Some rigidity results for non-commutative Bernoulli shifts

Sorin Popa∗,1

Mathematical Department, University of California, Los Angeles, CA 90095-1555, USA

Received 10 June 2005; accepted 21 June 2005
Communicated by Alain Connes
Available online 19 October 2005

Abstract

We introduce the outer conjugacy invariants $\mathcal{I}(\sigma)$, $\mathcal{I}_s(\sigma)$ for cocycle actions $\sigma$ of discrete groups $G$ on type II$_1$ factors $N$, as the set of real numbers $t > 0$ for which the amplification $\sigma^t$ of $\sigma$ can be perturbed to an action, respectively, to a weakly mixing action. We calculate explicitly $\mathcal{I}(\sigma)$, $\mathcal{I}_s(\sigma)$ and the fundamental group of $\sigma$, $\mathcal{F}(\sigma)$, in the case $G$ has infinite normal subgroups with the relative property (T) (e.g., when $G$ itself has the property (T) of Kazhdan) and $\sigma$ is an action of $G$ on the hyperfinite II$_1$ factor by Connes–Størmer Bernoulli shifts of weights $\{t_i\}_i$. Thus, $\mathcal{I}_s(\sigma)$ and $\mathcal{F}(\sigma)$ coincide with the multiplicative subgroup $S$ of $\mathbb{R}_+^*$ generated by the ratios $\{t_i/t_j\}_i, j$, while $\mathcal{I}(\sigma) = \mathbb{Z}_+^*$ if $S = \{1\}$ (i.e. when all weights are equal), and $\mathcal{I}(\sigma) = \mathbb{R}_+^*$ otherwise. In fact, we calculate all the “1-cohomology picture” of $\sigma^t$, $t > 0$, and classify the actions $(\sigma, G)$ in terms of their weights $\{t_i\}_i$. In particular, we show that any 1-cocycle for $(\sigma, G)$ vanishes, modulo scalars, and that two such actions are cocycle conjugate iff they are conjugate. Also, any cocycle action obtained by reducing a Bernoulli action

∗ Fax: +1 310 206 6673.
E-mail address: popa@math.ucla.edu.
1 Supported in part by NSF Grants 9801324 and 0100883.

0022-1236/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jfa.2005.06.017
action of a group $G$ as above on $N = \bigotimes_{g \in G} (M_{r \times n}(\mathbb{C}), \text{tr})_g$ to the algebra $pNp$, for $p$ a projection in $N$, $p \neq 0, 1$, cannot be perturbed to a genuine action.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Cocycles; Bernoulli actions; Property (T) groups

0. Introduction

The study of automorphisms in the theory of von Neumann algebras has often followed problems and phenomena already present in commutative ergodic theory. But starting with the early 1970s, with the advent of Tomita–Takesaki theory and Connes’ ground breaking work on the classification of factors, the full importance and the distinguishing features of this “non-commutative ergodic theory” have become increasingly evident.

A typically non-commutative aspect of the study of automorphisms of a von Neumann algebra $N$ is that such automorphisms are regarded both as elements in the automorphism group of $N$, $\text{Aut} N$, and modulo perturbation by inner automorphisms, in the quotient group $\text{Aut} N / \text{Int} N$. This leads to the study of two types of classification up to conjugacy for automorphisms, in $\text{Aut} N$ and, respectively, in $\text{Aut} N / \text{Int} N$. It also leads to considering cocycles for the corresponding actions.

Along these lines, the more general problem is to study morphisms from groups $G$ into $\text{Aut} N$ and into $\text{Aut} N / \text{Int} N$, up to conjugacy. The former are called actions of $G$ on $N$. Not all faithful group morphisms $\sigma$ from $G$ into $\text{Aut} N / \text{Int} N$ can be lifted to actions. Thus, Nakamura and Takeda have already pointed out in [NT] that to any such morphism $\sigma$ one can associate a scalar 3-cocycle $\mu \in H^3(G, \mathbb{T})$ of the group $G$, and that $\mu$ gives an obstruction for $\sigma$ to be liftable to a genuine action. Also, it was shown in [NT] that the morphism $\sigma$ has trivial obstruction $\mu$ if and only if it can be lifted to a cocycle action.

The classification of cocycle actions up to outer conjugacy, i.e., conjugacy in $\text{Aut} N / \text{Int} N$, particularly in the case $N$ is isomorphic to the hyperfinite type II$_1$ factor $R$, is an important problem in the theory of von Neumann algebras. Part of it is the so-called “vanishing cohomology” problem, asking whether a cocycle action $\sigma$ can be perturbed by inner automorphisms to a genuine action. In other words, whether the 3-cocycle $\mu$ is the only obstruction for a $\sigma$ to be liftable to an action.

The classification (+vanishing cohomology) problem was completely solved in the case of countable amenable groups $G$ and $N \simeq R$: Starting with Connes [C2,C3], for cyclic groups, then Jones [J1], for finite groups, and finally Ocneanu [Oc], for arbitrary amenable groups, it was shown that any two cocycle actions of $G$ on $N = R$ are outer conjugate (in fact even cocycle conjugate). In particular, the vanishing cohomology problem has a positive answer in the case $G$ is amenable and $N \simeq R$ (cf. [Oc]). Furthermore, for the vanishing cohomology problem the condition $N \simeq R$ was removed in [Po1] (see also [Su] for the case $G$ is finite). Thus, all cocycle actions of amenable groups $G$ on arbitrary type II$_1$ factors $N$ can be perturbed by inner automorphisms to genuine actions.
At the opposite end, each non-amenable group $G$ was shown to have at least two non-outer conjugate actions on $N = R$ [J2]. Also, for any infinite group $G$ with the property (T) of Kazhdan, Connes and Jones have found examples of cocycle actions on the free group factor $N = L(\mathbb{F}_\infty)$ that cannot be perturbed to an action [CJ]. But altogether, the problem of classifying certain classes of actions of groups $G$ on $N = R$ beyond the case $G$ amenable (in particular the vanishing cohomology problem) remained wide open.

The purpose of this paper is to approach this problem by introducing and studying new invariants for cocycle actions and by applying them to answer some of the above questions in the case of actions of groups by non-commutative Bernoulli shifts. Thus, we define an outer conjugacy invariant for cocycle actions $\sigma$ of discrete groups $G$ on type $\text{II}_1$ factors $N$, called the spectrum of $\sigma$ and denoted $\mathcal{F}(\sigma)$, as follows: $\mathcal{F}(\sigma)$ is the set of all $t > 0$ for which the cocycle action $\sigma^t$, obtained by amplifying $\sigma$ by $t$, can be perturbed to an actual action. We also consider the strong spectrum, $\mathcal{F}_s(\sigma)$, as the set of all $t > 0$ for which $\sigma^t$ can be perturbed to a weakly mixing action. Thus, $\mathcal{F}(\sigma)$ and $\mathcal{F}_s(\sigma)$ measure the stability with respect to amplifications of the vanishing cohomology properties of $\sigma$.

The amplification of a (cocycle) action is defined only up to inner perturbations: If $0 < t \leq 1$ then $\sigma^t$ is the “compression” of $\sigma$ to $pNp$, for some projection $p \in N$ of normalized trace $\tau(p)$ equal to $t$: choose $w_g \in N$ to be partial isometries with $w_gw_g^*p = p$, $w_g^*w_g = \sigma_g(p)$ then let $\sigma_g^t(x) = w_g(\sigma_g(x))w_g^* \forall x \in pNp$. If $t > 1$ then $\sigma^t$ is the compression of $\text{id}_n \otimes \sigma \in \text{Aut}(\mathbb{C}^n \otimes N)$ to a projection of trace $t/n$, for some $n \geq t$. It is trivial to see that $\sigma^t$ are cocycle actions, i.e., $\sigma_g^t \sigma_h^t$ differs from $\sigma_{gh}^t$ by an inner automorphism $\text{Ad} u_{g,h}$ $\forall g, h \in G$, with the unitary elements $u_{g,h}$ satisfying a 2-cocycle relation with respect to $\sigma$.

The $\mathcal{F}$-invariants satisfy the relations $\mathcal{F}(\sigma^t) = \mathcal{F}(\sigma)/t$ and $\mathcal{F}_s(\sigma^t) = \mathcal{F}_s(\sigma)/t$, for all $t > 0$. These invariants are related to the fundamental group of $\mathcal{F}(\sigma)$, defined as the set of $t > 0$ for which $\sigma^t$ is outer conjugate to $\sigma$. Thus, both $\mathcal{F}(\sigma), \mathcal{F}_s(\sigma)$ are invariant to multiplication by elements in the group $\mathcal{F}(\sigma)$. In particular, if $\mathcal{F}_s(\sigma) = \{1\}$ then $\mathcal{F}(\sigma) = \{1\}$.

Our main result in this paper gives explicit calculations of all these invariants in the case the groups $G$ satisfy a certain “weak rigidity” property and $\sigma$ are actions of $G$ on the hyperfinite type $\text{II}_1$ factor by Connes–Størmer Bernoulli shifts. As an application, we obtain classification results for the actions $\sigma$ and their amplifications $\sigma^t$. For instance, we show that two such actions are cocycle conjugate if and only if they are conjugate. In particular, this shows that any such action has vanishing 1-cohomology, modulo scalars. Until now, such a rigid situation was only known to hold true for actions of finite groups [C3, C4, J1].

