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# Convergence to strong nonlinear diffusion waves for solutions to $p$-system with damping on quadrant

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**Abstract**

In this paper, we consider the so-called $p$-system with linear damping on quadrant. We show that for a certain class of given large initial data $(v_0(x), u_0(x))$, the corresponding initial–boundary value problem admits a unique global smooth solution $(v(x,t), u(x,t))$ and such a solution tends time-asymptotically, at the $L^p$ $(2 \leqslant p \leqslant \infty)$ optimal decay rates, to the corresponding nonlinear diffusion wave $(\bar{v}(x,t), \bar{u}(x,t))$ which satisfies (1.9) provided the corresponding prescribed initial error function $(V_0(x), U_0(x))$ lies in $(H^3(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)) \times (H^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+))$.


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*Keywords:* Linear damping; Boundary effect; Nonlinear diffusion waves; Convergence rates; Energy estimates

## 1. Introduction

In this paper, we consider the asymptotic behavior and the convergence rates of solutions to the so-called $p$-system with linear damping

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

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with the initial data

$$\big(v(x,0),u(x,0)\big)=\big(v_0(x),u_0(x)\big)\to(v_+,u_+),\quad v_+>0,\ \text{as } x\to+\infty, \tag{1.2}$$

and with the null-Dirichlet boundary condition

$$u|_{x=0}=0. \tag{1.3}$$

The system (1.1) can be viewed as the isentropic Euler equations in Lagrangian coordinates with frictional term $-\alpha u$ in the momentum equation and it can be used to model the compressible flow through porous media. Here, $v>0$ is the specific volume, $u$ is the velocity, the pressure $p(v)$ is a decreasing smooth function, $\alpha$ is a positive constant.

For the Cauchy problem to $p$-system with linear damping, the global existence of smooth solutions with small initial data has been studied by many authors, cf. [2,9,16,26], and the large time behavior of the solutions was carried out by Hsiao and Liu in [4,5] firstly. Precisely, they showed that the solutions of the Cauchy problem to (1.1) with the initial data

$$\big(v(x,0),u(x,0)\big)=\big(v_0(x),u_0(x)\big)\to(v_\pm,u_\pm),\quad \text{as } x\to\pm\infty,\ v_\pm>0, \tag{1.4}$$

tend time-asymptotically to the nonlinear self-similar diffusion wave solutions $(\bar{\bar{v}},\bar{\bar{u}})(x,t)$ of the porous media equation

$$\begin{cases}\bar{\bar{v}}_t-\bar{\bar{u}}_x=0,\\ p(\bar{\bar{v}})_x=-\alpha\bar{\bar{u}},\quad x\in\mathbb{R},\ t>0,\end{cases} \tag{1.5}$$

or

$$\begin{cases}\bar{\bar{v}}_t=-\dfrac{1}{\alpha}p(\bar{\bar{v}})_{xx},\\ \bar{\bar{u}}=-\dfrac{1}{\alpha}p(\bar{\bar{v}})_x,\quad x\in\mathbb{R},\ t>0,\end{cases} \tag{1.6}$$

with the same end states as $v_0(x)$:

$$\bar{\bar{v}}(\pm\infty,t)=v_\pm. \tag{1.7}$$

Here the well-known porous media equation is obtained by Darcy's law. And a better convergence rate and the optimal convergence rate when $v(+\infty,0)=v(-\infty,0)$ were obtained by Nishihara in [17,18]. For the other related results we refer to [20,25]. For the case of the large initial data, Zhao in [24] showed that for a certain class of given large initial data, the Cauchy problem (1.1), (1.4) admitted a unique global smooth solution and such a solution tended time-asymptotically, at the $L^p$ $(2\leqslant p\leqslant\infty)$ decay rates to the nonlinear diffusion wave $(\bar{\bar{v}}(x,t),\bar{\bar{u}}(x,t))$ but without any further smallness assumptions on the strength of the nonlinear diffusion wave and the initial error. For the Cauchy problem to $p$-system with nonlinear damping, we refer to [29]. For other results, see [1,3,6,8,21,23,28].

For the initial–boundary value problems on $\mathbb{R}^+$ to the equations of viscous conservation laws, the global existence and the asymptotic behavior of the solution have been investigated by several authors, cf. [10–12,22]. For the initial–boundary value problems on $\mathbb{R}^+$ to $p$-system with linear damping, see [13–15,19], Nishihara and Yang in [19] considered (1.1)–(1.3), and they got the asymptotic behavior and the convergence rates by perturbing the initial value around the linear diffusion waves $(\tilde{v}, \tilde{u})(x,t)$ which satisfies

$$\begin{cases} \tilde{v}_t - \tilde{u}_x = 0, \quad x \in \mathbb{R}^+,\ t > 0, \\ p'(v_+)\tilde{v}_x = -\alpha \tilde{u}, \\ \tilde{u}|_{x=0} = 0, \quad (\tilde{v}, \tilde{u})|_{x=\infty} = (v_+, 0). \end{cases} \tag{1.8}$$

Marcati, Mei and Rubino in [14] also considered (1.1)–(1.3), and they got the asymptotic behavior and improved the convergence rates in [19] by perturbing the initial value around the nonlinear diffusion waves $(\bar{v}, \bar{u})(x,t)$ which satisfies the porous media equation

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \quad x \in \mathbb{R}^+,\ t > 0, \\ p(\bar{v})_x = -\alpha \bar{u}, \\ \bar{u}|_{x=0} = 0, \quad (\bar{v}, \bar{u})|_{x=\infty} = (v_+, 0). \end{cases} \tag{1.9}$$

In the above two papers, they all asked that the initial disturbance data under their considerations to be small, but for the large initial data, there are few results. In this paper, we also consider (1.1)–(1.3) and obtained the same asymptotic behavior and the convergence rates as in [14] but only under a rather weaker smallness assumption on the initial disturbance.

The rest of this paper is organized as follows. In Section 2, we reformulate the problem (1.1)–(1.3) and state the main theorem. In Sections 3 and 4, the proof of the main theorem will be given, much of Section 4 is based on the paper [24].

**Notations.** Hereafter, we denote several generic positive constants depending on $a, b, \dots$ by $C_{a,b,\dots}$ or only by $C$ or $O(1)$ without any confusion and $\varepsilon$ will always be used to represent sufficiently small positive constants. $L^p = L^p(\mathbb{R}^+)$ $(1 \leqslant p \leqslant \infty)$ denotes usual Lebesgue space with the norm

$$\|f\|_{L^p} = \left( \int_{\mathbb{R}^+} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

$$\|f\|_{L^\infty} = \sup_{\mathbb{R}^+} |f(x)|,$$

and the integral region $\mathbb{R}^+$ will be omitted without any confusion. $H^l$ $(l \geqslant 0)$ denotes the usual $l$th-order Sobolev space with the norm

$$\|f\|_l = \left( \sum_{j=0}^{l} \|\partial_x^j f\|^2 \right)^{\frac{1}{2}},$$

where $\|\cdot\| = \|\cdot\|_0 = \|\cdot\|_{L^2}$. For simplicity, $\|f(\cdot,t)\|_{L^p}$ and $\|f(\cdot,t)\|_l$ are denoted by $\|f(t)\|_{L^p}$ and $\|f(t)\|_l$, respectively.

## 2. Reformulation of the problem and theorems

We first reformulate the problem (1.1)–(1.3). To eliminate the value of $u(x,t)$ at $x=+\infty$, we introduce the following auxiliary functions as in [4,5,19],

$$\begin{cases} \hat{v}(x,t) = \dfrac{u_+ m_0(x)}{-\alpha} e^{-\alpha t}, \\ \hat{u}(x,t) = u_+ e^{-\alpha t} \displaystyle\int_0^x m_0(y)\,dy, \end{cases} \tag{2.1}$$

where $m_0(x)$ is a smooth function with compact support such that

$$\int_0^\infty m_0(y)\,dy = 1, \quad \operatorname{supp} m_0 \subset \mathbb{R}^+, \quad m_0'(0) = 0. \tag{2.2}$$

Then $(\hat{v}, \hat{u})(x,t)$ satisfies

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \quad x \in \mathbb{R}^+,\ t > 0, \\ \hat{u}_t = -\alpha \hat{u}, \\ \hat{u}|_{x=0} = 0, \quad (\hat{v}, \hat{u})|_{x=\infty} = \big(0, u_+ e^{-\alpha t}\big). \end{cases} \tag{2.3}$$

**Lemma 2.1.** *For $k \geqslant 0$, $j \geqslant 0$, $p \in [1,\infty]$, we have*

$$\big\| \partial_x^k \partial_t^j \hat{v}(\cdot,t) \big\|_{L^p} = O(1) e^{-\alpha t}. \tag{2.4}$$

For the solution $(\bar{v}, \bar{u})(x,t)$ of (1.9), from [14], we have

**Lemma 2.2.** *(See [14].) Let $\bar{v}(x,t)$ be a solution of (1.9) with the initial data $\bar{v}(x,0) = v_+ + \bar{\delta}_0 \varphi_0(x)$, where $\bar{\delta}_0$ is a constant such that*

$$\int_0^{+\infty} \big(v_0(x) - v_+\big)\,dx - \bar{\delta}_0 \int_0^{+\infty} \varphi_0(x)\,dx + \frac{u_+}{\alpha} = 0, \tag{2.5}$$

*and $\varphi_0(x)$ is a given smooth function such that*

$$\varphi_0(x) \in L^1\big(\mathbb{R}^+\big), \quad \int_0^{+\infty} \varphi_0(x)\,dx \neq 0,$$

