A generalized real-valued measure of the inequality associated with a fuzzy random variable

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Abstract

Fuzzy random variables have been introduced by Puri and Ralescu as an extension of random sets. In this paper, we first introduce a real-valued generalized measure of the “relative variation” (or inequality) associated with a fuzzy random variable. This measure is inspired in Csiszár’s $f$-divergence, and extends to fuzzy random variables many well-known inequality indices. To guarantee certain relevant properties of this measure, we have to distinguish two main families of measures which will be characterized. Then, the fundamental properties are derived, and an outstanding measure in each family is separately examined on the basis of an additive decomposition property and an additive decomposability one. Finally, two examples illustrate the application of the study in this paper. © 2001 Elsevier Science Inc. All rights reserved.

1. Introduction

Commonly, random experiments are assumed to involve a quantification process which can be identified with a real- or vectorial-valued random
variable. However, random experiments may have an associated imprecision in this quantification process, so that this process can be identified either with a set-valued or a fuzzy set-valued (measurable) mapping.

Random sets formalize measurable set-valued mappings (see, for instance, [13,23,24], and others) which convert experimental outcomes into sets of certain spaces. Fuzzy random variables in Puri and Ralescu’s sense (also referred to in the literature as random fuzzy sets) have been introduced [31] to model fuzzy set-valued measurable mappings converting experimental outcomes into fuzzy sets of certain spaces (often, Euclidean ones).

Fuzzy random variables extend the concept of random sets and, hence, that of random variables. Furthermore, the expected value of a fuzzy random variable has been defined ([31] – see also [14]) as an extension of the Aumann integral of a random set (cf. [2]).

The expected value of a fuzzy random variable has been introduced as a summary fuzzy-valued measure of the “central tendency” of its values, and extends the well-known mean value of the real-valued random variables.

Another useful summary measure which is considered for many real-valued random variables (especially for those concerning certain economic attributes, like income, wealth, and so on) is the inequality associated with them in a population. In the inequality measurement, the relative variation of variables is quantified by considering “how many times each variable value has as much as a referential value” (usually, the mean value), and therefore it distinguishes between values being above and below the referential one.

Atkinson [1] has emphasized that the inequality measures “… are widely used for two purposes:

(i) to compare (income, wealth, etc.) distributions;
(ii) to attach some measure to the degree of inequality, or to give some idea whether the inequality is ‘large’ or ‘small’…”

To achieve the purpose (ii) some studies have been already developed [3,6,7,11,19,20,27–29], in which several fuzzy-valued measures of the degree of inequality of a population with respect to a (real- or fuzzy-valued) random variable have been stated.

Nevertheless, if we want to give an appropriate answer to the purpose (i), one can either make use of a real-valued measure of inequality, or apply a ranking of fuzzy sets after a fuzzy-valued measurement of inequality.

The generalized measure we will present in Section 3 (after some preliminaries in Section 2) combines both options. This measure is based on Csiszár’s $f$-divergence and includes as a particular case the extension to fuzzy random variables of the additively decomposable indices (see, for instance [32]).

Section 4 presents the characterization of the functions on which the most operational and valuable inequality indices are based.

In Section 5, we will examine some fundamental properties of the generalized measure. The discussion of some relevant properties allows us to justify
why we have paid special attention in Section 4 to two main families of inequality measures, and also why we have examined separately an outstanding measure in each of these families (the hyperbolic index and the index of the Shannon type) for some concrete properties.

Two examples illustrate in Section 6 the application of the introduced measures.

Finally, a few comments are included to end the paper.

2. Preliminary concepts

Throughout this paper, the involved data are assumed to be imprecise, and the model for these imprecise data are certain fuzzy sets of the space $\mathbb{R}$ of real numbers. We will consider the fuzzy subsets of $\mathbb{R}$ to satisfy the following conditions:

Definition 2.1. $F(\mathbb{R})$ denotes the class of fuzzy subsets of $\mathbb{R}$, $\tilde{V} : \mathbb{R} \rightarrow [0, 1]$, satisfying that

(i) the $\alpha$-level set of $\tilde{V}$, $\tilde{V}_\alpha = \{x \in \mathbb{R} | \tilde{V}(x) \geq \alpha\}$, is compact for all $\alpha \in (0, 1]$, 
(ii) $\tilde{V}_1 = \{x \in \mathbb{R} | \tilde{V}(x) = 1\} \neq \emptyset$ (i.e., $\tilde{V}$ is normal), 
(iii) $\tilde{V}$ is a convex fuzzy subset, that is, for any $\alpha \in (0, 1]$ the $\alpha$-level set $\tilde{V}_\alpha$ is a convex subset of $\mathbb{R}$, 
(iv) the closed convex hull of the support of $\tilde{V}$ (where supp $\tilde{V} = \{x \in \mathbb{R} | \tilde{V}(x) > 0\}$), which in this case coincides with the closure of supp $\tilde{V}$ and is denoted by $\tilde{V}_0$, is compact.

Obviously, $F(\mathbb{R})$ can be briefly described as the class of fuzzy subsets $\tilde{V}$ of $\mathbb{R}$ such that $\tilde{V}_\alpha \in \mathcal{K}(\mathbb{R})$ for all $\alpha \in [0, 1]$, with $\mathcal{K}(\mathbb{R})$ being the class of non-empty compact intervals contained in $\mathbb{R}$.

On $F(\mathbb{R})$ we can establish the extension of the algebraic operations based on Zadeh’s principle [34]. On the basis of some conclusions (see, for instance [17]) derived from Nguyen’s results in [26], the arithmetic of elements in $F(\mathbb{R})$ reduces to the interval arithmetic. In particular, the fuzzy sum of $\tilde{V}$, $\tilde{W} \in F(\mathbb{R})$ is given by the element $\tilde{V} \oplus \tilde{W} \in F(\mathbb{R})$ such that for all $\alpha \in [0, 1]$

$$
(\tilde{V} \oplus \tilde{W})_\alpha = \text{Minkowski's sum of } \tilde{V}_\alpha \text{ and } \tilde{W}_\alpha 
=$$ 

$$
\left[ \inf \tilde{V}_\alpha + \inf \tilde{W}_\alpha, \sup \tilde{V}_\alpha + \sup \tilde{W}_\alpha \right],
$$

the fuzzy product of $\tilde{V} \in F(\mathbb{R})$ by a real number $\lambda \geq 0$ is given by the element $\lambda \odot \tilde{V} \in F(\mathbb{R})$ such that for all $\alpha \in [0, 1]$

$$
\lambda \odot \tilde{V}_\alpha = \lambda \tilde{V}_\alpha
$$
The fuzzy quotient of $\tilde{V} \in \mathcal{F}_c((0, +\infty))$ and $\tilde{W} \in \mathcal{F}_c((0, +\infty))$ (i.e., $\tilde{V}, \tilde{W} \in \mathcal{F}_c(\mathbb{R})$ and $\tilde{V}_0, \tilde{W}_0 \subset (0, +\infty))$ is given by the element $\tilde{V} \circ \tilde{W} \in \mathcal{F}_c(\mathbb{R})$ such that for all $z \in [0, 1]$

$$\left( \tilde{V} \circ \tilde{W} \right)_z = \left[ \inf \tilde{V}_z, \sup \tilde{V}_z \right].$$

Some studies and results on the above operations can be found, for instance, in [9,10,12,22].

