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Gelfand pairs on the Heisenberg group and Schwartz functions ☆

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Abstract

Let H_n be the $(2n+1)$-dimensional Heisenberg group and K a compact group of automorphisms of H_n such that $(K \ltimes H_n, K)$ is a Gelfand pair. We prove that the Gelfand transform is a topological isomorphism between the space of K-invariant Schwartz functions on H_n and the space of Schwartz function on a closed subset of $\mathbb{R}^s$ homeomorphic to the Gelfand spectrum of the Banach algebra of K-invariant integrable functions on H_n.

Keywords: Gelfand pair; Schwartz space; Heisenberg group

1. Introduction

A fundamental fact in harmonic analysis on $\mathbb{R}^n$ is that the Fourier transform is a topological isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ onto itself.

Various generalizations of this result for different classes of Lie groups exist in the literature, in particular in the context of Gelfand pairs, where the operator-valued Fourier transform can be replaced by the scalar-valued spherical transform. Most notable is the case of a symmetric pair

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of the noncompact type, with Harish-Chandra's definition of a bi-K-invariant Schwartz space on the isometry group (cf. [17, p. 489]).

The definition of a Schwartz space on a Lie group becomes quite natural on a nilpotent group N (say connected and simply connected). In that case one can define the Schwartz space by identifying N with its Lie algebra via the exponential map.

The image of the Schwartz space on the Heisenberg group H_n under the group Fourier transform has been described by D. Geller [14]. Let K be a compact group of automorphisms of H_n such that convolution of K-invariant functions is commutative, in other words assume that $(K \ltimes H_n, K)$ is a Gelfand pair . Then a scalar-valued spherical transform $\mathcal{G}_K$ (where $\mathcal{G}$ stands for "Gelfand transform") of K-invariant functions is available, and Geller's result can be translated into a characterization of the image under $\mathcal{G}_K$ of the space $\mathcal{S}_K(H_n)$ of K-invariant Schwartz functions. In the same spirit, a characterization of $\mathcal{G}_K(\mathcal{S}_K(H_n))$ is given in [6] for closed subgroups K of the unitary group $\mathrm{U}(n)$.

In [1] we have proved that, for K equal to $\mathrm{U}(n)$ or $\mathbb{T}^n$ (i.e. for radial—respectively polyradial—functions), an analytically more significant description of $\mathcal{G}_K(\mathcal{S}_K(H_n))$ can be obtained by making use of natural homeomorphic embeddings of the Gelfand spectrum of $L^1_K(H_n)$ in Euclidean space. The result is that $\mathcal{G}_K(\mathcal{S}_K(H_n))$ is the space of restrictions to the Gelfand spectrum of the Schwartz functions on the ambient space. This condition of "extendibility to a Schwartz function on the ambient space" subsumes the rather technical condition on iterated differences in discrete parameters that are present in the previous characterizations.

In this article we extend the result of [1] to general Gelfand pairs $(K \ltimes H_n, K)$, with K a compact group of automorphisms of H_n. Some preliminary notions and facts are required before we can give a precise formulation of our main theorem.

Let G be a connected Lie group and K a compact subgroup thereof such that (G, K) is a Gelfand pair and denote by $L^1(G//K)$ the convolution algebra of all bi-K-invariant integrable functions on G. The Gelfand spectrum of the commutative Banach algebra $L^1(G//K)$ may be identified with the set of bounded spherical functions with the compact-open topology. Spherical functions are characterized as the joint eigenfunctions of all G-invariant differential operators on G/K, normalized in the L^∞-norm. G-invariant differential operators on G/K form a commutative algebra $\mathbb{D}(G/K)$ which is finitely generated [16].

Given a finite set of generators $\{V_1, \dots, V_s\}$ of $\mathbb{D}(G/K)$, we can assign to each bounded spherical function ϕ the s-tuple $\widehat{V}(\phi) = (\widehat{V_1}(\phi), \dots, \widehat{V_s}(\phi))$ of its eigenvalues with respect to these generators. In this way, the Gelfand spectrum is identified with a closed subset Σ_K^V of $\mathbb{C}^s$. When all bounded spherical functions are of positive type and the operators V_j self-adjoint, $\Sigma_K^V \subset \mathbb{R}^s$. As proved in [10], the Euclidean topology induced on Σ_K^V coincides with the compact-open topology on the set of bounded spherical functions (see also [4] for $G = K \ltimes H_n$ and $K \subset \mathrm{U}(n)$). When the Gelfand spectrum is identified with Σ_K^V, the spherical transform will be denoted by $\mathcal{G}_K^V$.

Let K be a compact group of automorphisms of H_n such that $(K \ltimes H_n, K)$ is a Gelfand pair and denote by $\mathbb{D}_K$ the commutative algebra of left-invariant and K-invariant differential operators on H_n. Let $V = \{V_1, \dots, V_s\}$ be a set of formally self-adjoint generators of $\mathbb{D}_K$. We denote by $\mathcal{S}(\Sigma_K^V)$ the space of restrictions to Σ_K^V of Schwartz functions on $\mathbb{R}^s$, endowed with the quotient topology of $\mathcal{S}(\mathbb{R}^s)/\{f\colon f|_{\Sigma_K^V} = 0\}$.

Our main result is the following:

Theorem 1.1. *The map $\mathcal{G}_K^V$ is a topological isomorphism between $\mathcal{S}_K(H_n)$ and $\mathcal{S}(\Sigma_K^V)$.*

As customary, for us H_n is understood as $\mathbb{R} \times \mathbb{C}^n$, with canonical coordinates of the first kind. It is well known that, under the action

$$k \cdot (t, z) = (t, kz) \quad \forall k \in \mathrm{U}(n),\ (t, z) \in H_n,$$

$\mathrm{U}(n)$ is a maximal compact connected group of automorphisms of H_n, and that every compact connected group of automorphisms of H_n is conjugated to a subgroup of $\mathrm{U}(n)$. Therefore, if K is a compact group of automorphisms of H_n, then its identity component in K is conjugated to a subgroup of $\mathrm{U}(n)$. For most of this article, we deal with the case of K connected and contained in $\mathrm{U}(n)$, leaving the discussion of the general case to the last section.

In Section 3 we show that it suffices to prove Theorem 1.1 for one particular set of generators of $\mathbb{D}_K$, and in Section 4 we choose a convenient set of generators. From a homogeneous Hilbert basis of K-invariant polynomials on $\mathbb{R}^{2n}$ we derive by symmetrization d differential operators $V_1, \ldots, V_d$, invariant under K; to these we add the central operator $V_0 = i^{-1}\partial_t$, obtaining in this way a generating system of $d + 1$ homogeneous operators. Here we benefit from the deep study of the algebraic properties of the multiplicity-free actions of subgroups of $\mathrm{U}(n)$, developed in [2–4,6–8]. In particular, rationality of the "generalized binomial coefficients" is a crucial point in our argument (see Proposition 7.5 below). It must be noticed that the proof of rationality in [8] is based on the actual classification of multiplicity-free actions. In this respect, our proof depends on the actual classification of the groups K giving rise to Gelfand pairs.

After these preliminaries, we split the proof of Theorem 1.1 into two parts.

In the first part we show that, if m is a Schwartz function on $\mathbb{R}^{d+1}$, its restriction to Σ_K^V is the Gelfand transform of a function f in $\mathcal{S}_K(H_n)$ (see Theorem 5.5 below). The argument is based on Hulanicki's theorem [19], stating that Schwartz functions on the real line operate on positive Rockland operators on graded nilpotent Lie groups producing convolution operators with Schwartz kernels. We adapt the argument in [26] to obtain a multivariate extension of Hulanicki's theorem (see Theorem 5.2 below).

In the second part we prove that the Gelfand transform $\mathcal{G}_K^V f$ of a function f in $\mathcal{S}_K(H_n)$ can be extended to a Schwartz function on $\mathbb{R}^{d+1}$ (see Theorem 7.1 below). The proof begins with an extension to the Schwartz space of the Schwarz–Mather theorem [23,25] for C^∞ K-invariant functions (see Theorem 6.1 below). This allows us to extend to a Schwartz function on $\mathbb{R}^d$ the restriction of $\mathcal{G}_K^V f$ to the "degenerate part" Σ_0 of the Gelfand spectrum (that corresponding to the one-dimensional representations of H_n, or equivalently corresponding to the eigenvalue 0 for V_0). Then we associate to f a Schwartz jet on Σ_0. As in [1], the key tool here is the existence of "Taylor coefficients" at points of Σ_0, proved by Geller [14] (see Theorem 7.2 below). The Whitney extension theorem (adapted to Schwartz jets in Proposition 7.4) gives therefore a Schwartz extension to $\mathbb{R}^{d+1}$ of the jet associated to f. To conclude the proof, it remains to prove that if $f \in \mathcal{S}_K(H_n)$ and the associated jet on Σ_0 is trivial, then $\mathcal{G}_K^V f$ admits a Schwartz extension. This is done by adapting an explicit interpolation formula already used in [1] (see Proposition 7.5). For a nonconnected group K, we remark that, calling K_0 the connected component of the identity, one can view the K-Gelfand spectrum as the quotient of the K_0-Gelfand spectrum under the action of the finite group $F = K/K_0$ (it is known that if $(K \ltimes H_n, K)$ is a Gelfand pair, so is $(K_0 \ltimes H_n, K_0)$, cf. [5]). Starting from an F-invariant generating system of K_0-invariant differential operators, the K-Gelfand spectrum is then conveniently embedded in a Euclidean space by means of the Hilbert map associated to the action of F on the linear span of these generators.

The paper is structured as follows. In Section 2 we recall some facts about Gelfand pairs $(K \ltimes H_n, K)$ and the associated Gelfand transform. In Section 3 we show that it suffices to prove Theorem 1.1 for one particular set of generators of $\mathbb{D}_K$. In Section 4 we choose a convenient set of generators in the case where K is a connected closed subgroup of $\mathrm{U}(n)$. In Section 5 we show that every function in $\mathcal{S}(\mathbb{R}^{d+1})$ gives rise to the Gelfand transform of a function in $\mathcal{S}_K(H_n)$ via functional calculus. In Section 6 we extend the Schwarz–Mather theorem [23,25] to Schwartz spaces. Section 7 is devoted to define a Schwartz extension on $\mathbb{R}^{d+1}$ of the Gelfand transform of a function in $\mathcal{S}_K(H_n)$. In Section 8 we show that our result holds for all compact groups of automorphisms of H_n.