*and the compatibility condition*

$$\phi_0'(0) = 0.$$

*Then for any $k \geqslant 0$, $j \geqslant 0$, we have*

$$\begin{cases} \left\| \partial_x^k \partial_t^j (\bar{v} - v_+)(t) \right\| \leqslant O(1)\bar{\delta}_0 (1+t)^{-\frac{4j+2k+1}{4}}, \\ \left\| \bar{v}_{xt}(t) \right\|_{L^1} \leqslant O(1)\bar{\delta}_0 (1+t)^{-\frac{3}{2}}. \end{cases} \tag{2.6}$$

From Lemma 2.2 and the Sobolev's inequality, we have

**Corollary 2.3.** *For any $k \geqslant 0$, $j \geqslant 0$, we have*

$$\left\| \partial_x^k \partial_t^j (\bar{v} - v_+)(t) \right\|_{L^\infty} \leqslant O(1)\bar{\delta}_0 (1+t)^{-\frac{2j+k+1}{2}}.$$

Combining (1.1), (1.9), (2.3), we have

$$\begin{cases} (v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0, \\ (u - \bar{u} - \hat{u})_t + \big(p(v) - p(\bar{v})\big)_x = -\alpha(u - \bar{u} - \hat{u}) - \bar{u}_t. \end{cases} \tag{2.7}$$

Integrating $(2.7)_1$ over $[0, \infty) \times [0, t]$, and recalling that $\bar{v}(x, 0) = v_+ + \bar{\delta}_0 \varphi_0(x)$ from [14], we get from (2.5) that

$$\int_0^{+\infty} (v - \bar{v} - \hat{v})(y, t)\, dy = \int_0^{+\infty} \big(v_0(x) - v_+\big)\, dx - \bar{\delta}_0 \int_0^{+\infty} \varphi_0(x)\, dx + \frac{u_+}{\alpha} = 0, \tag{2.8}$$

then it is reasonable to introduce the following perturbation

$$\begin{cases} V(x, t) := -\displaystyle\int_x^{\infty} (v - \bar{v} - \hat{v})(y, t)\, dy, \\ U(x, t) := u(x, t) - \bar{u}(x, t) - \hat{u}(x, t). \end{cases} \tag{2.9}$$

From (2.7) and (2.9), we deduce that $(V, U)(x, t)$ solves the following problem

$$\begin{cases} V_t - U = 0, \quad x \in \mathbb{R}^+,\ t > 0, \\ U_t + \big(p(V_x + \bar{v} + \hat{v}) - p(\bar{v})\big)_x + \alpha U = \dfrac{1}{\alpha} p(\bar{v})_{xt}, \end{cases} \tag{2.10}$$

with the initial data

$$\begin{aligned} (V, U)(x, 0) &= (V_0, U_0)(x) \\ &:= \left( -\int_x^{\infty} \big(v_0(y) - \bar{v}(y, 0) - \hat{v}(y, 0)\big)\, dy, u_0(x) - \bar{u}(x, 0) - \hat{u}(x, 0) \right) \end{aligned} \tag{2.11}$$

and the boundary condition

$$V|_{x=0} = 0. \tag{2.12}$$

Before stating our main results, we list some further notations and assumptions.

We assume that the pressure $p(v)$ satisfies the following assumptions ($\mathrm{P}_1$) or ($\mathrm{P}_2$):

($\mathrm{P}_1$) $p(v) \in C^3(0,\infty)$, $p'(v) < 0$, $p''(v) > 0$, $4p'(v)p'''(v) \geqslant 5(p''(v))^2$, for $v \in (0,\infty)$, and

$$\lim_{v\to 0}\int_v^1 \sqrt{-p'(\tau)}\,d\tau = +\infty, \qquad \lim_{v\to+\infty} p'(v) = 0;$$

($\mathrm{P}_2$) $p(v) \in C^2(\mathbb{R})$, $p'(v) < 0$, for $v \in \mathbb{R}$.

Under the above assumptions, it is easy to see that the system (1.1) has two eigenvalues

$$\lambda = -\sqrt{-p'(v)}, \qquad \mu = \sqrt{-p'(v)} \tag{2.13}$$

and the corresponding Riemann invariants are taken as

$$r = u + h(v), \qquad s = u - h(v), \tag{2.14}$$

where

$$h(v) = \int_a^v \mu(\tau)\,d\tau, \quad a \in (0,\infty) \text{ is any fixed constant.} \tag{2.15}$$

Under the above notations, our result on the asymptotic behavior and the decay rates of the solution to the initial–boundary problem (1.1)–(1.3) can be summarized as the following.

**Theorem 2.4.** *Under the assumption* ($\mathrm{P}_1$) (*respectively* ($\mathrm{P}_2$))*, and for arbitrarily given positive constants* $v_1, v_2$ (*respectively* $M_1$) *and* $M_2$*, there exists a sufficiently small positive constant* $M_3$ *such that* $(r_0(x), s_0(x)) \in C_b^1(\mathbb{R}^+)$ *with*

$$\begin{cases} v_1 \leqslant v_0(x) \leqslant v_2 & \big(\text{respectively } |v_0(x)| \leqslant M_1\big), \quad |u_0(x)| \leqslant M_2, \\ |r_0'(x)| \leqslant \alpha M_3, & |s_0'(x)| \leqslant \alpha M_3, \end{cases} \tag{2.16}$$

*where*

$$r_0(x) = u_0(x) + h\big(v_0(x)\big), \qquad s_0(x) = u_0(x) - h\big(v_0(x)\big), \tag{2.17}$$

*then the initial–boundary value problem* (1.1)–(1.3) *admits a unique global in time smooth solution* $(v(x,t), u(x,t))$.

*Moreover, if* $(V_0, U_0)$ *lies in* $(H^3(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)) \times (H^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+))$*, then the solution* $(v(x,t), u(x,t))$ *tends to the solution* $(\bar{v}(x,t), \bar{u}(x,t))$ *of* (1.9) *and satisfies the following decay estimates*

$$v - \bar{v} - \hat{v} \in C^k\big(0,\infty; H^{2-k}\big(\mathbb{R}^+\big)\big), \quad k = 0,1,2,$$
$$u - \bar{u} - \hat{u} \in C^k\big(0,\infty; H^{2-k}\big(\mathbb{R}^+\big)\big), \quad k = 0,1,2,$$

*and*

$$\begin{cases} \left\|\partial_x^k(v-\bar{v}-\hat{v})(t)\right\| \leqslant O(1)(1+t)^{-\frac{2k+3}{4}}, & k=0,1, \\ \left\|(v-\bar{v}-\hat{v})(t)\right\|_{L^p} \leqslant O(1)(1+t)^{-(1-\frac{1}{2p})}, & 2\leqslant p\leqslant +\infty, \\ \left\|(u-\bar{u}-\hat{u})(t)\right\| \leqslant O(1)(1+t)^{-\frac{5}{4}}. \end{cases} \tag{2.18}$$

**Remarks.** 1. Unlike those in [14,19] in which they asked $\|v_0-v_+\|_{L^1}+\|V_0\|_3+\|U_0\|_2+|u_+|\ll 1$, here we get the same results as [14] without any smallness conditions on the initial error.

2. When the pressure function $p(v)$ satisfies the $\gamma$-law, the assumption $(\mathrm{P}_1)$ corresponds to the case $1\leqslant\gamma\leqslant 3$.

3. In this paper, we will only prove Theorem 2.4 for the case when the assumption $(\mathrm{P}_1)$ is satisfied, for the assumption $(\mathrm{P}_2)$, the proof is less complicated and the details will be omitted.

## 3. The proof of the main results

In this section, we will prove our main Theorem 2.4. To do this, we need the following lemmas.

The global smooth solvability results for the initial–boundary problem (1.1)–(1.3) has been considered in [7], the results can be restated as in the following

**Lemma 3.1.** *(See [7].) Under the assumption* $(\mathrm{P}_1)$ *(respectively* $(\mathrm{P}_2)$*), and for arbitrarily given positive constants* $v_1, v_2$ *(respectively* $M_1$*) and* $M_2$*, there exists a sufficiently small positive constant* $M_3$ *such that* $(r_0(x), s_0(x))\in C_b^1(\mathbb{R}^+)$ *and the assumption* (2.16) *is satisfied, then the initial–boundary value problem* (1.1)–(1.3) *admits a unique global smooth solution* $(v(x,t),u(x,t))$ *which satisfies*

$$\begin{cases} v_*\leqslant v(x,t)\leqslant v^* \quad \big(\text{respectively } \left|v(x,t)\right|\leqslant M_1\big), \quad \left|u(x,t)\right|\leqslant M_2, \\ \left|r_x(x,t)\right|\leqslant \alpha M_4, \quad \left|s_x(x,t)\right|\leqslant \alpha M_4, \end{cases} \tag{3.1}$$

*where* $v_*, v^*$ *and* $M_4$ *are positive constants depending only on* $v_1$*,* $v_2$*,* $M_2$ *and* $M_3$ *but independent of* $x$ *and* $t$*, and* $M_4$ *can be chosen sufficiently small.*

From Lemma 3.1, we can get the following corollary directly.

**Corollary 3.2.** *Under the assumptions of Theorem* 2.4*, the initial–boundary value problem* (2.10)–(2.12) *admits a unique global smooth solution* $(V(x,t),U(x,t))$ *which satisfies*

$$\left|V_x(x,t)\right|\leqslant M_5, \qquad \left|U(x,t)\right|\leqslant M_5, \tag{3.2}$$

*and*

$$\begin{cases} \left|\partial_x^i\partial_t^j\big(V_x(x,t)+\bar{v}(x,t)+\hat{v}(x,t)\big)\right|\leqslant M_6, \quad i\geqslant 0,\ j\geqslant 0,\ i+j=1, \\ \left|U_x(x,t)+\bar{u}_x(x,t)+\hat{u}_x(x,t)\right|\leqslant M_6, \end{cases} \tag{3.3}$$

*where* $M_i$ $(i=5,6)$ *are time-independent positive constants and* $M_6$ *can be chosen as small as we wanted.*

**Remark.** The constant $M_6$ can be chosen sufficiently small will play an important role in the proof of Theorem 4.3, and this is one of the keys that we can prove Theorem 2.4 without the *a priori* assumption (4.1).