To formalize the notion of measurable fuzzy set-valued mapping, a metric must be considered on the space $\mathcal{F}_c(\mathbb{R})$. This metric has been introduced by Puri and Ralescu [30] as an extension of the Hausdorff metric from $\mathcal{K}_c(\mathbb{R})$ to $\mathcal{F}_c(\mathbb{R})$. This extended metric is defined as follows.

**Definition 2.2.** If $\tilde{V}, \tilde{W} \in \mathcal{F}_c(\mathbb{R})$, the “generalized Hausdorff metric” between $\tilde{V}$ and $\tilde{W}$, $d_\infty(\tilde{V}, \tilde{W})$, is given by

$$d_\infty(\tilde{V}, \tilde{W}) = \sup_{z \in [0, 1]} d_H(\tilde{V}_z, \tilde{W}_z),$$

where $d_H$ denotes the well-known Hausdorff metric on $\mathcal{H}_c(\mathbb{R})$, which for $K, K' \in \mathcal{H}_c(\mathbb{R})$ is given by

$$d_H(K, K') = \max \left\{ \sup_{a \in K} \inf_{b \in K'} |a - b|, \sup_{b \in K'} \inf_{a \in K} |a - b| \right\},$$

and because of the convexity of elements in $\mathcal{H}_c(\mathbb{R})$, it can be alternatively defined by

$$d_H(K, K') = \max \{|\inf K - \inf K'|, |\sup K - \sup K'|\}.$$

On the basis of the above concepts, we can now recall the notion of fuzzy random variable. Consider a random experiment which is modelled by means of a probability space $(\Omega, \mathcal{A}, P)$. Then,

**Definition 2.3.** A mapping $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c(\mathbb{R})$ is said to be a fuzzy random variable associated with $(\Omega, \mathcal{A})$ if, and only if, $\mathcal{X}$ is $(\mathcal{A}, \mathcal{B}_d)$-measurable, $\mathcal{B}_d$ being the $\sigma$-field generated by the topology induced from $d_\infty$ on $\mathcal{F}_c(\mathbb{R})$.

**Remark 2.1.** An important fact to be emphasized in working with fuzzy random variables is the connection of this concept with that of random sets. Thus, if $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c(\mathbb{R})$ is a fuzzy random variable, then for all $z \in [0, 1]$ the $z$-level
function $X: \Omega \rightarrow \mathcal{K}(\mathbb{R})$ is defined so that $X_\omega = (X(\omega))_\omega$ for all $\omega \in \Omega$, is a convex compact random set (see [14,31]), that is, $X$ is a Borel-measurable set-valued mapping.

The *expected value* of a fuzzy random variable has been introduced by Puri and Ralescu [31] as follows.

**Definition 2.4.** If $X: \Omega \rightarrow \mathcal{F}_c(\mathbb{R})$ is a fuzzy random variable, the expected value of $X$ is the unique fuzzy subset of $\mathbb{R}$ (if it exists), $E(X)$ such that for all $\alpha \in [0,1]$ we have that $(E(X))_\alpha = E(X_\alpha)$, that is, $(E(X))_\alpha$ equals the Aumann integral of the random set $X_\alpha$.

**Remark 2.2.** Since $X$ is $\mathcal{F}_c(\mathbb{R})$-valued, then $(E(X))_\alpha = [E(\inf X_\alpha), E(\sup X_\alpha)]$ for all $\alpha \in [0,1]$ (where $E$ denotes the expectation of a real-valued random variable). Furthermore, if $X$ takes on a finite number of different values, $\bar{x}_1, \ldots, \bar{x}_k$, with induced probabilities $p_1, \ldots, p_k$, respectively ($p_i > 0, i = 1, \ldots, k, \sum_{i=1}^k p_i = 1$), then

$$E(X) = p_1 \odot \bar{x}_1 \oplus \cdots \oplus p_k \odot \bar{x}_k,$$

(i.e., $(E(X))_\alpha = [\sum_{i=1}^k p_i \inf(\bar{x}_i)_\alpha, \sum_{i=1}^k p_i \sup(\bar{x}_i)_\alpha]$ for all $\alpha \in [0,1]$).

Throughout the present paper we will assume that fuzzy random variables take on values on $\mathcal{F}_c((0, +\infty))$, that is, we will consider positive fuzzy random variables, where

**Definition 2.5.** A fuzzy random variable $X: \Omega \rightarrow \mathcal{F}_c((0, +\infty))$ (i.e., $X: \Omega \rightarrow \mathcal{F}_c(\mathbb{R})$ with $X_0(\omega) \subset \mathcal{P}((0, +\infty))$), is said to be a positive fuzzy random variable.

The assumption above extends the one commonly made for real-valued random variables.

**Remark 2.3.** In accordance with Puri and Ralescu [31], in the case of positive fuzzy random variables $X$, the condition $E(\sup X_0) < \infty$ guarantees the existence of $E(X)$.

A special type of fuzzy random variable, we will also refer sometimes to the following present work:

**Definition 2.6.** A fuzzy random variable $X: \Omega \rightarrow \mathcal{F}_c(\mathbb{R})$ is said to be degenerate, if there exists a fuzzy number $\bar{V} \in \mathcal{F}_c(\mathbb{R})$ such that $X = \bar{V}$ almost surely $[P]$. In particular, if $\bar{V}$ reduces to the indicator function of an interval of
$X_c(\mathbb{R})$, $X$ is said to be a fuzzy random variable degenerate at an interval value, and if $\widetilde{V}$ reduces to the indicator function of a singleton in $\mathbb{R}$, $X$ is said to be a fuzzy random variable degenerate at a real value.

3. The real-valued $f$-inequality indices for fuzzy random variables

Assume that we consider a general population $\Omega$ and let $(\Omega, \mathcal{A}, P)$ be a probability space defined on it. Let $X: \Omega \rightarrow \mathcal{F}_c((0, +\infty))$ be a fuzzy random variable associated with $(\Omega, \mathcal{A}, P)$ such that $E(\sup X) < \infty$.