The groups $G$ that we consider are required to contain infinite, normal subgroups $H$ with the relative property (T) of Kazhdan–Margulis [Ma] (see A.1 in the Appendix). We call such groups $G$ weakly rigid (w-rigid). Any product of an infinite property (T) group with an arbitrary group is w-rigid. Another classical example is the group $G = \mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z})$, with its subgroup $H = \mathbb{Z}^2$ (cf. [K, Ma]).
A Connes–Størmer (left) Bernoulli action \( \sigma \) of a discrete group \( G \) on the hyperfinite type \( \text{II}_1 \) factor is defined as follows: Let \( (\mathcal{N}, \varphi) = \bigotimes_{g \in G} (M_{k \times k}(\mathbb{C}), \varphi_0)_g \), where \((M_{k \times k}(\mathbb{C}), \varphi_0)_g\) are identical copies of \((M_{k \times k}(\mathbb{C}), \varphi_0)\), \( \varphi_0 \) being a faithful state on \( M_{k \times k}(\mathbb{C}) \) of weights \( \{t_j\}_j \). If \( x = \bigotimes_g x_g \in \mathcal{N} \) and \( h \in G \) then define \( \sigma_h(x) = \bigotimes_g x'_g \), where \( x'_g = x_{h^{-1} g} y_g \). \( \sigma \) is clearly an action of \( G \) on \( \mathcal{N} \), which leaves invariant the state \( \varphi = \bigotimes_g \varphi_g \). Let \( \mathcal{N} = \{ x \in \mathcal{N} \mid \varphi(xy) = \varphi(yx) \ \forall y \in \mathcal{N} \} \) be the centralizer of this state. Thus \( \sigma_g(N) = N \forall g \). The restriction of \( \sigma \) to \( N \), still denoted \( \sigma \), is called the Connes–Størmer Bernoulli \( G \)-action of weights \( \{t_j\}_j \).

Note that if \( D_0 \subset M_{k \times k}(\mathbb{C}) \) is the algebra of diagonal \( k \times k \) matrices then the restriction of \( \sigma \) to \( A = \bigotimes_{g \in G} (D_0, \varphi|_{D_0})_g \subset N \) is equal to the “classical” commutative Bernoulli shift action of \( G \) of weights \( \{t_j\}_j \).

Such actions \( \sigma \) were introduced in [CS], where it was proved that \( N \) is isomorphic to the hyperfinite type \( \text{II}_1 \) factor \( R \) and that the automorphisms \( \sigma_g \) are outer on \( N \ \forall g \neq e \). Note that in case \( k < \infty \) and \( \varphi_0 \) is equal to the trace, i.e., when all \( t_j \) are equal, then \( \mathcal{N} = N \) and \( \sigma \) becomes the usual non-commutative Bernoulli \( G \)-action of multiplicity \( k \). If the weights \( t_j \) are not all equal, then \( \mathcal{N} \) is an ITPF\(1 \) factor of type III (see [P, AW]), with \( \varphi \) an almost periodic state on it. Moreover, \((\mathcal{N}, \varphi)\) has a discrete decomposition with spectrum \( S(\mathcal{N}, \varphi) \) equal to the multiplicative group \( S \subset \mathbb{R}^*_+ \) generated by \( \{t_i/t_j\}_{i,j} \).

0.1. Theorem. Let \( G \) be a \( w \)-rigid group. Let \( \sigma \) be the Connes–Størmer Bernoulli \( G \)-action of weights \( \{t_j\}_j \), on the hyperfinite \( \text{II}_1 \) factor. Let \( S \) be the multiplicative subgroup of \( \mathbb{R}^*_+ \) generated by \( \{t_i/t_j\}_{i,j} \). Then we have:

1. \( \mathcal{F}(\sigma) = \mathcal{F}_s(\sigma) = S \), so that \( \mathcal{F}_s(\sigma') = S/t \forall t > 0 \).
2. If \( S \neq \{1\} \) then \( \mathcal{F}(\sigma) = \mathbb{R}^*_+ \). If \( S = \{1\} \) then \( \mathcal{F}(\sigma) = \mathbb{Z}^*_+ \) and more generally \( \mathcal{F}(\sigma') = \mathbb{Z}^*_+/t \ \forall t > 0 \).

Thus, the above result shows that the outer conjugacy class of \( \sigma \) “remembers” the discrete decomposition of the factor \((\mathcal{N}, \varphi)\) underlying the construction of the Connes–Størmer Bernoulli action \( \sigma \), (which is of type III whenever \( S \neq \{1\} \)), and a fortiori the isomorphism class of the factor \( \mathcal{N} \).

Noticing that any countable subgroup \( S \) of \( \mathbb{R}^*_+ \) can be realized as the multiplicative group generated by some weights \( t_j > 0 \) with \( \Sigma_j t_j = 1 \), Theorem 0.1 allows listing a large class of distinct (up to outer conjugacy) actions of \( G \):

0.2. Corollary. Let \( G \) be a \( w \)-rigid group. Let \( \sigma, \sigma' \) be two Connes–Størmer Bernoulli \( G \)-actions and let \( S, S' \) denote the multiplicative groups generated by the ratios of their corresponding weights. If \( \sigma' \) is outer conjugate to \( \sigma'' \), for some \( t, t' > 0 \), then \( S = S' \) and \( t/t' \in S \). Moreover, if \( \sigma, \sigma' \) are outer conjugate themselves, then they follow conjugate.

Note that if in the construction of the Connes–Størmer Bernoulli shifts \( \sigma_g \) the group \( G \) one starts with is equal to \( \mathbb{Z} \), then \( \sigma \) become the actions studied in [CS], in relation to which Connes and Størmer introduced their notion of entropy, proving that it is a
conjugacy invariant and calculating it in terms of the weights \( \{t_j\}_j \), as \(-\Sigma_j t_j \log t_j\).

Thus, the entropy can distinguish between some of the Connes–Størmer Bernoulli \( \mathbb{Z} \)-actions \( \sigma \), up to conjugacy, but by Connes [C2] these actions are all outer conjugate. Same for \( G \) an arbitrary amenable group, by Kawahigashi [Ka] and respectively, [Oc]. Our result shows that if \( G \) is w-rigid then the Connes–Størmer Bernoulli \( G \)-actions \( \sigma \) become much more rigid, as they are not even outer conjugate. Also, rather than entropical, our invariants \( \mathcal{J}(\sigma), \mathcal{J}_s(\sigma) \) come from the cohomological properties of the action.

When all weights \( t_j \) are taken equal, Theorem 0.1 allows us to provide examples of cocycle actions on the hyperfinite \( \mathbb{II}_1 \) factor that cannot be perturbed to genuine actions, thus answering a problem posed by Connes and Jones [CJ]:

0.3. Corollary. Let \( G \) be a w-rigid group and \( \sigma \) the action of \( G \) on \( N = \mathcal{B}_g(M_n \otimes \mathbb{C}) \), \( \text{tr}_g \) by left Bernoulli shifts, for some \( 2 \leq n < \infty \). If \( t > 0 \) is not an integer then the cocycle action \( \sigma^t \) of \( G \) on \( N^t \cong \mathbb{R} \) cannot be perturbed by inner automorphisms to an action.

To prove Theorem 0.1, we in fact calculate explicitly all \( 1 \) cocycles of amplifications \( \sigma^t, t > 0 \), of Connes–Størmer Bernoulli actions \( \sigma \) of w-rigid groups \( G \). In particular we show that, like in the case of finite groups [Su,J1], any \( 1 \)-cocycle for \( (\sigma,G) \) is equivalent to a character of \( G 

0.4. Corollary. Let \( (\sigma,G), N \) be as in 0.3. Then any weak \( 1 \)-cocycle \( w \) for \( \sigma \) is coboundary modulo scalars, i.e. if \( w : G \to \mathcal{U}(N) \) is so that \( \text{Ad}(w_g) \circ \sigma_g \) is an action of \( G \) on \( N \), then there exists a unitary element \( v \in \mathcal{U}(N) \) such that \( w_g = v^* \sigma_g(v) \mod \mathbb{C} \forall g \in G \). Also, any \( 1 \)-cocycle for \( \sigma \) is cohomologous to a character of \( G \). In particular, if \( G \) has no characters, e.g., if \( G = \text{SL}(n, \mathbb{Z}), n \geq 3 \), then any \( 1 \)-cocycle for \( \sigma \) is trivial.

We also obtain results similar to 0.1–0.4 for actions of w-rigid groups \( G \) by free Bernoulli shifts on the free group factor \( L(\mathbb{F}_\infty) \cong L(\mathcal{F}_G) \).

There are two key properties that enable us to prove these results for non-commutative Bernoulli shifts \( \sigma \) (and similarly for free shifts): On the one hand, the rigidity properties of the groups \( G \) that we consider ensure that the representations of \( G \) obtained from the perturbations of \( \sigma \) by \( 1 \)-cocycles are always “somewhat rigid”. On the other hand, Bernoulli actions \( \sigma \) have good deformation properties, a feature we call mal-leability. More precisely, the “quantum double” \( \sigma \otimes \sigma \) of a Bernoulli action \( \sigma \) admits a “transversal” deformation of that displaces continuously \( N \otimes \mathbb{C} \), from its initial position, until it is “flipped” to \( \mathbb{C} \otimes N \). If one assumes there exist partial isometries \( w_g \) making \( \text{Ad}(w_g) \circ \sigma_g \) an action, then the w-rigidity of \( G \) implies that the deformation of \( w_g \) is locally given by an intertwiner. Patching such “local” intertwiners, one obtains that the flip on \( w \) is implemented by an intertwiner. When properly interpreted, this shows that \( w \) is coboundary.

The paper is organized as follows: In Section 1 we recall some basic definitions of actions, cocycle actions and \( 1 \)-cocycles. In Section 2 we give some examples of
actions, that are to be used in the paper. We also discuss the non-commutative Bernoulli actions and their generalized version considered by Connes and Størmer. In Section 3 we construct cocycle actions by reducing actions by projections (Proposition 3.1). Also, we introduce a class of non-trivial “generalized” 1-cocycles for Connes–Størmer Bernoulli shifts that are needed in the sequel (Theorem 3.2).

In Section 4 we prove the key technical result of this paper, showing that the generalized 1-cocycles constructed in Section 3 give the list of all 1-cocycles of amplifications of Connes–Størmer Bernoulli \( G \)-actions, when \( G \) is w-rigid (4.1, 4.10). In Section 5 we obtain various applications, calculating the spectra \( \mathcal{S} \) and \( \mathcal{S}_s \) of such actions in terms of the weights of shifts (5.5). We also derive a classification result for Connes–Størmer Bernoulli actions (5.6), the existence of cocycle actions that cannot be perturbed to actions (5.10), the vanishing of the 1-cohomology for Bernoulli actions (5.9). In Section 6 we prove the analogue result for free shifts.

