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On the characterization of the domination of a diameter-constrained network reliability model

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Abstract

Let $G=(V, E)$ be a digraph with a distinguished set of terminal vertices $K \subseteq V$ and a vertex $s \in K$. We define the s, K-diameter of G as the maximum distance between s and any of the vertices of K. If the arcs fail randomly and independently with known probabilities (vertices are always operational), the diameter-constrained s, K-terminal reliability of G, $R_{s,K}(G, D)$, is defined as the probability that surviving arcs span a subgraph whose s, K-diameter does not exceed D.

The diameter-constrained network reliability is a special case of coherent system models, where the domination invariant has played an important role, both theoretically and for developing algorithms for reliability computation. In this work, we completely characterize the domination of diameter-constrained network models, giving a simple rule for computing its value: if the digraph either has an irrelevant arc, includes a directed cycle or includes a dipath from s to a node in K longer than D, its domination is 0; otherwise, its domination is -1 to the power $|E|-|V|+1$. In particular this characterization yields the classical source-to-K-terminal reliability domination obtained by Satyanarayana.

Based on these theoretical results, we present an algorithm for computing the reliability.

Keywords: Graph theory; Domination; Diameter; Network reliability

1. Introduction and reliability model

In this paper we are concerned with digraphs (directed digraphs) $G=(V, E)$, where V and E are the set of vertices and arcs of G, respectively. A subgraph $G'=(V', E')$ of a digraph $G=(V, E)$ is a digraph such that $V' \subseteq V$, and $E \subseteq E$, and G' is a spanning subgraph of G if $V'=V$.

For a digraph $G=(V, E)$, we denote a dipath P from vertex u to vertex v (also called a u, v-dipath) in G as $P=\langle(u_1=u, u_2), (u_2, u_3), \ldots, (u_{r-1}, u_r=v)\rangle$, where the vertices of P are distinct, and (u_j, u_{j+1}) is an arc of G. Moreover, let $r-1$ be the length of P. A directed cycle (dicycle) C in G is obtained from P by allowing $u=v$. Furthermore, we say that a digraph G is cyclic if it contains a directed cycle; otherwise G is acyclic.

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The distance between vertex u and vertex v of a digraph G is defined as the length of the shortest dipath from vertex u to vertex v in G, and in the case that no dipath exists between u and v, the distance is infinite.

A communication network can be modeled by a digraph (also called its underlying graph) G, where the set of nodes (e.g., packet switches) and communication links of the network are the vertices and arcs of G, respectively. Moreover, since a communication network could be subject to random failures of its components, we represent this probabilistic behavior of the network by assigning failure probabilities to the vertices and/or arcs of its underlying digraph. A widely used probabilistic model is the one where arcs fail randomly and independently with known probabilities, and where vertices are always operational; when we mention a probabilistic digraph, we will refer to this model.

Let $G = (V, E)$ be a probabilistic digraph, with terminal vertex set $K \subseteq V$, and distinguished vertex $s \in K$ (called the source node), and diameter bound D. The s, K-diameter of G is defined as the maximum distance from s to any other vertex $u \in K$. By the definition of distance, in the case that no dipath exists from vertex s to some vertex $u \in K$, then the s, K-diameter is infinite.

The diameter-constrained s, K-terminal reliability $R_{s,K}(G, D)$ is defined [13] as the probability that the surviving arcs span a subgraph of G whose s, K-diameter does not exceed D, or equivalently, as the probability that for each vertex $u \in K$, there exists an operating dipath (i.e., a dipath composed of surviving arcs of G) from s to u of at most D arcs. This reliability measure subsumes the classical source-to-K-terminal reliability $R_{s,K}(G)$ of a probabilistic digraph G (see [18] for a complete discussion on this subject), which is the probability that the surviving arcs span a subgraph where there exists an operational dipath between s and u, $u \in K$: noting that the longest dipath in G has at most $n - 1$ arcs, where n is the number of nodes of G, we have that $R_{s,K}(G)$ is equal to $R_{s,K}(G, D)$ for $D = n - 1$.

As the classical reliability does not take into account the length of the dipaths connecting the terminal nodes of a digraph G, this reliability model was extended to assess the probability that there are short-enough dipaths from the source vertex s to a set of terminal vertices of G. That is the case in multicasting-routing with end-to-end delay constraints, where a source node must broadcast messages to a set of destination nodes in a network (e.g., teleconference), while these messages must meet certain delay constraints. This problem can be modeled as a digraph with a source node s, a set M of destination nodes, and where each arc is assigned a weight corresponding to the delay to be experienced by a packet traveling along this arc.

A line of research in this area is the study of techniques to obtain diameter-constrained Steiner trees, in order to ensure that a packet traveling from the source to a terminal node can arrive within the allowed delays [5–7,11,15]. Another area of research consists in meeting with diameter and two-connectivity objectives by extending an existing topology with new arcs [4]. To our knowledge none of these previous works take into account the operational probability of the network components, thus the diameter-constrained reliability measure may be applied to determine the suitability of a network to meet end-to-end delay constraints.

The domination of a digraph is a graph-theoretic measure which appeared in alternative formulations for improving the efficiency of evaluating the classical reliability. In this paper, we give a characterization of the domination of a digraph, for the diameter-constrained reliability model, which in particular yields the domination for the classical case. For a digraph $G = (V, E)$, with terminal set K, source node $s \in K$, and diameter bound D, the diameter-constrained s, K-terminal reliability can be computed as a sum of terms involving the domination of some spanning subgraphs of G. Even though, for general graphs, the computation of the reliability remains a $\#P$-complete problem, since the evaluation of the classical source-to-K-terminal reliability belongs to this computational class (see [14]), the application of the domination for calculating the diameter-constrained reliability substantially reduces the computational effort.

In Section 2, we present some preliminary definitions and notation that will be used in the following sections, and introduce the domination for general systems. In Section 3 we give a complete characterization of the diameter-constrained reliability domination of a digraph. In Section 4 we formally prove this domination characterization, and finally, in Section 5, we present an algorithm based on the previous theoretical results.

The notation in this paper follows that of Harary [8], unless otherwise noted.

2. Preliminaries and domination

As we are considering digraphs, we use the notation $ind_G(u)$ and $outd_G(u)$ to denote the indegree and outdegree of vertex u in G, respectively, where the indegree of u is the number of arcs directed into u, while the outdegree is the number of arcs emanating from u.