Let $f: (0, +\infty) \rightarrow \mathbb{R}$ be a strictly convex (intended as strictly convex downward, that is, for all $\lambda \in (0, 1)$ and $x$, $y \in (0, +\infty)$ with $x \neq y$, we have that $f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$) and monotonic function satisfying that $f(u) + f(1/u) \geq 0$ for all $u \in (0, +\infty)$ and $f(1) = 0$.

To quantify the inequality associated with $X$ in $\Omega$ by means of a real-valued measurement, we suggest the following indices:

**Definition 3.1.** The real-valued $f$-inequality index associated with $X$ in the population $\Omega$ given by the value (if it exists)

$$I_f(X) = F\left(\tilde{I}_f(X)\right),$$

where $F$ is the ranking function in $\mathcal{F}_c(\mathbb{R})$ introduced by Yager [33] and associating with any $\widetilde{V} \in \mathcal{F}_c(\mathbb{R})$ the real value

$$F(\widetilde{V}) = \frac{1}{2} \int_{[0,1]} \left[ \sup \tilde{V}_x + \inf \tilde{V}_x \right] dx,$$

and

$$\tilde{I}_f(X) = \tilde{E}\left[ f(X \otimes \tilde{E}(X)) \right]$$

is the fuzzy $f$-inequality index associated with $X$ in $\Omega$ [6,7], where $f(X \otimes \tilde{E}(X))$ denotes the image of $X \otimes \tilde{E}(X)$ induced from $f$ on the basis of Zadeh’s extension principle (that is),

$$\left(f(X \otimes \tilde{E}(X))\right)_x = \min \left\{ f\left(\frac{\inf X}{E(\sup X)}\right), f\left(\frac{\sup X}{E(\inf X)}\right) \right\},$$

$$\max \left\{ f\left(\frac{\inf X}{E(\sup X)}\right), f\left(\frac{\sup X}{E(\inf X)}\right) \right\}$$

for all $x \in [0, 1]$.

The conditions assumed for $f$ are satisfied by the functions associated with the additively decomposable indices for $x \neq 0, 1$, and the function
\[ f(x) = - \log x \] associated with the index of the Shannon type (see Section 4). However, the function associated with Theil's index \( f(x) = x \log x \) is non-monotonic and it would be removed from the present study, although a weak extra assumption on \( X \) and properties in Section 5 will allow us to include it in this study.

**Remark 3.1.** The index above introduced can be viewed as an extension of Csiszár's \( f \)-divergence for a real valued random variable \( X \) between the probability distributions \( P \) and \( Q = P : X/E(X) \), with respect to \( P \) (see [21] for a detailed explanation of this point).

The \( f \)-inequality index is not necessarily defined for a fuzzy random variable in any population, and conditions guaranteeing the existence of \( \bar{I}_f \) depend on the function \( f \).

The following conditions, which ensure the existence of \( \bar{I}_f(X) \) (see [6]), would also guarantee that \( I_f(X) \in \mathbb{R} \).

Consider a probability space \((\Omega, \mathcal{A}, P)\), and assume that \( X : \Omega \to \mathcal{F}_c((0, +\infty)) \) is a positive fuzzy random variable associated with \((\Omega, \mathcal{A}, P)\) and such that \( E(\sup X_0) < \infty \). Let \( f : (0, +\infty) \to \mathbb{R} \) be a strictly convex and monotonic function belonging to \( C^1 \) (class of real-valued functions with continuous derivative functions) and satisfying that \( f(u) + f(1/u) \geq 0 \) for all \( u \in (0, +\infty) \) and \( f(1) = 0 \).

Then, the real-valued \( f \)-inequality index \( I_f(X) \) is well-defined and belongs to \( \mathbb{R} \) if, and only if,

1. \( f(\inf X_0/E(\sup X_0)) \in L^1(\Omega, \mathcal{A}, P) \) if \( f \) is non-increasing,
2. \( f(\sup X_0/E(\inf X_0)) \in L^1(\Omega, \mathcal{A}, P) \) if \( f \) is non-decreasing.

Furthermore, and due to the results from López-Díaz and Gil [18] (which have been really established for the more general criterion of Campos and González [5]), one can conclude that if \( X \) is a fuzzy random variable with expected value \( \bar{E}(X) \in \mathcal{F}_c(\mathbb{R}) \), then

\[
F(\bar{E}(X)) = E(F \circ \bar{X}),
\]

so that the value of the ranking function \( F \) for the (fuzzy) expected value of the fuzzy random variable \( \bar{X} \) reduces to the expected value of the real-valued random variable \( F \circ \bar{X} \).

Consequently,

**Theorem 3.1.** Let \((\Omega, \mathcal{A}, P)\) be a probability space and let \( X : \Omega \to \mathcal{F}_c((0, +\infty)) \) be a fuzzy random variable such that \( E(\sup X_0) < \infty \). Consider a mapping \( f : (0, +\infty) \to \mathbb{R} \) strictly convex and monotonic, belonging to \( C^1 \), and satisfying that \( f(u) + f(1/u) \geq 0 \) for all \( u \in (0, +\infty) \) and \( f(1) = 0 \).
If either (1) or (2) are also satisfied, then,

\[ I_f(\mathcal{X}) = \frac{1}{2} \int_{[0,1]} \mathbb{E} \left[ f\left( \frac{\inf \mathcal{X}_z}{\max \mathcal{X}_z} \right) + f\left( \frac{\sup \mathcal{X}_z}{\min \mathcal{X}_z} \right) \right] \, dz. \]

Obviously, if we forget about either the convexity condition for the values of \( \mathcal{X} \) or the monotonicity of the function \( f \), we could not characterize the \( f \)-indices as presented in Theorem 3.1, which would mean an important (both, practical and theoretical) inconvenience for most of the studies in this paper.

In the following section, we will develop characterizations for two relevant families of functions satisfying the conditions we have assumed for \( f \). These two families cover the extension of the most useful inequality indices, and the differences between them will be valuable for purposes of discussing some of the fundamental properties of the corresponding indices in Section 5.

4. Characterizing two families of functions

In the definition of the \( f \)-inequality indices, the function \( f \) is assumed to satisfy several conditions, some of them not being required in the real-valued case (see [21]).

The constraints imposed by the new conditions (and, especially, by the ones assumed in the analysis of the minimality properties of the inequality measures in Section 5), allow us to establish certain characterizations of the corresponding functions \( f \).

First of all, we are going to look for the subfamily of the twice-differentiable functions \( f \) satisfying that \( f(u) + f(1/u) = 0 \) for all \( u \in (0, +\infty) \).

**Theorem 4.1.** The set of the solutions of the functional equation \( f(u) + f(1/u) = 0 \) for all \( u \in (0, +\infty) \) with \( f : (0, +\infty) \to \mathbb{R} \) twice-differentiable, strictly convex and monotonic and satisfying that \( f(1) = 0 \), is given by the family of functions \( f(u) = h(\log u) \) with \( h : \mathbb{R} \to \mathbb{R} \) such that for all \( x \in \mathbb{R} \) we have that

(i) \( h(x) = -h(-x) \),

(ii) \( h \) (strictly) decreasing, and

(iii) \( h''(x) \geq h'(x) \).

