda}(k_1)|\,|G^{\mu}(k_2)|}{L_{\alpha}^{n+1}}$$

$$\times\,|\psi_n(l,k_1,\dots,k_n)|\,|\psi_n(l,k_3,\dots,k_{n+2})|\,\mathrm{d}l\,\mathrm{d}^{n+2}k$$

$$\leqslant C\,\||D|H_{\mathrm{f}}^{-1/2}|D||\psi_n|\|^2\leqslant C(\psi_n,H_{\mathrm{f}}\psi_n).$$

The mixed terms then are estimated by means of Schwarz' inequality. □

Finally, we recover the last contributing term by proving the following.

Lemma B.4 (Evaluating the term in (2.46c)). *For every* $n\geqslant 0$,

$$|\,\mathrm{Re}(L_{\alpha}^{-1}F^*L_{\alpha}^{-1}F^*\psi_n,D^*D^*\psi_n)-\|\psi_n\|^2\langle 0|E\mathscr{A}^{-1}E\mathscr{A}^{-1}D^*D^*|0\rangle|$$

$$\leqslant C[\alpha^{-1/2}\ln(1/\alpha)\,(\psi_n,L\psi_n)+\sqrt{\alpha}\|\psi_n\|^2]. \tag{B.28}$$

Proof. *Step* 1: We first observe that, by Schwarz' inequality,

$$|(L_{\alpha}^{-1}F^*L_{\alpha}^{-1}\mathscr{P}\cdot D^*\psi_n,D^*D^*\psi_n)|$$

$$\leqslant C\|L_{\alpha}^{-1}\mathscr{P}\cdot D^*\psi_n\|\,\|FL_{\alpha}^{-1}D^*D^*\psi_n\|$$

$$\leqslant C\alpha^{-1/2}\|L_{\alpha}^{-1}\mathscr{P}\cdot D^*\psi_n\|^2+C\sqrt{\alpha}\,\|FL_{\alpha}^{-1}D^*D^*\psi_n\|^2$$

$$\leqslant C[\alpha^{-1/2}\ln(1/\alpha)\,(\psi_n,L\psi_n)+\sqrt{\alpha}\,\|\psi_n\|^2+\sqrt{\alpha}(\psi_n,H_{\mathrm{f}}\psi_n)],$$

thanks to (B.9) and since the other $\mathscr{L}^2$ norm is easily checked to be bounded due to the fact that

$$F^*F\leqslant C\,(H_{\mathrm{f}}+|\mathscr{P}|^2\,H_{\mathrm{f}})$$

in virtue of Griesemer et al. [4, Lemma A.4].

Step 2: We now look at the term

$$\mathrm{Re}(L_{\alpha}^{-1}\mathscr{P}\cdot D^*L_{\alpha}^{-1}\sigma\cdot E^*\psi_n,D^*D^*\psi_n)$$

$$=\ 2\sum_{\lambda,\mu=1,2}\ \sum_{\gamma=1}^{3}$$

$$\times\,\mathrm{Re}\int\frac{\mathscr{P}_{n+2}\cdot G^{\lambda}(k_{n+2})\overline{H}_{\gamma}^{\mu}(k_{n+1})\ \sum_{i=1}^{n+1}\ \sum_{j=i+1}^{n+2}\ G^{\lambda}(k_i)\cdot G^{\mu}(k_j)}{[|\mathscr{P}_{n+1}|^2+H_{\mathrm{f}}^{n+1}+\alpha^3][|\mathscr{P}_{n+2}|^2+H_{\mathrm{f}}^{n+2}+\alpha^3]}$$

$$\times\,(\sigma_{\gamma}\psi_n(l,k_1,\dots,k_n),\psi_n(l,k_1,\dots,\check{k}_i,\dots,\check{k}_j,\dots,k_{n+2}))\,\mathrm{d}l\,\mathrm{d}^{n+2}k.$$

The diagonal term, when $i = n+1$ and $j = n+2$, vanishes since H is purely imaginary while G is real. We then have three off-diagonal terms to deal with, I_n, II_n and III_n, which correspond respectively to the cases $j = n+2$, $j = n+1$ and $j \notin \{n+1, n+2\}$.

