MATHEMATICS

ON SLIPPAGE TESTS

II. SLIPPAGE TESTS FOR DISCRETE VARIATES

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5. *Slippage tests for some discrete variables*

In this section slippage tests will be discussed for variates which follow the Poisson, the binomial or the negative binomial law. These are special cases of a general class of variates determined by the condition of theorem 5.1. First we shall consider the *Poisson* case in some detail. Suppose we have a set of independent random variables

$$(5.1) \qquad \mathbf{z}_1, \ldots, \mathbf{z}_k$$

distributed according to Poisson distributions, i.e.:

$$(5.2) \qquad P[\mathbf{z}_i = z_i] = \frac{e^{-\mu_i} \mu_i^{z_i}}{z_i!}, \qquad (i = 1, \ldots, k), \mu_i > 0.$$

Now we want to test the hypothesis H_0 that the means μ_i have known ratios

$$(5.3) \qquad H_0: \frac{\mu_i}{\sum_j \mu_j} = p_i, \qquad (i = 1, \ldots, k).$$

This situation occurs for instance if from k Poisson-populations with, under H_0, equal means, known unequal numbers of observations are present and $z_1, \ldots, z_k$ represent the sums of the values obtained in these observations. In this case the p_i are proportional to the numbers of observations. Also k Poisson processes with the same parameter may be observed during different lengths of time. Then the p_i are proportional to these lengths of time.

We want to test H_0 against the alternatives

$$(5.4) \quad H_{1i}: \frac{\mu_i}{\sum_j \mu_j} = c p_i, \ \frac{\mu_l}{\sum_j \mu_j} = \frac{1 - c p_i}{1 - p_i} p_l \quad (l \neq i), 1 < c < \frac{1}{p_i}, \ c \text{ unknown},$$

for one unknown value of i or

(5.5) $H_{2i}: \frac{\mu_i}{\sum_j \mu_j} = c p_i, \ \frac{\mu_l}{\sum_j \mu_j} = \frac{1-c p_j}{1-p_i} p_l \quad (l \neq i), 0 < c < 1, c$ unknown,

for one unknown of i.

A well known property of Poisson-variates is: If $\mathbf{z}_1, \ldots, \mathbf{z}_k$ are independent Poisson-variates with means $\mu_1, \ldots, \mu_k$, then the simultaneous conditional distribution of $\mathbf{z}_1, \ldots, \mathbf{z}_k$ given their sum (i.e. $\sum \mathbf{z}_i = N$, N a constant), is a multinomial distribution with probabilities $p_i = \mu_i / \sum \mu_j$ and number of trials $\sum \mathbf{z}_i = N$. As the hypotheses (5.3), (5.4) and (5.5) only contain the ratios p_i it seems natural to use a conditional test for H_0, using only the multinomial distribution

(5.6) $P[\mathbf{z}_1 = z_1, \ldots, \mathbf{z}_k = z_k | \sum \mathbf{z}_i = N] = \frac{N!}{\prod z_i!} \prod p_i^{z_i}$, if $\sum z_i = N$ and 0 otherwise.

From this it is clear that a test against slippage for Poisson vairates is closely related to a similar test for a multinomial distribution. The reader may easily translate the tests stated here into tests for the multinomial case.

In the next section the following theorem will be proved.

Theorem 5.1. *Suppose the discrete, random variables*

(5.7) $$\mathbf{u}_1, \ldots, \mathbf{u}_k$$

are distributed independently and can take integer values only (the latter assumption is not essential but gives a much simpler notation).

If

(5.8) $$\frac{P[\sum \mathbf{u}_l - \mathbf{u}_i - \mathbf{u}_j = a]}{P[\sum \mathbf{u}_l - \mathbf{u}_i - \mathbf{u}_j = a+1]}$$

where a is an integer, is a non decreasing function of a, then

(5.9) $P[\mathbf{u}_i \geqq u_i \text{ and } \mathbf{u}_j \geqq u_j | \sum \mathbf{u}_l = N] \leqq P[\mathbf{u}_i \geqq u_i | \sum \mathbf{u}_l = N] \cdot P[\mathbf{u}_j \geqq \mathbf{u}_j | \sum \mathbf{u}_l = N]$,

for every pair of integers u_i and u_j and for every non-negative integer N.