**Proof.** The transformations \( x = \log u \) and \( h(x) = f(e^x) \) allow us to conclude that \( f(u) + f(1/u) = 0 \) for all \( u \in (0, +\infty) \) if, and only if, we have that \( f(u) = h(\log u) \) for all \( u \in (0, +\infty) \) and \( h(x) = -h(-x) \) for all \( x \in \mathbb{R} \). Furthermore, since \( h(0) = 0 \), we then ensure that \( f(1) = 0 \).
On the other hand, to prove that condition (ii) must hold we should take into account that because of the (strict) increasing of the function $g$ such that $g(x) = \log x$, then $f = h \circ g$ is strictly monotonic if, and only if, $h$ is also strictly monotonic and $f$ and $h$ are both increasing or decreasing.

In this case, $h$ must be decreasing since otherwise the convexity of $f$ would entail for all $u \in (0, +\infty)$ that

$$0 \leq f''(u) = \frac{1}{u^2} \left( h''(\log u) - h'(\log u) \right),$$

whence $h''(\log u) \leq 0$, so that $h$ should be convex. If $h$ should be convex, increasing and satisfying that $h(x) = -h(-x)$ for all $x \in \mathbb{R}$, $h$ has to be defined as $h(x) = ax$ with $a > 0$, and hence $f(u) = a \log u$ for all $u \in (0, +\infty)$ with $a > 0$ could not be convex.

Consequently, the differentiability of $f$ guarantees that of $h$, and therefore $h'(x) \leq 0$ for all $x \in \mathbb{R}$.

Finally, since $f$ is twice-differentiable, $h$ must be also twice-differentiable, and in virtue of the convexity of $f$ we will have that

$$0 \leq f''(u) = \frac{1}{u^2} \left( h''(\log u) - h'(\log u) \right),$$

so that we can conclude that $h''(\log u) \geq h'(\log u)$ for all $u \in (0, +\infty)$ must be satisfied. $\square$

**Remark 4.1.** It can be pointed out that the conditions (i) and (ii) are necessary, even if $f$ is not assumed to be differentiable. The index of the Shannon type, which corresponds to $f(x) = -\log x$ for all $x \in (0, +\infty)$ is the one associated with $h(x) = -x$, whence it is obviously an index associated with a solution of the functional equation in Theorem 4.1.

Secondly, we are interested in looking for the solutions of the family of the twice-differentiable functions $f$ satisfying that $f(u) + f(1/u) \geq 0$ for all $u \in (0, +\infty)$ with equality if, and only if, $u = 1$, (that is, $f(u) + f(1/u) = 0$ if $u = 1$, $> 0$ otherwise), we can obtain by following arguments similar to those in Theorem 4.1 that

**Theorem 4.2.** The set of the functions $f : (0, +\infty) \rightarrow \mathbb{R}$ being twice-differentiable, strictly convex and monotonic and satisfying that $f(u) + f(1/u) > 0$ for all $u \in (0, +\infty) \setminus \{1\}$ and $f(1) = 0$, is the family of functions $f(u) = h(\log u)$ with $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

(i) $h(x) > -h(-x)$ for all $x \in \mathbb{R} \setminus \{0\}$ and $h(0) = 0$,

(ii) $h$ strictly monotonic, and

(iii) $h''(x) \geq h'(x)$ for all $x \in \mathbb{R}$.
Remark 4.2. The extension of the additively decomposable indices for $x \neq 0, 1$, which correspond to $f(x) = x^2 - 1$ if $x \notin [0, 1]$ and $= 1 - x^2$ if $x \in (0, 1)$ (i.e., $h(x) = e^{2x} - 1$ if $x \notin [0, 1]$ and $= 1 - e^{2x}$ if $x \in (0, 1)$) is associated with a function $f$ being a solution of the functional equation in Theorem 4.2.

If we want now to find the solutions of the general functional inequation $f(u) + f(1/u) \geq 0$ for all $u \in (0, +\infty)$ with $f : (0, +\infty) \to \mathbb{R}$ being twice-differentiable, strictly convex and monotonic and satisfying that $f(1) = 0$, the transformations in the proof of Theorem 4.1 indicate that $h(x) \geq -h(-x)$ and $h(0) = 0$.

Since $h$ is strictly monotonic, then to determine functions $h$ satisfying the latter inequality we can reason as follows:

Let $h_1 : [0, +\infty) \to \mathbb{R}$ and $h_2 : (-\infty, 0] \to \mathbb{R}$ be (strictly) decreasing functions such that $h_1(0) = h_2(0) = 0$ and $h_2(x) \geq -h_1(-x)$. Consider $h : \mathbb{R} \to \mathbb{R}$, defined so that

$$h(x) = \begin{cases} h_1(x) & \text{if } x \geq 0, \\ h_2(x) & \text{otherwise}. \end{cases}$$

Function $h$ is clearly (strictly) decreasing and satisfies that $h(x) \geq -h(-x)$ and $h(0) = 0$. Therefore, the function $f(u) = h(\log u)$ is a solution for the considered problem, whenever $f$ is convex. On the other hand, if $f$ is assumed to be twice-differentiable, then $f''(u) \geq 0$ if, and only if, $h'' \geq h'$, and this happens if, and only if,

$$\begin{cases} h''_1 \geq h'_1 & \text{if } x \geq 0, \\ h''_2 \geq h'_2 & \text{otherwise}, \end{cases}$$

the function $f(u) = h(\log u)$, where $h$ is constructed in accordance with the suggested procedure, is a solution of the considered problem if, and only if, $h''_1 \geq h'_1$ and $h''_2 \geq h'_2$.

It should be emphasized that, since $h(0) = 0$ and $h$ must be strictly monotonic, then either $h(x) \geq 0$ for all $x \in (-\infty, 0]$ and $h(x) \leq 0$ for all $x \in [0, +\infty)$ (which is the considered case), or $h(x) \leq 0$ for all $x \in (-\infty, 0]$ and $h(x) \geq 0$ for all $x \in [0, +\infty)$. In this latter case, $h$ would be increasing, but since $h$ must hold that $h(x) \geq -h(-x)$, if as in the preceding case one constructs (strictly) increasing functions $h_1 : [0, +\infty) \to \mathbb{R}$ and $h_2 : (-\infty, 0] \to \mathbb{R}$ such that $h_1(0) = h_2(0) = 0$ and $h_1(x) \geq -h_2(-x)$, then the function $h : \mathbb{R} \to \mathbb{R}$, defined so that

$$h(x) = \begin{cases} h_1(x) & \text{if } x \geq 0, \\ h_2(x) & \text{otherwise}, \end{cases}$$

is clearly increasing and satisfies that $h(x) \geq -h(-x)$ and $h(0) = 0$.