Firstly, using (B.10) and $\frac{|H^\lambda(k_{n+1})|}{|\mathscr{P}_{n+1}|^2 + H_\mathrm{f}^{n+1} + \alpha^3} \leqslant |G^\lambda(k_{n+1})|$,

$$|\mathrm{I}_n| \leqslant n \sum_{\lambda=1,2} \int \frac{|\mathscr{P}_{n+2}|\,|G(k_{n+2})|^2\,|H^\lambda(k_{n+1})|\,|G^\lambda(k_1)|}{[|\mathscr{P}_{n+2}|^2 + H_\mathrm{f}^{n+2} + \alpha^3][|\mathscr{P}_{n+1}|^2 + H_\mathrm{f}^{n+1} + \alpha^3]}$$

$$\times |\psi_n(l, k_1, \ldots, k_n)|\,|\psi_n(l, k_2, \ldots, k_{n+1})|\,\mathrm{d}l\,\mathrm{d}^{n+2}k$$

$$\leqslant C \int \frac{|G(k_{n+2})|^2}{|k_{n+2}|^{1/2}}\,\mathrm{d}k_{n+2}\ \, \||D|\,|\psi_n|\|^2 \leqslant C(\psi_n, H_\mathrm{f}\psi_n),$$

thanks to Lemma A.1 and (B.10). Secondly, thanks again to (B.10) and Lemma A.1, we have

$$|\mathrm{II}_n| \leqslant n \sum_{\lambda,\mu=1,2} \int \frac{|G^\lambda(k_{n+2})|\,|H^\mu(k_{n+1})||G^\mu(k_{n+1})|\,|G^\lambda(k_1)|}{[H_\mathrm{f}^{n+2} + \alpha^3]^{1/2}[H_\mathrm{f}^{n+1} + \alpha^3]}$$
$$\times |\psi_n(l, k_1, \ldots, k_n)|\,|\psi_n(l, k_2, \ldots, \check{k}_{n+1}, k_{n+2})|\,\mathrm{d}l\,\mathrm{d}^{n+2}k$$
$$\leqslant C \sum_{\lambda=1,2} \int \frac{|G^\lambda(k_{n+1})|\,|H^\lambda(k_{n+1})|}{|k_{n+1}|^{3/2}}\,\mathrm{d}k_{n+1}\ \, \||D|\,|\psi_n|\|^2$$
$$\leqslant C(\psi_n, H_\mathrm{f}\psi_n).$$

Finally, the full off-diagonal term reads

$$|\mathrm{III}_n| \leqslant n\,(n-1) \sum_{\lambda,\mu=1,2} \int \frac{|G^\lambda(k_{n+2})|\,|H^\mu(k_{n+1})|\,|G^\lambda(k_1)|\,|G^\mu(k_2)|}{[H_\mathrm{f}^{n+2} + \alpha^3]^{1/2}[H_\mathrm{f}^{n+1} + \alpha^3]}$$

$$\times |\psi_n(l, k_1, \ldots, k_n)|\,|\psi_n(l, k_3, \ldots, k_{n+2})|\,\mathrm{d}l\,\mathrm{d}k_1 \ldots \mathrm{d}k_{n+2}$$

$$\leqslant C|(|X|\,H_\mathrm{f}^{-1/2}|D|\psi_n,\ |D|H_\mathrm{f}^{-1/2}\,|E|\ \psi_n)|$$

$$\leqslant C\ln(1/\alpha)^{1/2}(\psi_n, H_\mathrm{f}\psi_n).$$

Step 3: To conclude the proof of the lemma, we are thus lead to prove that

$$|\,\mathrm{Re}(L_\alpha^{-1}\sigma \cdot E^* L_\alpha^{-1}\sigma \cdot E^*\psi_n, D^*D^*\psi_n) - ||\psi_n||^2\ \langle 0|E\mathscr{A}^{-1}E\mathscr{A}^{-1}D^*D^*|0\rangle|$$
$$\leqslant C\sqrt{\alpha}||\psi_n||^2 + C\,\alpha^{-1/2}(\psi_n, L\,\psi_n).$$

