Topology Vol. 4, pp. 283–293. Pergamon Press, 1965. Printed in Great Britain

IMMERSIONS AND EMBEDDINGS OF DOLD MANIFOLDS

J. J. UCCI†

(*Received* 2 *July* 1965)

INTRODUCTION

THE MAIN RESULT of this paper is a non-immersion and non-embedding theorem for Dold manifolds $P(m, n)$ (see §1) in Euclidean space. $P(m, 0)$ and $P(0, n)$ are real and complex projective spaces of real dimensions m and $2n$, respectively, so that the set $\{P(m, n)\}$ contains the usual test spaces for immersion and embedding.

Let $\varphi(n)$ = the number of integers s with $0 < s \leqq n$ and $s \equiv 0, 1, 2$ or $4 \bmod 8$;

$\bar{\sigma}(m, n)$ = the largest integer s for which $2^{s-1}\binom{m+n+s}{s}$ is not divisible by $2^{\varphi(n)}$;

$$\sigma^*(m, n) = \begin{cases} \max\left(\bar{\sigma}(m, n), 2\left[\dfrac{n}{2}\right]\right) & \text{if} \quad m > 0, \\ 2\left[\dfrac{n}{2}\right] & \text{if} \quad m = 0, \end{cases}$$

where $\left[\dfrac{n}{2}\right]$ denotes the integral part of $\dfrac{n}{2}$. Then our result can be stated

THEOREM 2.12. (i) *$P(m, n)$ cannot be immersed in $R^{m-2n+\sigma^*(m,n)-1}$*, (ii) *$P(m, n)$ cannot be embedded in $R^{m+2n+\sigma^*(m,n)}$.*

Note that $\sigma^*(m, O) = \bar{\sigma}(m, O) = \sigma(m)$, where σ is defined in [2], and so (2.12) can be viewed as an extension of Atiyah's Theorem (5.1) of [2]. Indeed, we prove (2.12) using the methods of [2], i.e. K-theory.

The arrangement of the paper is as follows. In §1 we recall the basic properties of $P(m, n)$ and determine its stable tangent bundle in terms of two canonical bundles. In §2 the rings $KU(P(m, n))$ and $KO(P(m, n))$ are partially determined. The Grothendieck operators γ^i are computed and applied to give (2.12). In conclusion §3 collects together some remarks about the implications of characteristic classes for immersing and embedding $P(m, n)$ in Euclidean space.

Our notation is basically derived from [2], [4] and [10]. Minor changes are explicitly given as needed.

† The material of this paper represents a portion of a thesis submitted to the University of California, Berkeley, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

I wish to thank my thesis advisor, Professor E. H. Spanier for his encouragement and advice during the preparation of this work.

§1. DOLD MANIFOLDS

Cohomology

Let $S^m \subset R^{m+1}$ be the usual m-sphere and CP_n the usual complex projective n-space. Then $P(m, n)$ is the Dold manifold of dimension $m + 2n$ obtained from $S^m \times CP_n$ by identifying (x, z) with $(-x, \bar{z})$, where $(x, z) \in S^m \times CP_n$. $P(m, 0)$ and $P(0, n)$ are readily seen to be RP_m and CP_n, respectively.

The canonical map $\Phi : S^m \times CP_n \to P(m, n)$ defines a two-fold covering. The map $p : P(m, n) \to RP_m$ induced by the projection $S^m \times CP_n \to S^m$ defines an analytic fibration with fibre CP_n and structure group Z_2 (conjugation is the non-trivial element of Z_2). If $m' \leqq m$ and $n' \leqq n$, then there is an obvious inclusion $i' : P(m', n') \to P(m, n)$, which, for the cases $(m', n') = (m, 0)$ and $(m', n') = (0, n)$, we denote by i and j, respectively.

In [4] $P(m, n)$ is given a cell decomposition with a k-cell (C_i, D_j) for every pair (i, j), $i, j \geqq 0$, for which $i + 2j = k \leqq m + 2n$. Moreover, Φ, p and i' are cellular maps when $S^m \times CP_n$ is given an appropriate cell decomposition [4]. The boundary operator satisfies

$$\partial(C_i, D_j) = (1 + (-1)^{i+j})(C_{i-1}, D_j), \; i > 0, \qquad (1.1)$$
$$\partial(C_0, D_j) = 0.$$

Let $c^i d^j$ denote the cochain which assigns 1 to (C_i, D_j) and 0 to all other $(i + 2j)$-cells. Then $c^i d^j$ defines an $(i + 2j)$-dimensional mod 2 cohomology class which is natural with respect to the inclusion i'. In particular, c and d define 1-dimensional and 2-dimensional classes, respectively.

Dold's determination of the ring structure of $H^*(P(m, n); Z_2)$ [4] can be described:

$$H^*(P(m, n); Z_2) \cong \left(\frac{Z_2[c]}{c^{m+1}}\right) \otimes \left(\frac{Z_2[d]}{d^{n+1}}\right). \qquad (1.2)$$

Moreover, (1.1) determines the additive structure of $H^*(P(m, n); Z)$. We note only that $H^2(P(m, n); Z) \cong Z_2$ if $m \geqq 2$, with the generator reducing mod 2 to c^2.