Consequently, the function $f(u) = h(\log u)$ is a solution of the considered problem, whenever $f$ is convex. But since $h$ is increasing and $f$ is convex, then $h$
should be convex. In this way we are constrained (under the construction we have just suggested) to the functions

\[ h(x) = \begin{cases} h_1(x) & \text{if } x \geq 0, \\ h_2(x) & \text{otherwise}, \end{cases} \]

being increasing, convex and such that \( f(u) = h(\log u) \) is convex. As in the preceding case, if \( f \) is twice-differentiable, the function \( f(u) = h(\log u) \) is a solution of the considered problem if, and only if, \( h''_1 \geq h'_1 \) and \( h''_2 \geq h'_2 \).

Finally, it should be pointed out that there exist functions \( f \) satisfying the general conditions assumed in Section 3 for the extended real-valued inequality indices, although they do not fit the models in Theorems 4.1 and 4.2. In this way, the function \( f(x) = 1/\exp x \) if \( x \in (0, 1/e] \) and \( = -\log x \) if \( x \in (1/e, +\infty) \), is such that \( f \in C^1 \), it is strictly convex and monotonic, satisfies that \( f(1) = 0 \), and furthermore \( f(x) + f(1/x) = 0 \) if \( x \in [1/e, e] \) and \( > 0 \) otherwise.

5. Relevant properties of the real-valued \( f \)-inequality indices

From now on, and without mentioning it for each property, we will consider a probability space \((\Omega, \mathcal{A}, P)\), and a function \( f : (0, +\infty) \to \mathbb{R} \) which is strictly convex and monotonic, belongs to \( C^1 \), and satisfies that \( f(u) + f(1/u) \geq 0 \) for all \( u \in (0, +\infty) \) and \( f(1) = 0 \). We also will suppose that if \( f \) is non-increasing condition (1) in Section 3 is satisfied, and if \( f \) is non-decreasing condition (2) in Section 3 is satisfied. The strict convexity of \( f \) could be weakened by assuming \( f \) is convex, but in such a case we could not establish the conditions under which the equality would hold in several of the properties below.

The indices introduced in Section 3 are not changed by an equiproportional real-valued variation in the values of the fuzzy random variables. In other words, and in accordance with Kölm’s terminology [15,16] these indices are “rightist measures”, and following Blackorby and Donaldson [4] they are measures of “relative inequality”. In this way, the following result extends to fuzzy random variables, the mean independence property (also referred to as (positive) scale invariance or homogeneity of degree 0) of most of the classical inequality indices.

**Theorem 5.1** (Mean independence). If \( \mathcal{X} : \Omega \to \mathcal{F}_{c}((0, +\infty)) \) is a fuzzy random variable such that \( E(\sup \mathcal{X}_0) < \infty \), then for all \( k \in (0, +\infty) \) we have that \( I_f(k \circ \mathcal{X}) = I_f(\mathcal{X}) \).

**Proof.** Since \( k \circ \mathcal{X} \) is a fuzzy random variable, with \( k \circ \mathcal{X} : \Omega \to \mathcal{F}_{c}((0, +\infty)) \) defined so that \( (k \circ \mathcal{X})(\omega) = k \circ \mathcal{X}(\omega) \) for all \( \omega \in \Omega \) and \( E(\sup(k \cdot \mathcal{X})_0) = kE(\sup \mathcal{X}_0) < \infty \), then for all \( z \in [0, 1] \) and \( k \in (0, +\infty) \) we have that
\[ \frac{\inf(k \odot \mathcal{X})}{E(\sup(k \odot \mathcal{X}))} = \frac{k \inf \mathcal{X}}{kE(\sup \mathcal{X})} = \frac{\inf \mathcal{X}}{E(\sup \mathcal{X})}, \]

and

\[ \frac{\sup(k \odot \mathcal{X})}{E(\inf(k \odot \mathcal{X}))} = \frac{k \sup \mathcal{X}}{kE(\inf \mathcal{X})} = \frac{\sup \mathcal{X}}{E(\inf \mathcal{X})}, \]

whence

\[ I_f(k \odot \mathcal{X}) = I_f(\mathcal{X}). \quad \square \]

**Remark 5.1.** In virtue of the mean independence property, which would be also true for indices based in non-monotonic functions because of the positive scale invariance of the fuzzy quotient \( \mathcal{X} \circ \mathcal{E}(\mathcal{X}) \), whenever the positive fuzzy random variable \( \mathcal{X} \) is “upper bounded” (in the sense, that the \( \sup_{\omega \in \Omega} \mathcal{X}_0(\omega) \) is finite and, in particular, whenever \( \mathcal{X} \) takes on a finite number of different values over \( \Omega \), and due to the monotonicity of the function \( f(x) = x \log x \) in \( (0, 1/e) \), the extension of Theil’s index can be considered as an index satisfying the same properties of indices in Definition 3.1. In connection with the minimality property it should correspond to that in Theorem 5.5.

The *sign-preserving* (or *non-negativeness*) holds for all the indices in Definition 3.1. Thus,

**Theorem 5.2 (Non-negativeness).** Let \( \mathcal{X} : \Omega \to \mathcal{F}_c((0, +\infty)) \) be a fuzzy random variable such that \( E(\sup \mathcal{X}_0) < \infty \). Then, we have that \( I_f(\mathcal{X}) \geq 0 \).

**Proof.** Indeed, because of the convexity of \( f \) and in virtue of Jensen’s inequality and the conditions assumed for \( f \), we have that

\[
E\left( f \left( \frac{\inf \mathcal{X}}{E(\sup \mathcal{X})} \right) \right) + E\left( f \left( \frac{\sup \mathcal{X}}{E(\inf \mathcal{X})} \right) \right) \\
\geq f \left( E\left( \frac{\inf \mathcal{X}}{E(\sup \mathcal{X})} \right) \right) + f \left( E\left( \frac{\sup \mathcal{X}}{E(\inf \mathcal{X})} \right) \right) \\
= f \left( E\left( \frac{\inf \mathcal{X}}{E(\sup \mathcal{X})} \right) \right) + f \left( E\left( \frac{\sup \mathcal{X}}{E(\inf \mathcal{X})} \right) \right) \geq 0. \quad \square
\]

The *positiveness* (or *sensitivity out of equality*) is formalized in the following result:
Theorem 5.3 (Sensitivity out of equality). Let \( \mathcal{X} : \Omega \to \mathcal{F}_c((0, +\infty)) \) be a fuzzy random variable such that \( E(\sup \mathcal{X}_0) < \infty \). If \( I_f(\mathcal{X}) = 0 \), then \( \mathcal{X} \) has to be a degenerate fuzzy random variable (that is, if \( \mathcal{X} \) is non-degenerate we can ensure that \( I_f(\mathcal{X}) \) is positive).