**Lemma 3.3.** *Under the assumptions of Theorem* 2.4*, there exists a unique time-global solution* $(V,U)(x,t)$ *of the initial–boundary problem* (2.10)–(2.12) *satisfying*

$$V\in C^i\big([0,\infty),H^{3-i}\big),\quad i=0,1,2,3,$$

$$U\in C^i\big([0,\infty),H^{2-i}\big),\quad i=0,1,2,$$

$$\begin{aligned}&\sum_{k=0}^{3}(1+t)^k\big\|\partial_x^k V(t)\big\|^2+\sum_{k=0}^{2}(1+t)^{k+2}\big\|\partial_x^k U(t)\big\|^2\\&\quad+\int_0^t\Bigg(\sum_{j=1}^{3}(1+s)^{j-1}\big\|\partial_x^j V(s)\big\|^2+\sum_{j=0}^{2}(1+s)^{j+1}\big\|\partial_x^j U(s)\big\|^2\Bigg)\,ds\\&\leqslant O(1)\big(\|V_0\|_3^2+\|U_0\|_2^2+1\big),\end{aligned}\tag{3.4}$$

*and*

$$\begin{aligned}&(1+t)^4\big\|U_t(t)\big\|^2+(1+t)^5\big(\big\|U_{xt}(t)\big\|^2+\big\|U_{tt}(t)\big\|^2\big)\\&\quad+\int_0^t\big((1+s)^4\big\|U_{xt}(s)\big\|^2+(1+s)^5\big\|U_{tt}(s)\big\|^2\big)\,ds\\&\leqslant O(1)\big(\|V_0\|_3^2+\|U_0\|_2^2+1\big).\end{aligned}\tag{3.5}$$

The proof of Lemma 3.3 will be completed in the next section.

From Lemma 3.3, we can easily get the following corollary.

**Corollary 3.4.** *Under the assumptions of Theorem* 2.4*, there exists a unique time-global solution* $(V,U)(x,t)$ *of the initial–boundary problem* (2.10)–(2.12) *satisfying the following decay estimates*:

$$\big\|\partial_x^k V(t)\big\|\leqslant O(1)(1+t)^{-\frac{k}{2}},\quad k=0,1,2,3,\tag{3.6}$$

$$\big\|\partial_x^k U(t)\big\|\leqslant O(1)(1+t)^{-(1+\frac{k}{2})},\quad k=0,1,2,\tag{3.7}$$

$$(1+t)^4\big\|U_t(t)\big\|^2+(1+t)^5\big(\big\|U_{xt}(t)\big\|^2+\big\|U_{tt}(t)\big\|^2\big)\leqslant O(1).\tag{3.8}$$

In order to get Theorem 2.4, we need to improve the decay rates in Corollary 3.4. By using the method of Fourier transform and Corollary 3.4, we can get the following lemma.

**Lemma 3.5.** *Under the assumptions of Theorem* 2.4*, the solution* $(V,U)(x,t)$ *of* (2.10)–(2.12) *decays time-asymptotically as*

$$\left\|\partial_x^k V(t)\right\| \leqslant O(1)(1+t)^{-\frac{2k+1}{4}}, \quad k=0,1,2, \tag{3.9}$$

$$\left\|U(t)\right\| \leqslant O(1)(1+t)^{-\frac{5}{4}}. \tag{3.10}$$

Lemma 3.5 can be proved by using the same argument as in [14] where we can get the rigorous proof of (3.9) and (3.10), so we omit its details here.

Once we have the above lemmas, we can prove Theorem 2.4.

**Proof of Theorem 2.4.** From Lemmas 3.1 and 3.3, in order to prove Theorem 2.4, we only need to prove (2.18).

Noticing that $V_x = v - \bar{v} - \hat{v}$, $U = u - \bar{u} - \hat{u}$, by using (2.4), (3.9) and (3.10), we have

$$\left\|\partial_x^k (v-\bar{v}-\hat{v})(t)\right\| = \left\|\partial_x^k V_x(t)\right\| \leqslant O(1)(1+t)^{-\frac{2k+3}{4}} \tag{3.11}$$

and

$$\left\|(u-\bar{u}-\hat{u})(t)\right\| = \left\|U(t)\right\| \leqslant O(1)(1+t)^{-\frac{5}{4}}. \tag{3.12}$$

This proved $(2.18)_1$ and $(2.18)_3$.

For the proof of $(2.18)_2$, by using $(2.18)_1$ and the Sobolev's inequality, we have

$$\begin{aligned}
\left\|(v-\bar{v}-\hat{v})(t)\right\|_{L^p} &\leqslant \left\|(v-\bar{v}-\hat{v})(t)\right\|_{L^\infty}^{\frac{p-2}{p}} \left\|(v-\bar{v}-\hat{v})(t)\right\|^{\frac{2}{p}} \\
&\leqslant \left(\sqrt{2}\left\|(v-\bar{v}-\hat{v})(t)\right\|^{\frac{1}{2}} \left\|\partial_x(v-\bar{v}-\hat{v})(t)\right\|^{\frac{1}{2}}\right)^{\frac{p-2}{p}} \left\|(v-\bar{v}-\hat{v})(t)\right\|^{\frac{2}{p}} \\
&= 2^{\frac{p-2}{2p}} \left\|\partial_x(v-\bar{v}-\hat{v})(t)\right\|^{\frac{p-2}{2p}} \left\|(v-\bar{v}-\hat{v})(t)\right\|^{\frac{p+2}{2p}} \\
&\leqslant O(1)(1+t)^{-\frac{3}{4}\times\frac{p+2}{2p}}(1+t)^{-\frac{5}{4}\times\frac{p-2}{2p}} \leqslant O(1)(1+t)^{-(1-\frac{1}{2p})}.
\end{aligned} \tag{3.13}$$

This proved $(2.18)_2$. The proof of Theorem 2.4 is completed. □

## 4. The proof of Lemma 3.3

In this section, we will prove Lemma 3.3.

Recall that in [19], the authors got the estimates (3.4) and (3.5) under the *a priori* assumption

$$N(T) := \sup_{0<t<T} \left\{ \sum_{k=0}^{3} (1+t)^k \left\|\partial_x^k V(t)\right\|^2 + \sum_{k=0}^{2} (1+t)^{k+2} \left\|\partial_x^k U(t)\right\|^2 \right\} \leqslant \varepsilon_1^2, \tag{4.1}$$

where $0<\varepsilon_1 \ll 1$, which means the smallness assumption on the initial error, $\|V_0\|_3 + \|U_0\|_2 \ll 1$, is needed.

But in this paper, we want to get the estimates (3.4) and (3.5) without any smallness conditions on the initial error, so we cannot use (4.1). However, through some delicate energy estimates, we find that if we can get the following estimates

$$\begin{cases} \left\| V_{xx}(t) \right\|_{L^\infty} \leqslant O(1)(1+t)^{-\frac{1}{2}}, \\ \left\| V_{xt}(t) \right\|_{L^\infty} \leqslant O(1)(1+t)^{-1}, \end{cases} \tag{4.2}$$

which is a direct consequence of (4.1) in [19], then the techniques in [19] can still be used to deduce (3.4) and (3.5). Thanks to the argument developed by Zhao in [24], we can use the similar method to get the estimates (4.2), then we can use the techniques in [19] to get (3.4) and (3.5).

Now we give the main idea in deducing Lemma 3.3. In [19], the authors considered the energy estimates and the decay estimates (3.4) and (3.5) simultaneously, but in this paper, we first deduced certain energy estimates in Theorem 4.3 without any smallness conditions on the initial error by using Lemmas 2.1, 2.2 and Corollary 3.2. Secondly we prove (4.2) by using (4.3), then we can prove (3.4) and (3.5) by using the techniques developed by Nishihara and Yang in [19].

To make the proof easy to read, we divide it into the following two steps:

### *4.1. Energy estimates and asymptotic behavior*

Our main purpose in this subsection is to prove the following asymptotic behavior of the solution $(v(x,t), u(x,t))$ obtained in Lemma 3.1.

**Theorem 4.1.** *Under the assumptions of Theorem* 2.4*, the unique global smooth solution* $(V(x,t), U(x,t))$ *of* (2.10)–(2.12) *satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^+} \left\{ \left| \partial_x^i \partial_t^j V(x,t) \right| \right\} = 0, \quad i \geqslant 0,\ j \geqslant 0,\ i+j \leqslant 2. \tag{4.3}$$

Before proving Theorem 4.1, we first cite the following fundamental result, whose proof can be found in [27].

**Lemma 4.2.** *If there exists a constant $C > 0$, which is independent of $x$ and $t$, such that*

$$\begin{cases} \displaystyle\int u^2(x,t)\,dx \leqslant C, \quad \int u_x^2(x,t)\,dx \leqslant C, \\ \displaystyle\int_0^t\!\!\int u_x^2(x,t)\,dx\,dt \leqslant C, \quad \int u_t^2(x,t)\,dx \leqslant C, \end{cases}$$

*then we have*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^+} \left\{ \left| u(x,t) \right| \right\} = 0. \tag{4.4}$$

From Lemma 4.2, to prove Theorem 4.1, we only need to get the following energy estimates.