In the special case where $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are distributed according to the same type of distribution and this distribution has the property that a sum of k independent variates has again the same type of distribution, it is easy to verify whether condition (5.8) holds or not.

In our case the sum of $k-2$ of the variables $\mathbf{z}_i$ (given by (5.2)) has a Poisson-distribution with mean μ, say. So condition (5.8) states that

(5.10) $$\frac{e^{-\mu}\mu^a}{a!} \cdot \frac{(a+1)!}{e^{-\mu}\mu^{a+1}} = \frac{a+1}{\mu},$$

is non decreasing in a, which is clearly true.

Thus the inequality (5.9) holds for every pair $\mathbf{z}_i$, $\mathbf{z}_j$ and the procedure described in section 2 may be applied to the variables $\mathbf{z}_1, \ldots, \mathbf{z}_k$ under the condition $\sum z_i = N$ [1]). Now the marginal distribution of $\mathbf{z}_i$ under the condition $\sum \mathbf{z}_i = N$ is a binomial one, so when testing H_0 against H_{1i} $(i=1, \ldots, k)$ we compute, if $z_1, \ldots, z_k$ are the observed values and $\sum z_i = N$,

$$(5.11)\quad r_i \overset{\text{def}}{=} P[\mathbf{z}_i \geqq z_i | \sum \mathbf{z}_l = N] = \sum_{x=z_i}^{N} \binom{N}{x} p_i^x (1-p_i)^{N-x} = I_{p_i}(z_i, N-z_i+1),$$

where $I_{p_i}(z_i, N-z_i+1)$ stands for the incomplete **B**-function

$$\frac{N!}{(z_i-1)!(N-z_i)!} \int_0^{p_i} u^{z_i-1}(1-u)^{N-z_i}\, du.$$

Now H_0 is rejected if

$$(5.12)\qquad \min_i r_i \leqq \alpha/k$$

and then we decide that $\mu_j/\sum \mu_i > p_j$ if j is the smallest integer for which $r_j = \min r_i$.

If under $H_0 : \mu_1 = \ldots = \mu_k$, all p_i are equal and the smallest r_i corresponds to the largest value z_i.

The test for slippage to the left is completely analogous.

A table of critical values for max z_i is given in section 11 for the case $p_1 = \ldots = p_k$.

Along the same lines as followed by R. Doornbos and H. J. Prins (1956) in the case of Γ-variates it can be shown that the probability Q_j of making the correct decision when the j-th population has slipped to the right (i.e. H_{1i} is true with $i=j$) satisfies the inequality

$$(5.13)\quad \begin{cases} I_{cp_j}(G_{j,\alpha}, N-G_{j,\alpha}+1)\,[1 - \sum_{i \neq j} I_{\frac{1-cp_j}{1-p_j}p_i}(G_{i,\alpha}, N-G_{i,\alpha}+1)] \leqq \\ \qquad\qquad \leqq Q_j \leqq I_{cp_j}(G_{j,\alpha}, N-G_{j,\alpha}+1). \end{cases}$$

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[1]) The validity of (5.9) in the case of Poisson-variates can also be proved in the following way, using their relation with Γ-variates. The well known relation

$$(1)\quad \begin{cases} P[\mathbf{z}_1 \geqq z_1 | \sum z_i = N] = \sum_{x=z_1}^{N} \binom{N}{x} p_1^x (1-p_1)^{N-x} = \\ \qquad = \frac{N!}{(z_1-1)!(N-z_1)!} \int_0^{p_1} u^{z_1-1}(1-u)^{N-z_1}\, du, \end{cases}$$

can be generalized to

$$(2)\quad \begin{cases} \sum_{x_1=z_{i_1}}^{N} \cdots \sum_{x_r=z_{i_r}}^{N} \frac{N!}{x_1!\ldots x_r!(N-x_1\ldots-x_r)!} p_{i_1}^{x_1} \cdots p_{i_r}^{x_r} (1-p_{i_1} \cdots - p_{i_r})^{N-x_1\ldots-x_r} = \\ \qquad = \frac{N!}{(z_{i_1}-1)!\ldots(z_{i_r}-1)!(N-z_{i_1}\ldots-z_{i_r})!} \int_0^{p_{i_1}} \cdots \\ \qquad \cdots \int_0^{p_{i_r}} u_1^{z_{i_1}-1} \ldots u_r^{z_{i_r}-1}(1-u_1\ldots-u^r)^{N-z_{i_1}\ldots-z_{i_r}}\, du_1 \ldots du_r \\ \qquad (r \leqq k-1, (i_1, \ldots, i_r) \in (1, \ldots, k)), \end{cases}$$

which may be proved by induction or otherwise. Using (2) for $r=2$ it is seen immediately that inequality (4.10) in R. Doornbos and H. J. Prins (1956) is the same as (5.9) for Poisson variates.