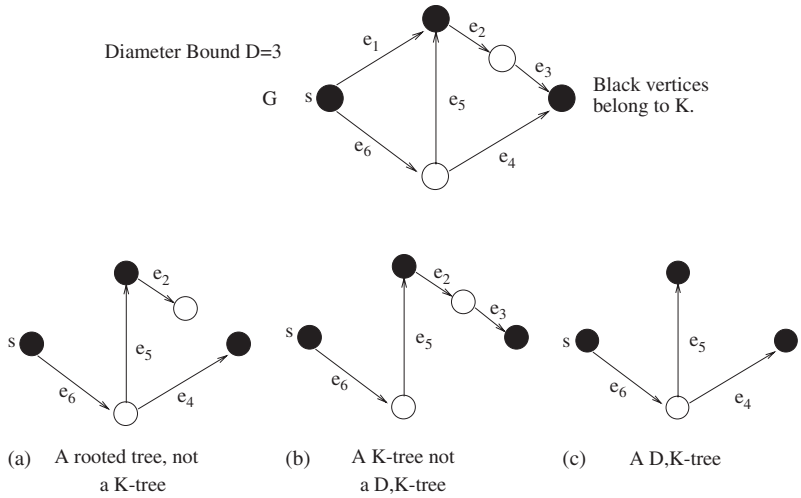

Fig. 1. Different types of trees of a digraph G.

The following definitions and notation will be used in the remainder of this paper:

(i) Let $G = (V, E, \mathbf{P}(E))$ be a probabilistic digraph with a distinguished set $K \subseteq V$, vertex $s \in K$, and $D \in Z^+$, where $n = |V|$, and where $\mathbf{P} : E \mapsto [0, 1]$ are the operational probabilities of the arcs in set E. We represent the operational probability of an arc (or arc reliability) $x \in E$ as $p(x)$ and we have $p(x) = 1 - q(x)$, where $q(x)$ is the probability of failure.

(ii) Let the sample space Ω represent the set of all possible subsets of E, corresponding to sets of operational arcs (i.e., $\Omega = 2^E$).

(iii) Under independent failures assumption each $H \in \Omega$ has probability

$$P(H) = \prod_{x \in H} p(x) \prod_{x \notin H} q(x).$$

(iv) $H \in \Omega$ is a pathset or operating state if H spans a subgraph whose s, K-diameter is at most D.

(v) Let $\mathbf{O}_K^D(E) = \{H \in \Omega : H \text{ is a pathset}\}$.

(vi) An operating state H of $\mathbf{O}_K^D(E)$ is called a minpath if $H - \{x\} \notin \mathbf{O}_K^D(E)$ for all $x \in H$ (i.e., a minpath is a minimal operating state).

(vii) A K-tree T of a digraph G, is a tree, rooted at s, covering all the vertices of K, and such that any pendant vertex u (i.e., u has indegree 1 and outdegree 0) of T belongs to K. In addition, a K-tree whose s, K-diameter is at most D is called a D, K-tree (see Fig. 1).

(viii) G is called a D, K-digraph, if every arc of G lies in some D, K-tree of G.

From the definition of $R_{s,K}(G, D)$ and definition (v) one gets

$$R_{s,K}(G, D) = Pr\left(\mathbf{O}_K^D(E)\right) = \sum_{H \in \mathbf{O}_K^D(E)} \prod_{x \in H} p(x) \prod_{x \notin H} q(x). \tag{1}$$

The following lemma gives a characterization of the minpaths M of $\mathbf{O}_K^D(E)$:

Lemma 1. *Let $G = (V, E)$ be a digraph with terminal set K, vertex $s \in K$, and bound D; M is a minpath of G if and only if M is a D, K-tree.*

We next discuss the definition of the domination invariant in the case of general systems, and in the case of the diameter-constrained network reliability.

A graph invariant called the reliability domination of a graph G was introduced by Satyanarayana and Prabhakar [17] for the classical network reliability models, and has since been explored by several researchers in reliability theory [1–3,9,10]. The reliability domination plays an important role, allowing to efficiently implement the principle of inclusion–exclusion of probability theory applied to the evaluation of reliability measures for general reliability systems.

Let E be a finite set, and 2^E be the power set of E. A non-empty subset $\mathbf{C} \subseteq 2^E$ is called a clutter of E if for any two elements $C_1, C_2 \in \mathbf{C}$, whenever $C_1 \subseteq C_2$, then $C_1 = C_2$. A pair $(E, \mathbf{C})$ will be referred to as a system and a system is coherent if each element of E is contained in some element of $\mathbf{C}$. A formation of $(E, \mathbf{C})$ is a collection of elements of $\mathbf{C}$ whose union yields E. The signed domination of the system $(E, \mathbf{C})$, denoted $d(E, \mathbf{C})$, is defined as the number of odd formations minus the number of even formations of E, where a formation is said to be odd or even if it is of odd or even cardinality, respectively. Trivially by the previous definitions, a non-coherent system has no formations, so its signed domination is 0.

The clutters associated with the operation and failure of a specific element $x \in E$ are defined as follows. Let $\mathbf{C} - x = \{C - x : C \in \mathbf{C}\}$ and $\mathbf{C}_{-x} = \{C \in \mathbf{C} : x \notin C\}$. Now $\mathbf{C}_{-x}$ is clearly a clutter but $\mathbf{C} - x$ may not be one. We define $\mathbf{C}_{+x}$ to be the collection of elements of $\mathbf{C} - x$ which are not proper supersets of some element of $\mathbf{C} - x$. For an element $x \in E$, $\mathbf{C}_{-x}$ and $\mathbf{C}_{+x}$ are called the minors with respect to x of $\mathbf{C}$. Huseby [9,10] showed the following result:

Theorem 1. *If $(E, \mathbf{C})$ is a system, with $x \in E$, and minors $\mathbf{C}_{-x}$ and $\mathbf{C}_{+x}$ of $\mathbf{C}$, then $d(E, \mathbf{C}) = d(E - \{x\}, \mathbf{C}_{+x}) - d(E - \{x\}, \mathbf{C}_{-x})$.*

We look now at the case of the diameter-constrained s, K-terminal reliability of a digraph $G = (V, E)$ with $K \subseteq V$, $s \in K$, and diameter bound D. The system underlying our model is $(E, \mathbf{F}_{D,K}(G))$, where E is the set of arcs of G, and where $\mathbf{F}_{D,K}(G)$ is the collection of D, K-trees of G. A formation F of G is then a collection of D, K-trees of G whose union is E, the set of arcs of G. The signed domination of a digraph $G = (V, E)$, simply called domination, and denoted as $d(E, \mathbf{F}_{D,K}(G))$, with respect to a given subset $K \subseteq V$, $s \in K$, and bound D, is the number of odd minus the number of even formations of G.