2. Preliminaries

For the content of this section we refer to [2–4,9,10,21].

2.1. The Heisenberg group and its representations

We denote by H_n the Heisenberg group, i.e., the real manifold $\mathbb{R} \times \mathbb{C}^n$ equipped with the group law

$$(t,z)(u,w) = \left(t + u + \frac{1}{2}\operatorname{Im} w \cdot \bar{z}, z + w\right), \quad t, u \in \mathbb{R},\ \forall z, w \in \mathbb{C}^n,$$

where $w \cdot \bar{z}$ is a short-hand writing for $\sum_{j=1}^n w_j \overline{z_j}$. It is easy to check that Lebesgue measure $dt\,dz$ is a Haar measure on H_n.

Denote by Z_j and $\bar{Z}_j$ the complex left-invariant vector fields

$$Z_j = \partial_{z_j} - \frac{i}{4}\bar{z}_j \partial_t, \qquad \bar{Z}_j = \partial_{\bar{z}_j} + \frac{i}{4} z_j \partial_t,$$

and set $T = \partial_t$.

For $\lambda > 0$, denote by $\mathcal{F}_\lambda$ the Fock space consisting of the entire functions F on $\mathbb{C}^n$ such that

$$\|F\|_{\mathcal{F}_\lambda}^2 = \left(\frac{\lambda}{2\pi}\right)^n \int_{\mathbb{C}^n} \left|F(z)\right|^2 e^{-\frac{\lambda}{2}|z|^2}\,dz < \infty,$$

equipped with the norm $\|\cdot\|_{\mathcal{F}_\lambda}$. Then H_n acts on $\mathcal{F}_\lambda$ through the unitary representation π_λ defined by

$$\left[\pi_\lambda(t,z)F\right](w) = e^{i\lambda t} e^{-\frac{\lambda}{2} w\cdot\bar{z} - \frac{\lambda}{4}|z|^2} F(w+z) \quad \forall (z,t) \in H_n,\ F \in \mathcal{F}_\lambda,\ w \in \mathbb{C}^n,$$

and through its contragredient $\pi_{-\lambda}(t,z) = \pi_\lambda(-t,\bar{z})$. These are the Bargmann representations of H_n.

The space $\mathcal{P}(\mathbb{C}^n)$ of polynomials on $\mathbb{C}^n$ is dense in $\mathcal{F}_\lambda$ ($\lambda > 0$) and an orthonormal basis of $\mathcal{P}(\mathbb{C}^n)$ seen as a subspace of $\mathcal{F}_\lambda$ is given by the monomials

$$p_{\lambda,\mathbf{d}}(w) = \frac{w^{\mathbf{d}}}{((2/\lambda)^{|\mathbf{d}|}\mathbf{d}!)^{1/2}}, \quad \mathbf{d} \in \mathbb{N}^n.$$

Besides the Bargmann representations, H_n has the one-dimensional representations

$$\tau_w(t,z) = e^{i\,\mathrm{Re}\, z\cdot\bar{w}}$$

with $w \in \mathbb{C}^n$. The π_λ ($\lambda \neq 0$) and the τ_w fill up the unitary dual of H_n.

2.2. Gelfand pairs $(K \ltimes H_n, K)$

Let K be a compact groups of automorphisms of H_n such that the convolution algebra $L^1_K(H_n)$ of integrable K-invariant functions on H_n is abelian, i.e., such that $(K \ltimes H_n, K)$ is a Gelfand pair. It is known [5] that this property holds for K if and only if it holds for its connected identity component K_0. On the other hand, every compact, connected group of automorphisms of H_n is conjugate, modulo an automorphism, to a subgroup of $\mathrm{U}(n)$, acting on H_n via

$$k\cdot(t,z) = (t,kz) \quad \forall (t,z) \in H_n,\ k \in \mathrm{U}(n).$$

For the remainder of this section, we assume that K is connected, contained in $\mathrm{U}(n)$, and $(K \ltimes H_n, K)$ is a Gelfand pair.

For the one-dimensional representation τ_w, we have $\tau_w(t,k^{-1}z) = \tau_{kw}(t,z)$. Therefore, $\tau_w(f)$ is a K-invariant function of w for every $f \in L^1_K(H_n)$.

As to the Bargmann representations, if $k \in \mathrm{U}(n)$, $\pi^k_{\pm\lambda}(t,z) = \pi_{\pm\lambda}(t,kz)$ is equivalent to $\pi_{\pm\lambda}$ for every $\lambda > 0$ and every choice of the $\pm$ sign. Precisely, we set

$$\nu_+(k)F(z) = F\bigl(k^{-1}z\bigr), \qquad \nu_-(k)F(z) = F\bigl(\bar{k}^{-1}z\bigr),$$

each of the two actions being the contragredient of the other. We then have

$$\pi_\lambda(kz,t) = \nu_+(k)\pi_\lambda(z,t)\nu_+(k)^{-1}, \qquad \pi_{-\lambda}(kz,t) = \nu_-(k)\pi_{-\lambda}(z,t)\nu_-(k)^{-1}.$$

By homogeneity, the decomposition of $\mathcal{F}_\lambda$ into irreducible invariant subspaces under ν_+ (respectively ν_-) is independent of λ and can be reduced to the decomposition of the dense subspace $\mathcal{P}(\mathbb{C}^n)$ of polynomials.

It is known since [2,9] that $(K \ltimes H_n, K)$ is a Gelfand pair if and only if ν_+ (equivalently ν_-) decomposes into irreducibles without multiplicities (in other words, if and only if it is *multiplicity free*). The subgroups of $\mathrm{U}(n)$ giving multiplicity free actions on $\mathcal{P}(\mathbb{C}^n)$ have been classified by Kač [21] and the resulting Gelfand pairs $(K \ltimes H_n, K)$ are listed in [8, Tables 1 and 2].

Under these assumptions, the space $\mathcal{P}(\mathbb{C}^n)$ of polynomials on $\mathbb{C}^n$ decomposes into ν_+-irreducible subspaces,

$$\mathcal{P}\bigl(\mathbb{C}^n\bigr) = \sum_{\alpha\in\Lambda} P_\alpha,$$

where Λ is an infinite subset of the unitary dual $\widehat{K}$ of K and α denotes the equivalence class of the action on P_α. The irreducible ν_--invariant subspaces of $\mathcal{P}(\mathbb{C}^n)$ are

$$P_{\alpha'} = \bigl\{\bar{p}(z) = \overline{p(\bar{z})}:\ p \in P_\alpha\bigr\},$$

with the action of K on $P_{\alpha'}$ being equivalent to the contragredient α' of α.

On the other hand, $\mathcal{P}(\mathbb{C}^n) = \sum_{m\in\mathbb{N}} \mathcal{P}_m(\mathbb{C}^n)$, where $\mathcal{P}_m(\mathbb{C}^n)$ is the space of homogeneous polynomials of degree m. Since $\nu_\pm$ preserves each $\mathcal{P}_m(\mathbb{C}^n)$, each P_α is contained in $\mathcal{P}_m(\mathbb{C}^n)$ for some m. We then say that $|\alpha| = m$, so that

$$\mathcal{P}_m(\mathbb{C}) = \sum_{|\alpha|=m} P_\alpha = \sum_{|\alpha|=m} P_{\alpha'}.$$

As proved in [2], all the bounded spherical functions are of positive type. Therefore there are two families of spherical functions. Those of the first family are

$$\eta_{Kw}(t,z) = \int_K e^{i\,\mathrm{Re}\langle z,kw\rangle}\,dk, \qquad w \in \mathbb{C}^n,$$

parametrized by K-orbits in $\mathbb{C}^n$ and associated with the one-dimensional representations of the Heisenberg group.

The elements of the second family are parametrized by pairs $(\lambda,\alpha) \in \mathbb{R}^* \times \Lambda$. If $\lambda > 0$ and $\{v_1^\lambda, \ldots, v_{\dim(P_\alpha)}^\lambda\}$ is an orthonormal basis of P_α in the norm of $\mathcal{F}_\lambda$, we have the spherical function

$$\phi_{\lambda,\alpha}(t,z) = \frac{1}{\dim(P_\alpha)} \sum_{j=1}^{\dim(P_\alpha)} \langle \pi_\lambda(t,z)v_j^\lambda, v_j^\lambda\rangle_{\mathcal{F}_\lambda}. \tag{2.1}$$

Taking, as we can, v_j^λ as $\lambda^{|\alpha|/2} v_j^1$, we find that

$$\phi_{\lambda,\alpha}(z,t) = \phi_{1,\alpha}(\sqrt{\lambda}z, \lambda t).$$

For $\lambda < 0$, the analogous matrix entries of the contragredient representation give the spherical functions

$$\phi_{\lambda,\alpha}(z,t) = \overline{\phi_{-\lambda,\alpha}(z,t)}.$$

Setting for simplicity $\phi_\alpha = \phi_{1,\alpha}$,

$$\phi_\alpha(z,t) = e^{it} q_\alpha(z,\bar z) e^{-|z|^2/4},$$

where $q_\alpha \in \mathcal{P}(\mathbb{C}^n) \otimes \overline{\mathcal{P}(\mathbb{C}^n)}$ is a real K-invariant polynomial of degree $2|\alpha|$ in z and $\bar z$ (cf. [3]).

Denote by $\mathbb{D}_K$ the algebra of left-invariant and K-invariant differential operators on H_n. The symmetrization map establishes a linear bijection from the space of K-invariant elements in the symmetric algebra over $\mathfrak{h}_n$ to $\mathbb{D}_K$. Therefore every element $D \in \mathbb{D}_K$ can be expressed as $\sum_{j=0}^m D_j T^j$, where D_j is the symmetrization of a K-invariant polynomial in $Z, \bar Z$.