The Tangent Bundle

The tilde notation "$\sim$" will be used throughout for objects (bundles, classes) defined for RP_m or CP_n. Define a line bundle ξ over $P(m, n)$ whose total space $E(\xi)$ is $S^m \times CP_n \times R$ mod the identification $(x, z, t) \sim (-x, \bar{z}, -t)$. For $n = 0$, ξ is just the canonical line bundle $\tilde{\xi}$ over $P(m, O) = RP_m$ [10] and so we obtain a bundle map (i, i_E)

$$\begin{array}{ccc} E(\tilde{\xi}) & \xrightarrow{i_E} & E(\xi) \\ \downarrow & & \downarrow \\ P(m, O) & \xrightarrow{i} & P(m, n) \end{array}$$

implying that $i^*\xi = \tilde{\xi}$. Since $w_1(\tilde{\xi}) = c$ [10], we have by (1.2), $w_1(\xi) = c$.

Let c and r denote the usual operations of complexification and decomplexification. Represent CP_n as the unit sphere $S^{2n+1} \subset C^{n+1}$ mod the identification $u \sim \lambda u$, $u \in S^{2n+1}$, $\lambda \in C$ with $|\lambda| = 1$. Define a real 2-plane bundle η over $P(m, n)$ whose total space $E(\eta)$ is $S^m \times S^{2n+1} \times C$ mod the identifications $(x, u, w) \sim (x, \lambda u, \lambda w) \sim (-x, \overline{\lambda u}, \overline{\lambda w}) \sim (-x, \bar{u}, \bar{w})$, λ as above. For $m = 0$, η is just the canonical complex line bundle $\tilde{\eta}$ over $P(0, n) = CP_n$ considered as a real bundle (denoted $r(\tilde{\eta})$); thus we obtain a bundle map (j, j_E)

$$\begin{array}{ccc} E(r(\tilde{\eta})) & \xrightarrow{j_E} & E(\eta) \\ \downarrow & & \downarrow \\ P(O, n) & \xrightarrow{j} & P(m, n) \end{array}$$

implying $j^*\eta = r(\tilde{\eta})$. From [10], $w_2(r(\tilde{\gamma})) = \tilde{d} = j^*w_2(\eta)$. The map $S^m \times S^{2n+1} \times R^2 \to S^m \times S^{2n+1} \times C$ given by $(x, u; t_1, t_2) \to (x, u; t_1 + it_2)$ induces a bundle map (i, i_E')

$$\begin{array}{ccc} E(1 \oplus \tilde{\xi}) & \xrightarrow{i_E'} & E(\eta) \\ \downarrow & & \downarrow \\ P(m, 0) & \xrightarrow{i} & P(m, n) \end{array}$$

whence $i^*\eta = 1 \oplus \tilde{\xi}$. So $w_1(\eta) = c$ by (1.2). The equivalences $i^*\eta = 1 \oplus \tilde{\xi}$ and $j^*\eta = r(\tilde{\eta})$ together with (1.2) imply that $w_2(\eta) = d$ and so $w(\eta) = 1 + c + d$.

For any line bundle β, $\beta \otimes \beta = 1$; in particular, $\xi \otimes \xi = 1$. Moreover, the map $S^m \times S^{2n+1} \times (R \otimes C) \to S^m \times S^{2n+1} \times C$ given by $(x, u; t \otimes w) \to (x, u; itw)$ induces a bundle equivalence (i', g_E)

$$\begin{array}{ccc} E(\xi \otimes \eta) & \xrightarrow{g_E} & E(\eta) \\ \downarrow & & \downarrow \\ P(m, n) & \overset{i'}{=} & P(m, n) \end{array}$$

and so $\xi \otimes \eta = \eta$.

The above remarks are summarized in the following:

PROPOSITION (1.4). *There exist a* 1*-plane bundle* ξ *and a* 2*-plane bundle* η *over* $P(m, n)$ *such that*

(i) $w(\xi) = 1 + c$, $w(\eta) = 1 + c + d$;

(ii) $i^*\xi = \tilde{\xi}$, $j^*\eta = r(\tilde{\eta})$, $i^*\eta = 1 \oplus \tilde{\xi}$;

(iii) $\xi \otimes \xi = 1$, $\xi \otimes \eta = \eta$.

Our main objective in this section is the following generalization of Theorems 2 and 27 of [10]. Let τ denote the tangent bundle of $P(m, n)$.

THEOREM (1.5). $\tau \oplus \xi \oplus 2 = (m + 1)\xi \oplus (n + 1)\eta$.