For brevity, in what follows we will use the standard notation $\mathbf{C}$ to represent $\mathbf{F}_{D,K}(G)$, which is the clutter set in the diameter-constrained model. Also we denote the domination $d(E, \mathbf{F}_{D,K}(G))$ as $d_{D,K}(G)$. In addition, we observe that if x is an arc of G, then T is a D, K-tree of G such that $x \notin T$ iff T is a D, K-tree of $G - x$. Therefore $d(E - \{x\}, \mathbf{C}_{-x}) = d_{D,K}(G - x)$. Using this notation, the equation in Theorem 1 can be re-written as

$$d_{D,K}(G) = d(E - \{x\}, \mathbf{C}_{+x}) - d(E - \{x\}, \mathbf{C}_{-x}) \tag{2}$$

$$= d(E - \{x\}, \mathbf{C}_{+x}) - d_{D,K}(G - x). \tag{3}$$

We next state the main results of this work, in which we characterize the domination for diameter-constrained reliability models, and we discuss how these results can be used to compute the reliability of a network.

3. Characterization of the domination and application to reliability evaluation

Let $G = (V, E)$ be a digraph with terminal set K, vertex $s \in K$, $e = |E|$ arcs, $n = |V|$ vertices, and let D be the diameter bound. We define the following operation:

- $\mathbf{LP}(G, s, K)$: if G is s, K connected (i.e., there exists a dipath from s to any vertex $u \in K$ in G), this operation returns the length of the longest dipath from s to any vertex $u \in K$; otherwise it returns ∞.

Recall that if G is not a D, K-digraph, there are some arcs in E which are not covered by any D, K-tree, thus the corresponding system is non-coherent, and there are no formations over the clutter $\mathbf{F}_{D,K}(G)$ able to cover E, and as a result the domination is zero.

For all digraphs, the domination is completely characterized by the following theorems (which are proved in Section 4):

Theorem 2. *Let $G = (V, E)$ be a cyclic digraph with terminal set K, source node $s \in K$, and diameter bound $D \geqslant 0$, then $d_{K,D}(G) = 0$.*

Theorem 3. *Let $G = (V, E)$ be an acyclic digraph with terminal set K, source node $s \in K$, $e = |E|$ arcs, $n = |V|$ vertices, and let $D \geqslant 0$ be the diameter bound, then*

$$d_{D,K}(G) = \begin{cases} (-1)^{e-n+1} & G \text{ is a } D, K\text{-digraph}, \text{ and } \mathbf{LP}(G, s, K) \leqslant D, \\ 0 & \text{otherwise}. \end{cases}$$

When $D = n - 1$, we obtain the classical source-to-K-terminal reliability model as a particular case. As all dipaths are of length smaller than n, then $\mathbf{LP}(G, s, K) \leqslant D$. Then this characterization reduces to the results in [16], that is the domination is 0 if there is a directed cycle in G or G is not a D, K-digraph, and $(-1)^{e-n+1}$ otherwise.

These results are useful for computing the reliability of a given network. For a digraph $G = (V, E)$, terminal set K, and vertex $s \in K$, let $\mathbf{M} = \{M_1, M_2, \ldots, M_l\}$ be the set of minpaths of $\mathbf{O}_K^D(E)$. The situation where all the arcs of M_i operate (survive), is a random event which will be denoted by E_i. By inclusion–exclusion we obtain

$$R_{s,K}(G, D) = Pr\left(\bigcup_{i=1}^{l} E_i\right) = \sum_i Pr(E_i) - \sum_{i<j} Pr(E_i E_j) + \cdots + (-1)^{l+1} Pr(E_1 E_2 \ldots E_l), \tag{4}$$

where the event $E_i E_j \ldots E_m$ is the event that all the arcs of the subgraph obtained by the union of $M_i, M_j, \ldots, M_m$ are operating.

In Eq. (4), the terms correspond to subgraphs obtained by the union of minpaths. As discussed previously, for the diameter-constrained s, K-terminal reliability of a digraph G, with terminal set K, vertex $s \in K$, and diameter bound D, the minpaths are D, K-trees, the formations are sets of minpaths, and the subgraphs are D, K-digraphs. The same D, K-digraph can be obtained from different formations; this means that it may appear more than once, sometimes with positive sign, and sometimes with negative sign, depending if the corresponding formation has an odd or an even number of D, K-trees. In fact, its net contribution will be exactly the number of odd minus the number of even formations of the graph, i.e., its domination invariant. Thus using these facts and the above definitions, we can rewrite Eq. (4) as

$$R_{s,K}(G, D) = \sum_{H \in \mathbf{H}} d_{D,K}(H) Pr(H), \tag{5}$$

where $\mathbf{H}$ is the class of all D, K-digraphs of G, and $Pr(H)$ is the probability that the arcs of H are operative.

4. Proofs

We define two operations on a digraph $G = (V, E)$ with distinguished terminal set K, and source vertex $s \in K$

- $\mathbf{OP}_1(G, x)$: suppose u is a vertex of G such that $ind_G(u) > 1$, and $x = (s, u)$ be an arc of G. This operation returns an arc $x' = (v, u)$ with $x' \neq x$.
- $\mathbf{OP}_2(G, K, s)$: let $V' = \{u \in V - \{s\} : (s, u) \in E\}$, and suppose that $\forall u \in V'$, $ind_G(u) = 1$. This operation returns a digraph G^* with terminal set $K^* = K - V' - \{s\} \cup s^*$, where G^* is obtained from G by identifying s and the vertices of V' (i.e., contracting the arcs emanating from s and deleting self-loops) into one vertex s^* (i.e., s^* is the new source of G^*).