Here $G_{l,\alpha}$ $(l=1, \ldots, k)$ is the smallest number which satisfies

$$P[\mathbf{z}_l \geqq G_{l,\alpha} | \sum \mathbf{z}_i = N, H_0] \leqq \alpha/k \tag{5.14}$$

or

$$I_{p_l}(G_{l,\alpha}, N - G_{l,\alpha}+1) \leqq \alpha/k. \tag{5.15}$$

Clearly Q_j converges towards its upper bound when $c \to 1/p_j$ and for each $c \geqq 1$ the factor between square brackets is larger than $1-(k-1)\alpha/k$ according to (5.15).

In the case of slippage to the left we have analogously

$$\begin{cases} [1-I_{cp_j}(g_{j,\alpha}, N-g_{j,\alpha}+1)]\,(1-\alpha) \leqq \\ [1-I_{cp_j}(g_{j,\alpha}, N-g_{j,\alpha}+1)]\,[1-\sum_{i \neq j}\{1-I_{\frac{1-cp_j}{1-p_j}p_i}(g_{i,\alpha}, N-g_{i,\alpha}+1)\}] \leqq \\ \leqq P_j \leqq 1-I_{cp_j}(g_{j,\alpha}, N-g_{j,\alpha}+1), \end{cases} \tag{5.16}$$

where $g_{l,\alpha}$ $(l=1, \ldots, k)$ is the largest number satisfying

$$1-I_{p_l}(g_{l,\alpha}+1, N-g_{l,\alpha}) \leqq \alpha/k. \tag{5.17}$$

We can apply theorem 5.1 also to the case of independent variables

$$\mathbf{v}_1, \ldots, \mathbf{v}_k \tag{5.18}$$

which are distributed according to *binomial* laws with numbers of trials $n_1, \ldots, n_k$ and probabilities of success $p_1, \ldots, p_k$. Now the hypothesis H_0 is

$$H_0 : p_1 = \ldots = p_k = p, \text{ say} \tag{5.19}$$

and the alternatives are

$$\begin{cases} H_{1i} : p_1 = \ldots = p_{i-1} = p_{i+1} = \ldots = p_k = p, \\ \qquad p_i = c \cdot p \quad (1 \leqq c \leqq 1/p), \end{cases} \tag{5.20}$$

for one unknown value of i and

$$\begin{cases} H_{2i} : p_1 = \ldots = p_{i-1} = p_{i+1} = \ldots = p_k = p, \\ \qquad p_i = c \cdot p \quad (0 \leqq c \leqq 1), \end{cases} \tag{5.21}$$

for one unknown value of i.

Because, under H_0, the sum of $(k-2)$ of the variates (5.18) has again a binomial distribution with number of trials, n say, and probability of a success in each trial p, the condition (5.8) of theorem 5.1 reads:

$$\frac{\binom{n}{a} p^a (1-p)^{n-a}}{\binom{n}{a+1} p^{a+1}(1-p)^{n-a-1}} = \frac{a+1}{n-a} \cdot \frac{1-p}{p} \tag{5.22}$$

is a non decreasing function of a, which is true. So in this case also the approximation procedure described in section 2 can be applied to obtain a conditional test for slippage under the condition that the sum of the variates $\sum \mathbf{v}_i$ has a constant value N. The conditional distribution of $\mathbf{v}_i$ is a hypergeometrical one

$$(5.23)\qquad P[\mathbf{v}_i = v_i | \sum \mathbf{v}_i = N] = \binom{n_i}{v_i}\binom{\sum n_j - n_i}{N - v_i}\binom{\sum n_j}{N}^{-1} \qquad (\mathbf{v}_i \geqq 0),$$

so with help of this distribution critical values for the tests with prescribed level of significance may be obtained, in the same way as was done with the Poisson variates.