**Theorem 4.3** *(Energy estimates). Under the assumptions of Theorem* 2.4*, we have that*

$$\begin{aligned}
&\|V(t)\|_3^2+\|V_t(t)\|_2^2+\|V_{tt}(t)\|_1^2+\|V_{ttt}(t)\|^2\\
&\quad+\int_0^t\left(\|V_x(s)\|_2^2+\|V_t(s)\|_2^2+\|V_{tt}(s)\|_1^2+\|V_{ttt}(s)\|^2\right)ds\\
&\leqslant O(1)\left(\|V_0\|_3^2+\|U_0\|_2^2+1\right).
\end{aligned}\tag{4.5}$$

Theorem 4.3 will be proved by the following a series of lemmas.
We first rewrite (2.10)–(2.12) as follows:

$$\begin{cases}
V_{tt}+\left[p(V_x+\bar v+\hat v)-p(\bar v)\right]_x+\alpha V_t=\dfrac{1}{\alpha}p(\bar v)_{xt},\\
(V,V_t)(x,0)=(V_0,U_0)(x),\qquad V|_{x=0}=0.
\end{cases}\tag{4.6}$$

Since it suffices to establish the estimates for sufficiently smooth solution, Eq. $(4.6)_1$ and (2.12) give the following boundary condition for higher order derivatives:

$$V(0,t)=V_{xx}(0,t)=V_t(0,t)=V_{txx}(0,t)=0,\quad \text{etc.}\tag{4.7}$$

Therefore, the following estimates are formally quite similar to those in [24].
Our first result is on the basic energy estimates.

**Lemma 4.4** *(Basic energy estimates). Under the assumptions of Theorem* 2.4*, we have*

$$\begin{aligned}
&\|V(t)\|_1^2+\|V_t(t)\|^2+\int_0^t\left(\|V_x(s)\|^2+\|V_t(s)\|^2\right)ds\\
&\quad\leqslant O(1)\left(\|V_0\|_1^2+\|U_0\|^2+1\right).
\end{aligned}\tag{4.8}$$

**Proof.** First, multiplying $(4.6)_1$ by $V$ and integrating the resulting equation with respect to $x$ and $t$ over $\mathbb{R}^+\times[0,t]$, after some integrations by parts, we can get by using (4.7)

$$\begin{aligned}
&\frac{\alpha}{2}\int V^2\,dx-\int_0^t\!\!\int V_x\left[p(V_x+\bar v+\hat v)-p(\bar v)\right]dx\,ds\\
&\quad=\frac{\alpha}{2}\int V_0^2\,dx+\int_0^t\!\!\int V_t^2\,dx\,ds-\int VV_t\,dx+\int V_0U_0\,dx\\
&\qquad+\frac{1}{\alpha}\int_0^t\!\!\int p(\bar v)_{xt}V\,dx\,ds.
\end{aligned}\tag{4.9}$$

From Lemmas 2.1, 2.2 and the Cauchy–Schwarz's inequality, we have

$$
\begin{aligned}
&-\int_0^t\!\!\int V_x\big[p(V_x+\bar v+\hat v)-p(\bar v)\big]\,dx\,ds \\
&\quad= -\int_0^t\!\!\int p'\big(\theta(V_x+\hat v)+\bar v\big)V_x^2\,dx\,ds-\int_0^t\!\!\int p'\big(\theta(V_x+\hat v)+\bar v\big)V_x\hat v\,dx\,ds \\
&\quad\geqslant C_2\int_0^t\!\!\int V_x^2\,dx\,ds-\int_0^t\!\!\int p'\big(\theta(V_x+\hat v)+\bar v\big)V_x\hat v\,dx\,ds,
\end{aligned}
\tag{4.10}
$$

where $\theta\in(0,1)$, $C_2:=\inf_{v\in M}\{-p'(v)\}>0$, and $M=[\min\{v_*,\min_{\xi\in\mathbb{R}^+}\bar v(\xi)\},\max\{v^*,\max_{\xi\in\mathbb{R}^+}\bar v(\xi)\}]$.

On the other hand, we have by the Cauchy–Schwarz's inequality

$$
\frac{\alpha}{2}\int V_0^2\,dx+\int V_0U_0\,dx\leqslant O(1)\big(\|V_0\|^2+\|U_0\|^2\big)
\tag{4.11}
$$

and

$$
-\int VV_t\,dx\leqslant\frac{\alpha-\varepsilon}{2}\int V^2\,dx+\frac{1}{2(\alpha-\varepsilon)}\int V_t^2\,dx,
\tag{4.12}
$$

where $\varepsilon$ is a suitably small constant to be determined later. Furthermore, we have

$$
\begin{aligned}
-\int_0^t\!\!\int p'\big(\theta(V_x+\hat v)+\bar v\big)V_x\hat v\,dx\,ds&\leqslant\frac{\varepsilon}{2}\int_0^t\!\!\int V_x^2\,dx\,ds+O(1)\int_0^t\!\!\int \hat v^2\,dx\,ds \\
&\leqslant\frac{\varepsilon}{2}\int_0^t\!\!\int V_x^2\,dx\,ds+O(1)
\end{aligned}
\tag{4.13}
$$

and

$$
\begin{aligned}
\frac{1}{\alpha}\int_0^t\!\!\int p(\bar v)_{xt}V\,dx\,ds&=-\frac{1}{\alpha}\int_0^t\!\!\int p(\bar v)_tV_x\,dx\,ds \\
&\leqslant\frac{\varepsilon}{2}\int_0^t\!\!\int V_x^2\,dx\,ds+O(1)\int_0^t\!\!\int \bar v_t^2\,dx\,ds \\
&\leqslant\frac{\varepsilon}{2}\int_0^t\!\!\int V_x^2\,dx\,ds+O(1).
\end{aligned}
\tag{4.14}
$$

Substituting (4.10)–(4.14) into (4.9), we have

$$\frac{\varepsilon}{2}\int V^2\,dx+(C_2-\varepsilon)\int_0^t\!\!\int V_x^2\,dx\,ds$$
$$\leqslant O(1)\big(\|V_0\|^2+\|U_0\|^2+1\big)+\frac{1}{2(\alpha-\varepsilon)}\int V_t^2\,dx+\int_0^t\!\!\int V_t^2\,dx\,ds. \tag{4.15}$$

Next, we multiply $(4.6)_1$ by $V_t$ and integrate the resulting equation with respect to $x$ and $t$ over $\mathbb{R}^+\times[0,t]$, then we have

$$\frac{1}{2}\int V_t^2\,dx+\alpha\int_0^t\!\!\int V_t^2\,dx\,ds$$
$$=\frac{1}{2}\int U_0^2\,dx+\frac{1}{\alpha}\int_0^t\!\!\int p(\bar v)_{xt}V_t\,dx\,ds-\int_0^t\!\!\int V_t\big[p(V_x+\bar v+\hat v)-p(\bar v)\big]_x\,dx\,ds. \tag{4.16}$$

We now estimate the terms of the right-hand side of (4.16). By using Lemma 2.2 and the Cauchy–Schwarz's inequality we have

$$\frac{1}{\alpha}\int_0^t\!\!\int p(\bar v)_{xt}V_t\,dx\,ds\leqslant\varepsilon\int_0^t\!\!\int V_t^2\,dx\,ds+O(1)\int_0^t\!\!\int\big(\bar v_{xt}^2+\bar v_x^2\bar v_t^2\big)\,dx\,ds$$
$$\leqslant\varepsilon\int_0^t\!\!\int V_t^2\,dx\,ds+O(1). \tag{4.17}$$

For the last term, noting (4.7) and after some integrations by parts, we have

$$-\int_0^t\!\!\int V_t\big[p(V_x+\bar v+\hat v)-p(\bar v)\big]_x\,dx\,ds=\int_0^t\!\!\int V_{xt}\big[p(V_x+\bar v+\hat v)-p(\bar v)\big]\,dx\,ds$$
$$=\int\Bigg[\int_{\bar v}^{V_x+\bar v+\hat v}p(\tau)\,d\tau-p(\bar v)V_x\Bigg]dx-\int\Bigg[\int_{\bar v_0}^{V_{0x}+\bar v_0+\hat v_0}p(\tau)\,d\tau-p(\bar v_0)V_{0x}\Bigg]dx$$
$$-\int_0^t\!\!\int\big[p(V_x+\bar v+\hat v)-p(\bar v)-p'(\bar v)V_x\big]\bar v_t\,dx\,ds-\int_0^t\!\!\int p(V_x+\bar v+\hat v)\hat v_t\,dx\,ds$$
$$:=\sum_{i=1}^4 I_i. \tag{4.18}$$

Now we estimate $I_i$ $(i = 1, 2, 3, 4)$ as follows:
Since

$$\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(\tau)\,d\tau = p(\bar{v})(V_x + \hat{v}) + \frac{1}{2} p'\big(\theta_1(V_x + \hat{v}) + \bar{v}\big)(V_x + \hat{v})^2, \tag{4.19}$$

where $\theta_1 \in (0, 1)$, this deduces

$$\begin{aligned} I_1 &\leqslant -\frac{C_2 - \varepsilon}{2} \int V_x^2\,dx + O(1) \int \big(\hat{v}^2 + |\hat{v}|\big)\,dx \\ &\leqslant -\frac{C_2 - \varepsilon}{2} \int V_x^2\,dx + O(1). \end{aligned} \tag{4.20}$$

Similarly, we have

$$I_2 \leqslant O(1)\big(\|V_{0x}\|^2 + 1\big). \tag{4.21}$$

By using Lemmas 2.1, 2.2 and the Cauchy–Schwarz's inequality we have

$$I_4 \leqslant O(1) \int_0^t\!\!\int |\hat{v}_t|\,dx\,ds \leqslant O(1) \tag{4.22}$$

and

$$\begin{aligned} I_3 &= -\int_0^t\!\!\int p'(\bar{v})\hat{v}\bar{v}_t\,dx\,ds - \frac{1}{2}\int_0^t\!\!\int p''\big(\theta_2(V_x + \hat{v}) + \bar{v}\big)\bar{v}_t(V_x + \hat{v})^2\,dx\,ds \\ &\leqslant O(1) + \varepsilon \int_0^t\!\!\int V_x^2\,dx\,ds + O(1)\int_0^t (1+s)^{-3}\big\|V_x(s)\big\|^2\,ds, \end{aligned} \tag{4.23}$$

where $\theta_2 \in (0, 1)$.