Proof. Write $\zeta = (m+1)\xi \oplus (n+1)\eta$ and $X = S^m \times S^{2n+1} \times R^{m+1} \times C^{n+1}$. Then $E(\zeta)$ is the set of all $(x, u, y, v) \in X$ mod the identifications $(x, u, y, v) \sim (x, \lambda u, y, \lambda v) \sim (-x, \overline{\lambda u}, -y, \overline{\lambda v}) \sim (-x, \bar{u}, -y, \bar{v})$, where $\lambda \in C$ with $|\lambda| = 1$. Let $\langle\ \rangle$ and $(\ )$ denote the real and complex inner products of R^{m+1} and C^{n+1}, respectively. Then $E(\tau)$ is the subset of $E(\zeta)$ of all (x, u, y, v) satisfying $\langle x, y\rangle = 0$ and $(u, v) = 0$. Since $E(\tau) \subset E(\zeta)$, we have $\tau \oplus v^3 = \zeta$, where v^3 (the orthogonal complement of τ in ζ) has total space $E(v^3)$ given by $\{(x, u, \alpha x, \beta u) \in X : \alpha \in R,\ \beta \in C\}$ mod the identifications above. Now $E(2 \oplus \xi)$ is given by $S^m \times CP_n \times R^3$ mod the identifications $(x, u; t_1, t_2, t_3) \sim (x, \lambda u; t_1, t_2, t_3) \sim (-x, \overline{\lambda u}; t_1, t_2, -t_3) \sim (-x, \bar{u}; t_1, t_2, -t_3)$ and so the map $S^m \times S^{2n+1} \times R^3 \to E(v^3)$ given by $(x, u; t_1, t_2, t_3) \to (x, u; t_1 x, (t_2 + it_3)u)$ induces a bundle equivalence (i', h_E)

$$\begin{array}{ccc} E(2 \oplus \xi) & \xrightarrow{h_E} & E(v^3) \\ \downarrow & & \downarrow \\ P(m, n) & \overset{i'}{=} & P(m, n) \end{array}$$

Thus $v^3 = 2 \oplus \xi$.

§2. $KU(P(m, n))$ AND $KO(P(m, n))$

Additive and Multiplicative Structures

$KU(P(m, n))$ and $KO(P(m, n))$ are completely determined in [14]. Here we compute only the summands containing the tangent bundle of $P(m, n)$; they will be invariant under the operators γ^i.

We make use of the following known results for $KU(X)$ and $KO(X)$, where $X = RP_m$, CP_n. Recall that $\tilde{\xi}$ denotes the canonical real line bundle over RP_m and $\tilde{\eta}$ the canonical complex line bundle over CP_n. Let $c(\alpha)$ be denoted α_C for any real bundle α. Write $\tilde{x} = \tilde{\xi} - 1$, $\tilde{x}_1 = c(\tilde{x})$, $\tilde{y}_1 = \tilde{\eta} - 1_C$ and $\tilde{y} = r(\tilde{y}_1)$. Set $g = \left[\frac{m}{2}\right]$, $h = \left[\frac{n}{2}\right]$ and $f = \varphi(m)$, (see the introduction for the definition of φ). Let Z_k denote the cyclic group of order k and Z^k the free abelian group of rank k.

THEOREM (2.1). *$\tilde{K}O(RP_m) = Z_{2^f}$, generated by $\tilde{x}$ with the multiplicative relation $\tilde{x}_1^2 = -2\tilde{x}$.*

THEOREM (2.2). *$\tilde{K}U(RP_m) = Z_{2^g}$, generated by $\tilde{x}_1$ with the multiplicative relation $\tilde{x}^2 = -2\tilde{x}_1$.*

THEOREM (2.3). *$KO(CP_n)$ is the truncated polynomial ring (over Z) with one generator $\tilde{y}$ satisfying the relations*

(i) *if $n = 2t$, then $\tilde{y}^{t+1} = 0$,*

(ii) *if $n = 4t + 1$, then $2\tilde{y}^{2t+1} = 0$ and $\tilde{y}^{2t+2} = 0$,*

(iii) *if $n = 4t + 3$, then $\tilde{y}^{2t+2} = 0$.*

THEOREM (2.4). $KU(CP_n) = \dfrac{Z[\tilde{y}_1]}{\tilde{y}_1^{n+1}}$.

(2.1), (2.2) and (2.4) are proven in [1]; (2.3) is proven in [11], with the use of the following lemma.

LEMMA (2.5) $c(\tilde{y}^s) = (\tilde{y}_1^2 - \tilde{y}_1^3 + \ldots + (-1)^n \tilde{y}_1^n)^s$.

As a consequence of (2.5) we have

LEMMA (2.6). *$1, \tilde{y}_1, \tilde{y}_1^3, \ldots, \tilde{y}_1^{2k-1}, \tilde{w}_1, \tilde{w}_1^2, \ldots, \tilde{w}_1^h$ forms an additive basis for $KU(CP_n)$, where $\tilde{w}_1 = \tilde{y}_1 + \tilde{y}_1^*$, $*$ denotes conjugation and $k = n - h$.*

Proof. By (2.4) $1, \tilde{y}_1, \tilde{y}_1^2, \ldots, \tilde{y}_1^n$ forms an additive basis for $KU(CP_n)$ and so we need only check that the (additive) homomorphism f defined by $f(\tilde{y}_1^{2i+1}) = \tilde{y}_1^{2i+1}$ and $f(\tilde{y}_1^{2i}) = \tilde{w}_1^i$ is a (group) automorphism of $KU(CP_n)$. This fact, however, is an immediate consequence of (2.5) which shows that the matrix defined by f (with respect to the basis $1, \tilde{y}_1, \tilde{y}_1^2, \ldots, \tilde{y}_1^n$) is triangular with every diagonal entry equal to 1.

Write $x = \xi - 1$, $x_1 = c(x)$, $z = \eta - 2$, $y = z - x$, $z_1 = c(z)$ and $y_1 = c(y)$.