Let D be the symmetrization of the K-invariant polynomial $P(Z,\bar Z,T)$. With $\widehat{D}(\phi)$ denoting the eigenvalue of $D \in \mathbb{D}_K$ on the spherical function ϕ, we have

$$\widehat{D}(\eta_{Kw}) = P(w,\bar w,0) \tag{2.2}$$

for the spherical functions associated to the one-dimensional representations. For $\lambda \neq 0$, $d\pi_\lambda(D)$ commutes with the action of K and therefore it preserves each P_α. By Schur's lemma, $d\pi_\lambda(D)_{|P_\alpha}$ is a scalar operator $c_{\lambda,\alpha}(D)I_{P_\alpha}$. It follows from (2.1) that

$$\widehat{D}(\phi_{\lambda,\alpha}) = c_{\lambda,\alpha}(D).$$

In particular, for $D = T$, we have

$$\widehat{T}(\eta_{Kw}) = 0, \qquad \widehat{T}(\phi_{\lambda,\alpha}) = i\lambda.$$

3. Embeddings of the Gelfand spectrum

Let K be a compact group of automorphisms of H_n such that $(K \ltimes H_n, K)$ is a Gelfand pair. The Gelfand spectrum of the commutative Banach algebra $L^1_K(H_n)$ is the set of bounded spherical functions endowed with the compact-open topology. Given a set $V = \{V_0, V_1, \ldots, V_d\}$ of formally self-adjoint generators of $\mathbb{D}_K$, we assign to each spherical function ϕ the $(d+1)$-tuple $\widehat{V}(\phi) = (\widehat{V_0}(\phi), \widehat{V_1}(\phi), \ldots, \widehat{V_d}(\phi))$. Since $d\pi(V_j)$ is formally self-adjoint for every irreducible representation π, $\widehat{V}(\phi)$ is in $\mathbb{R}^{d+1}$. It has been proved, in a more general context [10], that $\Sigma_K^V = \{\widehat{V}(\phi)$: ϕ spherical$\}$ is closed in $\mathbb{R}^{d+1}$ and homeomorphic to the Gelfand spectrum via $\widehat{V}$ (see also [4]). Once we have identified the Gelfand spectrum with Σ_K^V, the Gelfand transform of a function f in $L^1_K(H_n)$ can be defined on the closed subset Σ_K^V of $\mathbb{R}^{d+1}$ as

$$\big(\mathcal{G}_K^V f\big)\big(\widehat{V}(\phi)\big) = \int\limits_{H_n} f\phi.$$

In order to prove Theorem 1.1, we first show that different choices of the generating system V give rise to natural isomorphisms among the corresponding restricted Schwartz spaces $\mathcal{S}(\Sigma_K^V)$. It will then suffice to prove Theorem 1.1 for one particular set of generators.

On the Schwartz space $\mathcal{S}(\mathbb{R}^m)$ we consider the following family of norms, parametrized by a nonnegative integer p:

$$\|f\|_{(p,\mathbb{R}^m)} = \sup_{y\in\mathbb{R}^m, |\alpha|\leqslant p} \big(1+|y|\big)^p \big|\partial^\alpha f(y)\big|.$$

Lemma 3.1. *Let E and F be closed subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $P : \mathbb{R}^n \to \mathbb{R}^m$ and $Q : \mathbb{R}^m \to \mathbb{R}^n$ be polynomial maps such that $P(E) = F$ and $Q \circ P$ is the identity on E. Given f in $\mathcal{S}(F)$ we let $P^\flat f = f \circ P|_E$. Then $P^\flat$ maps $\mathcal{S}(F)$ in $\mathcal{S}(E)$ continuously.*

Proof. We show that if f is in $\mathcal{S}(\mathbb{R}^m)$, then $P^\flat f$ can be extended to a function $\widetilde{P^\flat f}$ in $\mathcal{S}(\mathbb{R}^n)$ in a linear and continuous way. Let Ψ be a smooth function on $\mathbb{R}^n$ such that $\Psi(t) = 1$ if $|t| \leqslant 1$ and $\Psi(t) = 0$ if $|t| > 2$. Define

$$\widetilde{P^\flat f}(x) = \Psi\big(x - Q \circ P(x)\big)(f \circ P)(x) \quad \forall x \in \mathbb{R}^n.$$

Clearly $\widetilde{P^\flat f}$ is smooth and $\widetilde{P^\flat f}|_E = P^\flat f$. Moreover $\widetilde{P^\flat f}$ is zero when $|x - Q \circ P(x)| > 2$, so it suffices to prove rapid decay for $|x - Q \circ P(x)| \leqslant 2$. Note that there exists ℓ in $\mathbb{N}$ such that

$$|x| \leqslant 2 + \big|Q\big(P(x)\big)\big| \leqslant C\big(1 + \big|P(x)\big|\big)^\ell \quad \forall x \in \mathbb{R}^n,\ \big|x - Q \circ P(x)\big| \leqslant 2.$$

Therefore given a positive integer p there exists a positive integer q such that $\|\widetilde{P^\flat f}\|_{(p,\mathbb{R}^n)} \leqslant C\|f\|_{(q,\mathbb{R}^m)}$. The thesis follows immediately from the definition of the quotient topology on $\mathcal{S}(F)$ and $\mathcal{S}(E)$. □

Corollary 3.2. *Suppose that $\{V_0, \dots, V_d\}$ and $\{W_0, \dots, W_s\}$ are two sets of formally self-adjoint generators of $\mathbb{D}_K$. Then the spaces $\mathcal{S}(\Sigma_K^V)$ and $\mathcal{S}(\Sigma_K^W)$ are topologically isomorphic.*

Proof. There exist real polynomials p_j, $j = 0, 1, \dots, s$, and q_h, $h = 0, 1, \dots, d$, such that $W_j = p_j(V_0, \dots, V_d)$ and $V_h = q_h(W_0, \dots, W_s)$.

Setting $P = (p_0, p_1, \dots, p_s) : \mathbb{R}^{d+1} \to \mathbb{R}^{s+1}$ and $Q = (q_0, q_1, \dots, q_d) : \mathbb{R}^{s+1} \to \mathbb{R}^{d+1}$, we can apply Lemma 3.1 in both directions. □

4. Choice of the generators

In this section K shall be a closed connected subgroup of $\mathrm{U}(n)$ such that $(K \ltimes H_n, K)$ is a Gelfand pair. The subject of the following lemma is the choice of a convenient set of formally self-adjoint generators of $\mathbb{D}_K$.

Lemma 4.1. *A generating system $\{V_0 = -iT, V_1, \dots, V_d\}$ can be chosen such that, for each $j = 1, \dots, d$,*

1. *V_j is homogeneous of even order $2m_j$;*
2. *V_j is formally self-adjoint and $\widehat{V}_j(\phi_\alpha)$ is a positive integer for every α in Λ;*
3. *$\widehat{V}_j(\eta_{Kw}) = \rho_j(w, \bar{w})$, for every w in $\mathbb{C}^n$, where ρ_j is a nonnegative homogenous polynomial of degree $2m_j$, strictly positive outside of the origin.*

Notice that (1) and (2) imply that when $j = 1, \dots, d$

$$\widehat{V}_j(\phi_{\lambda,\alpha}) = |\lambda|^{m_j} \widehat{V}_j(\phi_\alpha) \quad \forall \lambda \in \mathbb{R} \setminus \{0\},\ \forall \alpha \in \Lambda. \tag{4.1}$$

Proof. Let $\mathbb{C}^n_{\mathbb{R}}$ denote $\mathbb{C}^n$ with the underlying structure of a real vector space. We denote by $\mathcal{P}(\mathbb{C}^n_{\mathbb{R}}) \cong \mathcal{P}(\mathbb{C}^n) \otimes \overline{\mathcal{P}(\mathbb{C}^n)}$ the algebra of polynomials in z and $\bar{z}$, and by $\mathcal{P}^K(\mathbb{C}^n_{\mathbb{R}})$ the subalgebra of K-invariant polynomials.

The fact that the representation of K on $\mathcal{P}(\mathbb{C}^n)$ is multiplicity free implies that the trivial representation is contained in $P_\alpha \otimes \overline{P_\beta} \subset \mathcal{P}(\mathbb{C}^n_{\mathbb{R}})$ if and only if $\alpha = \beta$, and with multiplicity one in each of them. Therefore a linear basis of $\mathcal{P}^K(\mathbb{C}^n_{\mathbb{R}})$ is given by the polynomials

$$p_\alpha(z, \bar{z}) = \sum_{h=1}^{\dim(P_\alpha)} v_h(z)\, \overline{v_h(z)} = \sum_{h=1}^{\dim(P_\alpha)} \left|v_h(z)\right|^2, \tag{4.2}$$

where $\{v_1, \dots, v_{\dim(P_\alpha)}\}$ is any orthonormal basis of P_α in the $\mathcal{F}_1$-norm.

A result in [18] ensures that there exist $\delta_1, \dots, \delta_d$ in Λ such that the polynomials

$$\gamma_j = p_{\delta_j}, \quad j = 1, \dots, d,$$

freely generate $\mathcal{P}^K(\mathbb{C}^n_{\mathbb{R}})$. In [8] the authors prove that $\gamma_1,\dots,\gamma_d$ have rational coefficients. More precisely, setting $m_j=|\delta_j|$, each γ_j can be written in the form

$$\gamma_j(z,\bar z)=\sum_{|\mathbf{a}|=|\mathbf{b}|=m_j}\theta^{(j)}_{\mathbf{a},\mathbf{b}}z^{\mathbf{a}}\bar z^{\mathbf{b}},$$

where $\mathbf{a}$, $\mathbf{b}$ are in $\mathbb{N}^n$ and $\theta^{(j)}_{\mathbf{a},\mathbf{b}}$ are rational numbers.

The symmetrization L_{γ_j} of $\gamma_j(Z,\bar Z)$ is a homogenous operator of degree $2m_j$ in $\mathbb{D}_K$ with rational coefficients, and $\{-iT,L_{\gamma_1},\dots,L_{\gamma_d}\}$ generate $\mathbb{D}_K$ [3,4]. Moreover, the eigenvalues $\widehat{L}_{\gamma_j}(\phi_\alpha)$ are rational numbers [7,8].