On the one hand, using the explicit formulations of the operators E, D and their adjoints, we recall that

$$\langle 0|E\mathscr{A}^{-1}E\mathscr{A}^{-1}D^*D^*|0\rangle$$

$$= 2\sum_{\lambda,\mu=1,2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\overline{H}^{\lambda}(k_1)\cdot\overline{H}^{\mu}(k_2)G^{\lambda}(k_1)\cdot G^{\mu}(k_2)}{[|k_1|^2+|k_1|][|k_1+k_2|^2+|k_1|+|k_2|]}\,\mathrm{d}k_1\mathrm{d}k_2.$$

On the other hand

$$\mathrm{Re}(L_{\alpha}^{-1}\sigma\cdot E^*L_{\alpha}^{-1}\sigma\cdot E^*\psi_n, D^*D^*\psi_n)$$

$$= 2\sum_{\lambda,\mu=1,2}\mathrm{Re}\sum_{\gamma,\gamma'=1}^{3}\int\frac{\overline{H}_{\gamma}^{\lambda}(k_{n+2})\overline{H}_{\gamma'}^{\mu}(k_{n+1})\ \sum_{i=1}^{n+1}\sum_{j=i+1}^{n+2}\ G^{\lambda}(k_i)\cdot G^{\mu}(k_j)}{[|\mathscr{P}_{n+2}|^2+H_{\mathrm{f}}^{n+2}+\alpha^3][|\mathscr{P}_{n+1}|^2+H_{\mathrm{f}}^{n+1}+\alpha^3]}$$

$$\times(\sigma_{\gamma}\psi_n(l,k_1,\ldots,k_n),\sigma_{\gamma'}\psi_n(l,k_1,\ldots,\check{k}_i,\ldots,\check{k}_j,\ldots,k_{n+2}))\,\mathrm{d}l\,\mathrm{d}^{n+2}k.$$

This term may again be decomposed as a sum of three terms according to the same convention as above. Nevertheless, it is easily checked that only the first term, which corresponds to $i=n+1$ and $j=n+2$, contributes, while the other ones may be bounded from above by exactly the same method as before. Following the scheme of proof of Lemmas B.1 and B.4, we introduce further simplifying notation:

$$R_{n+2}=L_{\alpha}^{n+2}-\mathscr{Q}_{n+2}=-2\mathscr{P}_n\cdot(k_{n+1}+k_{n+2})+|\mathscr{P}_n|^2+H_{\mathrm{f}}^n+\alpha^3$$

and

$$R_{n+1}=L_{\alpha}^{n+1}-\mathscr{Q}_{n+1}=-2\mathscr{P}_n\cdot k_{n+1}+|\mathscr{P}_n|^2+H_{\mathrm{f}}^n+\alpha^3.$$

The following difference is then to be evaluated

$$\sum_{\lambda,\mu=1,2}\int\left[\frac{1}{L_{\alpha}^{n+2}\,L_{\alpha}^{n+1}}-\frac{1}{\mathscr{Q}_{n+2}\,\mathscr{Q}_{n+1}}\right]\overline{H}^{\mu}(k_{n+2})\cdot\overline{H}^{\lambda}(k_{n+1})$$
$$\times G^{\lambda}(k_{n+1})\cdot G^{\mu}(k_{n+2})\,|\psi_n(l,k_1,\ldots,k_n)|^2\,\mathrm{d}l\,\mathrm{d}k_1\ldots\mathrm{d}k_{n+2}. \tag{B.29}$$

It is straightforward to check that

$$\frac{1}{L_{\alpha}^{n+2}\,L_{\alpha}^{n+1}}-\frac{1}{\mathscr{Q}_{n+2}\,\mathscr{Q}_{n+1}}$$

$$=2\frac{\mathscr{P}_n\cdot(k_{n+2}+k_{n+1})}{L_{\alpha}^{n+1}\,\mathscr{Q}_{n+2}\,\mathscr{Q}_{n+1}}+2\frac{\mathscr{P}_n\cdot k_{n+1}}{L_{\alpha}^{n+2}\,\mathscr{Q}_{n+2}\,\mathscr{Q}_{n+1}} \tag{B.30a}$$

$$-\left[\frac{L_\alpha^n}{L_\alpha^{n+1}\,\mathcal{Q}_{n+2}\,\mathcal{Q}_{n+1}}+\frac{L_\alpha^n}{L_\alpha^{n+2}\,\mathcal{Q}_{n+2}\,\mathcal{Q}_{n+1}}\right] \tag{B.30b}$$

$$+\frac{R_{n+1}\,R_{n+2}}{L_\alpha^{n+1}\,L_\alpha^{n+2}\,\mathcal{Q}_{n+2}\,\mathcal{Q}_{n+1}}. \tag{B.30c}$$