Proof. In virtue of Theorem 4.1, and the left-continuity of the real-valued random variables \( \inf(f(\mathcal{X} \ominus \mathcal{E}(\mathcal{X}))) \) and \( \sup(f(\mathcal{X} \ominus \mathcal{E}(\mathcal{X}))) \) with respect to \( \alpha \), then \( I_f(\mathcal{X}) = 0 \) if, and only if, for all \( \alpha \in [0, 1] \)

\[
E \left[ f \left( \frac{\inf \mathcal{X}_\alpha}{E(\inf \mathcal{X}_\alpha)} \right) \right] + E \left[ f \left( \frac{\sup \mathcal{X}_\alpha}{E(\sup \mathcal{X}_\alpha)} \right) \right] = 0.
\]

To obtain this condition, the inequality relation obtained by applying Jensen’s inequality in Theorem 5.2 should be an equality. This happens if, and only if, for all \( \alpha \in [0, 1] \) the real-valued random variables \( \inf \mathcal{X}_\alpha \) and \( \sup \mathcal{X}_\alpha \) are degenerate and, therefore, all the \( \alpha \)-level functions \( \mathcal{X}_\alpha \) must be degenerate convex compact random sets. Consequently (see [19]), \( \mathcal{X} \) has to be a degenerate fuzzy random variable. \( \square \)

The insensitivity or nullity of the \( f \)-inequality indices cannot be guaranteed for a degenerate fuzzy random variable whatever \( f \) may be. The discussion on this property, and the two following ones, has motivated the consideration of the two families characterized in Section 4. The following result states that for fuzzy random variables that degenerate at a positive real number this insensitivity always holds.

Theorem 5.4 (Insensitivity). Let \( \mathcal{X} : \Omega \to \mathcal{F}_c((0, +\infty)) \) be a fuzzy random variable such that \( E(\sup \mathcal{X}_0) < \infty \). If \( \mathcal{X} \) is degenerate at a positive real value, then \( I_f(\mathcal{X}) = 0 \).

Proof. Indeed, if \( \mathcal{X} \) is a fuzzy random variable degenerate at a positive real value, then for all \( \alpha \in [0, 1] \) we have that \( \inf \mathcal{X}_\alpha = \sup \mathcal{X}_\alpha \) as \([P]\) and, hence, \( E(\inf \mathcal{X}_\alpha) = E(\sup \mathcal{X}_\alpha) \), so that \( \inf(I_f(\mathcal{X}))(\alpha) = \sup(I_f(\mathcal{X}))(\alpha) = 0 \), whence \( I_f(\mathcal{X}) = 0 \). \( \square \)

The next results present two different minimality properties for the \( f \)-inequality indices, depending on certain conditions the function \( f \) satisfies.

Theorem 5.5 (Minimality I). Let \( \mathcal{X} : \Omega \to \mathcal{F}_c((0, +\infty)) \) be a fuzzy random variable such that \( E(\sup \mathcal{X}_0) < \infty \). If \( f \) satisfies that \( f(u) + f(1/u) = 0 \) if, and only if, \( u = 1 \), then \( I_f(\mathcal{X}) = 0 \) if, and only if, \( \mathcal{X} \) is a fuzzy random variable degenerate at a positive real number.
Proof. In virtue of Theorems 5.2 and 5.3, $I_f(\mathcal{X}) = 0$ if, and only if, $\inf \mathcal{X}_z$ and $\sup \mathcal{X}_z$ are degenerate real-valued random variables and (because of the extra condition assumed for $f$) $E(\inf \mathcal{X}_z) = E(\sup \mathcal{X}_z)$, that is if, and only if, $\inf \mathcal{X}_z$ and $\sup \mathcal{X}_z$ are real-valued random variables degenerate at the same value, and this forces $\mathcal{X}$ to be a fuzzy random variable degenerate at this value. \qed

As indicated in Section 4, the additional condition $f(u) + f(1/u) = 0$ if, and only if, $u \in (0, +\infty)$ is satisfied by many functions $f$ (in particular, for those serving to extend the additively decomposable indices of order $\alpha \neq 0, 1$), although other valuable functions like $f(x) = -\log x$ (which is the basis of the index of the Shannon type) satisfy that $f(u) + f(1/u) = 0$ for all $u \in (0, +\infty)$ (see also Section 4). In this latter case, the necessary and sufficient condition for $I_f(\mathcal{X})$ being null is gathered in the following result:

**Theorem 5.6 (Minimality II).** If $\mathcal{X}: \Omega \rightarrow \mathcal{F}_c((0, +\infty))$ is a fuzzy random variable such that $E(\sup \mathcal{X}_z) < \infty$, and $f(u) + f(1/u) = 0$ for all $u \in (0, +\infty)$, then $I_f(\mathcal{X}) = 0$ if, and only if, $\mathcal{X}$ is degenerate at an element in $\mathcal{F}_c((0, +\infty))$.

Proof. The necessary condition has been already proved in Theorem 5.3. Conversely, if $\mathcal{X}$ is a degenerate fuzzy random variable, $\inf \mathcal{X}_z$ and $\sup \mathcal{X}_z$ are degenerate real-valued random variables, whence because of the extra condition assumed for $f$

$$\frac{1}{2} \left[ f\left( \frac{E(\inf \mathcal{X}_z)}{E(\sup \mathcal{X}_z)} \right) + f\left( \frac{E(\sup \mathcal{X}_z)}{E(\inf \mathcal{X}_z)} \right) \right] = 0. \quad \square$$

**Remark 5.2.** In accordance with Theorems 5.5 and 5.6, if $\mathcal{X}$ is a fuzzy random variable degenerate at a fuzzy number in $\mathcal{F}_c((0, +\infty))$, the $f$-inequality index does not necessarily equal 0. Thus, for instance, if $\mathcal{X}$ equals almost surely the value $\tilde{x} = \text{Tri}(1, 2, 3)$ on $\Omega$ and we consider $f(x) = x^{-1} - 1$ for all $x \in (0, 1)$, then we obtain that $I_f(\mathcal{X}) = 0.999$.