Substituting (4.17), (4.20)–(4.23) into (4.16), we can get

$$\begin{aligned} &\frac{1}{2}\int V_t^2\,dx + (\alpha - \varepsilon)\int_0^t\!\!\int V_t^2\,dx\,ds + \frac{C_2 - \varepsilon}{2}\int V_x^2\,dx \\ &\quad\leqslant O(1)\big(\|V_0\|_1^2 + \|U_0\|^2 + 1\big) + \varepsilon \int_0^t\!\!\int V_x^2\,dx\,ds + O(1)\int_0^t (1+s)^{-3}\big\|V_x(s)\big\|^2\,ds. \end{aligned} \tag{4.24}$$

Taking $k > 0$ sufficiently large and choosing $\varepsilon > 0$ suitably small such that

$$\begin{cases} (\alpha - \varepsilon)k > 1, \\ C_2 - \varepsilon > k\varepsilon, \end{cases} \tag{4.25}$$

we have from (4.24) $\times k +$ (4.15) that

$$\|V(t)\|_1^2 + \|V_t(t)\|^2 + \int_0^t \left(\|V_x(s)\|^2 + \|V_t(s)\|^2\right) ds$$
$$\leqslant O(1)\left(\|V_0\|_1^2 + \|U_0\|^2 + 1\right) + O(1)\int_0^t (1+s)^{-3}\|V_x(s)\|^2\, ds. \tag{4.26}$$

In fact, there exist many $\varepsilon$ and $k$ satisfying (4.25). For example,

$$\varepsilon = \frac{C_2}{2k}, \quad k > \max\left(1, \frac{1}{\alpha}\left(1 + \frac{C_2}{2}\right)\right).$$

By using the Gronwall's inequality, from (4.26) we can get (4.8). This completes the proof of Lemma 4.4. □

**Lemma 4.5** *(Higher order energy estimates). Under the assumptions of Theorem* 2.4*, we have*

$$\|V_{xx}(t)\|^2 + \|V_{xt}(t)\|^2 + \|V_{tt}(t)\|^2 + \int_0^t \left(\|V_{xx}(s)\|^2 + \|V_{xt}(s)\|^2 + \|V_{tt}(s)\|^2\right) ds$$
$$\leqslant O(1)\left(\|V_0\|_2^2 + \|U_0\|_1^2 + 1\right) \tag{4.27}$$

*and*

$$\|V_{xxx}(t)\|^2 + \|V_{xxt}(t)\|^2 + \|V_{xtt}(t)\|^2 + \|V_{ttt}(t)\|^2$$
$$+ \int_0^t \left(\|V_{xxx}(s)\|^2 + \|V_{xxt}(s)\|^2 + \|V_{xtt}(s)\|^2 + \|V_{ttt}(s)\|^2\right) ds$$
$$\leqslant O(1)\left(\|V_0\|_3^2 + \|U_0\|_2^2 + 1\right). \tag{4.28}$$

**Proof.** We only prove (4.27). (4.28) can be treated in the same way.

Differentiating $(4.6)_1$ with respect to $x$, multiplying the resulting equation by $V_{xt}$, and integrating the results with respect to $x$ and $t$ over $\mathbb{R}^+ \times [0, t]$, we have

$$\frac{1}{2}\int V_{xt}^2\, dx + \alpha \int_0^t\int V_{xt}^2\, dx\, ds$$
$$= \frac{1}{2}\int U_{0x}^2\, dx + \frac{1}{\alpha}\int_0^t\int V_{xt}\, p(\bar{v})_{xxt}\, dx\, ds - \int_0^t\int V_{xt}\left[p(V_x + \bar{v} + \hat{v}) - p(\bar{v})\right]_{xx} dx\, ds$$
$$:= \sum_{i=5}^{7} I_i. \tag{4.29}$$

By using Lemma 2.2 and the Cauchy–Schwarz's inequality, we have

$$I_5 + I_6 \leqslant O(1)\big(\|U_{0x}\|^2 + 1\big) + \frac{\varepsilon}{3}\int_0^t\!\!\int V_{xt}^2\,dx\,ds. \tag{4.30}$$

Now we turn to estimate $I_7$. Noticing

$$\begin{aligned}
&V_{xxt}\,p(V_x+\bar v+\hat v)_x\\
&\quad=\frac{d}{dt}\left[\frac12 p'(V_x+\bar v+\hat v)V_{xx}^2\right]-\frac12 V_{xx}^2 p''(V_x+\bar v+\hat v)(V_x+\bar v+\hat v)_t\\
&\qquad+\big[V_{xt}(\bar v_x+\hat v_x)p'(V_x+\bar v+\hat v)\big]_x - V_{xt}(\bar v_{xx}+\hat v_{xx})p'(V_x+\bar v+\hat v)\\
&\qquad- V_{xt}V_{xx}(\bar v_x+\hat v_x)p''(V_x+\bar v+\hat v) - V_{xt}(\bar v_x+\hat v_x)^2 p''(V_x+\bar v+\hat v),
\end{aligned} \tag{4.31}$$

we have

$$\begin{aligned}
I_7 ={}& \frac12\int p'(V_x+\bar v+\hat v)V_{xx}^2\,dx - \frac12\int p'(V_{0x}+\bar v_0+\hat v_0)V_{0xx}^2\,dx\\
&-\frac12\int_0^t\!\!\int V_{xx}^2 p''(V_x+\bar v+\hat v)(V_x+\bar v+\hat v)_t\,dx\,ds\\
&-\int_0^t\!\!\int V_{xt}(\bar v_{xx}+\hat v_{xx})p'(V_x+\bar v+\hat v)\,dx\,ds\\
&-\int_0^t\!\!\int V_{xt}V_{xx}(\bar v_x+\hat v_x)p''(V_x+\bar v+\hat v)\,dx\,ds\\
&-\int_0^t\!\!\int V_{xt}(\bar v_x+\hat v_x)^2 p''(V_x+\bar v+\hat v)\,dx\,ds\\
&+\int_0^t\!\!\int V_{xt}\big[p'(v)\bar v_{xx}+p''(\bar v)\bar v_x^2\big]\,dx\,ds\\
:={}&\sum_{j=1}^{7} J_j.
\end{aligned} \tag{4.32}$$

Next we estimate $J_j$ $(j=1,\ldots,7)$. By employing Corollary 3.2, we have

$$J_3 \leqslant O(1)M_6\int_0^t\!\!\int V_{xx}^2\,dx\,ds. \tag{4.33}$$

By using Lemmas 2.1, 2.2 and the Cauchy–Schwarz's inequality, we have

$$
\begin{aligned}
J_4 &\leqslant \frac{\varepsilon}{6}\int_0^t\!\!\int V_{xt}^2\,dx\,ds + O(1)\int_0^t\!\!\int \left(\bar{v}_{xx}^2+\hat{v}_{xx}^2\right)dx\,ds \\
&\leqslant \frac{\varepsilon}{6}\int_0^t\!\!\int V_{xt}^2\,dx\,ds + O(1),
\end{aligned} \tag{4.34}
$$

$$
J_5 \leqslant \frac{\varepsilon}{6}\int_0^t\!\!\int V_{xt}^2\,dx\,ds + O(1)\int_0^t\!\!\int \left(\bar{v}_x^2+\hat{v}_x^2\right)V_{xx}^2\,dx\,ds, \tag{4.35}
$$

$$
\begin{aligned}
J_6 &\leqslant \frac{\varepsilon}{6}\int_0^t\!\!\int V_{xt}^2\,dx\,ds + O(1)\int_0^t\!\!\int \left(\bar{v}_x^4+\hat{v}_x^4\right)dx\,ds \\
&\leqslant \frac{\varepsilon}{6}\int_0^t\!\!\int V_{xt}^2\,dx\,ds + O(1),
\end{aligned} \tag{4.36}
$$

and

$$
\begin{aligned}
J_7 &\leqslant \frac{\varepsilon}{6}\int_0^t\!\!\int V_{xt}^2\,dx\,ds + O(1)\int_0^t\!\!\int \left(\bar{v}_x^4+\bar{v}_{xx}^2\right)dx\,ds \\
&\leqslant \frac{\varepsilon}{6}\int_0^t\!\!\int V_{xt}^2\,dx\,ds + O(1).
\end{aligned} \tag{4.37}
$$

Consequently,

$$
\begin{aligned}
I_7 &\leqslant \frac{2\varepsilon}{3}\int_0^t\!\!\int V_{xt}^2\,dx\,ds + O(1)\int_0^t\!\!\int \left(|\bar{v}_x|^2+|\hat{v}_x|^2\right)V_{xx}^2\,dx\,ds \\
&\quad + O(1)M_6\int_0^t\!\!\int V_{xx}^2\,dx\,ds + O(1)\left(\|V_0\|_2^2+\|U_0\|^2+1\right) - \frac{C_2}{2}\int V_{xx}^2\,dx.
\end{aligned} \tag{4.38}
$$

From (4.29), (4.30) and (4.38), we have

$$
\frac{1}{2}\int V_{xt}^2\,dx + \frac{C_2}{2}\int V_{xx}^2\,dx + (\alpha-\varepsilon)\int_0^t\!\!\int V_{xt}^2\,dx\,ds
$$

$$\leqslant O(1)\big(\|V_0\|_2^2+\|U_0\|_1^2+1\big)+O(1)\int_0^t\!\!\int\big(|\bar v_x|^2+|\hat v_x|^2\big)V_{xx}^2\,dx\,ds$$
$$+\,O(1)M_6\int_0^t\!\!\int V_{xx}^2\,dx\,ds. \tag{4.39}$$