1. Preliminaries and notations

Although the results in this paper are stated in the framework of type II\(_1\) factors, their proofs will require some considerations on von Neumann factors of type III\(_\lambda\), \( 0 < \lambda \leq 1 \), with discrete decomposition. Thus, most of the definitions and notations on actions of groups and their cocycles that we recall in this section will be stated in the more general context of arbitrary von Neumann factors.

1.1. Actions and crossed products

We denote by \((\mathcal{N}, \varphi)\) a pair consisting of a von Neumann factor \( \mathcal{N} \) (typically of type II\(_1\) or of type III\(_\lambda\), \( 0 < \lambda \leq 1 \)), with a normal faithful state \( \varphi \) on it (typically a trace or a generalized trace). The centralizer of \( \varphi \) is the set \( \mathcal{N}_\varphi = \{ x \in \mathcal{N} \mid \varphi(xy) = \varphi(yx) \forall y \in \mathcal{N} \} \).

In case the von Neumann algebra \( \mathcal{N} \) is specified to be a type II\(_1\) factor, we will use the notation \( \mathcal{N} \) instead of \( \mathcal{N} \). In this case the state \( \varphi \) will always be taken to be the unique trace \( \varphi \) on \( \mathcal{N} \).

An automorphism of \((\mathcal{N}, \varphi)\) is an automorphism of \( \mathcal{N} \) that preserves \( \varphi \). We denote the group of all such automorphisms of \( \mathcal{N} \) by \( \text{Aut}(\mathcal{N}, \varphi) \). Note that any automorphism of \((\mathcal{N}, \varphi)\) normalizes \( \mathcal{N}_\varphi \).

An automorphism \( \rho \) of \( \mathcal{N} \) is inner if there exists \( u \) in the unitary group of \( \mathcal{N} \), \( \mathcal{U}(\mathcal{N}) \), such that \( \rho(x) = \text{Ad} u(x) = u x u^* \forall x \in \mathcal{N} \). Note that the set \( \text{Int} (\mathcal{N}, \varphi) \) of inner automorphisms of \( \mathcal{N} \) that preserve \( \varphi \) coincides with the set of automorphisms \( \text{Ad} u \) with \( u \in \mathcal{U}(\mathcal{N}_\varphi) \). \( \text{Int} (\mathcal{N}, \varphi) \) is clearly a normal subgroup of \( \text{Aut} (\mathcal{N}, \varphi) \).

An automorphism \( \rho \) of the factor \( \mathcal{N} \) is properly outer (or simply outer) if it is not inner.

Let \( G \) be a discrete group. An action \( \sigma \) of \( G \) on \((\mathcal{N}, \varphi)\) is a group morphism \( \sigma : G \to \text{Aut}(\mathcal{N}, \varphi) \). Note that in case \( \mathcal{N} = \mathcal{N} \) is a type II\(_1\) factor, any automorphism of \( \mathcal{N} \) preserves the trace \( \tau \). We will simply denote by \( \text{Aut} \mathcal{N} \) the set of all automorphisms of \( \mathcal{N} \).
Recall that the crossed product algebra associated with the action $\sigma$ of $G$ on $(\mathcal{N}, \varphi)$, denoted $(\mathcal{N} \rtimes_\sigma G, \varphi)$, is the von Neumann subalgebra of $B(\ell^2(G, L^2(\mathcal{N}, \varphi)))$ generated by the unitaries $u_g \in B(\ell^2(G, L^2(\mathcal{N}, \varphi))), \; g \in G$, where

$$u_g(f)(h) = f(g^{-1}h) \quad \text{for} \; f \in \ell^2(G, L^2(\mathcal{N}, \varphi)),$$

and by a copy of the algebra $\mathcal{N}$ given by

$$(b \cdot f)(g) = \sigma_g^{-1}(b)f(g) \quad \text{for} \; b \in \mathcal{N}, \; f \in \ell^2(G, L^2(\mathcal{N}, \varphi)), \; g \in G.$$

together with the vector state $\varphi(X) = \langle X\xi_\varphi, \xi_\varphi \rangle$, where $\xi_\varphi \in \ell^2(G, L^2(\mathcal{N}, \varphi))$ is the function on $G$ that takes the value $\xi_\varphi$ at $e$ and 0 elsewhere.

The crossed product algebra $\mathcal{N} \rtimes_\sigma G$ can alternatively be viewed as the completion (on bounded sequences) of the Hilbert algebra of finite formal sums $\sum_{g \in G} u_g b_g u_g, \; b_g \in \mathcal{N}$, with multiplication rules $u_g u_h = u_{gh}, \; b u_g = u_g \sigma_g^{-1}(b), \; b = u_e b = 1b$, for $g, h \in G, \; b \in \mathcal{N}$, and $\ast$-operation $(u_g b)^* = u_{g^{-1}} \sigma_g(b^*)$, and with $\mathcal{N}$-valued expectation

$$E(\sum_{g \in G} u_g b_g) \overset{\text{def}}{=} b_e$$

and scalar expectation

$$\varphi\left(\sum u_g b_g\right) \overset{\text{def}}{=} \varphi\left(E\left(\sum u_g b_g\right)\right) = \varphi(b_e).$$

Recall that $(\mathcal{N} \rtimes_\sigma G, \varphi)$ this way defined is itself a von Neumann algebra, with $\varphi$ a faithful normal state. In case $\mathcal{N}$ is a finite von Neumann factor with $\varphi$ the trace, then the crossed product algebra is a finite von Neumann algebra itself, with $\varphi = \varphi \circ E$ a faithful trace on it.

If the action $\sigma$ is properly outer, i.e., if $\sigma_g$ is a properly outer automorphism $\forall g \neq e$, then $\mathcal{N}' \cap \mathcal{N} \rtimes_\sigma G = \mathbb{C}$. In particular, if $\sigma$ is properly outer then $\mathcal{M} = \mathcal{N} \rtimes_\sigma G$ is a factor.

### 1.2. Cocycle actions

A cocycle action $\sigma$ of $G$ on $(\mathcal{N}, \varphi)$ is a map $\sigma : G \to \text{Aut}(\mathcal{N}, \varphi)$ with the property that there exists a map $v : G \times G \to \mathcal{U}(\mathcal{N}_\varphi)$ such that:

$$\sigma_e = \text{id} \quad \text{and} \quad \sigma_g \sigma_h = \text{Ad} \; v_{g,h} \sigma_{gh} \; \forall g, h \in G, \; \text{(1.2.1)}$$

$$v_{g,h} v_{g,h,k} = \sigma_g(v_{h,k}) v_{g,hk} \; \forall g, h, k \in G. \; \text{(1.2.2)}$$

A map $v$ satisfying (1.2.2) is called a 2-cocycle for $\sigma$. The 2-cocycle is normalized if $v_{g,e} = v_{e,g} = 1 \; \forall g \in G$. Note that, since $\mathcal{N}$ is a factor, any 2-cocycle satisfies $v_{e,e} \in \mathbb{C}$. Thus any 2-cocycle $v$ can be normalized by replacing it, if necessary, by $v'_{g,h} = v_{g,e} v_{g,h}, \; g, h \in G$. All 2-cocycles considered from now on will be normalized. Furthermore, all cocycle actions (in particular all actions) that we will consider in this
paper are assumed to be properly outer, i.e., $\sigma_g$ cannot be implemented by unitary elements in $N \forall g \neq e$. Also, when given a cocycle action $\sigma$, we will sometimes specify from the beginning the 2-cocycle it comes with, thus considering it as a pair $(\sigma, v)$.

Note that the 2-cocycle $v$ is unique modulo perturbation by a scalar 2-cocycle $\mu$. More precisely, $v' : G \times G \to \mathcal{U}(N, \varphi)$, with $v'_{e,e} = 1$, satisfies conditions (1.2.1), (1.2.2) if and only if $v' = \mu v$ for some scalar valued function $\mu : G \times G \to \mathbb{T}$ satisfying $\mu_{e,e} = 1$ and

$$\mu_{g,h} \mu_{gh,k} = \mu_{h,k} \mu_{g,hk} \quad \forall g, h, k \in G. \quad (1.2.3)$$

When composed with the quotient map $\pi : \text{Aut}(N, \varphi) \to \text{Aut}(N, \varphi)/\text{Int}(N, \varphi)$ a cocycle action is clearly a (faithful) group morphism. Conversely, if a map $\sigma : G \to \text{Aut}(N, \varphi)$ is such that $\pi \circ \sigma$ is a faithful group morphism into $\text{Out}(N, \varphi) = \text{Aut}(N, \varphi)/\text{Int}(N, \varphi)$ then there exists some $v : G \times G \to \mathcal{U}(N, \varphi)$ such that condition (1.2.1) is satisfied. However, in general condition (1.2.2) may fail to hold true, no matter what $v$ we choose. More precisely, (1.2.2) holds true modulo a scalar-valued map $\nu : G \times G \times G \to \mathbb{T}$ which is a 3-cocycle for $G$. If $N = N_0$ is a II$_1$ factor with $\phi = \tau$ its trace, then by a result of Nakamura and Takeda the map $\sigma$ can be perturbed by inner automorphisms so that to admit a $v$ satisfying both (1.2.1) and (1.2.2) if and only if $v = 1$ in $H^3(G, \mathbb{T})$ [NT]. Also, this condition is proved equivalent to the existence of a crossed product construction $N \subset N \rtimes \sigma G$.