The following lemmas plays an important role:

Lemma 2. *Let $G = (V, E)$ be a digraph with terminal set K, and source node $s \in K$. Suppose that $x = (s, u)$ is an arc of G, and $ind_G(u) > 1$, then $d_{D,K}(G) = -d_{D,K}(G - x)$.*

Proof. Let $x' \neq (s, u)$ be the arc returned by $\mathbf{OP}_1(G, x)$, and suppose that $T' = (V', E')$ is a D, K-tree of G such that x' is an arc of T'. Considering Eq. (3), in order to show that $d(E - \{x\}, \mathbf{C}_{+x}) = 0$, is sufficient to prove that the system $(E - \{x\}, \mathbf{C}_{+x})$ is not coherent. Consider the tree T obtained from T' by deleting x' and adding the arc

x, and possibly deleting any pendant vertices which do not belong to K, created by this transformation. Since T' is a D, K-tree, then also T is a D, K-tree, since any path that uses the arc x' in T', is replaced by the arc x in T. Then we have $T - x \subseteq T' - x' \subset T'$, and both T' as well as $T - x$ belong to $\mathbf{C} - x$, but $T - x \subset T'$, thus $T' \notin \mathbf{C}_{+x}$. Therefore we conclude that no elements of $\mathbf{C}_{+x}$ contain x', and the system $(E - x, \mathbf{C}_{+x})$ is not coherent. □

Lemma 3. *Let $G = (V, E)$ be a D, K-digraph with terminal set K, and source node $s \in K$ such that all nodes adjacent to s have indegree 1, and suppose $\mathbf{OP}_2(G, K, s)$ returns a digraph G^* with terminal set K^* and source node s^*, then $d_{D,K}(G) = d_{D-1,K^*}(G^*)$. Moreover, G^* is a $D - 1, K^*$-digraph.*

Proof. We must show the following:

(1) There exists a one-to-one correspondence between the D, K-trees of G and the $D - 1, K^*$-trees of G^*.
(2) There exists a one-to-one correspondence between the formations of G and the formations of G^*, moreover, a formation of G and its corresponding formation of G^* have the same cardinality.

Let $U = \{u_1, u_2, \ldots, u_r\}$ be the set of vertices of G such that $(s, u_i) \in E$. Let also $\mathbf{T}$ and $\mathbf{T}^*$ represent the D, K-trees of G and $D - 1, K^*$-trees of G^*, respectively.

We then construct a one-to-one correspondence $\Gamma : \mathbf{T} \mapsto \mathbf{T}^*$ from D, K-trees of G to the $D - 1, K^*$-trees of G^* as follows:

Let T^* be a tree obtained from $T \in \mathbf{T}$ by identifying the vertex s and the vertices adjacent to s in T, as a single vertex s^*. It is clear that T^* is a $D - 1, K^*$-tree of G^*, as the identification to one node reduces the s, K-diameter of T by one, moreover, the terminal set of T^* is K^*.

To show that Γ is one-to-one, suppose that $\Gamma(T_1) = \Gamma(T_2) = T^*$. Let $E'(T^*) = \{(s^*, v_1), (s^*, v_2), \ldots, (s^*, v_t)\}$ be the set of arcs of T^* emanating from s^* in T^*. Moreover, suppose that the arc (s^*, v_i) corresponds to the arc (u_{j_i}, v_i) of G, and let $U' = \{u_{j_i} : (s^*, v_i) \in E'(T^*)\}$.

Let V_1 and V_2 be the set of vertices adjacent to s in T_1 and T_2, respectively. Since identifying s with V_1 in T_1 and identifying s with V_2 in T_2 yield the same tree T^*, it must be the case that U' is a subset of V_1 and V_2.

Moreover, partition V_1 into U' and V_1', and V_2 into U' and V_2', but the vertices of V_1' and V_2' must then have outdegree 0 in T_1 and T_2, respectively; this can only happen if both sets are terminal vertices of G, as T_1 and T_2 are D, K-trees of G. Furthermore both sets can only be reached from s in G (i.e., $V_1' \subseteq U$ and $V_2' \subseteq U$), concluding that $V_1' = V_2'$, as T_1 and T_2 have the same terminal set. Thus $V_1 = V_2$, and since the vertices of V_1 and V_2 have indegree 1 in G, then $T_1 = T_2$.

To show that Γ is onto, suppose that T^* is a $D-1, K^*$-tree of G^* with arc-set $E(T^*) = \{(s^*, v_1), (s^*, v2), \ldots, (s^*, v_t), x_1, x_2, \ldots, x_p\}$.

In addition, suppose that the arc (s^*, v_i) corresponds to the arc (u_{j_i}, v_i) of G, and let $U' = \{u_{j_i} : (s*, v_i) \in E(T^*)\}$.

Moreover, let $K' = K - U'$ (i.e., the terminal vertices of G not covered by U').

We construct a D, K-tree, T, whose arc-set is $E(T) = \{(s, u) \in E : u \in K'\} \cup \bigcup_{i=1}^{t} \{(u_{j_i}, v_i)\} \bigcup \{x_1, x_2, \ldots, x_p\}$, and where $T = \Gamma^{-1}(T^*)$. This finalizes the proof for (1).

To show (2), consider a formation F of G, and let $F^* = \bigcup_{T \in F} \Gamma(T)$. As the D, K-trees of F cover all the arcs of G, then the $D - 1, K^*$-trees of F^* also cover all the arcs of G^*, thus F^* is a formation of G^*. Moreover, $|F^*| = |F|$, as Γ is a bijection.

Conversely let F^* be a formation of G^*, and let $F = \bigcup_{T^* \in F^*} \Gamma^{-1}(T^*)$. It is obvious that the arcs of G corresponding to the arcs of G^* are covered by F.

We must also show that the arcs $A = \{(s, u_i) : u_i \in U\}$ are also covered by D, K-trees of F.

Suppose that arc (s, u) of G is not covered by a tree of F. Then it must be the case that u has outdegree 0 in G, otherwise there exists a $T^* \in F^*$, containing the arc (s^*, v) corresponding to an arc (u, v) of G, and $\Gamma^{-1}(T^*)$ contains the arc (s, u).

Suppose that $outd_G(u) = 0$, then clearly $u \in K$, otherwise G is not a D, K-digraph. Since F^* contains at least one tree T^*, $\Gamma^{-1}(T^*)$ contains the arc (s, u). Therefore, since Γ is a bijection, we conclude that $|F| = |F^*|$, concluding the proof for (2).

Moreover, since G has at least one formation (i.e., G is a D, K-digraph), then from (2), it follows that G^* has also at least one formation, thus G^* is a $D - 1, K^*$-digraph.