Reasons justifying that some $f$-indices do not vanish for degenerate fuzzy random variables lie in the fact that several of these indices (in particular, those associated with functions $f$ such that $f(u) + f(1/u) = 0$ if, and only if, $u = 1$), in addition to quantifying the intervals inequality also measures the intravalues inequality. In this sense, and as a special case, one can prove the following additive decomposition property for the hyperbolic index $I_H$ on finite populations (that is, $I_f$ with $f(x) = x^{-1} - 1$ for all $x \in (0, +\infty)$), in accordance with which if $\Omega = \{\omega_1, \ldots, \omega_N\}$, $\mathcal{X}(\Omega) = \{\tilde{x}_1, \ldots, \tilde{x}_k\}$ and $p_i = P(\omega \in \Omega | \mathcal{X}(\omega) = \tilde{x}_i)$, $i = 1, \ldots, k$, we have that

$$I_H(\mathcal{X}) = \sum_{i=1}^{k} p_i^2 I_H(\{\tilde{x}_i\}) + I_H^{bv}(\mathcal{X})$$
(where $I^H_{IH}(\mathcal{X})$ represents a kind of interval value inequality index and $I^H_H(\{\tilde{x}_i\})$ denotes the intravalue inequality index for value $\tilde{x}_i$) with $I^H_{IH}(\mathcal{X}) = 0$ if, and only if, $\mathcal{X}$ is a degenerate fuzzy random variable. Nevertheless, this property is not true for all $f$.

Several other interesting properties of inequality indices (namely, symmetry, population homogeneity, continuity, strict Schur-convexity, compatibility with Lorenz’s criterion, and principles of transfers) can be proved for the indices in Definition 3.1 (see [21]). Nevertheless, the extension to fuzzy random variables does not preserve the additive decomposability of several $f$-indices for real-valued random variables. The following result means a more general but weaker conclusion formalizing the effects of the “grouping” of fuzzy data in quantifying the $f$-inequality index. More precisely, this property expresses the “ordering relation” between the inequality of the population and the inequality between the groups of a given (classical) partition of the population, when each of the groups is represented by the expected value of the fuzzy random variable in it. From this result we can conclude that grouping entails an increase in inequality.

**Theorem 5.7** (Grouping effects). Consider a finite population $\Omega = \{\omega_{11}, \ldots, \omega_{1N_1}, \ldots, \omega_{M1}, \ldots, \omega_{M_{NM}}\}$ (with $N = N_1 + \ldots + N_M$), which is divided into $M$ subpopulations $\Omega_m = \{\omega_{m1}, \ldots, \omega_{m_{Nm}}\}$, $m = 1, \ldots, M$, and assume that $(\Omega, \mathcal{P}(\Omega))$ is endowed with the uniform distribution $\mathcal{P}$ and that $\mathcal{P} = \{\Omega_m\}_{m=1}^M$ denotes the above partition. If $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c((0, +\infty))$ is a fuzzy random variable associated with $(\Omega, \mathcal{P}(\Omega), \mathcal{P})$, and $\mathcal{X}_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{F}_c((0, +\infty))$ is the fuzzy random variable such that $\mathcal{X}_{\mathcal{P}}(\Omega_m) = \text{expected value of } \mathcal{X} \text{ on } \Omega_m$ ($m = 1, \ldots, M$), and $\mathcal{X}_{\Omega_m}$ denotes the restriction of $\mathcal{X}$ from $\Omega$ to $\Omega_m$ ($m = 1, \ldots, M$), then we have that

$$I_f(\mathcal{X}) \geq I_f(\mathcal{X}_{\mathcal{P}}).$$

On the other hand, $I_f(\mathcal{X}) = I_f(\mathcal{X}_{\mathcal{P}})$ if, and only if, for each $m \in \{1, \ldots, M\}$ the fuzzy random variable $\mathcal{X}_{\Omega_m}$ is degenerate in $\Omega_m$.

**Proof.** Obviously, $\bar{E}(\mathcal{X}) = \bar{E}(\mathcal{X}_{\mathcal{P}}(\Omega_m))$. On the other hand, by applying Jensen’s inequality, we obtain that

$$\sum_{m=1}^{M} \frac{N_m}{N} f\left(\inf \mathcal{X}_{\mathcal{P}}(\Omega_m) \over \sup \bar{E}(\mathcal{X}_{\mathcal{P}}(\Omega_m))\right) = \sum_{m=1}^{M} \frac{N_m}{N} f\left(\frac{1/N_m}{\sum_{i=1}^{N_m} \inf \mathcal{X}_a(\omega_{mi})}{\sup \bar{E}(\mathcal{X}_a)}\right)$$

$$\leq \frac{1}{N} \sum_{m=1}^{M} \sum_{i=1}^{N_m} f\left(\inf \mathcal{X}_a(\omega_{mi}) \over \sup \bar{E}(\mathcal{X}_a)\right) = \frac{1}{N} \sum_{j=1}^{N} f\left(\inf \mathcal{X}_a(\omega_{mj}) \over \sup \bar{E}(\mathcal{X}_a)\right).$$
Analogously,
\[
\sum_{m=1}^{M} \frac{N_m}{N} f \left( \frac{\sup \mathcal{X} \mathcal{P}(\Omega_m)}{\inf E(\mathcal{X} \mathcal{P}(\Omega_m))} \right) \leq \frac{1}{N} \sum_{j=1}^{N} f \left( \frac{\sup \mathcal{X}(\omega_j)}{\inf E(\mathcal{X})} \right),
\]
whence
\[
I_f(\mathcal{X}) = \frac{1}{2} \int_{[0, 1]} \left[ \frac{1}{N} \sum_{j=1}^{N} f \left( \frac{\sup \mathcal{X}(\omega_j)}{\inf E(\mathcal{X})} \right) + \frac{1}{N} \sum_{j=1}^{N} f \left( \frac{\inf \mathcal{X}(\omega_j)}{\sup E(\mathcal{X})} \right) \right] \, dx
\]
\[
\geq \frac{1}{2} \int_{[0, 1]} \left[ \sum_{m=1}^{M} \frac{N_m}{N} f \left( \frac{\sup \mathcal{X} \mathcal{P}(\Omega_m)}{\inf E(\mathcal{X} \mathcal{P}(\Omega_m))} \right) \right. \\
\left. + \sum_{m=1}^{M} \frac{N_m}{N} f \left( \frac{\inf \mathcal{X} \mathcal{P}(\Omega_m)}{\sup E(\mathcal{X} \mathcal{P}(\Omega_m))} \right) \right] \, dx
\]
\[
= I_f(\mathcal{X} \mathcal{P}).
\]

The equality will be achieved if, and only if,
\[
\frac{1}{N} \sum_{j=1}^{N} f \left( \frac{\sup \mathcal{X}(\omega_j)}{\inf E(\mathcal{X})} \right) + \frac{1}{N} \sum_{j=1}^{N} f \left( \frac{\inf \mathcal{X}(\omega_j)}{\sup E(\mathcal{X})} \right) \\
= \sum_{m=1}^{M} \frac{N_m}{N} f \left( \frac{\sup \mathcal{X} \mathcal{P}(\Omega_m)}{\inf E(\mathcal{X} \mathcal{P}(\Omega_m))} \right) + \sum_{m=1}^{M} \frac{N_m}{N} f \left( \frac{\inf \mathcal{X} \mathcal{P}(\Omega_m)}{\sup E(\mathcal{X} \mathcal{P}(\Omega_m))} \right)
\]
for all \( x \in [0, 1] \), and this happens if, and only if, all the expressions to which we have applied Jensen’s inequality become equalities. This condition is equivalent to the fact that for each \( x \in [0, 1] \), and whatever \( \Omega_m \) may be \( m \in \{1, \ldots, M\} \), the values \( \inf \mathcal{X}(\omega_m) \) coincide for all different values of \( i = 1, \ldots, N_a \) and the values \( \sup \mathcal{X}(\omega_m) \) also coincide for different \( i = 1, \ldots, N_m \), that is, if the fuzzy random variable is degenerate in \( \Omega_m \).