We now insert this expression into (B.29) and simply bound $|G^\lambda(k_{n+1})|\times|H^\lambda(k_{n+1})|$ by $C\,\chi(|k_{n+1}|)$ and similarly for $|G^\mu(k_{n+2})|\,|H^\mu(k_{n+2})|$. It is then very easy to bound the two terms in (B.30a) by $C\,\|\psi_n\|\,\|\mathscr{P}\psi_n\|$ and the terms in (B.30b) by $C\,(\psi_n,L\psi_n)+C\,\alpha^3\|\psi_n\|^2$. Concerning (B.30c), the term involving $\frac{|\mathscr{P}_n|^2\,|k_{n+1}|\,|k_{n+1}+k_{n+2}|}{L_\alpha^{n+1}\,L_\alpha^{n+2}\,\mathcal{Q}_{n+2}\,\mathcal{Q}_{n+1}}$ is also easily bounded by $\|\mathscr{P}\psi_n\|^2$ while all the terms involving $H_{\mathrm{f}}^n+\alpha^3$ admit simple bounds by $C\,\|\psi_n\|\,\|\mathscr{P}\psi_n\|$ or $C\,(\psi_n,L\psi_n)+C\,\alpha^3\|\psi_n\|^2$. To deal with the remaining terms

$$\frac{2|\mathscr{P}_n|\,|k_{n+1}|\,|\mathscr{P}_n|^2}{L_\alpha^{n+1}\,L_\alpha^{n+2}\,\mathcal{Q}_{n+2}\,\mathcal{Q}_{n+1}},\ \frac{2|\mathscr{P}_n|\,|\mathscr{P}_n|^2\,|k_{n+1}+k_{n+2}|}{L_\alpha^{n+1}\,L_\alpha^{n+2}\,\mathcal{Q}_{n+2}\,\mathcal{Q}_{n+1}},\ \frac{|\mathscr{P}_n|^4}{L_\alpha^{n+1}\,L_\alpha^{n+2}\,\mathcal{Q}_{n+2}\,\mathcal{Q}_{n+1}}, \tag{B.31}$$

we observe that, from (B.26),

$$\frac{1}{L_\alpha^{n+2}}=\frac{1}{L_\alpha^n+\mathcal{Q}_{n+2}}-\frac{-2\mathscr{P}_n\cdot(k_{n+1}+k_{n+2})+|k_{n+1}+k_{n+2}|^2}{L_\alpha^{n+2}\,(L_\alpha^n+\mathcal{Q}_{n+2})}. \tag{B.32}$$

Since $L_\alpha^n=|\mathscr{P}_n|^2+H_{\mathrm{f}}^n+\alpha^3$, inserting (B.32) in (B.31) and using the two bounds

$$\frac{|\mathscr{P}_n|^2}{L_\alpha^n+\mathcal{Q}_{n+2}}\leqslant 1 \text{ and } \frac{|\mathscr{P}_n|}{L_\alpha^n+\mathcal{Q}_{n+2}}\leqslant\frac{1}{2(H_{\mathrm{f}}^n+\alpha^3+\mathcal{Q}_{n+2})^{1/2}},$$

it is a tedious but easy exercise to bound the contribution of all the terms in (B.31) by $\|\mathscr{P}\psi_n\|^2$, except for one term which comes from the last term in (B.31) and which is precisely bounded by

$$\frac{|\mathscr{P}_n|^5\,|k_{n+1}+k_{n+2}|}{L_\alpha^{n+1}\,L_\alpha^{n+2}(L_\alpha^n+\mathcal{Q}_{n+2})\,\mathcal{Q}_{n+2}\,\mathcal{Q}_{n+1}}.$$

To handle this term, we plug in (B.32) once more, and with the same two bounds as above, we again bound the contribution by $\|\mathscr{P}\psi_n\|^2$.