Finally let $\mathbf{f_o}$ and $\mathbf{f_e}$ represent the number of odd and the number of even formations of G, respectively. Similarly let $\mathbf{f_o^*}$ and $\mathbf{f_e^*}$ represent the number of odd and the number of even formations of G^*, respectively. Thus from (2) one gets $d_{D,K}(G) = \mathbf{f_o} - \mathbf{f_e} = \mathbf{f_o^*} - \mathbf{f_e^*} = d_{D-1,K^*}(G^*)$. □

The following lemma concerning cyclic D, K-digraphs can be easily proved:

Lemma 4. *Let $G = (V, E)$ be a cyclic D, K-digraph with terminal set K, and source node $s \in K$, then*

(1) *suppose that $x = (s, u)$ is an arc of G, and $ind_G(u) > 1$, then $G - x$ is also cyclic,*
(2) *suppose $\mathbf{OP}_2(G, K, s)$ returns a digraph G^* with terminal set K^* and source node s^*, then G^* is also cyclic.*

The following lemma is concerned with acyclic D, K-digraphs:

Lemma 5. *Let $G = (V, E)$ be an acyclic D, K-digraph with terminal set K, and source node $s \in K$, then*

(1) *suppose that $x = (s, u)$ is an arc of G, and $ind_G(u) > 1$, then $G - x$ is an acyclic digraph where $\mathbf{LP}(G - x, s, K) = \mathbf{LP}(G, s, K)$,*
(2) *suppose $\mathbf{OP}_2(G, K, s)$ returns a $D - 1$, K^*-digraph G^* with terminal set K^* and source node s^*, then G^* is also acyclic, and $\mathbf{LP}(G^*, s^*, K^*) = \mathbf{LP}(G, s, K) - 1$.*

Proof. To show (2), we first note from Lemma 3, that if G is a D, K-digraph, and $\mathbf{OP}_2(G, K, s)$ returns a digraph G^*, then G^* is a $D - 1$, K^*-digraph. The fact that G^* is acyclic is obvious, and since a longest dipath from s must use exactly one arc of the ones contracted to obtain G^*, and there are no arcs between the vertices adjacent to s, then $\mathbf{LP}(G^*, s^*, K^*) = \mathbf{LP}(G, s, K) - 1$.

To show (1), let $x' = (v, u)$ the arc returned by $\mathbf{OP}_1(G, x)$. Since G is a D, K-digraph, then there exists a D, K-tree containing x', thus there exists a dipath $P' = \langle (s, v_1), (v_1, v_2), \ldots, (v_{r-1}, v_r), (v_r, u)\rangle$, where $v_r = v$ in G.

If a longest dipath from s in G does not contain x, then the lemma follows trivially.

Suppose that $P = \langle (s, u), (u_1, u_2), \ldots, (u_{c-1}, u_c)\rangle$ is a longest dipath in G containing x. If $u_i = v_j$, for some i, $1 \leqslant i \leqslant c$, and some j, $1 \leqslant j \leqslant r$ (i.e., dipath P intercepts P'), then G contains a directed cycle, contradicting the hypothesis that G is acyclic. By this observation, if we replace the arc (s, u) in P by the dipath P', we obtain a dipath not containing x whose length is at least the length of P. □

In the next lemma, we characterize acyclic D, K-digraphs G for which $x = (s, u)$ is an arc with $ind_G(u) > 1$, and $G - x$ is not a D, K-digraph.

Lemma 6. *Let $G = (V, E)$ be an acyclic D, K-digraph with terminal set K, and source node $s \in K$. Suppose that $x = (s, u)$ is an arc of G, $ind_G(u) > 1$, and $G - x$ is not a D, K-digraph, then $\mathbf{LP}(G, s, K) > D$.*

Proof. We first note from Lemma 5(1) that it is sufficient to show that $\mathbf{LP}(G - x, s, K) > D$, as the length of the longest dipath is preserved whenever we delete x.

Suppose that $x' = (v, u)$ is the arc returned from $\mathbf{OP}_1(G, x)$. Consider a D, K-tree of G, T, containing x. As G is a D, K-digraph, then there exists a D, K-tree containing the arc x', assuring the existence of an s, u-dipath containing the arc x'. Then either $(T - x) + x'$ is a K-tree (not necessarily a D, K-tree) or it can be extended to a K-tree by adding missing arcs to establish a dipath from s to the tail of x' (i.e., v). Thus all the arcs of $G - x$ are either covered by the D, K-trees of G containing x', or from the K-trees mentioned in the previous observation. Thus we conclude that $G - x$ is a d, K-digraph, for some diameter bound d. But $G - x$ is not a D, K-digraph, thus $d > D$. This implies that $G - x$ has at least a K-tree with s, K-diameter greater than D, and therefore $G - x$ has a dipath from s of length greater than D. □

We are now ready to present the proofs of the main results, which were stated in the previous section.

Theorem 2. *Let $G = (V, E)$ be a cyclic digraph with terminal set K, source node $s \in K$, and diameter bound $D \geqslant 0$, then $d_{K,D}(G) = 0$.*

Proof. If G is not a D, K-digraph then the theorem follows trivially, thus we can assume that G is a D, K-digraph. The theorem will be established by induction on the number of arcs (i.e., $|E|$) of G.

Basis: let $|E|=0$ and $D \geqslant 0$. Since there do not exist cyclic D, K-digraphs with no arcs, then the assertion is vacuously true.

For the induction step, let G be a cyclic D, K-digraph with $|E| > 0$, terminal set K, source node $s \in K$, and diameter bound $D \geqslant 0$, and suppose that the theorem holds for all cyclic d, K'-digraphs, with $d \geqslant 0$, and fewer arcs than G.

If $D = 0$ then the only possible D, K-digraph is the solitary $G = (\{s\}, \emptyset)$, thus we can assume that $D > 0$ as $E > 0$.

Suppose that there exists an arc (s, u) in G, such that $ind_G(u) > 1$.

If $G - x$ is not a D, K-digraph, then $d_{D,K}(G - x) = 0$ and by Lemma 2, $d_{D,K}(G) = -d_{D,K}(G - x) = 0$.

If $G - x$ is a D, K-digraph, by Lemma 2, $d_{D,K}(G) = -d_{D,K}(G - x)$, but by Lemma 4(1), $G - x$ is also cyclic, and as it has fewer arcs than G, by the induction hypothesis $d_{D,K}(G - x) = 0$.