Provided that none of the values n_i, $\sum n_j - n_i$, N and $\sum n_j - N$ are very small, a good approximation to the sum of the tail terms of the hypergeometric series of equation (5.23) may be obtained from the integral under a normal curve, having the mean $n_i N / \sum n_j$ and variance

$$\frac{n_i(\sum n_j - n_i) N(\sum n_j - N)}{(\sum n_j)^2 (\sum n_j - 1)}.$$

In the special case $n_1 = \ldots = n_k = n$, the test procedure for slippage to the right reduces to comparing the largest variate $\mathbf{v}_m$ with a constant v_0 determined by the level of significance α, such that v_0 is the largest value satisfying

$$P[\mathbf{v}_i \geqq v_0 | \sum \mathbf{v}_i = N] \leqq \alpha / k.$$

The same holds for the variates

$$(5.24)\qquad \mathbf{w}_1, \ldots, \mathbf{w}_k,$$

which are independently distributed according to *negative binomial* laws, with parameters $r_1, \ldots, r_k$ and probabilities $p_1, \ldots, p_k$, i.e.

$$(5.25)\qquad P[\mathbf{w}_i = w_i] = \binom{w_i + r_i - 1}{r_i - 1} p_i^{r_i} q_i^{w_i}$$

where r_i is an integer $\geqq 1$ and $0 \leqq p_i \leqq 1$, whilst $p_i + q_i = 1$.

The hypothesis H_0 is

$$(5.26)\qquad H_0 : q_1 = \ldots = q_k = q, \text{ say}$$

and the alternatives are

$$(5.27)\qquad \begin{cases} H_{1i} : q_1 = \ldots = q_{i-1} = q_{i+1} = \ldots = q_k = q, \\ \qquad q_i = c \cdot q \quad (1 \leqq c \leqq 1/q), \end{cases}$$

for one unknown value of i or

$$(5.28)\qquad \begin{cases} H_{2i} : q_1 = \ldots = q_{i-1} = q_{i+1} = \ldots = q_k = q, \\ \qquad q_i = c \cdot q \quad (0 \leqq c \leqq 1), \end{cases}$$

for one unknown value of i.

The hypotheses are stated in terms of the q_i and not in terms of the p_i in order to obtain that slippage to the right of the i-th population corresponds to a large value of $\mathbf{w}_i$.

Under H_0, the sum of a set of independent negative binomial variates has again a negative binomial distribution with the same probability p (or q) and a parameter r, say, which is the sum of the r_i of the individual variates. So condition (5.8) gives here

$$(5.29)\qquad \frac{\binom{a+r-1}{r-1} p^r q^a}{\binom{a+r}{r-1} p^r q^{a+1}} = \frac{(a+1)}{(a+r)} \cdot \frac{1}{q}$$

is a non decreasing function of a, which is true if $r \geqq 1$. Thus again the method of section 2 may be applied. The conditional distribution of $\mathbf{w}_i$ under the condition $\sum \mathbf{w}_j = N$, has the form

$$(5.30)\quad P[\mathbf{w}_i = w_i | \sum \mathbf{w}_j = N] = \frac{\binom{w_i + r_i - 1}{r_i - 1}\binom{N + \sum r_j - w_i - r_i - 1}{\sum r_j - r_i - 1}}{\binom{N + \sum r_j - 1}{\sum r_j - 1}}, \quad (w_i = 0, \ldots, N).$$

The critical region for the test against H_{1i} $(i = 1, \ldots, k)$ (5.27) consists of large values of the variables $\mathbf{w}_i$. In the case where $r_1 = \ldots = r_k$ the test statistic is the largest variate $\mathbf{w}_m$, when testing against slippage to the right and the smallest when testing against slippage to the left.

· If in the Poisson case (5.1) $p_1 = \ldots = p_k$, then the following optimum property can be proved [2]). As in the case of the normal distribution we denote by D_0 the decision that H_0 is true and by Di $(i = 1, \ldots, k)$ the decision that H_{1i} is true, i.e. that H_{1i} is true and that the i-th population has slipped to the right. Now the procedure:

$$(5.31)\qquad \begin{cases} \text{if } \mathbf{z}_m > \lambda_{\alpha, N} \text{ select } D_m, \\ \text{if } \mathbf{z}_m \leqq \lambda_{\alpha, N} \text{ select } D_0, \end{cases}$$

under the condition that $\sum \mathbf{z}_i = N$, where m is the index of the maximum $\mathbf{z}$ value, maximizes the probability of making a correct decision when H_{1m} is true subject to the following restrictions:

(a) When H_0 is true, D_0 should be selected with probability $\geqq 1 - \alpha$.