A 2-cocycle $v \subset \mathcal{U}(N, \varphi)$ for the cocycle action $\sigma$ of $G$ on $(N, \varphi)$ is trivial (or it is a coboundary) if there exists a map $w : G \to \mathcal{U}(N, \varphi)$ such that $w_e = 1$ and $v = \partial w$, i.e.:

$$v_{g,h} = (\partial w)_{g,h} \overset{\text{def}}{=} \sigma_g(w_h^*)w_g^*w_{gh} \quad \forall g, h \in G. \quad (1.2.4)$$

The 2-cocycle $v$ is weakly trivial if there exists $w : G \to \mathcal{U}(N, \varphi)$ such that $w_e = 1$ and $v = \partial w$ modulo scalars, i.e.:

$$w_g \sigma_g(w_h) v_{g,h} w_{gh}^* \in \mathbb{C}1 \quad \forall g, h \in G. \quad (1.2.5)$$

Note that this is equivalent to

$$\left(\text{Ad} w_g \sigma_g \right) \left(\text{Ad} w_h \sigma_h \right) = \text{Ad} w_{gh} \sigma_{gh} \quad \forall g, h, \quad (1.2.5')$$

i.e., to $\sigma_g' = \text{Ad} w_g \sigma_g$ being an action.

A weakly trivial 2-cocycle is not necessarily trivial. In fact, if we take any scalar-valued 2-cocycle, then conditions (1.2.1), (1.2.2) and (1.2.5') are satisfied for any genuine action $\sigma$ of $G$ (taking $\mu$ for $v$ and $w = 1$). Thus $\mu$ is a weakly trivial 2-cocycle, in the above sense. But if we take $N = \mathbb{C}1$ it is not always true that
given any \( \mu \) like this there exists some \( w : G \to \mathbb{T} \) such that \( \mu = \hat{\omega} w \). In fact, for most groups \( G \) there do exist scalar 2-cocycles \( \mu \) that are not trivial (or coboundary).

Let us also recall here some well-known vanishing cohomology results. The first result along these lines is due to Connes, who considered the vanishing cohomology and classification problem for actions of the cyclic groups \( \mathbb{Z} \), \( \mathbb{Z}/n\mathbb{Z} \), \( n \geq 2 \), on factors [C2,C3]. A general vanishing cohomology result for arbitrary finite groups was obtained by Sutherland [Su], see also [J1], who proved that any 2-cocycle from a cocycle action of a finite group \( G \) on an arbitrary type II\(_1\) factor \( N \) is coboundary.

In the same vein, Ocneanu proved that any 2-cocycle from a cocycle action of a countable amenable group \( G \) on the hyperfinite type II\(_1\) factor \( R \) is coboundary [Oc]. It was then proved in [Po1] that the same vanishing cohomology result holds true for cocycle actions of amenable groups \( G \) on arbitrary type II\(_1\) factors \( N \). Thus, in particular, any cocycle action of a countable amenable group on an arbitrary type II\(_1\) factor can be perturbed to a genuine action.

Connes and Jones provided the first example of a cocycle action of a countable discrete group \( G \) on a type II\(_1\) factor \( N \) that cannot be perturbed to an action [CJ]. In their example, \( G \) had the property (T) of Kazhdan, with \( N \) being the free group factor \( L(\mathbb{F}_\infty) \) (see A.2 at the end of this paper).

1.3. 1-cocycles for actions

Let us now take \( \sigma \) to be a genuine action of \( G \) on \((N, \varphi)\). A map \( w : G \to \mathcal{U}(N_\varphi) \) satisfying condition

\[
w_g \sigma_g (w_h) = w_{gh} \quad \forall g, h
\]

is called a 1-cocycle for \( \sigma \). Such a 1-cocycle for \( \sigma \) is a coboundary (or it is trivial) if there exists a unitary element \( v \in \mathcal{U}(N_\varphi) \) such that \( w_g = v^* \sigma_g (v) \) \( \forall g \). (Clearly, such maps \( w_g \) do satisfy the 1-cocycle condition (1.3.1)).

The map \( w \) is called a weak 1-cocycle if it satisfies relation (1.3.1) modulo the scalars, i.e.,

\[
w_g \sigma_g (w_h) w_{gh}^* \in \mathbb{T} 1 \quad \forall g, h \in G.
\]

(1.3.1')

Note that this is equivalent to \( \text{Ad} w_g \circ \sigma_g \) being an action. Note also that if \( w \) is a weak 1-cocycle then \( \mu_{g,h} = w_g \sigma_g (w_h) w_{gh}^* \) is a scalar 2-cocycle.

A (weak) 1-cocycle \( w \) is weakly trivial (or weak coboundary) if there exists a unitary element \( v \in \mathcal{U}(N_\varphi) \) such that \( vw_g \sigma_g (v)^* \in \mathbb{T} 1 \forall g \).

Two (weak) 1-cocycles \( w, w' \) of the action \( \sigma \) are equivalent if there exists \( v \in N_\varphi \) such that \( w'_g = vw_g \sigma_g (v)^* \) \( \forall g \in G \) (resp. modulo scalars). Thus, a weak 1-cocycle is weakly trivial iff it is equivalent to a scalar valued weak 1-cocycle (N.B.: this are just plain scalar functions on \( G \)). Note that the scalar-valued genuine 1-cocycles are just characters of \( G \).
Recall that by Connes’ $2 \times 2$ matrix trick [C3] any 1-cocycle for an action of a finite group $G$ on an arbitrary type II$_1$ factor is trivial (the result in [C3] is for $G = \mathbb{Z}/n\mathbb{Z}$, but the proof there goes the same way for $G$ finite; for all the results on finite groups see also [J1]).

1.4. Generalized 1-cocycles

Like in 1.3, let $\sigma$ be a genuine action of the discrete group $G$ on the factor $(\mathcal{N}, \varphi)$. We consider the following general version of 1-cocycles: Let $p$ be a non-zero projection in $N$ and $w : G \rightarrow \mathcal{N}$ be so that $w_g \in \mathcal{N}$ are partial isometries satisfying $w_g w_g^* = p$, $w_g^* w_g = \sigma_g(p)$, $g \in G$, with $w_e = p$. If $w$ satisfies the condition

$$w_g \sigma_g(w_h) = w_{gh} \quad \forall g, h \in G$$

then $w$ is called a generalized 1-cocycle for $\sigma$. If $w$ satisfies the weaker condition

$$w_g \sigma_g(w_h) w_g^* = \mathbb{T} p \quad \forall g, h \in G,$$

then it is called a generalized weak 1-cocycle. Note that (1.4.1') is equivalent to $\sigma_g' = \text{Ad}(w_g) \sigma_g|_{pNp}$ being an action of $G$ on $pNp$. The projection $p$ is called the support of $w$. Also, like for weak 1-cocycles, note that if $w$ satisfies (1.4.1') then the scalar-valued function $\mu_{g,h}$ satisfying $w_g \sigma_g(w_h) = \mu_{g,h} w_{gh}$ is a 2-cocycle. Note that in case its support is equal to 1, a generalized (weak) 1-cocycle $w$ is a (weak) 1-cocycle.

A generalized 1-cocycle $w$ is trivial if there exists a partial isometry $v \in \mathcal{N}$ such that $vv^* = \sigma_g(vv^*)$ and $w_g = v^* \sigma_g(v)$, for all $g \in G$. If the partial isometry $v$ satisfies $vv^* = \sigma_g(vv^*) \quad \forall g$, but $w_g = v^* \sigma_g(v)$ holds true only modulo scalars $\forall g$, then $w$ is weakly trivial.

Note that if $\sigma$ is ergodic then a generalized 1-cocycle of support $\neq 1$ cannot be weakly trivial. Also, note that if $w$ is a generalized (weak) 1-cocycle of support $p$ and $q \in pNp$ is a projection in the fixed point algebra of $\sigma' = \text{Ad} w \circ \sigma$ then $\{qw_g\}_g$ is a generalized (weak) 1-cocycle for $\sigma$ as well.

1.5. Cocycle conjugacy of cocycle actions

Two cocycle actions $(\sigma, v; G)$ of $G$ on $(\mathcal{N}, \varphi)$ and $(\sigma', v'; G')$ of $G'$ on $(\mathcal{N}', \varphi')$ are cocycle conjugate if there exist isomorphisms $\Delta : (\mathcal{N}', \varphi) \simeq (\mathcal{N}', \varphi')$, $\delta : G \simeq G'$ and a map $w : G \rightarrow \mathcal{U}(\mathcal{N})$ such that the following conditions are satisfied:

$$\Delta(\text{Ad}(w_g) \sigma_g) \Delta^{-1} = \sigma'_g \quad \forall g \in G,$$

$$\Delta(w_g \sigma_g(w_h) v_g, h w_g^*) = v'_g, \delta(h) \quad \forall g, h \in G.$$  

We then write $\sigma \sim_c \sigma'$. 

By a well-known observation of Jones [J1], in case the factors $\mathcal{N}, \mathcal{N}'$ are of type II$_1$ with $\varphi, \varphi'$ their corresponding traces, then the cocycle actions $(\sigma, v; G)$, $(\sigma', v'; G')$ are cocycle conjugate if and only if the inclusions of factors $(\mathcal{N} \subset \mathcal{N} \rtimes_{\sigma, v} G)$ and $(\mathcal{N}' \subset \mathcal{N}' \rtimes_{\sigma', v'} G')$ are isomorphic.

The cocycle actions $\sigma, \sigma'$ are outer conjugate (or weakly cocycle conjugate) if condition (1.5.1) is satisfied. We then write $\sigma \sim_o \sigma'$. Note that if $(\mathcal{N}, \varphi) = (\mathcal{N}', \varphi')$ then the outer conjugacy of $\sigma, \sigma'$ is equivalent to the image morphisms of $\sigma, \sigma'$ in $\text{Out}(\mathcal{N}, \varphi) = \text{Aut}(\mathcal{N}, \varphi)/\text{Int}(\mathcal{N}, \varphi)$ being conjugate in $\text{Out}(\mathcal{N}, \varphi)$.

The cocycle actions $\sigma, \sigma'$ are conjugate if both conditions (1.5.1), (1.5.2) are satisfied with $w = 1$. We then write $\sigma \sim \sigma'$.