Next, we multiply $(4.6)_1$ by $-V_{xx}$ and integrate the results with respect to $x$ and $t$ over $\mathbb{R}^+\times[0,t]$, by using (4.7), we have after integration by parts that

$$\begin{aligned}\frac{\alpha}{2}\int V_x^2\,dx &= \frac{\alpha}{2}\int V_{0x}^2\,dx-\int V_xV_{xt}\,dx+\int V_{0x}U_{0x}\,dx\\ &\quad+\int_0^t\!\!\int V_{xt}^2\,dx\,ds-\frac{1}{\alpha}\int_0^t\!\!\int V_{xx}\,p(\bar v)_{xt}\,dx\,ds\\ &\quad+\int_0^t\!\!\int V_{xx}\big[p(V_x+\bar v+\hat v)-p(\bar v)\big]_x\,dx\,ds\\ &=\sum_{i=8}^{13}I_i.\end{aligned} \tag{4.40}$$

By employing Lemma 2.2, (4.8) and the Cauchy–Schwarz's inequality, we have

$$I_8+I_{10}\leqslant O(1)\big(\|V_{0x}\|^2+\|U_{0x}\|^2\big), \tag{4.41}$$

$$\begin{aligned}I_9&\leqslant \varepsilon\int V_{xt}^2\,dx+O(1)\int V_x^2\,dx\\ &\leqslant O(1)\big(\|V_0\|_1^2+\|U_0\|^2+1\big)+\varepsilon\int V_{xt}^2\,dx,\end{aligned} \tag{4.42}$$

$$I_{12}\leqslant O(1)\int_0^t\!\!\int\big(|\bar v_{xt}|+|\bar v_x\bar v_t|\big)|V_{xx}|\,dx\,ds\leqslant\frac{\varepsilon}{2}\int_0^t\!\!\int V_{xx}^2\,dx\,ds+O(1), \tag{4.43}$$

and

$$I_{13}\leqslant\int_0^t\!\!\int p'(V_x+\bar v+\hat v)V_{xx}^2\,dx\,ds+O(1)\int_0^t\!\!\int\big[|\bar v_x|\big(|V_x|+|\hat v|\big)+|\hat v_x|\big]|V_{xx}|\,dx\,ds$$
$$\leqslant\Big(\frac{\varepsilon}{2}-C_2\Big)\int_0^t\!\!\int V_{xx}^2\,dx\,ds+O(1)\big(\|V_0\|_1^2+\|U_0\|^2+1\big). \tag{4.44}$$

Consequently, substituting (4.41)–(4.44) into (4.40), and using (4.8), we get

$$\frac{\alpha}{2}\int V_x^2\,dx+(C_2-\varepsilon)\int_0^t\!\!\int V_{xx}^2\,dx\,ds$$
$$\leqslant O(1)\big(\|V_0\|_1^2+\|U_0\|_1^2+1\big)+\varepsilon\int V_{xt}^2\,dx+\int_0^t\!\!\int V_{xt}^2\,dx\,ds. \tag{4.45}$$

Let $k,\varepsilon$ be the positive constants chosen in Lemma 4.4, we have from (4.39) $\times\, k+$ (4.45) that

$$\left(\frac{k}{2}-\varepsilon\right)\int V_{xt}^2\,dx+\frac{kC_2}{2}\int V_{xx}^2\,dx+\big[(\alpha-\varepsilon)k-1\big]\int_0^t\!\!\int V_{xt}^2\,dx\,ds$$
$$+(C_2-\varepsilon)\int_0^t\!\!\int V_{xx}^2\,dx\,ds$$
$$\leqslant O(1)\big(\|V_0\|_2^2+\|U_0\|_1^2+1\big)+O(1)\int_0^t\!\!\int\big(|\bar{v}_x|^2+|\hat{v}_x|^2\big)V_{xx}^2\,dx\,ds$$
$$+O(1)M_6k\int_0^t\!\!\int V_{xx}^2\,dx\,ds. \tag{4.46}$$

Thus if we further choose $\frac{k}{2}>\varepsilon$ and $M_6$ sufficiently small such that

$$C_2-\varepsilon>O(1)M_6k, \tag{4.47}$$

we can deduce that

$$\int\big(V_{xt}^2+V_{xx}^2\big)\,dx+\int_0^t\!\!\int\big(V_{xt}^2+V_{xx}^2\big)\,dx\,ds$$
$$\leqslant O(1)\big(\|V_0\|_2^2+\|U_0\|_1^2+1\big)+O(1)\int_0^t\!\!\int\big(|\bar{v}_x|^2+|\hat{v}_x|^2\big)V_{xx}^2\,dx\,ds. \tag{4.48}$$

For the last term of the right of (4.48), by using the Cauchy–Schwarz's inequality, Lemmas 2.1 and 2.2, we have

$$\int_0^t\!\!\int\big(|\bar{v}_x|^2+|\hat{v}_x|^2\big)V_{xx}^2\,dx\,ds\leqslant\varepsilon\int_0^t\!\!\int V_{xx}^2\,dx\,ds+O(1)\int_0^t\!\!\int\big(\bar{v}_x^4+\hat{v}_x^4\big)V_{xx}^2\,dx\,ds$$
$$\leqslant\varepsilon\int_0^t\!\!\int V_{xx}^2\,dx\,ds+O(1)\int_0^t(1+s)^{-4}\big\|V_{xx}(s)\big\|^2\,ds. \tag{4.49}$$

From (4.48) and (4.49), we can get

$$\int \left(V_{xt}^2 + V_{xx}^2\right) dx + \int_0^t\!\!\int \left(V_{xt}^2 + V_{xx}^2\right) dx\, ds$$
$$\leqslant O(1)\left(\|V_0\|_2^2 + \|U_0\|_1^2 + 1\right) + O(1)\int_0^t (1+s)^{-4}\left\|V_{xx}(s)\right\|^2 ds. \tag{4.50}$$

By using the Gronwall's inequality, from (4.50), we have

$$\int \left(V_{xt}^2 + V_{xx}^2\right) dx + \int_0^t\!\!\int \left(V_{xt}^2 + V_{xx}^2\right) dx\, ds \leqslant O(1)\left(\|V_0\|_2^2 + \|U_0\|_1^2 + 1\right). \tag{4.51}$$

On the other hand, from $(4.6)_1$,

$$V_{tt} = \frac{1}{\alpha} p(\bar{v})_{xt} - \alpha V_t - \left[p(V_x + \bar{v} + \hat{v}) - p(\bar{v})\right]_x, \tag{4.52}$$

we can get from (4.51) that

$$\int V_{tt}^2\, dx + \int_0^t\!\!\int V_{tt}^2\, dx\, ds \leqslant O(1)\left(\|V_0\|_2^2 + \|U_0\|_1^2 + 1\right). \tag{4.53}$$

Then (4.27) follows from (4.51) and (4.53). This completes the proof of Lemma 4.5 and so does Theorem 4.3. Consequently, Theorem 4.1 follows also. □

### *4.2. Decay estimates (3.4) and (3.5)*

In this subsection, we devote to prove the decay estimates (3.4) and (3.5). We first give the following energy estimates.

**Lemma 4.6.** *Under the assumptions of Theorem* 2.4*, we have*

$$\frac{d}{dt}\int \left\{\frac{1}{2}V_t^2 - \left(\int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)\, ds - p(\bar{v})V_x\right)\right\} dx + \frac{\alpha}{2}\int V_t^2\, dx$$
$$\leqslant O(1)(1+t)^{-\frac{7}{2}} + O(1)(1+t)^{-\frac{3}{2}}\int V_x^2\, dx. \tag{4.54}$$

**Proof.** Similar to the proof of Lemma 4.4, we multiply $(4.6)_1$ by $V_t$ and integrate the results with respect to $x$ over $\mathbb{R}^+$, after some integrations by parts, we have

$$\frac{d}{dt}\int\left\{\frac{1}{2}V_t^2\,dx-\left(\int\limits_{\bar v}^{V_x+\bar v+\hat v}p(s)\,ds-p(\bar v)V_x\right)\right\}dx+\alpha\int V_t^2\,dx$$
$$=\frac{1}{\alpha}\int p(\bar v)_{xt}V_t\,dx-\int\left[p(V_x+\bar v+\hat v)-p(\bar v)-p'(\bar v)V_x\right]\bar v_t\,dx$$
$$-\int p(V_x+\bar v+\hat v)\hat v_t\,dx. \tag{4.55}$$

By using the Cauchy–Schwarz's inequality and Lemma 2.2, we have

$$\frac{1}{\alpha}\int p(\bar v)_{xt}V_t\,dx\leqslant\frac{\alpha}{2}\int V_t^2\,dx+O(1)\int\left(\bar v_{xt}^2+\bar v_x^2\bar v_t^2\right)dx$$
$$\leqslant\frac{\alpha}{2}\int V_t^2\,dx+O(1)(1+t)^{-\frac{7}{2}}. \tag{4.56}$$

Similar to (4.22) and (4.23), we have

$$-\int\left[p(V_x+\bar v+\hat v)-p(\bar v)-p'(\bar v)V_x\right]\bar v_t\,dx-\int p(V_x+\bar v+\hat v)\hat v_t\,dx$$
$$\leqslant O(1)\int|\hat v_t|\,dx+O(1)\int|\hat v||\bar v_t|\,dx+O(1)\int|\bar v_t|V_x^2\,dx$$
$$\leqslant O(1)e^{-\alpha t}+O(1)(1+t)^{-\frac{3}{2}}\int V_x^2\,dx. \tag{4.57}$$

Substituting (4.56) and (4.57) into (4.55), we can get (4.54). This proves Lemma 4.6. □

Similarly, performing $(4.6)_{1x}\times V_{xt}$, $(4.6)_1\times V_{xx}$, $(4.6)_{1t}\times V_{tt}$, $(4.6)_{1t}\times V_t$, $(4.6)_{1x}\times V_{xxx}$, $(4.6)_{1xx}\times V_{xxt}$, $(4.6)_{1tt}\times V_{ttt}$, and integrating the resulting equations with respect to $x$ over $\mathbb{R}^+$ respectively, by using the method of integration by parts, the Cauchy–Schwarz's inequality, Lemmas 2.1 and 2.2, we can get the following estimates, the details are omitted.