We now turn to the bound on the non-contributing terms. Using first that $L_\alpha^{n+1}\,L_\alpha^{n+2}\geqslant|k_{n+1}|^2$, we check that

$$|\mathrm{II}_n|\leqslant n\sum_{\lambda,\mu=1,2}\int\frac{|H^\lambda(k_{n+1})||G^\lambda(k_{n+1})|}{|k_{n+1}|^2}\,|H^\mu(k_{n+2})|\,|G^\mu(k_1)|$$
$$\times\,|\psi_n(l,k_2,\ldots,\check{k}_{n+1},k_{n+2})|\,|\psi_n(l,k_1,\ldots,k_n)|\ \mathrm{d}l\,\mathrm{d}k_1\ldots\mathrm{d}k_{n+2}$$

$$\leqslant C \sum_{\lambda=1,2} \int \frac{|G^\lambda(k_{n+1})|\,|H^\lambda(k_{n+1})|}{|k_{n+1}|^2}\,\mathrm{d}k_{n+1}|(|D|\,|\psi_n|,\ |E|\,|\psi_n|)|$$
$$\leqslant C\,(\psi_n, H_\mathrm{f}\psi_n),$$

while, with $L_\alpha^{n+1}\,L_\alpha^{n+2} \geqslant |k_{n+2}|\,(\sum_{i=3,\ \neq n+1}^{n+2} |k_i|)^{1/2}\,(\sum_{i=2}^{n} |k_i|)^{1/2}$, we have

$$|\mathrm{III}_n| \leqslant n(n-1) \sum_{\lambda,\mu=1,2} \int \frac{|H^\lambda(k_{n+2})|\,|H^\mu(k_{n+1})|}{|k_{n+2}|\,(\sum_{i=3}^{n+2} |k_i|)^{1/2}}\,|\psi_n(l, k_3, \dots, k_{n+2})|$$
$$\times \frac{|G^\lambda(k_1)|\,|G^\mu(k_2)|}{(\sum_{i=2}^{n} |k_i|)^{1/2}}\,|\psi_n(l, k_1, \dots, k_n)|\,\mathrm{d}l\,\mathrm{d}k_1 \dots \mathrm{d}k_{n+2}$$
$$\leqslant C\,(|D| H_\mathrm{f}^{-1/2} |E|\,\psi_n, |D| H_\mathrm{f}^{-1/2} |D|\,\psi_n)|$$
$$\leqslant C\,(\psi_n, H_\mathrm{f}\psi_n). \qquad \square$$

Appendix C. Evaluation of the terms of higher order in (2.46)

First, we investigate the cross-terms in (2.46) which appear with a factor $\alpha^{3/2}$.

Lemma C.1 (Bound on (2.46d)).

$$|(L_\alpha^{-1} F^* L_\alpha^{-1} F^* \psi_n, F^* h_{n+1})| \leqslant C[\alpha||\psi_n||^2 + \alpha(\psi_n, H_\mathrm{f}\psi_n) + \alpha||\mathscr{P}\psi_n||^2 + \alpha^{-1}(h_{n+1}, H_\mathrm{f} h_{n+1})]. \tag{C.1}$$

Proof. For shortness we restrict ourselves to the case $F = 2\mathscr{P}\cdot D$, which is the most delicate one. The other cases work similarly.

By permutational symmetry the first part of the l.h.s. of (C.1) is bounded from above by

$$\sqrt{n+1} \sum_{\lambda,\mu=1,2} \int \frac{[G^\lambda(k_{n+2})\cdot\mathscr{P}_{n+2}]^2 |G^\mu(k_{n+1})\cdot\mathscr{P}_{n+1}|}{[|\mathscr{P}_{n+2}|^2 + H_\mathrm{f}^{n+2} + \alpha^3][|\mathscr{P}_{n+1}|^2 + H_\mathrm{f}^{n+1} + \alpha^3]}$$
$$\times |\psi_n(l, k_1, \dots, k_n)||h_{n+1}(l, k_1, \dots, k_{n+1})|\mathrm{d}l\,\mathrm{d}^{n+2}k$$
$$\leqslant \sum_{\lambda=1,2} \int \frac{|G^\lambda(k_{n+2})|^2}{|k_{n+2}|}\,\mathrm{d}k_{n+2}|(\mathscr{P}\psi_n, |D| h_{n+1})|$$
$$\leqslant C\,||\mathscr{P}\psi_n||\,(h_{n+1}, H_\mathrm{f} h_{n+1})^{1/2}, \tag{C.2}$$

since $G^{\lambda}(k_{n+1}) \cdot \mathscr{P}_{n+1} = G^{\lambda}(k_{n+1}) \cdot \mathscr{P}_n$ and where we used (B.10) and additionally $\frac{\mathscr{P}^2}{\mathscr{P}^2 + H_{\mathrm{f}}} \leqslant 1$.