Remark 5.3. The additive decomposability of several inequality indices for real-valued random variables is lost in the extension to the fuzzy case, except for Shannon’s type index. In this way, if \( f(x) = -\log x \) for all \( x \in (0, +\infty) \), it can be easily proven for \( I_{Sh}(\mathcal{X}) \) (or extended Shannon’s type index) that
\[
I_{Sh}(\mathcal{X}) = I_{Sh}(\mathcal{X} \mathcal{P}) + \sum_{m=1}^{M} \frac{N_m I_{Sh}(\mathcal{X} \Omega_m)}{N},
\]
that is, the inequality in the population coincides with the sum of the inequality between groups (more precisely, between the expected values of \( \mathcal{X} \) in different groups) and the average of the inequality within groups.
6. Illustrative examples

In the following examples we illustrate the computation and use of certain \( f \)-inequality indices in comparing populations.

**Example 6.1.** A phone poll is carried out on the population \( \Omega \) of the 105 male members of a sports center who are requested to classify themselves into one of the following four groups: SHORT, NOT TALL, TALL and VERY TALL. Assume that the obtained answers are 12 ‘SHORT’, 23 ‘NOT TALL’, 57 ‘TALL’ and 13 ‘VERY TALL’. This type of classification can be identified with a fuzzy random variable whose values are the preceding four groups.

Suppose that to describe these values we use the characterization given by Norwich and Turksen [25], which is based on mean direct rating for certain referential points in the interval \([54, 88]\) (where units are inches) and on a linear interpolation for the remaining points in \( \mathbb{R} \), that supplies us the polygonals in Fig. 1.

If we want to quantify the inequality of heights in this population on the basis of the performed poll, we can consider, for instance, the \( f \)-inequality index extending the normalized variance for the real-value case (corresponding to \( f(x) = x^2 - 1 \)), which takes on the value

\[
I_{\text{NVar}}(\mathcal{X}, \Omega) = 0.04.
\]

**Example 6.2.** Consider the variable ANNUAL INCOME, \( \mathcal{X} \), in accordance with the classification which is adopted in some credit assessment systems. Following Cox [8], this variable can be viewed as a variable whose (fuzzy) values are \( \tilde{x}_1 = \text{SOMewhat HIGH} \), \( \tilde{x}_2 = \text{moderately HIGH} \), \( \tilde{x}_3 = \text{HIGH} \) and \( \tilde{x}_4 = \text{very HIGH} \), where \( \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \) and \( \tilde{x}_4 \) are described (in US thousand dollars) by means of the following \( S \)- and \( H \)-curves:

![Graphical representation of the value TALL and related values.](image)

Fig. 1. Graphical representation of the value TALL and related values.
\[ \bar{x}_1 = 1 - S(100, 125), \]
\[ \bar{x}_2 = \Pi(100, 125, 150), \]
\[ \bar{x}_3 = \Pi(125, 147.5, 170), \]

and
\[ \bar{x}_4 = S(147.5, 170), \]

and \( \text{supp } \bar{x}_i \subset [90, 180], \ i = 1, 2, 3, 4 \), for all candidates for a credit in the considered system (see Fig. 2), where

\[
S(a, b)(t) = \begin{cases} 
0 & \text{if } t \leq a, \\
2 \left( \frac{t-a}{b-a} \right)^2 & \text{if } t \in \left[ a, \frac{a+b}{2} \right], \\
1 - 2 \left( \frac{t-b}{b-a} \right)^2 & \text{if } t \in \left[ \frac{a+b}{2}, b \right], \\
1 & \text{otherwise,}
\end{cases}
\]

\[
\Pi(a, (a+b)/2, b)(t) = \begin{cases} 
S(a, (a+b)/2) & \text{if } t \leq (a+b)/2, \\
1 - S((a+b)/2, a) & \text{otherwise.}
\end{cases}
\]

Assume that a bank adopting the above system wishes to compare two different towns by means of the income inequality, and to this purpose we observe the values of \( \mathcal{X} \) in the central offices of these two towns.

If there are 125 candidates for a credit in one of the offices (\( \Omega_1 \)) during a certain period, 28 of them having a SOMEWHAT HIGH annual income, 43 MODERATELY HIGH, 31 HIGH and 23 VERY HIGH, whereas there are 178 candidates for a credit in the other office (\( \Omega_2 \)) during the same period, 63 of them having SOMEWHAT HIGH, 79 MODERATELY HIGH, 27 HIGH and 9 VERY HIGH, and we employ the \( f \)-inequality index with \( f(x) = -\log x \), we obtain that

![Fig. 2. Fuzzy values of the variable ANNUAL INCOME.](image-url)
whence we can conclude that the two towns have a close inequality of annual income.

7. Concluding remarks

In contrast with the computation of the fuzzy-valued inequality indices (see [6,19]), the \(f\)-inequality indices in the present work are definitely much easier to be computed, and we do not need to supply a fuzzy number (usually its graphical representation) as the answer to the problem of quantifying the inequality. Nevertheless, most of the software developed by López García and Colubi Cervero, in what concerns the characterization of fuzzy data, and the calculus of the induced images of the functions \(f\) and the expected values of fuzzy random variables, are useful for the aim of the present work because of the complementary information it provides us with, although the final calculus of the \(f\)-inequality indices only requires quite simple approximation techniques.

Most of the studies in this paper could be immediately extended by considering a more general index, obtained by composing the \(\lambda\)-average ranking function by Campos and González [5] with \(\lambda \in [0, 1]\), which when particularized to \(\lambda = 0.5\) leads to the function \(F\) by Yager [33]. The only (even though determinant) inconvenience to deal with this more general ranking is that for values \(\lambda \in [0, 0.5]\) the non-negativeness and minimality properties could be lost (actually, Theorem 5.6 could not be stated for \(\lambda \neq 0.5\)). Consequently, the comparison of populations/samples and/or variables through indices which could take on non-negative values would often lead to wrong conclusions.

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