Fix any positive integer m and denote by $M_{j,m}$ the matrix which represents the restriction of $d\pi_1(L_{\gamma_j})$ to $\mathcal{P}_m(\mathbb{C}^n)$ in the basis of monomials w^α, $|\alpha|=m$. For every F in $\mathcal{F}_1(\mathbb{C}^n)$,

$$\big[d\pi_1(Z_h)F\big](w)=[\partial_{w_h}F](w),\qquad \big[d\pi_1(\bar Z_h)F\big](w)=-\frac{1}{2}w_hF(w)\quad \forall w\in\mathbb{C}^n,$$

for $h=1,\dots,n$. Therefore $M_{j,m}$ has rational entries, with denominators varying in a finite set independent of m. We can then take N such that the matrices $NM_{j,m}$ have integral entries, for all m and $j=1,\dots,d$. Thus the characteristic polynomial of $NM_{j,m}$ is monic, with integral coefficients and rational zeroes $N\widehat{L_{\gamma_j}}(\phi_\alpha)$; therefore these zeroes must be integers. They all have the same sign, independently of m, equal to $(-1)^{m_j}$ (cf. [4]).

We then define

$$V_j=N(-1)^{m_j}L_{\gamma_j}+\mathcal{L}^{m_j},\quad j=1,\dots,d,$$

where $\mathcal{L}=-2\sum_{j=1}^n(Z_j\overline{Z}_j+\overline{Z}_jZ_j)$ is the U(n)-invariant subLaplacian, satisfying $\widehat{\mathcal{L}}(\phi_\alpha)=2|\alpha|+n$. We show that $\{V_0=-iT,V_1,\dots,V_d\}$ is a set of generators satisfying the required conditions.

Since $\sum_{m_j=1}\gamma_j=\frac12|z|^2$ (cf. [4]), we have

$$\sum_{m_j=1}L_{\gamma_j}=\sum_{j=1}^n(Z_j\overline{Z}_j+\overline{Z}_jZ_j)=-\frac12\mathcal{L},$$

and then

$$\sum_{m_j=1}V_j=\sum_{m_j=1}(-NL_{\gamma_j}+\mathcal{L})=\left(\frac{N}{2}+r\right)\mathcal{L},\tag{4.3}$$

where r is the cardinality of the set $\{\delta_j\colon m_j=|\delta_j|=1\}$. Therefore each L_{γ_j} is a polynomial in $V_1,\dots,V_d$. Since $-iT,L_{\gamma_1},\dots,L_{\gamma_d}$ generate $\mathbb{D}_K$, the same holds for $-iT,V_1,\dots,V_d$.

Condition (1) follows from the homogeneity of L_{γ_j} and $\mathcal{L}^{m_j}$, and from (4.2). Since the polynomials in (4.2) are real-valued, then the $V_j's$ are formally self-adjoint and condition (2) is easily verified. Finally condition (3) follows from (2.2), which gives

$$\widehat{L_{\gamma_j}}(\eta_{Kw})=(-1)^{m_j}\gamma_j(w,\bar w),\qquad \widehat{\mathcal{L}^{m_j}}(\eta_{Kw})=|w|^{2m_j}$$

for all w in $\mathbb{C}^n$ and $j=1,\dots,d$. □

Let $V = \{V_0, V_1, \dots, V_d\}$ denote the privileged set of generators chosen in Lemma 4.1. We set $\rho = (\rho_1, \dots, \rho_d)$ the polynomial map in (3) of Lemma 4.1.

Coordinates in $\mathbb{R}^{d+1}$ will be denoted by (λ, ξ), with λ in $\mathbb{R}$ and ξ in $\mathbb{R}^d$. So, if $(\lambda, \xi) = \widehat{V}(\phi)$, then either $\lambda = -i\widehat{T}(\phi) = 0$, in which case $\phi = \eta_{Kw}$ and $\xi = \rho(w, \bar{w})$, or $\lambda = -i\widehat{T}(\phi) \neq 0$, in which case $\phi = \phi_{\lambda,\alpha}$ and $\xi_j = \widehat{V}_j(\phi_{\lambda,\alpha}) = |\lambda|^{m_j}\widehat{V}_j(\phi_\alpha)$.

The spectrum Σ_K^V consists therefore of two parts. The first part is $\Sigma_0 = \{0\} \times \rho(\mathbb{C}^n)$, a semi-algebraic set. The second part, Σ', is the countable union of the curves $\Gamma_\alpha(\lambda) = \widehat{V}(\phi_{\lambda,\alpha})$, $\lambda \neq 0$. Each Γ_α and Σ_0 are homogeneous with respect to the dilations

$$(\lambda, \xi_1, \dots, \xi_d) \mapsto \bigl(t\lambda, t^{m_1}\xi_1, \dots, t^{m_d}\xi_d\bigr) \quad (t > 0) \tag{4.4}$$

and to the symmetry $(\lambda, \xi_1, \dots, \xi_d) \mapsto (-\lambda, \xi_1, \dots, \xi_d)$. By our choice of V, $\Sigma' \cap \{\lambda = 1\}$ is contained in the positive integer lattice. Moreover Σ' is dense in Σ_K^V (cf. [4,14]).

For the sake of brevity, we denote by $\widehat{f}$ the Gelfand transform $\mathcal{G}_K^V f$ of the integrable K-invariant function f.

5. Functional calculus

In this section we prove one of the two implications of Theorem 1.1 for a closed connected subgroup K of $\mathrm{U}(n)$ such that $(K \ltimes H_n, K)$ is a Gelfand pair. More precisely we prove that if m is a Schwartz function on $\mathbb{R}^{d+1}$, its restriction to Σ_K^V is the Gelfand transform of a function f in $\mathcal{S}_K(H_n)$ (see Theorem 5.5 below).

The proof is based on a result of Hulanicki [19] (see Theorem 5.1 below) on functional calculus for Rockland operators on graded groups and a multi-variate extension of it.

A Rockland operator D on a graded Lie group N is a formally self-adjoint left-invariant differential operator on N which is homogeneous with respect to the dilations and such that, for every nontrivial irreducible representation π of N, the operator $d\pi(D)$ is injective on the space of C^∞ vectors.

As noted in [20], it follows from [24] and [15] that a Rockland operator D is essentially self-adjoint on the Schwartz space $\mathcal{S}(N)$, as well as $d\pi(D)$ on the Gårding space for every unitary representation π. We keep the same symbols for the self-adjoint extensions of such operators.

Let N be a graded Lie group, and let $|\cdot|$ be a homogeneous gauge on it. We say that a function on N is Schwartz if and only if it is represented by a Schwartz function on the Lie algebra $\mathfrak{n}$ in any given set of canonical coordinates. The fact that changes of canonical coordinates are expressed by polynomials makes this condition independent of the choice of the coordinates. Given a homogeneous basis $\{X_1, \dots, X_n\}$ of the Lie algebra $\mathfrak{n}$ we keep the same notation X_j for the associated left-invariant vector fields on N. Following [12], we shall consider the following family of norms on $\mathcal{S}(N)$, parametrized by a nonnegative integer p:

$$\|f\|_{(p,N)} = \sup\bigl\{\bigl(1 + |x|\bigr)^p \bigl|X^I f(x)\bigr|\colon x \in N,\ \deg X^I \leqslant p\bigr\}, \tag{5.1}$$

where $X^I = X_1^{i_1} \cdots X_n^{i_n}$ and $\deg X^I = \sum i_j \deg X_j$. Note that the Fréchet space structure induced on $\mathcal{S}(N)$ by this family of norms is independent of the choice of the X_j and is equivalent to that induced from $\mathcal{S}(\mathfrak{n})$ via composition with the exponential map.

Theorem 5.1. *(See [19].) Let D a positive Rockland operator on a graded Lie group N and let $D = \int_0^{+\infty} \lambda\, dE(\lambda)$ be its spectral decomposition. If m is in $\mathcal{S}(\mathbb{R})$ and*

$$m(D) = \int_0^{+\infty} m(\lambda)\, dE(\lambda),$$

then there exists M in $\mathcal{S}(N)$ such that

$$m(D)f = f * M \quad \forall f \in \mathcal{S}(N).$$

Moreover for every p there exists q such that

$$\|M\|_{(p,N)} \leqslant C\|m\|_{(q,\mathbb{R})}.$$

Suppose that $D_1, \dots, D_s$ form a commutative family of self-adjoint operators on N (in the sense that they have commuting spectral resolutions). Then they admit a joint spectral resolution and one can define the bounded operator $m(D_1, \dots, D_s)$ for any bounded Borel function m on their joint spectrum in $\mathbb{R}^s$. The following theorem was proved in [26] in a special situation.

Theorem 5.2. *Suppose that N is a graded Lie group and $D_1, \dots, D_s$ form a commutative family of positive Rockland operators on N. If m is in $\mathcal{S}(\mathbb{R}^s)$, then there exists M in $\mathcal{S}(N)$ such that*

$$m(D_1, \dots, D_s)f = f * M.$$

Moreover, for every p there exists q such that

$$\|M\|_{(p,N)} \leqslant C\|m\|_{(q,\mathbb{R}^d)}.$$

Proof. We prove the theorem by induction on s. By Theorem 5.1 the thesis holds when $s = 1$. Let $s \geqslant 2$ and suppose that the thesis holds for $s - 1$. Let $m(\lambda_1, \dots, \lambda_s)$ be in $\mathcal{S}(\mathbb{R}^s)$. Then there exist sequences $\{\psi_k\}$ in $\mathcal{S}(\mathbb{R}^{s-1})$ and $\{\varphi_k\}$ in $\mathcal{S}(\mathbb{R})$ such that $m(\lambda_1, \dots, \lambda_{s-1}, \lambda_s) = \sum_k \psi_k(\lambda_1, \dots, \lambda_{s-1})\varphi_k(\lambda_s)$ and $\sum_k \|\psi_k \otimes \varphi_k\|_{N,\mathbb{R}^s} < \infty$ for every N.