Recall that Connes proved in [C3] that if $\mathcal{N}$ is isomorphic to the hyperfinite type II$_1$ factor $R$ then any two actions of $\mathbb{Z}/n\mathbb{Z}$ on it are conjugate. Then Jones proved that any two actions of an arbitrary finite group $G$ on $R$ are conjugate. Also, it was proved in [C2] that any two actions of $\mathbb{Z}$ on $R$ are cocycle conjugate. Finally, Ocneanu proved in [Oc] that any two actions of an arbitrary amenable group $G$ on $R$ are cocycle conjugate.

2. Examples of actions

Let us mention some interesting classes of examples of actions and cocycle actions, that will intervene in the sequel.

2.1. Non-commutative Bernoulli actions

Let $(N_0, \tau_0)$ be a finite von Neumann factor with a faithful normal trace $\tau_0$, $\tau_0(1) = 1$. Let $G$ be a discrete group and $(N, \tau) \overset{\text{def}}{=} \bigotimes_{g \in G} (N_0, \tau_0)_g$, where $(N_0, \tau_0)_g = (N_0, \tau_0) \forall g \in G$. We let $\sigma : G \to \text{Aut } N$ be defined by $\sigma_g (\bigotimes_{h \in G} b_{h}) = \bigotimes_{h \in G} b'_h$, where $b_h$ are all but finitely many equal to 1 and $b'_h = b_{e^{-h}} \forall h \in G$, and call it non-commutative (left)-Bernoulli $G$-action. When we need to be more specific, we will say that $\sigma$ is the $(N_0, \tau_0)$-Bernoulli $G$-action. Note that, with this terminology, $\sigma \otimes \Delta$ is the $N_0 \overline{\otimes} N_0$-Bernoulli $G$-action.

It is well known that if either $G$ is infinite and $N_0 \not\cong \mathbb{C}$, or if $G$ is finite and $N_0$ has no atoms, then the Bernoulli $G$-action $\sigma$ is properly outer. Moreover, if $G$ is infinite then $\sigma$ is also ergodic (it is even strongly mixing: see 2.4 hereafter). Also, note that if $N_0$ is a factor (respectively, an approximately finite-dimensional finite von Neumann algebra) then so is $N$. In particular, by taking $N_0$ to be isomorphic to the hyperfinite II$_1$ factor, this shows that any discrete countable group $G$ admits a properly outer (+ ergodic, if $|G| = \infty$) action on $R \simeq \bigotimes_{g \in G} R_g$.

2.2. Connes–Størmer Bernoulli actions

Let us now consider a generalized version of the above example, as introduced in [CS] and as already used in [GoNe,Ka]. To this end, we first need to recall some
basic facts about ITPFI factors and, more generally, about factors having a discrete decomposition [AW,C4,C5,T3].

2.2.0. Factors with discrete decomposition

We consider von Neumann factors with normal faithful states on them, \((N, \varphi)\), satisfying the following conditions:

(a) The centralizer \(N = N_\varphi\) is a \(\Pi_1\) factor and \(N' \cap N = \mathbb{C}\).
(b) If one denotes \(\mathcal{V} = \mathcal{V}(N, \varphi)\), where

\[
\mathcal{V}(N, \varphi) = \{v \in N \mid v^*v = 1, vNv^* = vv^*Nv^*, \varphi(vX) = \varphi(vv^*)\varphi(Xv) \forall X \in N\},
\]

then \(\mathcal{V}\) generates \(N\) as a von Neumann algebra.

If \((N, \varphi)\) satisfies these conditions, then we say that it has a factorial discrete decomposition [C5].

Notice that if condition (a) is satisfied then an isometry \(v \in N\) is in \(\mathcal{V}\) if and only if \(\varphi(vX) = \varphi(vv^*)\varphi(Xv) \forall X \in N\) (see e.g. [C4,C5]). Further, denote \(S_1 = S_1(N, \varphi) = \{\varphi(vv^*) \mid v \in V\}\) and for each \(\beta \in S_1\), put \(\mathcal{V}_\beta = \{v \in V \mid \varphi(vv^*) = \beta\}\). Thus, \(\mathcal{V} = \cup_\beta \mathcal{V}_\beta\). We clearly have \(\mathcal{V}_\beta \mathcal{V}_\gamma = \mathcal{V}_{\beta\gamma}\) and \(\mathcal{V}_1 = \mathcal{U}(N)\). In particular, \(\mathcal{V}_1 \mathcal{V}_\beta \mathcal{V}_1 = \mathcal{V}_\beta\). Moreover, if \(\beta, \beta' \in \mathcal{V}, \beta \leq \beta', \) and \(v \in \mathcal{V}_\beta\), \(v' \in \mathcal{V}_{\beta'}\) are so that \(vv^* \leq v'v'^*\) (which can always be done by multiplying \(v'\) or \(v\) from the left by an appropriate unitary in \(\mathcal{V}_1 = \mathcal{U}(N)\)), then \(v'v^* \) is an isometry in \(\mathcal{V}_{\beta/\beta'}\). In particular, this shows that if \(S = S(N, \varphi)\) denotes the multiplicative group generated by \(S_1\) then \(S = S_1 \cup S_1^{-1}\).

We call \(S(N, \varphi)\) the spectrum of the factorial discrete decomposition \((N, \varphi)\).

Let now \(E = E_N^\varphi\) denote the \(\varphi\)-preserving conditional expectation of \(N\) onto \(N\). If \(v \in \mathcal{V}_\beta, \ v' \in \mathcal{V}_{\beta'}\), with \(\beta \neq \beta'\), then \(E(vv^*) = 0 = E(v'v'^*)\). In turn, if both \(v, v'\) lie in the same \(\mathcal{V}_\beta\) then \(v'v^*\) is a partial isometry lying in \(N\). If in addition, \(v, v'\) have the same left supports then \(v^*v' \in \mathcal{V}_1 = \mathcal{U}(N)\).

With these notations at hand, the discrete decomposition of \((N, \varphi)\) can be made more explicit, as a crossed product type decomposition: For each \(\beta \in S_1\) choose an isometry \(v_\beta \in \mathcal{V}_\beta\); then any \(X \in \mathcal{N}\) has the expansion \(X = \sum_\beta E(Xv_\beta^*)v_\beta + \sum_\beta v_\beta^*E(v_\beta X)\).

The above crossed product type decomposition becomes an actual crossed product if we consider the amplified inclusion \((N^\infty \subset N^\infty) = (N \subset N) \otimes B(\ell^2(N))\): For each \(\beta \in S_1\) there exists a unitary element \(u_\beta \in N^\infty\) such that \(u_\beta N^\infty u_\beta^* = N^\infty\) and \(p_0u_\beta p_0 = v_\beta\), where \(p_0 = 1_N \otimes q_0 \in N^\infty = N \otimes B(\ell^2(N))\), with \(q_0 \in B(\ell^2(N))\) a one-dimensional projection. If we also put \(u_{\beta^{-1}} = u_\beta^*\), for \(\beta \in S_1\), then for each \(\beta \in S\), \(\text{Ad}(u_\beta)\) gives an automorphism \(\theta_\beta\) on \(N^\infty\) that scales the trace \(\text{Tr}\) on \(N^\infty\) by \(\beta\). Moreover, \(\theta_\beta \theta_{\beta'}\) differs from \(\theta_{\beta\beta'}\) by an inner automorphism. Thus, \(\theta\) implements a cocycle action of \(S\) on \(N^\infty\). Since \(N^\infty\) is properly infinite, by perturbing if necessary each \(\theta_\beta\) by an inner automorphism, we may assume \(\theta\) is in fact a genuine action of \(S\) on \(N^\infty\) such that

\[
(N^\infty \subset N^\infty) = (N^\infty \subset (N^\infty \cup \{u_\beta\}_{\beta \in S})^\prime) \simeq (N^\infty \subset N^\infty \rtimes \theta S).
\]
Moreover, via this isomorphism, the state $\varphi$ on $\mathcal{N} = p_0 \mathcal{N}^\infty p_0$ corresponds to $\text{Tr} \circ E(p_0 \cdot p_0)$.

Conversely, if $\mathcal{N}^\infty$ is a type II$_\infty$ factor, $S$ is a countable multiplicative subgroup of $\mathbb{R}_{+}^*$ and $\theta : S \to \text{Aut}\mathcal{N}^\infty$ is an action of $S$ on $\mathcal{N}^\infty$ such that $\text{Tr} \circ E(p_0 \cdot p_0) = \beta \text{Tr}$, then $(\mathcal{N}, \varphi) = (p_0(\mathcal{N}^\infty \times_\theta S)p_0, \text{Tr} \circ E(p_0 \cdot p_0))$, where $p_0 \in \mathcal{N}^\infty$ is a projection of trace 1, has a discrete decomposition.

In terms of modular theory, it is easy to see that a factor $(\mathcal{N}, \varphi)$ has a factorial discrete decomposition if and only if the modular automorphism group $\sigma^\varphi$ associated with $\varphi$ is almost periodic (equivalently, the modular operator $\Delta_\varphi$ has pure point spectrum, i.e., its eigenvectors generate all the Hilbert space $\mathcal{H}_\varphi$, equivalently $\Delta_\varphi$ is diagonalizable) and has factorial fixed point algebra $\mathcal{N}^{\sigma^\varphi} = \mathcal{N}_\varphi$ [C4,C5,A,T2,T3]. Moreover, in this case $S = S(\mathcal{N}, \varphi)$ coincides with the point spectrum of $\Delta_\varphi$.