(b) The probability of making a correct decision when the i-th population has slipped by an amount c must be the same for $i = 1, \ldots, k$.

The constant $\lambda_{\alpha, N}$ in (5.31) is determined by the level of significance α and depends on N, the sum of the variates. A proof will be given in the next section.

6. *Proofs of the results stated in section* 5

Starting with the proof of theorem 5.1 we have that

$$(6.1)\qquad \frac{P[\mathbf{u}_i = y] \cdot P[\mathbf{u}_j = x] \cdot P[\sum \mathbf{u}_l - \mathbf{u}_i - \mathbf{u}_j = N - x - y]}{P[\mathbf{u}_i = y] \cdot P[\mathbf{u}_j = x+1] \cdot P[\sum \mathbf{u}_l - \mathbf{u}_i - \mathbf{u}_j = N - x - y - 1]}$$

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[2]) In the sequel only the case of slippage to the right is considered but all statements may be easily translated for the other case.

is non-increasing in y, according to (5.8). Dividing (6.1) by the factor

$$\frac{P[\sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x+1]}{P[\sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x]} \tag{6.2}$$

which does not depend on y, (6.1) changes into

$$\frac{P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x]}{P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x+1]}. \tag{6.3}$$

Thus also (6.3) is non increasing in y for all values of x. This means that there exists a value y_0, which may depend on x, which has the property that

$$\begin{cases} P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x] \geqq P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x+1], \text{if } y \geqq y_0 \\ P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x] \leqq P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x+1], \text{if } y < y_0 \end{cases}. \tag{6.4}$$

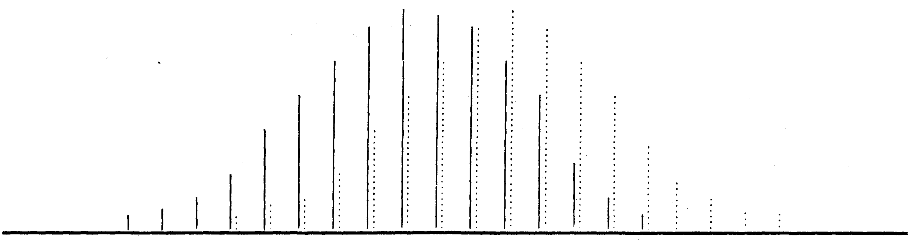

Fig. 6.1. $P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N$ and $\mathbf{u}_j = x]$ (dotted lines), and $P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N$ and $\mathbf{u}_j = x+1]$ (full lines).

This situation is sketched in figure 6.1. It follows that for each value of u_i

$$P(x) \stackrel{\text{def}}{=} \sum_{y=u_i}^{\infty} P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x] \tag{6.5}$$

is a non increasing function of x. Now

$$\begin{cases} \dfrac{P[\mathbf{u}_i \geqq u_i \text{ and } \mathbf{u}_j \geqq u_j | \sum \mathbf{u}_l = N]}{P[\mathbf{u}_j \geqq u_j | \sum \mathbf{u}_l = N]} = \\ = \dfrac{\sum_{x=u_j}^{\infty} P[\mathbf{u}_j = x | \sum \mathbf{u}_l = N] \sum_{y=u_i}^{\infty} P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = x]}{\sum_{x=u_j}^{\infty} P[\mathbf{u}_j = x | \sum \mathbf{u}_l = N]} \leqq \\ \leqq \sum_{y=u_i}^{\infty} P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = u_j]. \end{cases} \tag{6.6}$$

In the same way we have

$$\frac{P[\mathbf{u}_i \geqq u_i \text{ and } \mathbf{u}_j < u_j | \sum \mathbf{u}_l = N]}{P[\mathbf{u}_j < u_j | \sum \mathbf{u}_l = N]} \geqq P[\mathbf{u}_i = y | \sum \mathbf{u}_l = N \text{ and } \mathbf{u}_j = u_j]. \tag{6.7}$$

From (6.6) and (6.7) it follows that, in the notation of (2.6), where $u_i = g_i + 1$ and $u_j = g_j + 1$, whilst $\mathbf{u}_i$ under the condition $\sum \mathbf{u}_l = N$ stands for $\mathbf{x}_i$ and $\mathbf{u}_j$ under the condition $\sum \mathbf{u}_l = N$ for $\mathbf{x}_j$

$$(6.8) \qquad \frac{q_{i,j}}{q_j} \leqq \frac{q_i - q_{i,j}}{1 - q_j}$$

or

$$(6.9) \qquad q_{i,j} \leqq q_i \cdot q_j$$

which proves the theorem, because (6.9) is the same as (5.9).