THEOREM (2.7). *$\tilde{K}U(P(m, n))$ contains a summand isomorphic to $G_U = Z_{2^g} + Z^h$ generated by $x_1, y_1, y_1^2, \ldots, y_1^h$ with the relation $2^g x_1 = 0$. The multiplicative structure of G_U is given by $x_1^2 = -2x_1$ and $x_1 y_1 = 0$.*

Proof. Since the composition $p \circ i : P(m, 0) \to P(m, n) \to RP_m$ is the identity. (2.2) implies that $p^! \tilde{x}_1 = x_1$ is a generator of order 2^g. From (1.4) we have $j^! y_1 = cr(\tilde{y}_1) = \tilde{y}_1 + \tilde{y}_1^* = \tilde{w}_1$ and so $j^! y_1^i = \tilde{w}_1^i$. Since $\tilde{w}_1, \ldots, \tilde{w}_1^h$ is a subset of a basis for $\tilde{K}U(CP_n)$ by (2,6), then we see that $y_1, \ldots, y_1^h$ generate a summand isomorphic to Z^h. We use (1.4) to determine the ring structure: $x_1^2 = (\xi_C - 1_C)^2 = \xi_C \otimes \xi_C - 2\xi_C + 1_C = -2(\xi_C - 1_C) = -2x_1$ and $x_1 y_1 = (\xi_C - 1_C)(\eta_C - 2_C) - x_1^2 = \xi_C \otimes \eta_C - \eta_C - 2(\xi_C - 1_C) + 2x_1 = -2x_1 + 2x_1 = 0$.

THEOREM (2.8). *$\tilde{K}O(P(m, n))$ contains a summand isomorphic to $G_O = Z_{2^f} + Z^h$ generated by $x, y, y^2, \ldots, y^h$ with the relation $2^f x = 0$. The multiplicative structure of G_O is given by $x^2 = -2x$ and $xy = 0$.*

Proof. (2.1) implies that $p^! \tilde{x} = x$ is a generator of order 2^f. Moreover, since $c(y) = y_1$, we have that $y, y^2, \ldots, y^h$ generate a summand isomorphic to Z^h. The multiplicative structure is determined as in (2.7), again using (1.4).

Using the Chern character it can be shown [14] that y^{h+1} (and y_1^{h+1} also) vanishes. However, as it is not needed for (2.12) we omit the proof.

γ_t-structure of G_O

Let M^n be a compact n-dimensional differentiable (C^∞) manifold. Set $v_0 = -\tau_0(M^n) = n - \tau(M^n)$ in $\tilde{K}O(M^n)$, where $\tau(M^n)$ denotes the tangent bundle of M^n. Let λ^i, γ^i and g. dim (i.e. geometrical dimension) be as defined in [2]. Let $\subseteq$ and $\subset$ denote (C^∞) immersion and embedding, respectively, and $\not\subseteq$ and $\not\subset$ their negations. Then the main (general) results of [2] can be stated

(2.9) (i) If $M^n \subseteq R^{n+k}$, then $\gamma^i(v_0) = 0$ for $i > k$,

(ii) If $M^n \subset R^{n+k}$, then $\gamma^i(v_0) = 0$ for $i \geqq k$.

Now let $M^{m+2n} = P(m, n)$ and recall that for $P(m, n)$ we have $\tau \oplus \xi \oplus 2 = (m+1)\xi \oplus (n+1)\eta$ so that in $\tilde{K}O(P(m, n))$ we have

$$v_0 = -\tau_0 = -mx - (n+1)z = -(m+n+1)x - (n+1)y$$

where $\tau = \tau(P(m, n))$. Since γ_t is a homomorphism, we have

$$\gamma_t(v_0) = \gamma_t(x)^{-(m+n+1)}\gamma_t(y)^{-(n+1)}.$$

Clearly the geometrical dimensions of x and y are 1 and 2, respectively, so $\gamma_t(x) = 1 + xt$ and $\gamma_t(y) = 1 + yt + \gamma^2(y)t^2$. To compute $\gamma^2(y)$ we make use of the following two elementary lemmas.

LEMMA (2.10). *If ζ^2 is any 2-plane bundle over X, then $\lambda^2(\zeta^2)$ is the determinant bundle. In particular, $\lambda^2(\zeta^1 \oplus 1) = \zeta^1$ for any line bundle ζ^1.*

Proof. Let $A : R^2 \to R^2$ be a linear transformation which, with respect to a given basis e_1, e_2 is represented by the matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

For the second exterior power of A, $\Lambda^2 A : \Lambda^2 R^2 \to \Lambda^2 R^2$, we have

$$\Lambda^2 A(e_1 \wedge e_2) = A(e_1) \wedge A(e_2) = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})e_1 \wedge e_2 = (\det A)e_1 \wedge e_2.$$

Hence $\lambda^2\zeta^2$ is the determinant bundle. The second statement is an immediate consequence of the first.

Let v^k denote the universal k-plane bundle over the classifying space $B_{O(k)}$. There is an inclusion map $i : B_{O(1)} \to B_{O(2)}$ such that $i^*v^2 = v^1 \oplus 1$. Together with Theorem 8 of [10] this equivalence implies that $i^* : H^1(B_{O(2)}; Z_2) \to H^1(B_{O(1)}; Z_2)$ is a monomorphism.