The second, off-diagonal, part can be estimated by

$$\begin{aligned} |(|D|\psi_n, |D|H_{\mathrm{f}}^{-1/2}|D|h_{n+1})| &\leqslant C(\psi_n, H_{\mathrm{f}}\psi_n)^{1/2}\,(h_{n+1}, H_{\mathrm{f}}h_{n+1})^{1/2} \\ &\leqslant C[\alpha(\psi_n, H_{\mathrm{f}}\psi_n) + \alpha^{-1}(h_{n+1}, H_{\mathrm{f}}h_{n+1})], \end{aligned} \tag{C.3}$$

again with Schwarz' inequality and Lemma A.1. □

Lemma C.2 (Bound on (2.46e)).

$$\begin{aligned} |(L_{\alpha}^{-1}D^*D^*\psi_n, F^*h_{n+1})| \leqslant C[\alpha\|\psi_n\|^2 + \sqrt{\alpha}(\psi_n, H_{\mathrm{f}}\psi_n) \\ + \alpha^{-1}(h_{n+1}, H_{\mathrm{f}}h_{n+1})]. \end{aligned} \tag{C.4}$$

Proof. We restrict once again to $F = 2\mathscr{P} \cdot D$. The absolute value of the diagonal part is bounded by

$$\begin{aligned} &\sqrt{n+1} \sum_{\lambda,\mu=1,2} \int \frac{|G^{\lambda}(k_{n+2}) \cdot \mathscr{P}_{n+2}|\,|G^{\lambda}(k_{n+2})|\,|G^{\mu}(k_{n+1})|}{[|\mathscr{P}_{n+2}|^2 + H_{\mathrm{f}}^{n+2} + \alpha^3]} \\ &\quad \times |\psi_n(l, k_1, \dots, k_n)||h_{n+1}(l, k_1, \dots, k_{n+1})|\mathrm{d}l\,\mathrm{d}^{n+2}k \\ &\leqslant \sum_{\lambda=1,2} \int \frac{|G^{\lambda}(k_{n+2})|^2}{|k_{n+2}|^{1/2}}\,\mathrm{d}k_{n+2}|(\psi_n, |D|h_{n+1})| \\ &\leqslant C\,\|\psi_n\|\,(h_{n+1}, H_{\mathrm{f}}h_{n+1})^{1/2}, \end{aligned} \tag{C.5}$$

with the help of (B.10), whereas the off-diagonal term can again be bounded by

$$|(|D|\psi_n, |D|H_{\mathrm{f}}^{-1/2}|D|h_{n+1})| \leqslant C\,(\psi_n, H_{\mathrm{f}}\psi_n)^{1/2}(h_{n+1}, H_{\mathrm{f}}h_{n+1})^{1/2}. \quad \square \tag{C.6}$$

For the term appearing with α^2 in (2.46h) we derive

Lemma C.3 (Bound on (2.46h)).

$$\begin{aligned} |(L_{\alpha}^{-1}F^*L_{\alpha}^{-1}D^*D^*\psi_{n-1}, F^*h_{n+1})| \leqslant C\,[\alpha||\psi_{n-1}||^2 + (\psi_{n-1}, H_{\mathrm{f}}\psi_{n-1}) \\ + \alpha^{-1}\ln(1/\alpha)(h_{n+1}, H_{\mathrm{f}}h_{n+1}) + (h_{n+1}, H_{\mathrm{f}}h_{n+1})]. \end{aligned} \tag{C.7}$$