**Lemma 4.7.** *Under the assumptions of Theorem* 2.4*, we have*

$$\frac{d}{dt}\int\left\{V_{xt}^2-p'(V_x+\bar v+\hat v)V_{xx}^2\right\}dx+\alpha\int V_{xt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{9}{2}}+O(1)(1+t)^{-3}\int V_x^2\,dx$$
$$+O(1)\left(\|V_{xt}\|_{L^\infty}+(1+t)^{-\frac{3}{2}}\right)\int V_{xx}^2\,dx, \tag{4.58}$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_x^2+V_xV_{xt}\right\}dx+\frac{C_2}{2}\int V_{xx}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{7}{2}}+O(1)(1+t)^{-2}\int V_x^2\,dx+\int V_{xt}^2\,dx, \tag{4.59}$$

$$\frac{d}{dt}\int\big\{V_{tt}^2-p'(V_x+\bar v+\hat v)V_{xt}^2\big\}\,dx+\alpha\int V_{tt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{11}{2}}+O(1)(1+t)^{-4}\int V_x^2\,dx+O(1)(1+t)^{-3}\int V_{xx}^2\,dx$$
$$+O(1)\big(\|V_{xt}\|_{L^\infty}+(1+t)^{-\frac{3}{2}}\big)\int V_{xt}^2\,dx, \tag{4.60}$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_t^2+V_tV_{tt}\right\}dx+\frac{C_2}{2}\int V_{xt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{9}{2}}+O(1)(1+t)^{-3}\int V_x^2\,dx+\int V_{tt}^2\,dx, \tag{4.61}$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_{xx}^2+V_{xx}V_{xxt}\right\}dx+\frac{C_2}{2}\int V_{xxx}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{9}{2}}+O(1)(1+t)^{-3}\int V_x^2\,dx+\int V_{xxt}^2\,dx$$
$$+O(1)\big(\|V_{xx}\|_{L^\infty}^2+(1+t)^{-2}\big)\int V_{xx}^2\,dx, \tag{4.62}$$

$$\frac{d}{dt}\int\big\{V_{xxt}^2-p'(V_x+\bar v+\hat v)V_{xxx}^2\big\}\,dx+\alpha\int V_{xxt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{11}{2}}+O(1)(1+t)^{-4}\int V_x^2\,dx$$
$$+O(1)\big(\|V_{xx}\|_{L^\infty}^4+(1+t)^{-3}+(1+t)^{-2}\|V_{xx}\|_{L^\infty}^2\big)\int V_{xx}^2\,dx$$
$$+O(1)\big(\|V_{xt}\|_{L^\infty}+\|V_{xx}\|_{L^\infty}^2+(1+t)^{-\frac{3}{2}}\big)\int V_{xxx}^2\,dx, \tag{4.63}$$

$$\frac{d}{dt}\int\big\{V_{ttt}^2-p'(V_x+\bar v+\hat v)V_{xtt}^2\big\}\,dx+\alpha\int V_{ttt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{15}{2}}+O(1)(1+t)^{-6}\int V_x^2\,dx+O(1)(1+t)^{-5}\int V_{xx}^2\,dx$$
$$+O(1)\big(\|V_{xt}\|_{L^\infty}^2+(1+t)^{-3}\big)\int V_{xxt}^2\,dx+O(1)\big((1+t)^{-\frac{3}{2}}+\|V_{xt}\|_{L^\infty}\big)\int V_{xtt}^2\,dx$$
$$+O(1)\big(\|V_{xt}\|_{L^\infty}^2\big(\|V_{xx}\|_{L^\infty}^2+(1+t)^{-2}\big)+(1+t)^{-3}\|V_{xx}\|_{L^\infty}^2+(1+t)^{-4}\big)\int V_{xt}^2\,dx. \tag{4.64}$$

Having obtained Lemmas 4.6 and 4.7, if we want to get the decay estimates (3.4) and (3.5) by employing the techniques developed by Nishihara and Yang in [19], we need only to get the following estimates

$$\begin{cases}\big\|V_{xx}(t)\big\|_{L^\infty}\leqslant O(1)(1+t)^{-\frac{1}{2}},\\ \big\|V_{xt}(t)\big\|_{L^\infty}\leqslant O(1)(1+t)^{-1}.\end{cases} \tag{4.65}$$

In [19], (4.65) is a direct consequence of the *a priori* assumption (4.1), but in this paper, we do not ask the initial error to be small, so the techniques in [19] cannot be used directly and we have to get (4.65) by using another method which is the main novelty in this paper.

First, if we let

$$\delta := \sup_{t \geqslant T^*} \left\{ \|V_{xt}\|_{L^\infty} + \frac{1}{1+t} + \|V_{xx}\|_{L^\infty}^2 \right\} \tag{4.66}$$

for some fixed positive constant $T^*$, from the estimates (4.3), then we can choose a fixed $T^* > 0$ sufficiently large such that $\delta$ can be chosen as small as we wanted. For such a $\delta$, we can also find a suitably small positive constant $\lambda$ such that

$$\begin{cases} \dfrac{1}{2} > \dfrac{\lambda}{\alpha}, \\ \dfrac{C_2}{2}\lambda > O(1)\delta. \end{cases} \tag{4.67}$$

For $\delta, \lambda, T^*$ chosen as above, we have the following result:

**Lemma 4.8.** *Under the assumptions of Theorem* 2.4*, there exists a positive constant* $\beta > 0$ *such that for each* $t \geqslant T^*$*, we have*

$$\frac{d}{dt}\int \left\{ \frac{1}{2}V_t^2 - \left( \int_{\bar{v}}^{V_x+\bar{v}+\hat{v}} p(s)\,ds - p(\bar{v})V_x \right) \right\}(t)\,dx + \frac{\alpha}{2}\int V_t^2(t)\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{7}{2}} + O(1)(1+t)^{-\frac{3}{2}} \int V_x^2(t)\,dx, \tag{4.68}$$

$$\frac{d}{dt}\int \left\{ \frac{1}{2}V_{xt}^2 - \frac{1}{2}p'(V_x+\bar{v}+\hat{v})V_{xx}^2 + \frac{\lambda\alpha}{2}V_x^2 + \lambda V_x V_{xt} \right\}(t)\,dx + \beta \int \left(V_{xt}^2 + V_{xx}^2\right)(t)\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{7}{2}} + O(1)(1+t)^{-2} \int V_x^2(t)\,dx, \tag{4.69}$$

$$\frac{d}{dt}\int \left\{ \frac{1}{2}V_{tt}^2 - \frac{1}{2}p'(V_x+\bar{v}+\hat{v})V_{xt}^2 + \frac{\lambda\alpha}{2}V_t^2 + \lambda V_t V_{tt} \right\}(t)\,dx + \beta \int \left(V_{xt}^2 + V_{tt}^2\right)(t)\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{9}{2}} + O(1)(1+t)^{-3} \int \left(V_{xx}^2 + V_x^2\right)(t)\,dx, \tag{4.70}$$

$$\frac{d}{dt}\int \left\{ \frac{1}{2}V_{xxt}^2 - \frac{1}{2}p'(V_x+\bar{v}+\hat{v})V_{xxx}^2 + \frac{\lambda\alpha}{2}V_{xx}^2 + \lambda V_{xx} V_{xxt} \right\}(t)\,dx$$
$$+ \beta \int \left(V_{xxx}^2 + V_{xxt}^2\right)(t)\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{9}{2}} + O(1)(1+t)^{-3} \int V_x^2(t)\,dx$$
$$+ O(1)\left((1+t)^{-2} + \|V_{xx}\|_{L^\infty}^2\right) \int V_{xx}^2(t)\,dx, \tag{4.71}$$

*where*

$$\beta=\min\left\{\frac{C_2\lambda}{2}-O(1)\delta,\ \frac{\alpha}{2}-\lambda\right\}. \tag{4.72}$$

**Proof.** We only prove (4.69), the rest can be treated similarly. In fact, multiplying (4.58) by $\frac{1}{2}$, (4.59) by $\lambda$, and adding the result inequalities, then we can get (4.69) by using (4.66), (4.67) and (4.72). This completes the proof of Lemma 4.8. □

Now we turn to prove the estimates (4.65).