For a proof of this, one can first decompose m as a sum of C^∞-functions supported in a sequence of increasing balls and with rapidly decaying Schwartz norms, and then separate variables in each of them by a Fourier series expansion (see also [26]).

By the inductive hypothesis, for every k there exist Ψ_k and Φ_k in $\mathcal{S}(N)$ such that $\psi_k(D_1, \dots, D_{s-1})f = f * \Psi_k$ and $\varphi_k(D_s)f = f * \Phi_k$ for every f in $\mathcal{S}(N)$. Then

$$\begin{aligned}\psi_k \otimes \varphi_k(D_1, \dots, D_{s-1}, D_s)f &= \psi_k(D_1, \dots, D_{s-1})\, \varphi_k(D_s)f \\ &= f * \Phi_k * \Psi_k.\end{aligned}$$

By straightforward computations (cf. [12, Proposition 1.47]) and the inductive hypothesis, we obtain that

$$\begin{aligned}\|\Psi_k * \Phi_k\|_{(p,N)} &\leqslant C_p \|\Psi_k\|_{(p',N)} \|\Phi_k\|_{(p''+1,N)} \\ &\leqslant C_p \|\psi_k\|_{(q,\mathbb{R}^{s-1})} \|\varphi_k\|_{(q',\mathbb{R})} \\ &\leqslant C_p \|\psi_k \otimes \varphi_k\|_{(q'',\mathbb{R}^s)}.\end{aligned}$$

Since the series $\sum_k \psi_k \otimes \varphi_k$ is totally convergent in every Schwartz norm on $\mathbb{R}^s$, there exists a function F in $\mathcal{S}(N)$ such that

$$\sum_k \Psi_k * \Phi_k = F$$

and hence

$$\sum_k f * \Psi_k * \Phi_k = f * F$$

for every f in $\mathcal{S}(N)$.

On the other hand, if $M \in \mathcal{S}'(N)$ is the convolution kernel of $m(D_1, \dots, D_{s-1}, D_s)$, by the Spectral Theorem,

$$\begin{aligned}m(D_1, \dots, D_{d-1}, D_d) f &= \sum_k \psi_k(D_1, \dots, D_{d-1}) \otimes \varphi_k(D_d) f \\ &= \sum_k f * \Psi_k * \Phi_k,\end{aligned}$$

for every f in $\mathcal{S}(N)$, with convergence in $L^2(N)$. Therefore $\sum_k \Psi_k * \Phi_k$ converges to M in $\mathcal{S}'(N)$, i.e. $F = M$ and M is in $\mathcal{S}(N)$.

Finally, given p there exists q such that

$$\|M\|_{(p,N)} \leqslant \|m\|_{(q,\mathbb{R}^d)}.$$

This follows from the Closed Graph Theorem. Indeed, we have shown that there is a linear correspondence $m \mapsto M$ from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(N)$; moreover, reasoning as before, if $m_h \to m$ in $\mathcal{S}(\mathbb{R}^d)$ and $M_h \to \varphi$ in $\mathcal{S}(N)$ then $M_h \to M$ by the Spectral Theorem in $\mathcal{S}'(N)$. Therefore $\varphi = M$. □

Going back to our case, we prove that the operators $V_1, \dots, V_d$ satisfy the hypotheses of Theorem 5.2.

Lemma 5.3. *The differential operators $V_1, \dots, V_d$ defined in Lemma* 4.1 *form a commutative family of positive Rockland operators on H_n.*

Proof. Suppose that $j = 1, \dots, d$. The formal self-adjointness of V_j follows directly from its definition and the identity $\int_{H_n} Z_k f \bar{g} = -\int_{H_n} f Z_k \bar{g}$, for every pair of Schwartz functions f and g on H_n.

We check now the injectivity condition on the image of V_j in the nontrivial irreducible unitary representations of H_n.

By (3) in Lemma 4.1, $d\tau_w(V_j) = \rho_j(w) > 0$ for $w \neq 0$.

As to the Bargmann representations π_λ, the Gårding space in $\mathcal{F}_{|\lambda|}$ can be characterized as the space of those $F = \sum_{\alpha\in\Lambda} F_\alpha$ (with F_α in P_α for $\lambda > 0$ and in $P_{\alpha'}$ for $\lambda < 0$) such that

$$\sum_{\alpha\in\Lambda} \bigl(1 + |\alpha|\bigr)^N \|F_\alpha\|^2_{\mathcal{F}_{|\lambda|}} < \infty$$

for every integer N. For such an F,

$$V_j F = \sum_{\alpha\in\Lambda} \widehat{V_j}(\phi_{\lambda,\alpha}) F_\alpha,$$

where the series is convergent in norm, and therefore it is zero if and only if $F_\alpha = 0$ for every α.

Positivity of V_j follows from Plancherel's formula: for $f \in \mathcal{S}(H_n)$,

$$\int_{H_n} V_j f \bar{f} = \left(\frac{1}{2\pi}\right)^{n+1} \int_0^{+\infty} \sum_{\alpha\in\Lambda} \widehat{V_j}(\phi_{\lambda,\alpha})\bigl(\bigl\|\pi_\lambda(f)_{|P_\alpha}\bigr\|^2_{HS} + \bigl\|\pi_{-\lambda}(f)_{|P_{\alpha'}}\bigr\|^2_{HS}\bigr)\lambda^n\, d\lambda \geqslant 0,$$

since the eigenvalues $\widehat{V_j}(\phi_{\lambda,\alpha})$ are positive.

Given a Borel subset ω of $\mathbb{R}^+$, define the operator $E_j(\omega)$ on $L^2(H_n)$ by

$$\pi_\lambda\bigl(E_j(\omega)f\bigr) = \sum_{\alpha\in\Lambda} \chi_\omega\bigl(\widehat{V_j}(\phi_{\lambda,\alpha})\bigr)\pi_\lambda(f)\Pi_{\lambda,\alpha}, \tag{5.2}$$

where χ_ω is the characteristic function of ω and $\Pi_{\lambda,\alpha}$ is the orthogonal projection of $\mathcal{F}_{|\lambda|}$ onto P_α if $\lambda > 0$, or onto $P_{\alpha'}$ if $\lambda < 0$. Then $E_j = \{E_j(\omega)\}$ defines, for each j, a resolution of the identity, and, for $f \in \mathcal{S}(H_n)$,

$$\int_0^{+\infty} \xi\, dE_j(\xi) f = V_j f.$$

Therefore E_j is the spectral resolution of the self-adjoint extension of V_j. It is then clear that $E_j(\omega)$ and $E_k(\omega')$ commute for every ω, ω' and j, k. □

Corollary 5.4. *Let $V_0, \ldots, V_d$ be the differential operators defined in Lemma* 4.1. *If m is in $\mathcal{S}(\mathbb{R}^{d+1})$, then there exists M in $\mathcal{S}_K(H_n)$ such that*

$$m(V_0, \ldots, V_d) f = f * M \quad \forall f \in \mathcal{S}(H_n).$$

Moreover, for every p there exists q such that

$$\|M\|_{(p,H_n)} \leqslant C\|m\|_{(q,\mathbb{R}^{d+1})}.$$

Proof. We replace $V_0 = -iT$ by $\tilde{V}_0 = -iT + 2\mathcal{L}$. By (4.3), $\tilde{V}_0$ is a linear combination of the V_j. Therefore $m(V_0, \dots, V_d) = \tilde{m}(\tilde{V}_0, \dots, V_d)$, where $\tilde{m}$ is the composition of m with a linear transformation of $\mathbb{R}^{d+1}$.

Moreover, $\tilde{V}_0$ is a positive Rockland operator (cf. [11]), which commutes with the other V_j because so do V_0 and $\mathcal{L}$. Applying Theorem 5.2,

$$\tilde{m}(\tilde{V}_0, \dots, V_d)f = f * M,$$

with $M \in \mathcal{S}_K(H_n)$ and

$$\|M\|_{(p,H_n)} \leqslant C\|\tilde{m}\|_{(q,\mathbb{R}^{d+1})} \leqslant C'\|m\|_{(q,\mathbb{R}^{d+1})}. \qquad \square$$

Theorem 5.5. *Suppose that m is a Schwartz function on $\mathbb{R}^{d+1}$. Then there exists a function M in $\mathcal{S}_K(H_n)$ such that $\widehat{M} = m_{|_{\Sigma_K^V}}$. Moreover the map $m \mapsto M$ is a continuous linear operator from $\mathcal{S}(\mathbb{R}^{d+1})$ to $\mathcal{S}_K(H_n)$.*

Proof. It follows from (5.2) that the joint spectrum of $V_0, \dots, V_d$ is Σ_K^V (cf. [13, Theorem 1.7.10]). Therefore the continuous map $m \mapsto M$ of Corollary 5.4 passes to the quotient modulo $\{m\colon\ m = 0 \text{ on } \Sigma_K^V\}$.

On the other hand, by (5.2),

$$\pi_\lambda\big(m(V_0, \dots, V_d)f\big) = \sum_{\alpha \in \Lambda} m\big(\widehat{V}(\phi_{\lambda,\alpha})\big)\pi_\lambda(f)\,\mathrm{proj}_{\lambda,\alpha},$$

and this must coincide with $\pi_\lambda(f)\pi_\lambda(M)$. It follows that $\widehat{M}(\widehat{V}(\phi_{\lambda,\alpha})) = m(\widehat{V}(\phi_{\lambda,\alpha}))$ for every λ, α. By density, $\widehat{M} = m_{|_{\Sigma_K^V}}$. $\square$

6. Extension of Schwartz invariant functions on $\mathbb{R}^m$

Suppose that K is a compact Lie group acting orthogonally on $\mathbb{R}^m$. It follows from Hilbert's Basis Theorem [27] that the algebra of K-invariant polynomials on $\mathbb{R}^m$ is finitely generated. Let $\rho_1, \dots, \rho_d$ be a set of generators and denote by $\rho = (\rho_1, \dots, \rho_d)$ the corresponding map from $\mathbb{R}^m$ to $\mathbb{R}^d$. The image $\Sigma = \rho(\mathbb{R}^m)$ of ρ is closed in $\mathbb{R}^d$.