Proof. Consider again $F = 2\mathscr{P} \cdot D$. The main term reads

$$(n+1) \sum_{\lambda,\mu,\nu=1,2} \int \frac{[G^\lambda(k_{n+2}) \cdot \mathscr{P}_{n+2}]^2 |G^\mu(k_{n+1})||G^\nu(k_n)|}{[|\mathscr{P}_{n+1}|^2 + H_{\rm f}^{n+1} + \alpha^3][|\mathscr{P}_{n+2}|^2 + H_{\rm f}^{n+2} + \alpha^3]}$$

$$\times |\psi_{n-1}(l, k_1, \dots, k_{n-1})||h_{n+1}(l, k_1, \dots, k_{n+1})| {\rm d}l\, {\rm d}^{n+2}k$$

$$\leqslant C(|X|^* \psi_{n-1}, |X| h_{n+1})$$

$$\leqslant C \ln(1/\alpha)^{1/2} (h_{n+1}, H_{\rm f} h_{n+1})^{1/2}$$

$$\times [\|\psi_{n-1}\| + \ln(1/\alpha)^{1/2} (\psi_{n-1}, H_{\rm f} \psi_{n-1})^{1/2}],$$

whereas the totally off-diagonal term can be estimated by

$$|(|D|\psi_{n-1}, |D| H_{\rm f}^{-1/2} |D| H_{\rm f}^{-1/2} |D| h_{n+1})|$$

$$\leqslant C\, (\psi_{n-1}, H_{\rm f} \psi_{n-1})^{1/2}\, (h_{n+1}, H_{\rm f} h_{n+1})^{1/2}. \qquad \square$$

In the following, we consider the cross terms in (2.46) which appear with a factor $\alpha^{5/2}$, for which a rough estimate is enough. Therefore we merely indicate the proofs.

Lemma C.4 (Bound on (2.46f)).

$$|(L_\alpha^{-1} F^* L_\alpha^{-1} F^* \psi_n, F^* L_\alpha^{-1} D^* D^* \psi_{n-1})| \leqslant C[\sqrt{\alpha}\, ||\psi_{n-1}||^2 + \sqrt{\alpha}\, \|\psi_n\|^2 + \alpha^{-1/2}\, (\psi_n, H_{\rm f} \psi_n) + \alpha^{-1/2}\, (\psi_{n-1}, H_{\rm f} \psi_{n-1})]. \tag{C.8}$$

Proof. We restrict again to $F = 2\mathscr{P} \cdot D$ and regard only one diagonal term, namely

$$(n+1)^{1/2} \sum_{\lambda,\mu,\nu=1,2} \int \frac{[G^\lambda(k_{n+2}) \cdot \mathscr{P}_{n+2}]^2 |G^\mu(k_{n+1}) \cdot \mathscr{P}_{n+1}|}{[|\mathscr{P}_{n+1}|^2 + H_{\rm f}^{n+1} + \alpha^3]^2 [|\mathscr{P}_{n+2}|^2 + H_{\rm f}^{n+2} + \alpha^3]}$$
$$\times |G^\mu(k_{n+1})||G^\nu(k_n)||\psi_n(l, k_1, \dots, k_n)||\psi_{n-1}(l, k_1, \dots, k_{n-1})| {\rm d}l\, {\rm d}^{n+2}k$$
$$\leqslant \sum_{\lambda,\mu=1,2} \int \frac{|G^\mu(k_{n+1})|^2}{|k_{n+1}|^{3/2}} |G^\lambda(k_{n+2})|^2\, {\rm d}k_{n+1}\, {\rm d}k_{n+2} |(\psi_{n-1}, |D|\psi_n)|$$
$$\leqslant C\, \|\psi_{n-1}\|\, (\psi_n, H_{\rm f} \psi_n)^{1/2}. \tag{C.9}$$

The remaining terms are estimated similarly. $\square$

By similar methods the following concluding lemma concerning the error term (2.46g) is obtained.

Lemma C.5 (Bound on (2.46g)).

$$|(L_\alpha^{-1}F^*L_\alpha^{-1}D^*D^*\psi_{n-1}, D^*D^*\psi_n)| \leqslant C\,[\sqrt{\alpha}\,||\psi_{n-1}||^2 + \sqrt{\alpha}\,\|\psi_n\|^2 + \alpha^{-1/2}\,(\psi_n, H_\mathrm{f}\psi_n) + \alpha^{-1/2}\,(\psi_{n-1}, H_\mathrm{f}\psi_{n-1})]. \tag{C.10}$$

Notice that in the last two lemmas simple Schwarz' estimates would suffice.

Note in proof. Another, non-perturbative proof of enhanced binding for particles with spin was recently announced by Chen et al. (ArXiv: math-ph 0209062).

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