If $\sigma^\varphi$ is almost periodic and $\mathcal{N}_\varphi$ is not necessarily factorial, then $\mathcal{N}$ is still generated by $\mathcal{N}_\varphi$ and partial isometries $v \in \mathcal{N}$ with $vv^*, v^*v \in \mathcal{N}_\varphi$, $v\mathcal{N}_\varphi v^* = vv^*\mathcal{N}_\varphi vv^*$ and $\varphi(vXv^*) = \beta \varphi(v^*Xv^*)vX \in \mathcal{N}$, where $\beta = \varphi(vv^*)/\varphi(v^*v)$ runs over the spectrum of $\Delta_\varphi$. We still denote by $S(\mathcal{N}, \varphi)$ the spectrum of $\Delta_\varphi$ and call it the spectrum of $(\mathcal{N}, \varphi)$. Examples of factors $(\mathcal{N}, \varphi)$ with almost periodic modular automorphism group $\sigma^\varphi$ but non-factorial fixed point algebra $\mathcal{N}_\varphi$ can be obtained by taking $\mathcal{N}$ to be type I factors and $\varphi$ to be faithful, non-tracial states on $\mathcal{N}$ of weights $\{t_i\}_i$. Thus, $t_i > 0 \forall i$, $\Sigma_i t_i = 1$ and $\varphi(X) = \text{Tr}(XT_{\varphi})$ where $T_{\varphi}$ is a diagonal operator with entries $t_i$. In this case, $S(\mathcal{N}, \varphi)$ is equal to the set of ratios $\{t_i/t_j\}_{i,j}$ and in general may not be a group.

2.2.1. Tensor products of factors with discrete decomposition

The simplest case of a factor with discrete decomposition is $(\mathcal{N}, \varphi) = (\mathcal{N}, \tau)$, with $\mathcal{N}$ a type II$_1$ factor and $\tau$ its trace. Note that in this case $S = \{1\}$ and $\mathcal{N}$ equals the set of unitary elements in $\mathcal{N}$. Conversely, if $(\mathcal{N}, \varphi)$ has a discrete decomposition with $S(\mathcal{N}, \varphi) = \{1\}$, then $\mathcal{N}$ is a type II$_1$ factor and $\varphi$ its unique trace.

Another example is when $(\mathcal{N}, \varphi)$ is a type III$_\lambda$ factor, $0 < \lambda < 1$, with $\varphi$ its generalized trace (cf. [C4,C5]). In this case $S = \{\lambda^n \mid n \in \mathbb{Z}\}$ and $(\mathcal{N}, \varphi)$ is of the form $(p_0(\mathcal{N}^\infty \times_\theta S)p_0, \text{Tr} \circ E(p_0 \cdot p_0))$, for some type II$_\infty$ factor $\mathcal{N}^\infty$, $\theta$ an automorphism of $\mathcal{N}^\infty$ which scales the trace $\text{Tr}$ by $\lambda$ and $p_0 \in \mathcal{N}^\infty$ a projection with $\text{Tr}(p_0) = 1$. Conversely, if $(\mathcal{N}, \varphi)$ has a discrete decomposition and $S = \lambda^Z$ then $\mathcal{N}$ is of type III$_\lambda$ and $\varphi$ is its generalized trace.

In case $(\mathcal{N}, \varphi)$ has a discrete decomposition but $S \subset \mathbb{R}_{+}^*$ is not single generated, then $\mathcal{N}$ is of type III$_1$. As we saw in 2.2.0, such factors are of the form $\mathcal{N}^\infty \times S$ with $S$ acting on the type II$_\infty$ factor $\mathcal{N}^\infty$ by trace scaling automorphisms. Such examples can also be obtained by taking tensor products of type III$_\lambda$ factors:

**Proposition.** Let $(\mathcal{N}_j, \varphi_j)$, $j \geq 1$, be a finite or infinite sequence of von Neumann factors.

(a) If the modular automorphism groups $\sigma^{\varphi_j}$ are almost periodic $\forall j \geq 1$, then $(\mathcal{N}, \varphi) = \overline{\otimes}_j (\mathcal{N}_j, \varphi_j)$ has almost periodic modular automorphism group as well. Moreover, the spectrum $S(\mathcal{N}, \varphi)$ is equal to product set $S(\mathcal{N}_1, \varphi_1) \cdot S(\mathcal{N}_2, \varphi_2) \cdot \ldots$. 

(b) If \((N_j, \varphi_j)\) have factorial discrete decomposition \(\forall j\), then \((N, \varphi) = \bigotimes_j (N_j, \varphi_j)\) has factorial discrete decomposition as well.

**Proof.** (a) It is easy to see that the modular operator \(\Delta_{\varphi}\) is equal to the (possibly infinite) tensor product operator \(\bigotimes_j \Delta_{\varphi_j}\), which is defined in the obvious way on the Hilbert space \(\overline{\bigotimes}_j (L^2(N_j, \varphi_j), \hat{I}_{N_j})\) (we use that \(\Delta_{\varphi_j}(\hat{I}_{N_j}) = \hat{I}_{N_j} \forall j\), to make sense of this infinite tensor product of positive unbounded operators). Since each \(\Delta_{\varphi_j}\) is a diagonal operator, their tensor product is diagonal as well. Moreover, the diagonal entries of \(\Delta_{\varphi_j}\) are obtained as finite products of diagonal entries of the \(\Delta_{\varphi_j}'s\). This shows that \(S(N, \varphi) = S(N_1, \varphi_1) \cdot S(N_2, \varphi_2) \cdots\).

(b) For each \(j \geq 1\) let \(N_j = (N_j)_{\varphi_j}\) and \(S^j = S(N_j, \varphi_j)\). Let also \(N = \bigotimes_j N_j\) (this is a tensor product of type \(\text{II}_1\) factors with respect to their tracial states). We clearly have \(N \subset N^\varphi\). Since \(N_j' \cap N_j = 0 \forall j\), this implies that \(N_0' \cap N' \subset N_j' \cap N = \mathbb{C}\). In particular, \(N_0^\varphi\) is a factor. By (a), this finishes the proof. □

### 2.2.2. ITPF1 factors and their tensor powers

We recall in this section a well-known construction of von Neumann factors due to Powers [P] and Araki–Woods [AW]. A particular case of this construction will provide us with a large class of concrete examples of factors with a discrete decomposition. Thus, an ITPF1 factor is a von Neumann algebra \((N_0, \varphi_0)\) of the form \(\bigotimes_{j \in J} (N_{0,j}, \varphi_{0,j})\), with \(J\) a finite or infinite set of indices and \((N_{0,j}, \varphi_{0,j})\) a type \(\text{I}_1\) von Neumann factor with a normal faithful state on it \(\forall j \in J\). (N.B. Such tensor products von Neumann algebras \((N_0, \varphi_0)\) are factors by [P] or [AW]; their factoriality follows also by part (b) of the Proposition below).

From now on, when given an infinite countable set of indices \(K\) (typically, \(K = G\) for some group \(G\)) and a von Neumann algebra with a normal faithful state \((M_0, \psi_0)\), we denote by \((M_0, \psi_0)_k, k \in K, \) identical copies of \((M_0, \psi_0)\) indexed by the elements of the set \(K\). Moreover, we denote by \(\bigotimes_{k \in K} (M_0, \psi_0)_k\) the von Neumann tensor product of infinitely many identical copies of \((M_0, \psi_0)\) indexed by \(K\). The next result shows that such a tensor product of infinitely many copies of an ITPF1 factor always has factorial discrete decomposition.

**Proposition.** Let \(J\) be a set (finite or infinite) and for each \(j \in J\) let \((N_{0,j}, \varphi_{0,j})\) be a type \(\text{I}_{k_j}\) factor, for some \(2 \leq k_j \leq \infty\), with \(\varphi_{0,j}\) a normal, faithful state on it. Let \((N_0, \varphi_0) = \bigotimes_J (N_{0,j}, \varphi_{0,j})\) be the corresponding ITPF1 factor. Then we have:

(a) \((N_0, \varphi_0)\) has almost periodic modular automorphism group \(\sigma^{\varphi_{0,j}}\) and it is an approximately finite dimensional (or hyperfinite) factor. Moreover, its spectrum \(S(N_0, \varphi_0)\) is equal to the product set \(\prod_{j} S(N_{0,j}, \varphi_{0,j})\).

(b) If \(K\) is an infinite set, then \((N, \varphi) = \bigotimes_{k \in K} (N_0, \varphi_0)_k\) has factorial discrete decomposition and \(S(N, \varphi)\) is equal to the multiplicativ group generated by \(S(N_0, \varphi_0)\). Moreover, \((N, \varphi)\) is an ITPF1 factor itself and if \(K\) is countable then its centralizer \(N^\varphi\) is isomorphic to the hyperfinite \(\text{II}_1\) factor \(R\).
Proof. (a) The von Neumann algebra \((\mathcal{N}_0, \varphi_0)\) is clearly approximately finite dimensional by construction. By part (a) of Proposition 2.2.1, \(\sigma^{\varphi_0}\) is almost periodic, with \(S(\mathcal{N}_0, \varphi_0)\) equal to the product of the sets \(S(\mathcal{N}_0, \varphi_0, j), j \in J\).

(b) If \((\mathcal{N}_0, \varphi_0)\) is of type I, then \((\mathcal{N}, \varphi) = \bigotimes_{k \in K}(\mathcal{N}_0, \varphi_0)_k\) has factorial centralizer by Connes and Størmer [CS]. Since by part (a) the modular group \(\sigma^\varphi\) is almost periodic, by 2.2.0 it follows that \((\mathcal{N}, \varphi)\) has factorial discrete decomposition in this case.

Assume now that we have no restrictions on \((\mathcal{N}_0, \varphi_0)\). Since one has the isomorphism

\[
(\mathcal{N}, \varphi) = \bigotimes_{k \in K}(\mathcal{N}_0, \varphi_0, j)_k \cong \bigotimes_{j}(\bigotimes_{k \in K}(\mathcal{N}_0, \varphi_0, j)_k)_j
\]

and since by the first part of the proof each \(\bigotimes_{k \in K}(\mathcal{N}_0, \varphi_0, j)_k\) is a factor with factorial discrete decomposition, by part (b) of Proposition 2.2.1 it follows that \((\mathcal{N}, \varphi)\) has factorial discrete decomposition as well.