If h is a smooth function on $\mathbb{R}^d$, then $h \circ \rho$ is in $C_K^\infty(\mathbb{R}^m)$, the space of K-invariant smooth functions on $\mathbb{R}^m$. G. Schwarz [25] proved that the map $h \mapsto h \circ \rho$ is surjective from $C^\infty(\mathbb{R}^d)$ to $C_K^\infty(\mathbb{R}^m)$, so that, passing to the quotient, it establishes an isomorphism between $C^\infty(\Sigma)$ and $C_K^\infty(\mathbb{R}^m)$.

J. Mather [23] proved that the map $h \mapsto h \circ \rho$ is split-surjective, i.e. there is a continuous linear operator $\mathcal{E} : C_K^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^d)$ such that $(\mathcal{E}f) \circ \rho = f$ for every $f \in C_K^\infty(\mathbb{R}^m)$.

From this one can derive the following analogue of the Schwarz–Mather theorem for $\mathcal{S}_K(\mathbb{R}^m)$.

Theorem 6.1. *There is a continuous linear operator $\mathcal{E}' : \mathcal{S}_K(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^d)$ such that $(\mathcal{E}'g) \circ \rho = g$ for every $g \in \mathcal{S}_K(\mathbb{R}^m)$. In particular, the map $g \mapsto g \circ \rho$ is an isomorphism between $\mathcal{S}(\Sigma)$ and $\mathcal{S}_K(\mathbb{R}^m)$.*

Proof. It follows from Lemma 3.1 that the validity of the statement is independent of the choice of the Hilbert basis ρ. We can then assume that the polynomials ρ_j are homogeneous of degree α_j.

On $\mathbb{R}^d$ we define anisotropic dilations by the formula

$$\delta_r(y_1, \dots, y_k) = \left(r^{\alpha_1} y_1, \dots, r^{\alpha_d} y_d\right) \quad \forall r > 0,$$

and we shall denote by $|\cdot|_\alpha$ a corresponding homogeneous gauge, e.g.

$$|y|_\alpha = c \sum_{j=1}^{d} |y_j|^{1/\alpha_j}, \tag{6.1}$$

satisfying $|\delta_r y|_\alpha = r|y|_\alpha$.

On $\mathbb{R}^m$ we keep isotropic dilations, given by scalar multiplication. Clearly, ρ is homogeneous of degree 1 with respect to these dilations, i.e.,

$$\rho(rx) = \delta_r\left(\rho(x)\right) \quad \forall r > 0,\ x \in \mathbb{R}^m,$$

and Σ is δ_r-invariant for all $r > 0$.

Since ρ is continuous, the image under ρ of the unit sphere in $\mathbb{R}^m$ is a compact set not containing 0 (in fact, since $|x|^2$ is a polynomial in the ρ_j, cf. [23], $\rho_j(x) = 0$ for every j implies that $x = 0$).

Choosing the constant c in (6.1) appropriately, we can assume that $1 \leqslant |\rho(x)|_\alpha \leqslant R$ for every x in the unit sphere in $\mathbb{R}^m$. It follows by homogeneity that for every a, b, $0 \leqslant a < b$,

$$\rho\left(\left\{x\colon a \leqslant |x| \leqslant b\right\}\right) \subset \left\{y\colon a \leqslant |y|_\alpha \leqslant Rb\right\}. \tag{6.2}$$

Fix $\mathcal{E} : C^\infty(\mathbb{R}^m)_K \to C^\infty(\mathbb{R}^d)$ a continuous linear operator satisfying the condition $(\mathcal{E} f) \circ \rho = f$, whose existence is guaranteed by Mather's theorem.

Denote by B_s the subset of $\mathbb{R}^d$ where $|y|_\alpha < s$. For every $p \in \mathbb{N}$ there is $q \in N$ such that, for f supported in the unit ball,

$$\|\mathcal{E} f\|_{C^p(B_{R^2})} < C_p \|f\|_{C^q}.$$

Given $r > 0$, set $f_r(x) = f(rx)$ and

$$\mathcal{E}_r f = (\mathcal{E} f_r) \circ \delta_{r^{-1}}.$$

By the homogeneity of ρ, $(\mathcal{E}_r f) \circ \rho = f$. For $r > 1$ and f supported on the ball of radius r, we have

$$\|\mathcal{E}_r f\|_{C^p(B_{rR^2})} < C_p r^q \|f\|_{C^q}. \tag{6.3}$$

Let $\{\varphi_j\}_{j \geqslant 0}$ be a partition of unity on $\mathbb{R}^m$ consisting of radial smooth functions such that

(a) φ_0 is supported on $\{x\colon |x| < 1\}$;
(b) for $j \geqslant 1$, φ_j is supported on $\{x\colon R^{j-2} < |x| < R^j\}$;

(c) for $j \geqslant 1$, $\varphi_j(x) = \varphi_1(R^{-(j-1)}x)$.

Similarly, let $\{\psi_j\}_{j\geqslant 0}$ be a partition of unity on $\mathbb{R}^d$ consisting of smooth functions such that

(a′) ψ_0 is supported on $\{y\colon |y|_\alpha < R\}$;
(b′) for $j \geqslant 1$, ψ_j is supported on $\{y\colon R^{j-1} < |y|_\alpha < R^{j+1}\}$;
(c′) for $j \geqslant 1$, $\psi_j(y) = \psi_1(\delta_{R^{-(j-1)}} y)$.

For $f \in \mathcal{S}_K(\mathbb{R}^m)$ define

$$\mathcal{E}' f(y) = \sum_{j=0}^{\infty} \sum_{\ell=-2}^{1} \psi_{j+\ell}(y) \mathcal{E}_{R^j}(\varphi_j f) = \sum_{j=0}^{\infty} \psi_j(y) \sum_{\ell=-2}^{1} \mathcal{E}_{R^{j-\ell}}(\varphi_{j-\ell} f),$$

with the convention that $\psi_{-1} = \psi_{-2} = \varphi_{-1} = 0$. Then

$$\mathcal{E}' f\big(\rho(x)\big) = \sum_{j=0}^{\infty} \sum_{\ell=-2}^{1} \psi_{j+\ell}\big(\rho(x)\big)\varphi_j(x) f(x).$$

By (6.2), $\sum_{\ell=-2}^{1} \psi_{j+\ell}(\rho(x)) = 1$ on the support of φ_j, hence $\mathcal{E}' f \circ \rho = f$.

We have the following estimate for the Schwartz norms in (5.1):

$$\|f\|_{(p,\mathbb{R}^m)} \sim \sum_{j=0}^{\infty} R^{jp} \|f\varphi_j\|_{C^p}.$$

On $\mathbb{R}^d$ we adapt the Schwartz norms to the dilations δ_r by setting

$$\|g\|'_{(p,\mathbb{R}^d)} = \sup_{y\in\mathbb{R}^d,\, \sum a_j\alpha_j \leqslant p} \big(1 + |y|_\alpha\big)^p \big|\partial^\alpha g(y)\big|.$$

We then have

$$\|g\|'_{(p,\mathbb{R}^d)} \sim \sum_{j=0}^{\infty} R^{jp} \sup_{\sum a_j\alpha_j \leqslant p} \big\|\partial^\alpha g\psi_j\big\|_\infty \lesssim \sum_{j=0}^{\infty} R^{jp} \|g\psi_j\|_{C^p}.$$

Therefore

$$\begin{aligned}
\|\mathcal{E}' f\|'_{(p,\mathbb{R}^m)} &\leqslant C \sum_{j=0}^{\infty} R^{jp} \left\| \psi_j \sum_{\ell=-2}^{1} \mathcal{E}_{R^{j-\ell}}(\varphi_{j-\ell} f) \right\|_{C^p} \\
&\leqslant C \sum_{j=0}^{\infty} \sum_{\ell=-2}^{1} R^{jp} \big\|\mathcal{E}_{R^{j-\ell}}(\varphi_{j-\ell} f)\big\|_{C^p(B_{R^{j+1}})} \\
&= C \sum_{j=0}^{\infty} \sum_{\ell=-2}^{1} R^{jp} \big\|\mathcal{E}_{R^j}(\varphi_j f)\big\|_{C^p(B_{R^{j+\ell+1}})}
\end{aligned}$$

$$= C \sum_{j=0}^{\infty} R^{jp} \left\| \mathcal{E}_{R^j}(\varphi_j f) \right\|_{C^p(B_{R^{j+2}})}.$$

By (6.3), since $\varphi_j f$ is supported on the ball of radius R^j,

$$\|\mathcal{E}' f\|'_{(p,\mathbb{R}^m)} \leqslant C_p \sum_{j=0}^{\infty} R^{j(p+q)} \|\varphi_j f\|_{C^q} \leqslant C_p \|f\|_{(p+q,\mathbb{R}^m)}. \qquad \square$$

7. Schwartz extensions of the Gelfand transform of $f \in \mathcal{S}_K(H_n)$

In this section we suppose that K is a closed connected subgroup of $\mathrm{U}(n)$. The following theorem settles the proof of Theorem 1.1 in this case.

Theorem 7.1. *Let f be in $\mathcal{S}_K(H_n)$. For every p in $\mathbb{N}$ there exist F_p in $\mathcal{S}(\mathbb{R}^{d+1})$ and q in $\mathbb{N}$, both depending on p, such that $F_p|_{\Sigma_K^V} = \hat{f}$ and $\|F_p\|_{(p,\mathbb{R}^{d+1})} \leqslant C_p \|f\|_{(q,H_n)}$.*

Notice that this statement implies the existence of a continuous map from $\mathcal{S}(\Sigma_K^V)$ to $\mathcal{S}_K(H_n)$ that inverts the Gelfand transform, even though its formulation is much weaker than that of Theorem 6.1. We do not claim that for each f a single F can be found, all of whose Schwartz norms are controlled by those of f. In addition, our proof does not show if F_p can be chosen to be linearly dependent on f.

