NOTE

An Elementary Proof for the \(O_R\)-Theorem of Hardy and Littlewood

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1. INTRODUCTION

The series \(\sum a_n\) is called \(A\)-summable to 0 (\(A\)=summation \(\sum a_n = 0\)), if \(\sum a_n x^n\) converges for \(0 \leq x < 1\) and

\[\lim_{x \to 1^-} \sum a_n x^n = 0\]

holds. The following \(O_R\)-Theorem for (this \(Abel\) method) \(A\) is due to Hardy and Littlewood [2, Theorem 11].

**THEOREM HL.** \(A\)-\(\sum a_n = 0\) and \(na_n \leq 1 \Rightarrow \sum a_n = 0\).

Thus we pass from summability to convergence. There are several proofs of Theorem HL; see [1, 7, 6, 10, 8]. In this paper we give an elementary and short proof for Theorem HL. For this we combine the difference method (which was used by Ingham [4] to prove the High Indices Theorem of Hardy and Littlewood [3]) with ideas of Wielandt [9]. But we do not employ the Weierstrass approximation theorem.

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2. PRELIMINARIES

Let \( \alpha := 1/2 \). We write \( \Sigma, \lim, \liminf, \limsup \) instead of

\[
\sum_{n=1}^{\infty}, \ \lim_{x \to 1^-}, \ \limsup_{x \to 1^-}.
\]

Let \( \mathcal{W} \) be the set of all functions \( f: [0,1] \to \mathbb{R} \) such that (for the given series) \( \sum a_n f(x^n) \) converges for \( 0 \leq x < 1 \) and \( \lim \sum a_n f(x^n) = 0 \). If \( A - \sum a_n = 0 \), then every polynomial belongs to \( \mathcal{W} \). Let the step \( \psi \) be defined by \( \psi(x) := 0 \) for \( 0 \leq x < \alpha \) and \( \psi(x) := 1 \) for \( \alpha \leq x \leq 1 \). Then

\[
\sum a_n \psi(x^n) = \sum_{1 \leq n \leq n(x)} a_n \quad (0 < x < 1, n(x) := (\ln \alpha)/\ln x). \quad (2.1)
\]

Hence, for the proof of Theorem HL, it suffices to show \( \psi \in \mathcal{W} \).

Let \( g(x) := x(1-x) \) for \( 0 \leq x \leq 1 \). It is essential that \( g \) is symmetric about \( x = \alpha \). With \( R \in \mathbb{N} \) the polynomial ramp \( r \) is defined by

\[
r(x) := \frac{(2R+1)!}{R!R!} \int_0^x g^R(t) \ dt \quad (0 \leq x \leq 1).
\]

Then \( r \) is symmetric with respect to \( (\alpha, \alpha) \), \( r(\alpha) = \alpha \), \( r(1) = 1 \), and

\[
p_1 := r - \alpha \cdot 4^{R+1} g^{R+1} \leq \varphi \leq r + \alpha \cdot 4^{R+1} g^{R+1} = p_2 \quad (0 \leq x \leq 1).
\]

(2.2)

3. PROOF OF THEOREM HL

Let \( \varepsilon \in (0,1) \) be fixed. We choose \( \delta \in (0, \alpha) \) such that

\[
\ln\{[\ln(\alpha - \delta)]/\ln(\alpha + \delta)\} < \varepsilon. \quad (3.1)
\]

For each \( x > \alpha + \delta \) we define the indices \( p = p(x), q = q(x) \) by

\[
x^p \leq \alpha + \delta < x^{p-1}, \quad x^{q+1} < \alpha - \delta \leq x^q \quad (3.2)
\]

and we choose \( R \in \mathbb{N} \) such that

\[
4^{R+1} g^R(\alpha + \delta) < \varepsilon. \quad (3.3)
\]

Then, by (3.2) and (3.1),

\[
\lim \sum_{n=p}^{q} \frac{1}{n} = \lim \int_{\ln(\alpha - \delta)/\ln x}^{\ln(\alpha + \delta)/\ln x} \frac{dx}{x} < \varepsilon \quad (3.4)
\]
and, by (3.2) and (3.3),
\[
\sum_{n=1}^{p-1} n^{-1} 4^{R+1} g^{R+1}(x^n) + \sum_{n=q+1}^{\infty} n^{-1} 4^{R+1} g^{R+1}(x^n) \\
\leq 4^{R+1} g^R (\alpha + \delta) \sum n^{-1} x^n (1 - x^n) < \varepsilon \quad (\alpha + \delta < x < 1).
\]
(3.5)

Since the polynomials \( p_1, p_2 \) belong to \( \mathcal{W} \) we obtain, by (2.1) and (2.2),
\[
\lim_{0 < n \leq n(x)} \sum a_n = \lim_{0 < n \leq n(x)} \sum a_n (\varphi - p_1)(x^n) \leq \lim_{0 < n \leq n(x)} \sum n^{-1} (p_2 - p_1)(x^n),
\]
\[
\lim_{0 < n \leq n(x)} a_n = -\lim_{0 < n \leq n(x)} \sum a_n (p_2 - \varphi)(x^n) \geq -\lim_{0 < n \leq n(x)} \sum n^{-1} (p_2 - p_1)(x^n).
\]

From this we get, using (3.4) and (3.5),
\[
-2 \varepsilon \leq \lim_{0 < n \leq n(x)} \sum a_n \leq \lim_{0 < n \leq n(x)} \sum a_n \leq 2 \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, we have \( \sum a_n = 0 \).

4. CONCLUDING REMARKS

The following result of Littlewood [5] is a corollary of Theorem HL: \( A \cdot \sum a_n = 0 \) and \( n |a_n| \leq 1 \Rightarrow \sum a_n = 0 \). If one wants only this result, the above proof can be simplified. On the other hand, one needs some refinement to cover Tauberian conditions concerning oscillation or decrease.

REFERENCES

2. G. H. Hardy and J. E. Littlewood, Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive, \( J. London Math. Soc. (2) 13 \) (1914), 174–191.