First, multiplying (4.68) by $(1+t)$ and integrating the results with respect to $t$ over $[T^*,t]$, we have

$$\begin{aligned}
&(1+t)\int\left\{\frac{1}{2}V_t^2-\left(\int_{\bar v}^{V_x+\bar v+\hat v}p(s)\,ds-p(\bar v)V_x\right)\right\}(s)\,dx+\frac{\alpha}{2}\int_{T^*}^{t}(1+s)\int V_t^2(s)\,dx\,ds\\
&\quad\leqslant\left(1+T^*\right)\int\left\{\frac{1}{2}V_t^2-\left(\int_{\bar v}^{V_x+\bar v+\hat v}p(s)\,ds-p(\bar v)V_x\right)\right\}\left(T^*\right)dx+O(1)\int_{T^*}^{t}(1+s)^{-\frac{3}{2}}\,ds\\
&\qquad+O(1)\iint_{T^*}^{t}V_x^2(s)\,dx\,ds+\int_{T^*}^{t}\int\left\{\frac{1}{2}V_t^2-\left(\int_{\bar v}^{V_x+\bar v+\hat v}p(s)\,ds-p(\bar v)V_x\right)\right\}(s)\,dx\,ds.
\end{aligned} \tag{4.73}$$

The above inequality together with Theorem 4.3 implies that

$$(1+t)\left(\left\|V_x(t)\right\|^2+\left\|V_t(t)\right\|^2\right)+\int_0^t(1+s)\left\|V_t(s)\right\|^2ds\leqslant O(1). \tag{4.74}$$

Applying the same process to (4.69) and (4.70), we get

$$\begin{aligned}
&(1+t)\left(\left\|V_{xx}(t)\right\|^2+\left\|V_{xt}(t)\right\|^2+\left\|V_x(t)\right\|^2\right)+\int_0^t(1+s)\left(\left\|V_{xx}(s)\right\|^2+\left\|V_{xt}(s)\right\|^2\right)ds\\
&\quad\leqslant O(1)
\end{aligned} \tag{4.75}$$

and

$$\begin{aligned}
&(1+t)\left(\left\|V_{tt}(t)\right\|^2+\left\|V_{xt}(t)\right\|^2+\left\|V_t(t)\right\|^2\right)+\int_0^t(1+s)\left(\left\|V_{xt}(s)\right\|^2+\left\|V_{tt}(s)\right\|^2\right)ds\\
&\quad\leqslant O(1).
\end{aligned} \tag{4.76}$$

Furthermore, multiplying (4.70) by $(1+t)^2$, and using (4.74)–(4.76), we can get

$$(1+t)^2\big(\|V_{tt}(t)\|^2+\|V_{xt}(t)\|^2+\|V_t(t)\|^2\big)+\int_0^t(1+s)^2\big(\|V_{xt}(s)\|^2+\|V_{tt}(s)\|^2\big)\,ds \leqslant O(1). \tag{4.77}$$

By using (4.5), (4.75) and the Sobolev's inequality, we have

$$\|V_{xx}(t)\|_{L^\infty}\leqslant\|V_{xx}(t)\|^{\frac{1}{2}}\|V_{xxx}(t)\|^{\frac{1}{2}}\leqslant O(1)(1+t)^{-\frac{1}{4}}. \tag{4.78}$$

Substituting (4.78) into (4.71), we get

$$\begin{aligned}&\frac{d}{dt}\int\left\{\frac{1}{2}V_{xxt}^2-\frac{1}{2}p'(V_x+\bar v+\hat v)V_{xxx}^2+\frac{\lambda\alpha}{2}V_{xx}^2+\lambda V_{xx}V_{xxt}\right\}(t)\,dx\\&\quad+\beta\int\big(V_{xxx}^2+V_{xxt}^2\big)(t)\,dx\\&\leqslant O(1)(1+t)^{-\frac{9}{2}}+O(1)(1+t)^{-3}\int V_x^2(t)\,dx+O(1)(1+t)^{-\frac{1}{2}}\int V_{xx}^2(t)\,dx.\end{aligned} \tag{4.79}$$

Multiplying (4.79) by $(1+t)$, and using Theorem 4.3, (4.67) and (4.75), we have

$$(1+t)\big(\|V_{xxt}(t)\|^2+\|V_{xxx}(t)\|^2+\|V_{xx}(t)\|^2\big)+\int_0^t(1+s)\big(\|V_{xxx}(s)\|^2+\|V_{xxt}(s)\|^2\big)\,ds \leqslant O(1). \tag{4.80}$$

Then from (4.75), (4.80) and the Sobolev's inequality, we can deduce that

$$\|V_{xx}(t)\|_{L^\infty}\leqslant O(1)(1+t)^{-\frac{1}{2}}. \tag{4.81}$$

This proves $(4.65)_1$.

Now substituting (4.81) into (4.71), we have

$$\begin{aligned}&\frac{d}{dt}\int\left\{\frac{1}{2}V_{xxt}^2-\frac{1}{2}p'(V_x+\bar v+\hat v)V_{xxx}^2+\frac{\lambda\alpha}{2}V_{xx}^2+\lambda V_{xx}V_{xxt}\right\}(t)\,dx\\&\quad+\beta\int\big(V_{xxx}^2+V_{xxt}^2\big)(t)\,dx\\&\leqslant O(1)(1+t)^{-\frac{9}{2}}+O(1)(1+t)^{-3}\int V_x^2(t)\,dx+O(1)(1+t)^{-1}\int V_{xx}^2(t)\,dx.\end{aligned} \tag{4.82}$$

Multiplying (4.82) by $(1+t)^2$ and employing (4.5), (4.75) and (4.80), we get

$$
\begin{aligned}
&(1+t)^2\big(\|V_{xxt}(t)\|^2+\|V_{xxx}(t)\|^2+\|V_{xx}(t)\|^2\big)+\int_0^t (1+s)^2\big(\|V_{xxx}(s)\|^2+\|V_{xxt}(s)\|^2\big)\,ds\\
&\quad\leqslant O(1)+O(1)\int_0^t (1+s)\big(\|V_{xxt}(s)\|^2+\|V_{xxx}(s)\|^2+\|V_{xx}(s)\|^2\big)\,ds\\
&\qquad+O(1)\int_0^t (1+s)^{-\frac{5}{2}}\,ds+O(1)\int_0^t \|V_x(s)\|^2\,ds\\
&\quad\leqslant O(1).
\end{aligned}
\tag{4.83}
$$

Thus from (4.77), (4.83) and the Sobolev's inequality, we can deduce that

$$
\|V_{xt}(t)\|_{L^\infty}\leqslant \|V_{xt}(t)\|^{\frac{1}{2}}\|V_{xxt}(t)\|^{\frac{1}{2}}\leqslant O(1)(1+t)^{-1}. \tag{4.84}
$$

This proves $(4.65)_2$.

Combining (4.65), Lemmas 4.6 and 4.7, we have

**Lemma 4.9.** *Under the assumptions of Theorem* 2.4*, we have*

$$
\begin{aligned}
&\frac{d}{dt}\int\left\{\frac{1}{2}V_t^2-\left(\int_{\bar v}^{V_x+\bar v+\hat v} p(s)\,ds-p(\bar v)V_x\right)\right\}dx+\frac{\alpha}{2}\int V_t^2\,dx\\
&\quad\leqslant O(1)(1+t)^{-\frac{7}{2}}+O(1)(1+t)^{-\frac{3}{2}}\int V_x^2\,dx,
\end{aligned}
\tag{4.85}
$$

$$
\begin{aligned}
&\frac{d}{dt}\int\left\{V_{xt}^2-p'(V_x+\bar v+\hat v)V_{xx}^2\right\}dx+\alpha\int V_{xt}^2\,dx\\
&\quad\leqslant O(1)(1+t)^{-\frac{9}{2}}+O(1)(1+t)^{-3}\int V_x^2\,dx+O(1)(1+t)^{-1}\int V_{xx}^2\,dx,
\end{aligned}
\tag{4.86}
$$

$$
\begin{aligned}
&\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_x^2+V_xV_{xt}\right\}dx+\frac{C_2}{2}\int V_{xx}^2\,dx\\
&\quad\leqslant O(1)(1+t)^{-\frac{7}{2}}+O(1)(1+t)^{-2}\int V_x^2\,dx+\int V_{xt}^2\,dx,
\end{aligned}
\tag{4.87}
$$

$$
\begin{aligned}
&\frac{d}{dt}\int\left\{V_{tt}^2-p'(V_x+\bar v+\hat v)V_{xt}^2\right\}dx+\alpha\int V_{tt}^2\,dx\\
&\quad\leqslant O(1)(1+t)^{-\frac{11}{2}}+O(1)(1+t)^{-4}\int V_x^2\,dx+O(1)(1+t)^{-3}\int V_{xx}^2\,dx\\
&\qquad+O(1)(1+t)^{-1}\int V_{xt}^2\,dx,
\end{aligned}
\tag{4.88}
$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_t^2+V_tV_{tt}\right\}dx+\frac{C_2}{2}\int V_{xt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{9}{2}}+O(1)(1+t)^{-3}\int V_x^2\,dx+\int V_{tt}^2\,dx, \tag{4.89}$$

$$\frac{d}{dt}\int\left\{\frac{\alpha}{2}V_{xx}^2+V_{xx}V_{xxt}\right\}dx+\frac{C_2}{2}\int V_{xxx}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{9}{2}}+O(1)(1+t)^{-3}\int V_x^2\,dx+\int V_{xxt}^2\,dx+O(1)(1+t)^{-1}\int V_{xx}^2\,dx, \tag{4.90}$$

$$\frac{d}{dt}\int\left\{V_{xxt}^2-p'(V_x+\bar{v}+\hat{v})V_{xxx}^2\right\}dx+\alpha\int V_{xxt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{11}{2}}+O(1)(1+t)^{-4}\int V_x^2\,dx+O(1)(1+t)^{-2}\int V_{xx}^2\,dx$$
$$+O(1)(1+t)^{-1}\int V_{xxx}^2\,dx, \tag{4.91}$$

$$\frac{d}{dt}\int\left\{V_{ttt}^2-p'(V_x+\bar{v}+\hat{v})V_{xtt}^2\right\}dx+\alpha\int V_{ttt}^2\,dx$$
$$\leqslant O(1)(1+t)^{-\frac{15}{2}}+O(1)(1+t)^{-6}\int V_x^2\,dx+O(1)(1+t)^{-5}\int V_{xx}^2\,dx$$
$$+O(1)(1+t)^{-2}\int V_{xxt}^2\,dx+O(1)(1+t)^{-1}\int V_{xtt}^2\,dx+O(1)(1+t)^{-3}\int V_{xt}^2\,dx. \tag{4.92}$$

Having obtained Lemma 4.9, then using the arguments developed by Nishihara and Yang in [19], we can get the decay estimates (3.4) and (3.5).

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