Suppose that every vertex adjacent to s has indegree 1, and let G^* the graph with terminal set K^*, and source node s^*, obtained from operation $\mathbf{OP}_2(G, K, s)$. Since G is a D, K-digraph, then by Lemma 3, G^* is a $D - 1, K^*$-digraph, and $d_{D,K}(G) = d_{D-1,K^*}(G^*)$. But from Lemma 4(2), G^* is also cyclic, and as G^* has fewer arcs than G, by the induction hypothesis $d_{D-1,K^*}(G^*) = 0$, thus the theorem follows. □

Theorem 3. *Let $G = (V, E)$ be an acyclic digraph with terminal set K, source node $s \in K$, $e = |E|$ arcs, $n = |V|$ vertices, and let $D \geqslant 0$ be the diameter bound, then*

$$d_{D,K}(G) = \begin{cases} (-1)^{e-n+1} & G \text{ is a } D, K\text{-digraph}, \text{ and } \mathbf{LP}(G, s, K) \leqslant D, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. If G is not a D, K-digraph then the theorem follows trivially, thus we can assume that G is a D, K-digraph. The theorem will be established by induction on the number of arcs (i.e.,$|E|$) of G.

Basis: when $|E| = 0$, the only acyclic D, K-digraph is the $G = (\{s\}, \emptyset)$ and $d_{D,K}(G) = 1 = (-1)^{e-n+1}$, for $D \geqslant 0$.

For the induction step, let G be an acyclic D, K-digraph with $|E| > 0$, terminal set K, source node $s \in K$, and diameter bound $D \geqslant 0$; suppose that the theorem holds for all acyclic d, K'-digraphs, with $d \geqslant 0$, and fewer arcs than G.

If $D = 0$ then the only possible D, K-digraph is the solitary $G = (\{s\}, \emptyset)$, thus we can assume that $D > 0$ as $E > 0$.

Suppose that there exists an arc (s, u) in G, such that $ind_G(u) > 1$. Consider the following cases:

(a) If $G - x$ is not a D, K-digraph, then $d_{D,K}(G - x) = 0$ and by Lemma 2, $d_{D,K}(G) = -d_{D,K}(G - x) = 0$. But in this case, Lemma 6 tell us that $\mathbf{LP}(G, s, K) > D$.

(b) If $G - x$ is a D, K-digraph, from Lemma 5(1), $G - x$ is also acyclic and $\mathbf{LP}(G, s, K) = \mathbf{LP}(G - x, s, K)$.
If $\mathbf{LP}(G, s, K) \leqslant D$ then $\mathbf{LP}(G - x, s, K) \leqslant D$, and by the induction hypothesis, $d_{D,K}(G - x) = (-1)^{e-1-n+1}$ as $G - x$ has one arc fewer that G, but from Lemma 2, $d_{D,K}(G) = -d_{D,K}(G - x) = (-1)^{e-n+1}$.
Similarly if $\mathbf{LP}(G, s, K) > D$ then $\mathbf{LP}(G - x, s, K) > D$, then by the induction hypothesis, $d_{D,K}(G - x) = 0$, but it follows from Lemma 2 that $d_{D,K}(G) = -d_{D,K}(G - x) = 0$.

Suppose that every vertex adjacent to s has indegree 1, and let $G^* = (V^*, E^*)$ be the graph with terminal set K^*, and source node s^*, obtained from operation $\mathbf{OP}_2(G, K, s)$. Since G is a D, K-digraph, then by Lemma 3, G^* is a $D - 1, K^*$-digraph, and $d_{D,K}(G) = d_{D-1,K^*}(G^*)$. But from Lemma 5(2), G^* is also acyclic and $\mathbf{LP}(G^*, s^*, K^*) = \mathbf{LP}(G, s, K) - 1$.

If $\mathbf{LP}(G, s, K) > D$ then $\mathbf{LP}(G^*, s^*, K^*) > D - 1$ and as G^* has fewer arcs than G, then from the induction hypothesis, $d_{D-1,K^*}(G^*) = 0$, and $d_{D,K}(G) = d_{D-1,K^*}(G^*) = 0$.

If $\mathbf{LP}(G, s, K) \leqslant D$ then $\mathbf{LP}(G^*, s^*, K^*) \leqslant D - 1$. Let $e^* = |E^*|$, and $n^* = |V^*|$. Let also $U = \{u_1, u_2, \ldots, u_r\}$ be the set of vertices of G such that $(s, u_i) \in E$. As $ind_G(u) = 1, u \in U$, then $e^* = e - |U|$ and $n^* = n - |U|$. Since $\mathbf{LP}(G^*, s^*, K^*) \leqslant D - 1$, and as G^* has fewer arcs than G, then from the induction hypothesis, $d_{D-1,K^*}(G^*) = (-1)^{e^*-n^*+1}$. But from Lemma 3, $d_{D,K}(G) = d_{D-1,K^*}(G^*) = (-1)^{e^*-n^*+1} = (-1)^{e-n+1}$. □

5. Algorithm

In this section we present an algorithm for the computation of the diameter-constrained s, K-terminal reliability based upon Eq. (5) and the characterization of the domination stated in Theorems 2 and 3.

It is easy to see that in any digraph G containing a set of parallel arcs $\{x_1, x_2, \ldots, x_m\}$ emanating from a node u, and directed into a node v, and with corresponding reliabilities $\{p(x_1), p(x_2), \ldots, p(x_m)\}$, the parallel arcs can be replaced by a single arc $x = (u, v)$ with reliability

$$p(x) = 1 - \prod_{i=1}^{m} (1 - p(x_i)), \tag{6}$$

without affecting the reliability of G; thus we are only concerned with digraphs without parallel arcs.

For a digraph $G = (V, E)$, with terminal set $K \subseteq V$, and distinguished vertex $s \in K$, we say that G is s, K-connected if there exists in G an s, u-dipath for every $u \in K$. If $ind_G(s) = 0$, we will denominate this graph s-rooted, and from this point on we will be only concerned with s-rooted digraphs, since if that is not the case, then $d_{D,K}(G) = 0$, as stated in the following claim:

Claim 1. *Suppose that $G = (V, E)$ is a digraph with terminal set $K \subseteq V$, vertex $s \in K$ and diameter bound D. If $ind_G(s) > 0$ then $d_{D,K}(G) = 0$.*

We next need to define irrelevant arcs.

Definition (ix) Given a graph $G = (V, E)$, with terminal set $K \subseteq V$, vertex $s \in K$, an arc $x = (u, v) \in E$ is an irrelevant arc if

(a) the arc x belongs to a connected component $G' = (V', E')$ of G, where $V' \subseteq V - K$,
(b) the vertex $u \in V - K$ has $ind_G(u) = 0$,
(c) the vertex $v \in V - K$ has $outd_G(u) = 0$.