LEMMA (2.11). $w_1(\lambda^2(v^2)) = w_1(v^2)$.

Proof. Since i^* is a monomorphism, it suffices to show that $i^*w_1(\lambda^2(v^2)) = i^*w_1(v^2)$. By (2.10) and naturality we have $i^*w_1(v^2) = w_1(i^*v^2) = w_1(v^1) = w_1(\lambda^2(v^1 \oplus 1)) = w_1(\lambda^2(i^*v^2) = i^*w_1(\lambda^2(v^2))$.

Hence for any 2-plane bundle ζ^2 $w_1(\lambda^2(\zeta^2)) = w_1(\zeta^2)$; moreover, $\lambda^2(\zeta^2)$ is determined by this formula since it is a line bundle. In particular, $\lambda^2(\eta) = \xi$ since $w_1(\eta) = c = w_1(\xi)$.

To simplify our computation, we collect together the following elementary facts:

(1) $\gamma^2 = \lambda^1 + \lambda^2$.

(2) $\lambda^i(a + b) = \sum_{j=0}^{i} \lambda^{i-j}(a)\lambda^j(b)$.

(3) $\gamma^i(a + b) = \sum_{j=0}^{i} \gamma^{i-j}(a)\gamma^j(b)$.

(4) $\gamma^2(-x) = x^2 = -2x$.

(5) $\lambda^2(-2) = 3$.

(1) follows from inspection of the coefficients of t^2 in the defining equation

$$\gamma_t = \sum \lambda^i t^i = \sum \gamma^i t^i (1-t)^{-i} = \lambda_{(t/(1-t))}.$$

(2) is property (*C*) in [2; §2], and (3), as noted in [2] is a consequence of (2) and the definition of γ_t. Finally, (4) and (5) may be verified directly from

$$\gamma_t(-x) = (1 + xt)^{-1} = 1 - xt + x^2t^2 - \cdots = 1 - xt - 2xt^2 - \cdots$$

$$\lambda_t(-2) = (1 + t)^{-2} = 1 - 2t + 3t^2 - \cdots.$$

Returning to the computation of $\gamma^2(y)$ we have

$$\begin{aligned}\gamma^2(y) &= \gamma^2(z - x) = \gamma^2(z) - xz + \gamma^2(-x) = \gamma^2(z) + 2x - 2x\\ &= \lambda^2(z) + \lambda^1(z) = \lambda^2(\eta - 2) + z = \lambda^2(\eta) - 2\eta + \lambda^2(-2) + z\\ &= \xi - 2\eta + 3 + z = x - 2z + z = x - z = -y.\end{aligned}$$

The full application of (2.9) for the non-immersion and non-embedding of $P(m, n)$ in R^{m+2n+k} requires the identification of the highest power of t having non-zero coefficient $\gamma^i(v_0)$ in the expansion of $\gamma_t(v_0)$. We make use of the relations $xy = 0$ and $x^2 = -2x$. Substituting we have

$$\gamma_t(v_0) = (1 + xt)^{-(m+n+1)}(1 + yt - yt^2)^{-(n+1)}$$

and so the coefficient of t^i is

$$\gamma^i(v_0) = \binom{m+n+i}{i}x^i + \sum_{j=\left[\frac{i+1}{2}\right]}^{i} \alpha_{ij}y^j = \pm 2^{i-1}\binom{m+n+i}{i}x + \sum_{j=\left[\frac{i+1}{2}\right]}^{i} \alpha_{ij}y^j$$

where the α_{ij} are non-zero integers (the relation $xy = 0$ kills all mixed product terms). Writing $1 + yt - yt^2 = 1 + yt(1 - t)$, we see that $\alpha_{hh}y^h$ appears in the coefficient of t^{2h}. Although $y^{h+1} = 0$ is shown in [14], $j^! y = \tilde{y}$ and Theorem (2.3) easily imply that y^h is independent of y^{h+r}, $r > 0$. Hence the coefficient of t^{2h} is non-zero since $\alpha_{2h,h} \neq 0$.
Using the function $\sigma^*(m, n)$ defined in the introduction, we obtain

THEOREM (2.12).

(i) $P(m, n) \nsubseteq R^{m+2n+\sigma^*(m,n)-1}$

(ii) $P(m, n) \not\subset R^{m+2n+\sigma^*(m,n)}$.

§3. REMARKS

(1) Many known results relating immersions and embeddings to Stiefel–Whitney classes may be applied to $P(m, n)$. In particular, we shall consider

(3.1) (i) If $M^n \subseteq R^{n+k}$, then $\bar{w}_i = 0$ for $i > k$.

(ii) If $M^n \subset R^{n+k}$, then $\bar{w}_i = 0$ for $i \geqq k$.

(iii) [5] If $0 \leqq k < 2(n - 4)$, then $M^n \subset R^{2n-k-1}$ if and only if $\overline{W}_{n-k-1} = 0$. ($\overline{W}_i$ denotes the—possibly twisted—i^{th} dual Stiefel–Whitney class of M^n.)