Finally, \(\mathcal{N}_\varphi\) follows injective, as a fixed point algebra of the action of the amenable group \(\sigma^\varphi\) on the injective algebra \(\mathcal{N}\). By Connes’ Theorem [C1], when separable \(\mathcal{N}_\varphi\) is isomorphic to the hyperfinite \(\text{II}_1\) factor. □

Note that \(\mathcal{N}\) above is a type \(\text{II}_1\) factor iff all \(\mathcal{N}_0, j\) are finite type I factors with \(\varphi_0, j\) their traces (i.e., \(k_j < \infty\) and \(t^j_1 = \cdots = t^j_{k_j}, \forall j\)).

2.2.3. Generalized Bernoulli actions

Let now \((\mathcal{N}_0, \varphi_0)\) be an \(\text{ITP}\) factor and \(G\) an infinite countable group and \(K\) a set on which \(G\) acts. Let \(\sigma : G \to \text{Aut}(\mathcal{N}, \varphi)\) be defined as follows: First define \(\sigma\) on the algebraic infinite tensor product \(\bigotimes_k(\mathcal{N}_0)_k\) by \(\sigma_x(\bigotimes_k x_k) = \bigotimes_k x'_k\), where \(x'_k = x_{g^{-1}k}\); then note that on this dense subalgebra of \(\mathcal{N}\) we have \(\varphi \circ \sigma_x = \varphi \forall g\); thus each \(\sigma_x\) can be extended to a \(\varphi\)-preserving automorphism \(\sigma_g\) on all \(\mathcal{N}\). We call \(\sigma\) the \((\mathcal{N}_0, \varphi_0)\)-Bernoulli \((G \curvearrowright K)\)-action. Generically, such actions are also referred to as generalized Bernoulli \(G\)-actions. In case \(K = G\) and \(G \curvearrowright G\) is the left multiplication, we simply call \(\sigma\) the \((\mathcal{N}_0, \varphi_0)\)-Bernoulli \(G\)-action.

Note that since a generalized Bernoulli action \(\sigma\) leaves \(\varphi\) invariant, it leaves invariant the discrete decomposition structure of \((\mathcal{N}, \varphi)\). Namely, it leaves invariant \(N = \mathcal{N}_\varphi\) and each one of the sets of isometries \(\mathcal{V}_\beta\forall \beta \in S_1(\mathcal{N}, \varphi)\). In fact, if \(v \in \mathcal{V}\) then \(w_\beta = v \sigma_\beta(v^*)\) are partial isometries in \(N\) with \(\sigma_\beta(v) = w_\beta v\).

Note also that a tensor product of generalized Bernoulli actions is a generalized Bernoulli action as well, more precisely, if \(\sigma_j\) are \((\mathcal{N}_0, \varphi_0, j)\)-Bernoulli \((G \curvearrowright K)\)-actions, \(j \geq 1\), then \(\bigotimes_j \sigma_j\) is the \(\bigotimes_j(\mathcal{N}_0, \varphi_0, j)\)-Bernoulli \((G \curvearrowright K)\)-action.

Finally, note that if \(\sigma\) is the \((\mathcal{N}_0, \varphi_0)\)-Bernoulli \((G \curvearrowright K)\)-action and \(H \subset G\) is an infinite subgroup, then \(\sigma|_H\) coincides with the \((\mathcal{N}_0, \varphi_0)\)-Bernoulli \((H \curvearrowright K)\)-action.

2.2.4. Connes–Størmer Bernoulli actions

Let \((\mathcal{N}_0, \varphi_0)\) be an \(\text{ITP}\) factor and \(\sigma\) be the \((\mathcal{N}_0, \varphi_0)\)-Bernoulli \((G \curvearrowright K)\)-action of the group \(G\) on the factor \((\mathcal{N}, \varphi) = \bigotimes_k(\mathcal{N}_0, \varphi_0)_k\), as in 2.2.3. Let \(N = \mathcal{N}_\varphi \subset \mathcal{N}\).
Since $\varphi$ is invariant to $\sigma$, $\sigma^g(N) = N \forall g$. By Proposition 2.2.2 $N$ is isomorphic to the hyperfinite type $II_1$ factor $R$.

Moreover, by arguing as in [CS] it is immediate to see that the action $\sigma_g = \sigma_{g|N}$, $g \in G$, is properly outer on $N$ whenever $G \rtimes K$ satisfies the following proper outerness condition:

$$||\{k \in K \mid gk \neq k\}| = \infty \ \forall g \in G \setminus \{e\}. \quad (2.2.4')$$

We call $\sigma|_N$ the $(N_0, \varphi_0)$-Connes–Størmer (abbreviated CS Bernoulli $(G \rtimes K)$-action (resp. $G$-action, if $K = G$ with $G$ acting by left multiplication). In the particular case when $N_0$ is a type I$_k$ factor and the state $\varphi_0$ on $N_0$ is given by the trace class operator with diagonal entries $\{t_j\}_j$, we call $\sigma|_N$ the CS Bernoulli $(G \rtimes K)$-action of weights $\{t_j\}_j$. The multiplicative group $S$ generated by $S(N_0, \varphi_0)$ (a set that coincides with $\{t_i/t_j\}_{i,j}$ when $N_0$ is type I) is called the ratio group of the $(N_0, \varphi_0)$-CS Bernoulli action $\sigma$.

Recall that if $\sigma$ is a $(N_0, \varphi_0)$-Bernoulli $(G \rtimes K)$-action and $\rho$ is a $(M_0, \psi_0)$-Bernoulli $(G \rtimes K)$-action, then $\sigma \otimes \rho$ is a $(N_0 \otimes M_0, \varphi_0 \otimes \psi_0)$-Bernoulli $(G \rtimes K)$-action. However, if we let $N$ (respectively $M$, $N_0$), be the centralizer of $\varphi$ (respectively $\psi$, $\varphi \otimes \psi$), and denote by $\theta_\varphi$ the CS Bernoulli $(G \rtimes K)$-action obtained by restricting $\sigma^g \otimes \rho^g$ to $\tilde{N}$, then the restriction of $\theta$ to $N \otimes M(\subset \tilde{N})$ equals $\sigma|_N \otimes \sigma|_M$. But in general $\tilde{N}$ is different from $N \otimes M$, and a fortiori $\theta \neq \sigma|_N \otimes \rho|_M$ as well.

In fact, it is easy to see that $\tilde{N} = N \otimes M$ iff $S(N_0, \varphi_0) \cap S(M_0, \psi_0) = \{1\}$. Thus, if we take $\rho = \sigma$, then we have $\tilde{N} = N \otimes N$ iff $S(N_0, \varphi_0) = \{1\}$, i.e., iff $N_0$ is a finite ITPF1 factor (so either $N_0 \simeq R$, or $N_0 \simeq M_{k \times k}(\mathbb{C})$, for some $2 \leq k < \infty$), with $\varphi_0$ its trace. Moreover, if $S(N_0, \varphi_0) = \{1\}$, then the $(N_0 \otimes M_0, \varphi_0 \otimes \psi_0)$ CS Bernoulli action coincides with the tensor product between the $(N_0, \varphi_0)$ CS Bernoulli action and the $(M_0, \psi_0)$ CS Bernoulli action, for any ITPF1 factor $(M_0, \psi_0)$.

Finally, note that if $H \subset G$ is an infinite subgroup then $\sigma|_H$ coincides with the $(N_0, \varphi_0)$ CS Bernoulli $(H \rtimes K)$-action on the hyperfinite factor, which is properly outer whenever $H \rtimes K$ satisfies $||(k \in K \mid hk \neq k)|| = \infty$.

2.3. Free Bernoulli shifts

Let $(B_0, \tau_0)$ be a finite von Neumann algebra with a finite faithful normal trace on it. As usual, let $\{(B_0, \tau_0)_g\}_g \in G$ be copies of $(B_0, \tau_0)$ indexed by the discrete group $G$. Define $(N, \tau) \equiv *_{g \in G} (B_0, \tau_0)_g$ to be the free product of the algebras $(B_0, \tau_0)_g$ (see [V1, Po2]). Then we define $\sigma^g (*_h b_h) = *_h b^g_h$, where $b^g_h = b^{-1}_g h$. Such an action is called a (left) free Bernoulli $G$-action.

Note that by Popa [Po5], Dykema [Dy2], if $G$ is infinite and $B_0 \neq \mathbb{C}$ then $N$ is a factor. Moreover, $\sigma$ is then easily seen to be properly outer and ergodic. Also, by Dykema [Dy1], if $B_0$ is AFD, or if $B_0$ is a free group factor, then $N$ is isomorphic to the free group factor $L(\mathbb{F}_\infty)$. Thus, any discrete countable group $G$ acts properly outer (+ ergodic, if $|G| = \infty$) on the free group factor $L(\mathbb{F}_\infty)$.
Note that one can also define a “free version” of the CS Bernoulli actions, by using Shlyakhtenko’s free version of the ITPF1 factors [Sh]. Thus, one starts with a fixed finite-dimensional von Neumann algebra with a faithful state on it, \((B_0, \varphi_0)\). One defines \((N, \varphi) \overset{\text{def}}{=} \bigotimes_{g \in G} (B_0, \varphi_0)_g\), noticing that \(\varphi\) is quasi-periodic on \(N\) [Sh], being a generalized trace if \(B_0 = M_{2 \times 2}(\mathbb{C})\). One takes \(N\) to be the centralizer of the state \(\varphi\). Then one considers the free Bernoulli action \(\sigma\) on \(N\) (which we could call a generalized free Bernoulli action), noticing that it preserves \(\varphi\). Finally, one takes the free CS Bernoulli \(G\)-action to be the restriction of \(\sigma\) to \(N\), which by [Dy2,Sh] is isomorphic to \(L(\mathbb{F}_\infty)\).