According to Theorems 2 and 3, the algorithm should only be concerned in identifying acyclic D, K-digraphs whose longest s, u-dipath, $u \in K$, is of length at most D. The following Lemma gives a sufficient condition for such digraphs.

Lemma 7. *Given a digraph $G = (V, E)$, with terminal set K, and vertex $s \in K$, suppose that G is an acyclic, s, K-connected digraph, with no irrelevant arcs, and $\mathbf{LP}(G, s, K) \leqslant D$, then G is a D, K-digraph.*

Proof. We will proceed by contradiction.

Suppose G is acyclic, s, K-connected digraph, with no irrelevant arcs, and whose longest s, u-dipath, $u \in K$, is of length at most D, and G is not a D, K-digraph. Then it must be the case that G has an arc $x = (u, v)$ that is not contained in any D, K-tree of G.

Since G is s, K-connected, and $\mathbf{LP}(G, s, K) \leqslant D$, then G contains a D, K-tree $T = (V', E')$.

Suppose next that $u, v \in K$. Consider the unique path in T, $P = \langle (s = u_1, u_2), (u_2, u_3), \ldots, (u_{r-1}, u_r = v) \rangle$. Consider the digraph T' obtained from T by deleting the arc (u_{r-1}, v) and possibly any pendant vertices which do not belong to K, and adding the arc (u, v). Since G is acyclic, clearly T' is a K-tree, but under the assumption that $\mathbf{LP}(G, s, k) \leqslant D$, then T' is also a D, K-tree, contradicting the notion that the arc $e = (u, v)$ does not belong to any D, K-tree.

In the case that $u \in K$, and $v \in V - K$, since G does not contain irrelevant arcs, $outd(v) \neq 0$, this condition assures the existence in G of a u, z-dipath, where $z \in K$, and (u, v) is an arc of this path. Since T contains all the vertices of K, then there exists a dipath in G, $P' = \langle (u = v_1, v_2 = v), (v_2, v_3), \ldots, (v_{t-1}, v_t) \rangle$, where $v_t \neq u$ (since G is acyclic), is the only vertex in P', where v_t is also a vertex in T. Consider the unique path in T, $P = \langle (s = u_1, u_2), (u_2, u_3), \ldots, (u_{r-1}, u_r = v_t) \rangle$, and let T' be the digraph obtained from T by deleting the arc (u_{r-1}, v_t) and possibly any pendant vertices which do not belong to K, and adding the path P'. Under the assumption that G is acyclic and $\mathbf{LP}(G, s, K) \leqslant D$, T' is a D, K-tree, contradicting the assumption that $x = (u, v)$ does not belong to any D, K-tree.

The cases where $u \in V - K$, $v \in K$, and $u \in V - K$, $v \in V - K$, are proved in a similar fashion, by also not allowing irrelevant arcs in which one of their end-points have indegree 0. □

We present now an algorithm for efficiently generating precisely all these digraphs having non-null domination. As a first step, we assume that G is s-rooted. If this is not the case we can simply delete all arcs directed into s, obtaining an s-rooted digraph. Moreover, parallel arcs are replaced by a single arc whose reliability is obtained as explained at the beginning of this section.

The algorithm has five stages:

(a) If G has irrelevant arcs, generate a digraph from G by deleting these arcs, and any isolated vertex $u \in V - K$ obtained from this deletion.
(b) If G is not s, K-connected, then do not generate any subgraphs from G.
(c) If G is s, K-connected and contains a dicycle, generate acyclic subgraphs of G.
(d) If G is s, K-connected, acyclic, and $\mathbf{LP}(G, s, K) > D$, then generate all possible acyclic subgraphs G' of G such that $\mathbf{LP}(G', s, K) \leqslant D$.
(e) If G is s, K-connected, acyclic and $\mathbf{LP}(G, s, K) \leqslant D$, then generate all possible subgraphs of G.

Generation of duplicate subgraphs at all stages is completely avoided by a simple check.

The algorithm grows a rooted directed tree with the following properties:

(1) Vertices represent non-empty subgraphs of G, the root vertex being G itself. Any vertex, say r, corresponds one-to-one with the subgraph G_r which is of one of the following five types: (a) G_r contains irrelevant arcs, (b) G_r is not s, K-connected, (c) G_r is s, K-connected and cyclic, (d) G_r is s, K-connected, acyclic, and $\mathbf{LP}(G, s, K) > D$, (e) G_r is s, K-connected, acyclic, and $\mathbf{LP}(G, s, K) \leqslant D$.
(2) A link directed from vertex i to vertex j of the tree is labeled X, where X represents the set of arcs deleted from G_i to obtain G_j.

The following additional definitions are needed to explain the directed tree generation:

(1) *Father* (*Child*): vertex $i\,(j)$ is the father (child) of $j\,(i)$ when there exists an link directed from i to j.
(2) *Ancestor*: vertex i is the ancestor to j when i is contained in the path from the root vertex to j ($i \neq j$).
(3) *Brother*: vertices having the same father are termed brothers.
(4) *Younger* (*Elder*) *Brother*: a vertex i is the younger (elder) brother of vertex j, if the algorithm generates the children of vertex i later (earlier) than the children of vertex j.

The algorithm starts at the root vertex and grows the tree progressively. There are five rules for generating the children of vertex r, depending on the nature of G_r.

Rule 1. *G_r has irrelevant arcs*. Let X' be the label corresponding to the set of irrelevant arcs of G_r. In this case generate a new node representing the digraph obtained from G_r by deleting these arcs (and possibly any isolated vertices obtained from this deletion), provided $X' \cap X = \emptyset$, where X is the label of the link incident into the elder brothers of r or elder brothers of an ancestor of r; otherwise do not generate any children from G_r.

Rule 2. *G_r is not s, K-connected*. In this case G_r does not generate any children.

Rule 3. *G_r is s, K-connected and cyclic*. Consider a dicycle C in G_r containing the arcs $x_1, x_2, \ldots, x_c$. Then $G_{r_j} = G_r - x_j$, $(j = 1, 2, \ldots, c)$, is a child of G_r, provided $\{x_j\} \cap X = \emptyset$, where X is the label of a link incident into the elder brothers of r or elder brothers of an ancestor of r. Determination of a dicycle is determined for example by application of Depth First Search. Clearly a state $G_r - x_j$ where x_j does not belong to the dicycle C, contains also C, thus by Theorem 2, $d_{D,K}(G_k - x_j) = 0$, so it is not necessary to generate this state.