(iv) [7] Let M^n be orientable, $n > 4$. If n is even, then $M^n \subseteq R^{2n-2}$ if and only if $\bar{w}_2 \cdot \bar{w}_{n-2} = 0$; if n is odd, then $M^n \subseteq R^{2n-2}$.

(v) [8] Let M^n be orientable. If $n \equiv 1 \bmod 4$, then $M^n \subseteq R^{2n-3}$ if and only if $\bar{w}_2 . \bar{w}_{n-3} \in$ Image (Sq^1); if $n \equiv 0 \bmod 4$ and $\bar{w}_2 = 0$, then $M^n \subseteq R^{2n-4}$ implies $\bar{w}_4 . \bar{w}_{n-4} = 0$.

The application of (3.1) to $P(m, n)$ requires some mod 2 arithmetic.

LEMMA (3.2). ([13]) $\binom{b}{a} \equiv 1$ *mod* 2 *if and only if the non-zero terms in the dyadic expansion of a are a subset of those of b.*

LEMMA (3.3).

(i) $\binom{2^s-1}{t} \equiv 1 \bmod 2 \quad \text{for} \quad 1 \leqq t \leqq 2^s - 1.$

(ii) *If* $\binom{2m-1}{m-1} \equiv 1 \bmod 2, \quad \text{then} \quad m = 2^s, s \geqq 0.$

(iii) *If* $\binom{2m-2}{m-2} \equiv 1 \bmod 2, \quad \text{then} \quad m = 2^s, s \geqq 1.$

(iv) $\binom{2m}{m} \equiv 0 \bmod 2 \quad \text{for} \quad m > 0.$

(v) $\binom{n+t}{t} \equiv 0 \bmod 2 \quad \text{for} \quad t = 1, \ldots, n$ *if and only if* $n = 2^s - 1, s \geqq 1.$

(3.2) implies (3.3) via the inspection of certain dyadic expansions.

Throughout the remainder of this section $w(\tau(P(m, n)))$ will be abbreviated $w(m, n)$ or w whenever the meaning is clear. Similarly, for w_i, etc.

COROLLARY (3.4). $w(m, n) = 1$ *if and only if* $(m, n) = (2^t - 2^s, 2^s - 1)$, $t \geqq s \geqq 0$.

Proof. For $(m, n) = (2^t - 2^s, 2^s - 1)$, $t \geqq s \geqq 0$, $w(m, n) = (1 + c)^{2^t - 2^s}(1 + c + d)^{2^s} = (1 + c)^{2^t} = 1 + c^{2^t} = 1$. Conversely, if $w(m, n) = 1$, then $(1 + c)^m = (1 + c + d)^{-(n+1)}$ and hence, the coefficient $\binom{n+t'}{t'}$ of $d^{t'}$ in the expansion of the right member is 0 mod 2 for all $t' = 1, \ldots, n$, whence $n = 2^s - 1$ by (3.3)(v). But this gives $(1 + c)^m = (1 + c + d)^{-2^s} = (1 + c)^{-2^s}$ and $(1 + c)^{m+2^s} = 1$, from which it follows that $m + 2^s = 2^t$, $t \geqq s$.

(3.4) is useful for comparing dual Stiefel–Whitney classes to γ^i operators for non-immersion, non-embedding results for $P(m, n)$; e.g. if $(m, n) = (2^t - 2^s, 2^s - 1)$, $\sigma^*(m, n) = \max(\sigma(2^t - 1), 2^s - 2) \geqq \max(2^{t-2}, 2^s - 2)$, $t \geqq 4$ since $\sigma(2^t - 1) \geqq 2^{t-2}$ if $t \geqq 4$ [2].

COROLLARY (3.5) $\bar{w}_{m+2n-1} \neq 0$ *if and only if* $(m, n) = (2^s, 0)$.

Proof. In one direction this is a well-known result about real projective spaces. Suppose $\bar{w}_{m+2n-1} \neq 0$. Then $\bar{w}_{m+2n-1} = c^{m-1}d^n$, the only non-zero class of dimension $m + 2n - 1$. But $\bar{w} = (1 + c)^{-m}(1 + c + d)^{-(n+1)}$ and the coefficient $\binom{2m-1}{m-1} \cdot \binom{2n}{n}$ of $c^{m-1}d^n$ is 0 unless $m = 2^s$, $s \geqq 0$ by (3.3)(ii) and $n = 0$ by (3.3)(iv).

Define sets

$$A_1 = \{(2^s, 0) : s \geqq 1\}$$

$$A_2 = \{(0, 2^s) : s \geqq 1\}$$

$$A_3 = \{(2^s, 2^t) : 0 \leqq s \leqq t\}$$

$$A_4 = \{(2^s - 1, 1) : s \geqq 1\}$$

$$A_5 = \left\{(m, 2^t) : 2^{t+1} \leqq m \quad \text{and} \quad \binom{2m + 2^{t+1} - 1}{m} \equiv 1 \bmod 2\right\}$$

$$A = \bigcup_{i=1}^{5} A_i$$

COROLLARY (3.6). *$\bar{w}_{m+2n-2} \neq 0$ if and only if $(m, n) \in A$. Moreover, if $\bar{w}_{m+2n-2} \neq 0$ with $n > 0$, then $\bar{w}_{m+2n-2} = c^m d^{n-1}$.*