The proof of Theorem 7.1 is modelled on that given in [1] for the cases $K = \mathrm{U}(n), \mathbb{T}^n$, but with some relevant differences. On one hand we present a simplification of the argument given there, disregarding the partial results concerning extensions of $\hat{f}$ with finite orders of regularity; on the other hand extra arguments are required in the general setting.

We need to show that the Gelfand transform $\hat{f}$ of $f \in \mathcal{S}_K(H_n)$ extends from Σ_K^V to a Schwartz function on $\mathbb{R}^{d+1}$. Our starting point is the construction of a Schwartz extension to all of $\{0\} \times \mathbb{R}^d$ of the restriction of $\hat{f}$ to

$$\Sigma_0 = \left\{ \widehat{V}(\eta_{Kw}) \colon w \in \mathbb{C}^n \right\} = \{0\} \times \rho\left(\mathbb{C}^n\right).$$

If $\mathcal{F} f$ denotes the Fourier transform in $\mathbb{C}^n \times \mathbb{R}$,

$$\mathcal{F} f(\lambda, w) = \int_{\mathbb{C}^n \times \mathbb{R}} f(t, z) e^{-i(\lambda t + \mathrm{Re}\, z \cdot \bar{w})}\, dw\, dt,$$

we denote

$$\tilde{f}(w) = \mathcal{F} f(0, -w) = \hat{f}\left(0, \rho(w)\right).$$

To begin with, we set

$$\hat{f}^{\sharp} = \mathcal{E}' \tilde{f} \in \mathcal{S}\left(\mathbb{R}^d\right). \tag{7.1}$$

Then $\hat{f}^{\sharp}(\xi) = \hat{f}(0, \xi)$ if $(0, \xi) \in \Sigma_0$.

The next step consists in producing a Taylor development of $\hat{f}$ at $\lambda = 0$. The following result is derived from [14]. In our setting the formula must take into account the extended functions in (7.1).

Proposition 7.2. *Let f be in $\mathcal{S}_K(H_n)$. Then there exist functions f_j, $j \geqslant 1$, in $\mathcal{S}_K(H_n)$, depending linearly and continuously on f, such that for any p in $\mathbb{N}$,*

$$\hat{f}(\lambda,\xi) = \sum_{j=0}^{p} \frac{\lambda^j}{j!} \widehat{f_j}^{\sharp}(\xi) + \frac{\lambda^{p+1}}{(p+1)!} \widehat{f_{p+1}}(\lambda,\xi), \quad \forall(\lambda,\xi) \in \Sigma_K,$$

where $f_0 = f$ and $\widehat{f_j}^{\sharp}$ is obtained from f_j applying (7.1).

Proof. For f in $\mathcal{S}_K(H_n)$, we claim that the restriction of $u(\lambda,\xi) = \hat{f}^{\sharp}(\xi)$ to Σ_K^V is in $\mathcal{S}(\Sigma_K^V)$. It is quite obvious that u is smooth. Let ψ be a smooth function on the line, equal to 1 on $[-2,2]$ and supported on $[-3,3]$. Define

$$\Psi(\lambda,\xi) = \psi\big(\lambda^2 + \xi_1^{2/m_1} + \cdots + \xi_d^{2/m_d}\big) + \psi\left(\frac{\lambda^2}{\xi_1^{2/m_1} + \cdots + \xi_d^{2/m_d}}\right)\big(1 - \psi\big(\lambda^2 + \xi_1^{2/m_1} + \cdots + \xi_d^{2/m_d}\big)\big).$$

By (4.1) and (2) in Lemma 4.1, Ψ is equal to 1 on a neighborhood of Σ_K^V. It is also homogeneous of degree 0 with respect to the dilations (4.4) outside of a compact set. Then Ψu is in $\mathcal{S}(\mathbb{R}^{d+1})$ and coincides with u on Σ_K^V.

It follows from Corollary 5.4 that there exists h in $\mathcal{S}_K(H_n)$ such that

$$\hat{f}(\lambda,\xi) - \hat{f}^{\sharp}(\xi) = \hat{h}(\lambda,\xi) \quad \forall(\lambda,\xi) \in \Sigma_K^V.$$

Since $\hat{h}(0,\rho(w)) = 0$ for every w, $\int_{-\infty}^{+\infty} h(z,t)\,dt = 0$ for every z. Therefore

$$f_1(z,t) = \int_{-\infty}^{t} h(z,s)\,ds$$

is in $\mathcal{S}_K(H_n)$ and

$$\hat{h}(\lambda,\xi) = \lambda \widehat{f_1}(\lambda,\xi) \quad \forall(\lambda,\xi) \in \Sigma_K^V.$$

It is easy to verify that the map $U : f \mapsto f_1$ is linear and continuous on $\mathcal{S}_K(H_n)$. We then define f_j, $j \geqslant 1$, by the recursion formula $f_j = jUf_{j-1}$ and the thesis follows by induction. □

We use now the Whitney Extension Theorem [22] to extend the C^∞-jet $\{\partial_\xi^\alpha \widehat{f_j}^{\sharp}\}_{(j,\alpha)\in\mathbb{N}^{d+1}}$ to a Schwartz function on $\mathbb{R}^{d+1}$. In doing so, we must keep accurate control of the Schwartz norms. For this purpose we use Lemma 4.1 in [1], which reads as follows.

Lemma 7.3. *Let $k \geqslant 1$ and let $h(\lambda, \xi)$ be a C^k-function on $\mathbb{R}^m \times \mathbb{R}^n$ such that*

(1) $\partial_\lambda^\alpha h(0,\xi) = 0$ *for* $|\alpha| \leqslant k$ *and* $\xi \in \mathbb{R}^n$;
(2) *for every* $p \in \mathbb{N}$,

$$\alpha_p(h) = \sup_{|\alpha|+|\beta| \leqslant k} \left\| \left(1 + |\cdot|\right)^p \partial_\lambda^\alpha \partial_\xi^\beta h \right\|_\infty < \infty.$$

Then, for every $\varepsilon > 0$ and $M \in \mathbb{N}$, there exists a function $h_{\varepsilon,M} \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ such that

(1) $\partial_\lambda^\alpha h_{\varepsilon,M}(0,\xi) = 0$ *for every* $\alpha \in \mathbb{N}^m$ *and* $\xi \in \mathbb{R}^n$;
(2) $\sup_{|\alpha|+|\beta| \leqslant k-1} \|(1 + |\cdot|)^M \partial_\lambda^\alpha \partial_\xi^\beta (h - h_{\varepsilon,M})\|_\infty < \varepsilon$;
(3) *for every* $p \in \mathbb{N}$ *there is a constant* $C_{k,p,M}$ *such that*

$$\|h_{\varepsilon,M}\|_{(p,\mathbb{R}^{m+n})} \leqslant C_{k,p,M}\left(1 + \alpha_M(h)^p \varepsilon^{-p}\right) \left\| \left(1 + |\cdot|\right)^p h \right\|_\infty.$$

The following proposition is in [1, Proposition 4.2] for $K = \mathbb{T}^n$. We give here a simplified proof.

Proposition 7.4. *Given $f \in \mathcal{S}_K(H_n)$ and $p \in \mathbb{N}$, there are $H \in \mathcal{S}(\mathbb{R}^{d+1})$ and $q \in \mathbb{N}$ such that $\partial_\lambda^j H(0,\xi) = \widehat{f_j}^\sharp(\xi)$ and*

$$\|H\|_{(p,\mathbb{R}^{d+1})} \leqslant C_p \|f\|_{(q,H_n)}.$$

Proof. Let η be a smooth function on $\mathbb{R}$ such that $\eta(t) = 1$ if $|t| \leqslant 1$ and $\eta(t) = 0$ if $|t| \geqslant 2$. By Theorem 6.1 and Proposition 7.2, for every k and r there exists $q_{k,r}$ such that

$$\left\| \widehat{f_k}^\sharp \right\|_{(r,\mathbb{R}^d)} \leqslant C_{k,r} \|f\|_{(q_{k,r},H_n)}. \tag{7.2}$$

We fix $p \in \mathbb{N}$ and apply Lemma 7.3 to

$$h_k(\lambda,\xi) = \eta(\lambda) \frac{\lambda^{k+1}}{(k+1)!} \widehat{f_{k+1}}^\sharp(\xi).$$

Hypothesis (1) is obviously satisfied and (2) also, because h_k is a Schwartz function. By (7.2),

$$\alpha_r(h_k) \leqslant C_{k,r} \|f\|_{(q_{k+1,r},H_n)}. \tag{7.3}$$

Let q be the maximum among the $q_{k,p}$ for $k \leqslant p+1$. Setting $\varepsilon_k = 2^{-k}\|f\|_{(q,H_n)}$, $M = p$, for each k there is a function $H_k \in \mathcal{S}(\mathbb{R}^{d+1})$ such that

(i) $\partial_\lambda^j H_k(0,\xi) = 0$ for all $j \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$;
(ii) $\sup_{|\alpha|+|\beta| \leqslant k-1} \|(1 + |\cdot|)^p \partial_\lambda^\alpha \partial_\xi^\beta (h_k - H_k)\|_\infty < 2^{-k}\|f\|_{(q,H_n)}$;
(iii) for $k \leqslant p$, using (7.3),

$$\|H_k\|_{(p,\mathbb{R}^{d+1})} \leqslant C_{k,p}\left(1 + \varepsilon_k^{-p} \|f\|_{(q,H_n)}^p\right) \left\| \left(1 + |\cdot|\right)^p h_k \right\|_\infty \leqslant C_p \|f\|_{(q,H_n)}.$$

Define

$$H = \sum_{k=0}^{p} h_k - \sum_{k=0}^{p} H_k + \sum_{k=p+1}^{\infty} (h_k - H_k).$$

By (7.2), (ii) and (iii), the pth Schwartz norm of H is finite and controlled by a constant times the qth Schwartz norm of f. Differentiating term by term, using (i) and the identity $\partial_\lambda^j h_k(0,\xi) = \delta_{j,k+1} \widehat{f_{k+1}}^{\sharp}(\xi)$, we obtain that $\partial_\lambda^j H(0,\xi) = \widehat{f_j}^{\sharp}(\xi)$ for every j. □

Let now φ be a smooth function on $\mathbb{R}$ such that $\varphi(t) = 1$ if $|t| \leqslant 1/2$ and $\varphi(t) = 0$ if $|t| \geqslant 3/4$. For h defined on Σ_K^V, we define the function Eh on $\mathbb{R}^{d+1}$ by

$$Eh(\lambda,\xi) = \begin{cases} \sum_{\alpha\in\Lambda} h(\lambda, \xi_{(\lambda,\alpha)}) \prod_{\ell=1}^{d} \varphi\big(\frac{\xi_\ell}{|\lambda|^{m_\ell}} - \widehat{V_\ell}(\phi_\alpha)\big), & \lambda \neq 0,\ \xi \in \mathbb{R}^d, \\ 0, & \lambda = 0,\ \xi \in \mathbb{R}^d, \end{cases}$$

where $\xi_{(\lambda,\alpha)} = (\widehat{V_1}(\phi_{\lambda,\alpha}), \ldots, \widehat{V_d}(\phi_{\lambda,\alpha})) = (|\lambda|^{m_1}\widehat{V_1}(\phi_\alpha), \ldots, |\lambda|^{m_d}\widehat{V_d}(\phi_\alpha))$.