Rule 4. *G_r is s, K-connected, acyclic, and* $\mathbf{LP}(G_r, s, K) > D$. Consider a longest s, u-dipath L in G_r containing the arcs $x_1, x_2, \ldots, x_l$. Then $G_{r_j} = G_r - x_j$, $(j = 1, 2, \ldots, l)$, is a child of G_r, provided $\{x_j\} \cap X = \emptyset$, where X is the label of a link incident into any elder brother of r or elder brother of an ancestor of r. A longest s, u-dipath can be found by application of a longest path algorithm with time complexity $\mathrm{O}(|V| + |E|)$ for acyclic digraphs (see for example [12]). It is not necessary to consider a state $G_r - x_j$ where x_j does not belong to the dipath L, because $G_r - x_j$ is either not s, K-connected and its domination is 0, or it is s, K-connected and contains the path L of length greater than D, and by Theorem 3 its domination is also 0.

Rule 5. *G_r is s, K-connected, acyclic, and* $\mathbf{LP}(G_r, s, K) \leqslant D$. Let $G_r = (V_k, E_k)$. Assuming that G_r does not have irrelevant arcs, it follows from Lemma 7 that G_r is a D, K-digraph, therefore contributing to the total reliability by $(-1)^{|E_r|-|V_r|+1} \prod_{x \in E_k} p(x)$. Moreover, let $G_{r_j} = G_r - x_j$, $x_j \in E_r$ be a child of G_r, provided $\{x_j\} \cap X = \emptyset$, where X is the label of a link incident into any elder brother of r or elder brother of an ancestor of r.

The algorithm applies the previous rules recursively, and employs a rooted tree, called *Auxt*, as an auxiliary data structure. This data structure is used to maintain the states already generated and to avoid state duplications. This is

done by determining, at each step, whether the arcs to be deleted from the digraph under consideration are contained in the label of a link incident into any brother or elder brother of an ancestor of this digraph.

We now present the pseudo-code of the algorithm:

Algorithm

Input: Original s-rooted digraph G, and diameter bound D.

Output: Source-to-all-terminal reliability R of G.

Data structures:

$\mathbf{P}(E)$. Represents the operational probabilities of the set of arcs E of the original digraph G, and the operational probability of an arc $x \in E$ is denoted as $p(x)$.

R. Global variable to represent the diameter-constrained reliability. Originally $R = 0$.

r. Current vertex being considered. This is a global variable and originally $r = 0$.

G_r. Current digraph under consideration. Originally $G_0 = G$.

n_r. Number of vertices of G_r.

e_r. Number of arcs of G_r.

Auxt. Rooted tree auxiliary data structure. Originally *Auxt* contains only the vertex $r = 0$, that represents the original graph $G_0 = G$.

Auxiliary procedures:

(1) *AddAuxt* (vertex l, vertex m, label X). This procedure will add a link from vertex l into a new vertex m of *Auxt*, whose label is X corresponding to a set of arcs deleted from G_l to obtain G_m.

(2) *bool CheckAuxt* (vertex l, label X). This procedure will backtrack from vertex l to find if any of the arcs represented by the label X, is an arc of set of arcs corresponding to a label incident into any elder brother or ancestor's elder brother of a vertex l (we assume that each vertex contains the label of its father). If that is the case will return *true*, otherwise will return *false*. This routine is computationally efficient, since the longest possible path from the root of *Auxt* is of at most $|E|$ links.

Main procedure:

CalcRel (**Graph** G_r)

1. Let $crntvrtx = r$; current vertex of the rooted tree;
2. If ($G_r = (V_r, E_r)$ contains a set of irrelevant arcs E')
    - 2.1. Let X the label of E';
    - 2.2. If ($CheckAuxt(crntvrtx, X) == false$)
        - 2.2.1. Let $r = r + 1$;
        - 2.2.2. $AddAuxt(crntvrtx, r, X)$;
        - 2.2.3. Let $G_r = G_{crntvrtx} - E'$;
        - 2.2.4. $CalcRel(G_r)$;
    - 2.3. return;
3. Apply Depth First Search to determine s, K-connectedness or detect dicycles.
4. If ($G_r = (V_r, E_r)$ is not s, K-connected) return;
5. If ($G_r = (V_r, E_r)$ is cyclic)
    - 5.1. Let $C = \{x_1, x_2, \ldots, x_c\}$ be the arcs of a dicycle of G_r.
    - 5.2. For ($x_i \in C$) do
        - 5.2.1. If ($CheckAuxt(crntvrtx, x_i) == false$)
            - 5.2.1.1. Let $r = r + 1$;
            - 5.2.1.2.$AddAuxt(crntvrtx, r, x_i)$;
            - 5.2.1.3. Let $G_r = G_{crntvrtx} - x_i$;
            - 5.2.1.4. $CalcRel(G_r)$;
    - 5.3. return;
6. Determine $\mathbf{LP}(G_r, s, K)$;

7. If ($G_r = (V_r, E_r)$ is acyclic and $\mathbf{LP}(G_r, s, K) > D$).
    7.1. Let $L = \{x_1, x_2, \ldots, x_l\}$ be the arcs of a longest s, u-dipath of G_r.
    7.2. For ($e_i \in L$) do
        7.2.1. If ($CheckAuxt(crntvrtx, x_i) == false$)
            7.2.1.1. Let $r = r + 1$;
            7.2.1.2. $AddAuxt(crntvrtx, r, x_i)$;
            7.2.1.3. Let $G_r = G_{crntvrtx} - x_i$;
            7.2.1.4. $CalcRel(G_r)$;
    7.3. return;
8. If ($G_r = (V_r, E_r)$ is acyclic and $\mathbf{LP}(G_r, s, K) \leqslant D$).
    8.1. Let $R = R + (-1)^{e_r - n_r + 1} \times \prod_{e \in E_r} p(e)$;
    8.2. For ($x_i \in E_r$) do
        8.2.1. If ($CheckAuxt(crntvrtx, x_i) == false$)
            8.2.1.1. Let $r = r + 1$;
            8.2.1.2. $AddAuxt(crntvrtx, r, x_i)$;
            8.2.1.3. Let $G_r = G_{crntvrtx} - x_i$;
            8.2.1.4. $CalcRel(G_r)$;
    8.3. return;

Of the possible $2^{|E|}$ states (i.e., digraphs) to be evaluated, steps 2, 5, and 6 of the above algorithm represent a significant reduction on the total number of executable operations performed, since many states are avoided, especially when the digraphs contain irrelevant arcs, when they contain several directed cycles, or when the diameter bound D is small.

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