Proof. In one direction the first statement can be verified directly, so we concern ourselves only with the converse. Suppose $\bar{w}_{m+2n-2} \neq 0$. If $n = 0$ or $m = 0$, it is easily seen that $(m, n) \in A_1$ or $(m, n) \in A_2$, respectively, so we may assume $m, n > 0$. Then $\bar{w}_{m+2n-2} = c^m d^{n-1}$ because the coefficient of $c^{m-2}d^n$ is $\alpha \cdot \binom{2n}{n}$ which is 0 mod 2 by (3.3)(iv). Let $2^t \leqq n + 1 < 2^{t+1}$. Then $\bar{w} = (1 + c)^{-(m+2^{t+1})}(1 + c + d)^{2^{t+1}-(n+1)}$ with $2^{t+1} - (n + 1) \geqq n - 1$. Hence $2^{t+1} - 2 \leqq 2n \leqq 2^{t+1}$ and $n = 2^t - 1$ or $n = 2^t$. If $n = 2^t - 1$, then $\bar{w} = (1 + c)^{-(m+2^t)}$ and so $n = 1$. If also $m < 2$, then $m = n = 1$ and $(m, n) \in A_4$. If $m \geqq 2$ and $n = 1$, then $\bar{w} = (1 + c)^{-(m+2)}$ and $\bar{w}_m = \binom{2m+1}{m}c^m = \binom{2p-1}{p-1}c^m$ where $m = p - 1$, thus $m = p - 1 = 2^s - 1$ by (3.3)(ii), i.e. $(m, n) \in A_4$. Finally if $n = 2^t$ and $m < 2^{t+1}$, then $\bar{w} = (1 + c)^{-m}(1 + c + d)^{2t-1}$ and $\bar{w}_{m+2n-2} = \binom{2m-1}{m}\binom{2^t-1}{2^t-1}c^m d^{n-1}$ and so by (3.3)(ii) $m = 2^s$ and $(m, n) \in A_3$. If $n = 2^t$, and $2^{t+1} \leqq m$, then $\bar{w} = (1 + c)^{-(m+2^{t+1})}(1 + c + d)^{2^t-1}$ and $\bar{w}_{m+2n-2} = \binom{2m+2^{t+1}-1}{m}\binom{2^t-1}{2^t-1}c^m d^{n-1}$ and so $(m, n) \in A_5$.

We may now apply (3.1)(iii)–(v) to $P(m, n)$.

THEOREM (3.7).

(i) *$P(m, n) \subset R^{2(m+2n)-1}$ if and only if $(m, n) \notin A_1$; moreover, if $(m, n) \in A — A_1$, this result is best possible.*

(ii) *An orientable $P(m, n) \subseteq R^{2(m+2n)-2}$ if and only if $(m, n) \notin A_2$.*

(iii) *If $(m, n) = (4s + 1, 2t)$, then $P(m, n) \subseteq R^{2(m+2n)-3}$.*

Proof. (i) If $P(m, n) \subset R^{2(m+2n)-1}$, then $\bar{w}_{m+2n-1} = 0$ by (3.1)(ii) and so $(m, n) \notin A_1$, for otherwise $\bar{w}_{m+2n-1} = \bar{w}_{m-1} = c^{m-1} \neq 0$. Conversely, if $m > 0$, $(m, n) \notin A_1$ implies that $\bar{w}_{m+2n-2} = c^m d^{n-1}$ (the only non-zero element in dimension $m + 2n - 2$) or 0. But if T^2 denotes the torsion subgroup of $H^2(P(m, n); Z)$ and ρ denotes reduction mod 2, then $\rho T^2 = Z_2$ with generator c^2 if $m \geqq 2$, or $\rho T^2 = 0$ if $m < 2$. Hence $\bar{w}_{m+2n-2}x = 0$ for any $x \in \rho T^2$ and so by Lemma 8 of [9] $\bar{W}_{m+2n-1} = 0$. Now (i) follows from (3.1)(iii).

(ii) If $(m, n) \in A_2$, then $\bar{w}_{2n-2} = d^{n-1}$ by (3.3)(i). Since $\bar{w}_2 = d$, then $\bar{w}_{2n-2}\,\bar{w}_2 = d^n \neq 0$, hence $P(m, n) \nsubseteq R^{2(m+2n)-2}$ by (3.1)(iv). Conversely, if m is odd and so $m + 2n$ is odd, then $P(m, n) \subseteq R^{2(m+2n)-2}$ by (3.1)(iv); if m is even, then n is odd and $(m, n) \notin A$, thus $\bar{w}_{m+2n-2} = 0$ and the result again follows from (3.1)(iv).

(iii) If $(m, n) = (4s + 1, 2t)$—thus $P(m, n)$ is orientable—then $Sq^1(c^{m-2}d^n) = c^{m-1}a^n$ thus implying that $Sq^1 : H^{m+2n-2}(P(m, n); Z_2) \to H^{m+2n-1}(P(m, n); Z_2) \cong Z_2$ is an epimorphism. Hence (iii) is a direct consequence of (3.1)(v).