Recall that, by Lemma 4.1, each $\widehat{V_\ell}(\phi_\alpha)$ is a positive integer. Therefore, if $\xi_\ell \leqslant 0$ for some ℓ, every term in the series vanishes, whereas, if ξ is in $\mathbb{R}_+^d$, the series reduces to at most one single term. Moreover, for every g in $\mathcal{S}(H_n)$, $E\,\widehat{g} = \widehat{g}$ on Σ'.

The proof of the following result goes as for [1, Lemma 3.1], using [6, p. 407] instead of [1, (2.2)]. In contrast with [1] we state it only for vanishing of infinite order of the Taylor development of the Gelfand transform on Σ_0.

Proposition 7.5. *Suppose that g in $\mathcal{S}_K(H_n)$ and $\widehat{g}_{j}{}_{|\Sigma_0} = 0$ for every j. Then*

(1) *$E\,\widehat{g}(\lambda,\xi) = \widehat{g}(\lambda,\xi)$ for all $(\lambda,\xi) \in \Sigma_K^V$;*
(2) *$\partial_\lambda^s(E\,\widehat{g})(0,\xi) = 0$ for all s and $\xi \in \mathbb{R}^d$;*
(3) *for every $p \geqslant 0$ there exist a constant C_p and an integer $q \geqslant 0$ such that*

$$\|E\,\widehat{g}\|_{(p,\mathbb{R}^{d+1})} \leqslant C_p \|g\|_{(q,H_n)}.$$

In particular, $E\,\widehat{g} \in \mathcal{S}(\mathbb{R}^{d+1})$.

To conclude the proof of Theorem 7.1, take f in $\mathcal{S}_K(H_n)$ and p in $\mathbb{N}$. Let H be the function in $\mathcal{S}(\mathbb{R}^{d+1})$, depending on p, defined as in Proposition 7.4. By Theorem 5.2, there exists h in $\mathcal{S}(H_n)$ such that

$$H_{|\Sigma_K^V} = \hat{h}.$$

Define

$$F = E(\hat{f} - \hat{h}) + H,$$

and the thesis follows easily.

8. General compact groups of automorphisms of H_n

We have discussed in the previous sections the Gelfand pairs associated with connected subgroups of $\mathrm{U}(n)$. In this section we only assume that K is a compact group of automorphisms of H_n. Let K_0 be the connected identity component of K. Then K_0 is a normal subgroup of K and $F = K/K_0$ is a finite group. Conjugating K with an automorphism if necessary, we may suppose that K_0 is a subgroup of $\mathrm{U}(n)$.

For D in $\mathbb{D}_{K_0}$ and $w = kK_0$ in F, define D^w by

$$D^w f = D\big(f \circ k^{-1}\big) \circ k \quad \forall f \in C^\infty(H_n).$$

Since K_0 is normal, D^w is in $\mathbb{D}_{K_0}$. It is also clear that $\mathbb{D}_{K_0}$ admits a generating set which is stable under the action of the group F. Indeed, it suffices to add to any given system of generators the F-images of its elements. Denoting by $\mathcal{V}$ the linear span of these generators, F acts linearly on $\mathcal{V}$. Let $V = \{V_1, \ldots, V_d\}$ be a basis of $\mathcal{V}$, orthonormal with respect to an F-invariant scalar product. Clearly, V is a generating set for $\mathbb{D}_{K_0}$.

Applying Hilbert's Basis Theorem as in Section 6, there exists a finite number of (homogeneous) polynomials $\rho_1, \ldots, \rho_r$ generating the subalgebra $P_F(\mathbb{R}^d)$ of F-invariant elements in $P(\mathbb{R}^d)$. Let $\rho = (\rho_1, \ldots, \rho_r) : \mathbb{R}^d \to \mathbb{R}^r$ be the corresponding Hilbert map and let $W_j = \rho_j(V_1, \ldots, V_d)$ for $j = 1, \ldots, r$.

When f is K_0-invariant and $w = kK_0$ in F, we set $f \circ w = f \circ k$.

Lemma 8.1. *The set $W = \{W_1, , \ldots, W_r\}$ generates $\mathbb{D}_K$. Moreover if ψ is a K-spherical function, then*

$$\psi = \frac{1}{|F|} \sum_{w \in F} \phi \circ w, \tag{8.1}$$

for some K_0-spherical function ϕ.

Proof. Take D in $\mathbb{D}_K$. As an element of $\mathbb{D}_{K_0}$, D is a polynomial in the V_j. Averaging over the action of F, we can express D as an F-invariant polynomial in the V_j. Hence D is a polynomial in $W_1 = \rho_1(V_1, \ldots, V_d), \ldots, W_r = \rho_r(V_1, \ldots, V_d)$.

Recall that all K_0- and K-spherical functions are of positive type [2]. Let $\mathcal{P}_K$ (respectively $\mathcal{P}_{K_0}$) denote the convex set of K-invariant (respectively K_0-invariant) functions of positive type equal to 1 at the identity element, and consider the linear map $J : L^\infty_{K_0} \to L^\infty_K$ defined by $J\varphi = \frac{1}{|F|} \sum_{w \in F} \varphi \circ w$. Since $\mathcal{P}_{K_0}$ and $\mathcal{P}_K$ are weak*-compact and J maps $\mathcal{P}_{K_0}$ to $\mathcal{P}_K$, the extremal points of $\mathcal{P}_K$ are images of extremal points of $\mathcal{P}_{K_0}$. This proves that every K-spherical function has the form (8.1).

Conversely, if ψ is given by (8.1) and $D \in \mathbb{D}_K$, then

$$D\psi = \frac{1}{|F|} \sum_{w \in F} D(\varphi \circ w) = \frac{1}{|F|} \sum_{w \in F} (D\varphi) \circ w = \widehat{D}(\varphi)\psi,$$

showing that ψ is K-spherical. $\square$

From Lemma 8.1 we derive the following property of the Gelfand spectra:

$$\rho\big(\Sigma_{K_0}^V\big) = \Sigma_K^W \subset \mathbb{R}^r.$$

If V is as above, the linear action of F on $\mathcal{V}$ leaves $\Sigma_{K_0}^V$ invariant. For a K_0-invariant function f and w in F,

$$\mathcal{G}_V(f \circ w) = (\mathcal{G}_V f) \circ w. \tag{8.2}$$

Let $\mathcal{S}_F(\Sigma_{K_0}^V)$ be the space of F-invariant elements in $\mathcal{S}(\Sigma_{K_0}^V)$.

Lemma 8.2. *The map $f \mapsto f \circ \rho$ is an isomorphism between $\mathcal{S}(\Sigma_K^W)$ and $\mathcal{S}_F(\Sigma_{K_0}^V)$.*

Proof. If f is in $\mathcal{S}(\Sigma_K^W)$, let $\tilde{f}$ be any Schwartz extension of f to $\mathbb{R}^r$. Then $g = \tilde{f} \circ \rho$ is an F-invariant Schwartz function on $\mathbb{R}^d$ and its restriction to $\Sigma_{K_0}^V$ is $f \circ \rho$. This proves the continuity of the map.

Conversely, given g in $\mathcal{S}_F(\Sigma_{K_0}^V)$, let $\tilde{g}$ be an F-invariant Schwartz extension of g to $\mathbb{R}^d$. Set $h = (\mathcal{E}'\tilde{g})_{|_{\Sigma_K^W}}$, where $\mathcal{E}'$ is the operator of Theorem 6.1 for the group F. The proof that the dependence of h on g is continuous is based on the simple observation that, for any Schwartz norm $\| \ \|_{(N)}$ on $\mathbb{R}^d$, the infimum of the norms of all extensions of g is the same as the infimum restricted to its F-invariant extensions. □

We can now prove Theorem 1.1 for general K. Assume that K is a compact group of automorphisms of H_n and let $K_0, F, V_1, \ldots, V_d, \rho$ be as above.

Take f in $L_K^1(H_n)$. Denote by $\mathcal{G}_V f$ (respectively $\mathcal{G}_W f$) its Gelfand transform as a K_0-invariant (respectively K-invariant) function. Then $\mathcal{G}_V f = \mathcal{G}_W f \circ \rho$. In particular, a K_0-invariant function is K-invariant if and only if $\mathcal{G}_V f$ is F-invariant.

If f is in $\mathcal{S}_K(H_n)$, then f is also K_0-invariant and $\mathcal{G}_V f$ is in $\mathcal{S}_F(\Sigma_{K_0}^V)$ by Theorem 7.1; therefore $\mathcal{G}_W f$ is in $\mathcal{S}(\Sigma_K^W)$ by Lemma 8.2.

Conversely, if $\mathcal{G}_W f$ is in $\mathcal{S}(\Sigma_K^W)$, it follows as before that $\mathcal{G}_V f$ is in $\mathcal{S}_F(\Sigma_{K_0}^V)$ and therefore f is in $\mathcal{S}_K(H_n)$ by (8.2).

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