The proof of the optimality of our procedure in the Poisson case is a straightforward application of the theory of A. WALD (1950). It consists mainly in showing that for any c and N there exists a set of non zero a priori probabilities $g_0, \ldots, g_k$, which are functions of N so that, when g_i is the probability that D_i is the correct decision the decision procedure described in section 5 maximizes the probability of making the correct decision. Assuming that this has been demonstrated, it follows easily that (5.31) is the optimum solution. For suppose there exists an allowable decision procedure, which for some c and N has a greater probability than (5.31) of making the correct decision when some category has slipped to the right by an amount c. Then this procedure will have a greater probability than (5.31) of making a correct decision (for these values of c and N) with respect to any set of a priori probabilities, with $\max_i g_i > 0$, which would be a contradiction.

According to A. WALD (1950), pp. 127–128 the optimum solution is given by the rule: "For each j ($j = 0, \ldots, k$) decide D_j for all points in the sample space where j is the smallest integer for which $g_j f_j = \max \{g_0 f_0, \ldots, g_k f_k\}$, where f_j is the joint elementary probability law of $\mathbf{z}_1, \ldots, \mathbf{z}_k$ under the hypothesis H_{1j}."

We consider the special a priori distribution $g_0 = 1 - g\ k$, $g_1 = \ldots = g_k = g$. For instance the region where D_1 is selected is given by the points in the sample space where $f_1 > f_i$ ($i = 2, \ldots, k$) and $g f_1 > (1 - gk) f_0$.

Here we have

$$(6.10) \quad \begin{cases} f_0(z_1, \ldots, z_k | \sum \mathbf{z}_l = N) = \dfrac{N!}{\prod z_l!} \left(\dfrac{1}{k}\right)^N \\ f_i(z_1, \ldots, z_k | \sum \mathbf{z}_l = N) = \dfrac{N!}{\prod z_l!} \left(\dfrac{1}{k}\right)^N c^{z_i} \left(\dfrac{k-c}{k-1}\right)^{N - z_i}, \qquad (1 < c < k). \end{cases}$$

As $c^{z_i} \left(\frac{k-c}{k-1}\right)^{N-z_i}$ is monotonously increasing in z_i for $1 < c < k$ the region where $f_1 > f_i$ is given by $z_1 > z_i$ and the region where $g f_1 > (1 - gk) f_0$ by $z_1 > L$, L depending on c and N.

Thus the Bayes solution is: if z_m is the maximum of $z_1, \ldots, z_k$ select D_m if $z_m > L$, otherwise select D_0. Define the function $F(g)$ by the equation

$$(6.11) \qquad F(g) = c^{\lambda_{\alpha,N}} \left(\frac{k-c}{k-1}\right)^{N - \lambda_{\alpha,N}} - \frac{1-gk}{g}$$

where $\lambda_{\alpha,N}$ is the constant used in (5.31). It is obvious that $F(g)$ is a continuous function of g, with $F(1/k)>0$ and that there exists a δ with $0<\delta<1/k$ such that $F(\delta)<0$. Hence there exists a value g^* with $0<\delta<g^*<1/k$ such that $F(g^*)=0$. To get the Bayes solution relative to $(1-kg^*, g^*, \ldots, g^*)$ it is only necessary in the solution given above to replace L by $\lambda_{\alpha,N}$. Thus the procedures (5.31) is the Bayes solution relative to $(1-kg^*, g^*, \ldots, g^*)$ which proves that it is an optimum one.

REFERENCES

DOORNBOS, R. and H. J. PRINS, Slippage tests for a set of gamma-variates, Indagationes Mathematicae, **18**, 329–337 (1956).

WALD, A., Statistical Decision Functions, New York and London, (1950).

(To be continued).