(2) From (1.5) we may compute the characteristic classes of $P(m, n)$. It is easily verified that the total rational Pontrjagin class of $P(m, n)$ is given by $p = (1 + d^2)^{n+1}$, $d^2 \in H^4(P(m, n); Q)$. Hence, applying (6.1) of [2] for $M^{m+2n} = P(m, n)$, we have that

$$P(m, n) \nsubseteq R^{m+2n+k} \quad \text{where} \quad k = 2\left[\frac{n}{2}\right] - 1$$

$$P(m, n) \not\subset R^{m+2n+k} \quad \text{where} \quad k = 2\left[\frac{n}{2}\right].$$

For $m = 0$ this result agrees with (2.12) thus verifying a remark of Atiyah [2]. Noting that $j^*p = \tilde{p}$, the total rational Pontrjagin class of CP_n, it follows that the methods of [3] (based on the $\hat{A}$-genus) applied directly to $P(m, n)$ are no better than those obtainable from [3] using the composite embedding $CP_n \to P(m, n) \to R^{m+2n+k}$.

(3) Since the two-fold covering $\Phi : S^m \times CP_m \to P(m, n)$ is itself an immersion, it can be used to translate non-immersion theorems for CP_n into the same for $P(m, n)$. More precisely, if $f : P(m, n) \to R^{m+2n+k}$ is an immersion, then so is $f \circ \Phi : S^m \times CP_n \to R^{m+2n+k}$. By a theorem of M. Hirsch [6], this implies that $CP_n \subseteq R^{2n+k}$. Hence the non-immersibility of CP_n with codimension k implies the same for $P(m, n)$. When m is small with respect to n, this idea together with (3.1)(iv) and some results of [12] should give better results than (2.12)(i). However, as m increases we should expect the reverse.

For example, $P(1, n) \subseteq R^{4n}$ for $n = 2^r$ by (3.1)(iv), but $P(1, n) \nsubseteq R^{4n-1}$ (more generally, $P(m, n) \nsubseteq R^{4n+m-2}$) because $CP_n \nsubseteq R^{4n-2}$ for this choice of n. Hence this is best possible.

(4) Our concluding remark, extending Theorem (4.1) of [11], solves the immersion problem for $P(m, 1)$, $m \leqq 8$.

THEOREM (3.8). *$P(m, 1) \subseteq R^{2^f}$ where $f = \varphi(m)$ and $m \neq 2, 6$, $P(2, 1) \subseteq R^5$ and $P(6, 1) \subseteq R^9$.*

Proof. For $P(m, n)$ we have by (1.5) that $\tau \oplus \xi \oplus 2 = (m + 1)\xi \oplus (n + 1)\eta$ and so in $\tilde{K}O(P(m, 1))$, $-T_0 = -(T - (m + 2n)) = -(m + 2)x = (2^f - (m + 2))x$ (here we use the fact that $2y = 0$; note for $n > 1$, this is not true). Hence $g\,.\dim(-T_0) \leqq (2^f - (m + 2))\cdot g.\dim(x) = 2^f - (m + 2)$. Thus by Theorem (2.1) of [11] $P(m, 1) \subseteq R^{2^f}$. $T_0 = 0$ for the exceptional cases $m = 2, 6$, and so the codimension is one. By explicit computation of $\bar{w}(m, 1)$, this is shown to be best possible for $m \leqq 8$.

REFERENCES

1. J. F. ADAMS: Vector fields on spheres, *Ann. Math.* **75** (1962), 603–632.
2. M. F. ATIYAH: Immersions and embeddings of manifolds, *Topology* **1** (1962), 125–132.
3. M. F. ATIYAH and F. HIRZEBRUCH: Quelques théorèmes de non-plongement pour les variétés différentiables, *Bull. Soc. Math. Fr.* **87** (1959), 383–396.
4. A. DOLD: Erzeugende der Thomschen Algebra 𝔑, *Math. Z.* **65** (1956), 25–35.
5. A. HAEFLIGER and M. HIRSCH: On the existence and classification of differentiable embeddings, *Topology* **2** (1963), 129–135.
6. M. HIRSCH: Immersions of manifolds, *Trans. Am. Math. Soc.* **93** (1959), 242–276.
7. M. MAHOWALD: On obstruction theory in orientable fiber bundles, *Trans. Am. Math. Soc.* **110** (1963), 420–429.
8. M. MAHOWALD and F. PETERSON: Secondary cohomology operations on the Thom class, *Topology* **2** (1963), 367–377.
9. W. S. MASSEY and F. PETERSON: On the dual Stiefel–Whitney classes of a manifold, *Bol. Soc. Mat. Mex.* **8** (1963), 1–13.
10. J. MILNOR: Lectures on characteristic classes (mimeographed notes) Princeton University, 1957.
11. B. J. SANDERSON: Immersions and embeddings of projective spaces, *Proc. London Math. Soc.* **14** (1964), 137–153.
12. B. J. SANDERSON and R. L. E. SCHWARZENBERGER: Non-immersion theorems for differentiable manifolds, *Proc. Camb. Phil. Soc.* **59** (1963), 319–322.
13. N. STEENROD and J. H. C. WHITEHEAD: Vector fields on the n-sphere, *Proc. Natn. Acad. Sci. U.S.A.* **37** (1951), 58–63.
14 J. J. UCCI: The K-theory of Dold manifolds (to appear).

Massachusetts Institute of Technology