2.4. Bernoulli actions are mixing

Both the “hyperfinite” (2.1, 2.2) and the “free” (2.3) Bernoulli actions are “very ergodic”. More precisely, they are weakly mixing, or even strongly mixing, under very general assumptions on \(G\). More precisely, they are weakly mixing, or even strongly mixing, under very general assumptions on \(G\) \(\cap\) \(K\). We recall here the appropriate definitions and prove this property holds for such shifts, for the reader’s convenience.

2.4.1. Definition. Let \((N, \varphi)\) be as usual a von Neumann algebra with a faithful normal state \(\varphi\). Let \(\sigma : G \to \text{Aut}(N, \varphi)\) be an action of the discrete group \(G\) on \((N, \varphi)\). The action \(\sigma\) is weakly mixing if \(\forall F \subset N\) finite and \(\forall \varepsilon > 0\), there exists \(g \in G\) such that

\[
|\varphi(\sigma_g(x)y) - \varphi(x)\varphi(y)| < \varepsilon \quad \forall x, y \in F.
\]

Weakly mixing actions are clearly ergodic. In fact their ergodicity is so “strong” that it goes through to tensor products. Moreover, weakly mixing actions have no invariant finite-dimensional subspaces other than \(\mathbb{C}1\):

2.4.2. Proposition. Let \(\sigma\) be an action of the group \(G\) on the von Neumann algebra \((N, \varphi)\). The following conditions are equivalent:

(i) \(\sigma\) is weakly mixing.

(ii) For any action \(\sigma_0\) of \(G\) on a von Neumann algebra \((N_0, \varphi_0)\), the action \(\theta'_g = \sigma_g \otimes \sigma_{0,g}, g \in G\), of \(G\) on \(N \otimes N_0\) satisfies \((N \otimes N_0)^{\theta'_g} = \mathbb{C} \otimes N_0^{\sigma_0}\).

(iii) The only finite dimensional vector subspace of \(L^2(N, \varphi)\) invariant to all \(\sigma_g, g \in G\), is \(\mathbb{C}1\).

Proof. (i) \(\Rightarrow\) (ii). Let \(\varphi' = \varphi \otimes \varphi_0\). Let \(x \in (N \otimes N_0)^{\varphi'}\), \(x \neq 0\) and \(\varepsilon > 0\). Denote by \(Y_0\) the \(\varphi'\)-preserving expectation of \(x\) onto \(\mathbb{C} \otimes N_0\) and \(X_0 = x - Y_0\). Note that \(Y_0 \in \mathbb{C} \otimes N_0^{\sigma_0} \subset (N \otimes N_0)^{\varphi'}\). Thus \(X_0\) belongs to \((N \otimes N_0)^{\varphi'}\) as well.

By the density of the algebra \(N \otimes N_0\) in \(N \otimes N_0\), there exists an orthonormal system \(x_0 = 1, x_1, x_2, \ldots, x_n \in N\) (with respect to the scalar product given by \(\varphi\)) and elements \(y_0 = Y_0, y_1, y_2, \ldots, y_n \in N_0\) such that if we denote \(x' = \sum_{i=0}^n x_i \otimes y_i\) then we have

\[
\|x - x'|_{\varphi'} < \varepsilon/(3\|x\|_{\varphi'}) \quad \text{and} \quad \|x'|_{\varphi'} \leq \|x\|_{\varphi'}.
\]
Since $\sigma$ is weakly mixing, there exists $g \in G$ such that

$$\sum_{i,j=1}^{n} |\phi(\sigma_{g}(x_{i})x_{j}^{*})| |\phi_{0}(\sigma_{0,g}(y_{i})y_{j}^{*})| < \varepsilon/3.$$ 

Thus, if we denote $X_{0}' = \sum_{i=1}^{n} x_{i} \otimes y_{i} = x' - Y_{0}$ then we have

$$|\phi'(\theta_{g}^{*}(X_{0}')X_{0}')| < \varepsilon/3 \quad \text{and} \quad \|X_{0} - X_{0}'\|_{\phi'} = \|x - x'\|_{\phi'} < \varepsilon/(3\|x\|_{\phi'}).$$

As a consequence we get

$$\|X_{0}\|_{\phi'}^{2} = \phi'(\theta_{g}^{*}(X_{0}')X_{0}) \leq |\phi'(\theta_{g}'(X_{0}')X_{0}')| + 2\|X_{0} - X_{0}'\|_{\phi'}\|X_{0}\|_{\phi'} < \varepsilon.$$ 

Since $\varepsilon > 0$ was arbitrary, it follows that $X_{0} = 0$ so that $x = Y_{0}$ belongs to $\mathbb{C} \otimes N_{0}^{\sigma_{0}}$.

(ii) $\implies$ (iii). By applying (ii) to $\sigma_{0} = \sigma$ and by taking into account that the Hilbert space of Hilbert–Schmidt operators on $L^{2}(N, \phi)$, with the action $\tilde{\sigma}$ implemented by $\sigma$ on it, can be naturally identified with $L^{2}(N, \phi) \otimes L^{2}(N, \phi)$, with the action $\sigma \otimes \sigma$ on it, it follows that the one-dimensional projection of $L^{2}(N, \phi)$ onto $\mathbb{C}1$ is the only fixed point for the former.

(iii) $\implies$ (i). It is clearly sufficient to check condition 2.4.1 for finite sets $F = F^{*} \subset N$ with $\phi(x) = 0\forall x \in F$. Let $\varepsilon > 0$. Let $H_{0} \subset L^{2}(N, \phi) \otimes \mathbb{C}1$ be the linear span of $F$ and $p_{0}$ the orthogonal projection onto $H_{0}$, regarded as an element in the Hilbert space $H$ of Hilbert–Schmidt operators on $L^{2}(N, \phi)$. Since $L^{2}(N, \phi) \otimes \mathbb{C}1$ has no non-zero finite-dimensional subspaces invariant to $\sigma$ it follows that $\forall \delta > 0$, $\exists g \in G$ such that $\text{Tr}(\tilde{\sigma}_{g}(p_{0})p_{0}) < \delta$.

Indeed, because if there would exist some $\delta_{0} > 0$ such that $\text{Tr}(\tilde{\sigma}_{g}(p_{0})p_{0}) \geq \delta_{0} \forall g \in G$, then for any $y$ in the weak closure of the convex hull $K_{p_{0}} \subset \mathcal{H}$ of $\{\tilde{\sigma}_{g}(p_{0})\}_{g}$ we would still have $\text{Tr}(yp_{0}) \geq \delta_{0}$. In particular, this would happen for the unique element $y_{0} \in K_{p_{0}}$ of minimal norm $\|\cdot\|_{2,\text{Tr}}$. But since $\|\tilde{\sigma}_{g}(y_{0})\|_{2,\text{Tr}} = \|y_{0}\|_{2,\text{Tr}}$, it follows that $\tilde{\sigma}_{g}(y_{0}) = y_{0} \forall g \in G$. This implies that any spectral projection of $y_{0} \geq 0$ is invariant to $\tilde{\sigma}$. By (iii) any such projection orthogonal to $\mathbb{C}1$ is equal to 0. Thus $y_{0} = 0$, contradicting $\text{Tr}(y_{0}p_{0}) \geq \delta_{0} > 0$. But if $\text{Tr}(\tilde{\sigma}_{g}(p_{0})p_{0}) < \delta$ for some $g \in G$ and for a sufficiently small $\delta > 0$, then this implies $\phi(\sigma_{g}(x)y) < \varepsilon \forall x, y \in F = F^{*}$.

2.4.3. Lemma. Let $(N_{0}, \phi_{0})$ be an ITPF1 factor. Let $G \curvearrowright K$ be an action of an infinite group $G$ on a countable set $K$ and denote by $\sigma$ the $(N_{0}, \phi_{0})$–Bernoulli $(G \curvearrowright K)$–action, as in 2.2.2. Then $\sigma$ is weakly mixing if and only if the following condition holds true:

$$\forall K_{0} \subset K \text{ finite, } \exists g \in G \text{ such that } gK_{0} \cap K_{0} = \emptyset. \quad (2.4.3')$$
Moreover, if this is the case then there exists a sequence $g_n \in G$ such that:

$$\lim_{n \to \infty} \phi(y_1 \sigma_{g_n}(x), y_2) = \phi(x) \phi(y_1, y_2) \quad \forall x, y_1, y_2 \in N$$

(a)

Also, $\sigma$ is strongly mixing if and only if the following condition holds true:

$$\forall K_0 \subset K \text{ finite}, \exists F \in G \text{ finite such that } gK_0 \cap K_0 = \emptyset \quad \forall g \in G \setminus F \quad (2.4.3')$$

and if this condition is satisfied then one actually has

$$\lim_{g \to \infty} \phi(y_1 \sigma_g(x), y_2) = \phi(x) \phi(y_1, y_2) \forall x, y_1, y_2 \in N.$$  

(b)

**Proof.** The fact that $\sigma$ weak mixing (respectively, strongly mixing) implies condition (2.4.3') (respectively (2.4.3'')) is trivial.

Let us prove that (2.4.3') implies $\sigma$ weak mixing. Let $(N, \phi) = \bigotimes_{k \in K} (N_0, \phi_0)_k$. Let $K_n \not\supset K$ be a sequence of finite subsets exhausting $K$ and for each $n$ let $g_n \in G$ be so that $g_nK_n \cap K_n = \emptyset$. We will show that $g_n$ satisfy (a). By using first the Kaplansky density theorem then the Cauchy–Schwartz inequality, it follows that it is sufficient to prove (a) for $y_{1,2}$ in the algebraic tensor product $\bigotimes_k (N_0, \phi_0)_k$.

Assume that in this infinite algebraic tensor product $y_{1,2}$ are supported by a finite subset $J \subset K$. By using Kaplansky’s theorem again, as well as the fact that $(N_0, \phi_0)$ has almost periodic modular automorphism group, we may assume each $y_i$ is a linear combination of elements of the form $\bigotimes_{k \in J} w_k^k$, where $w_k \in (N_0)_k \simeq N_0$ are of the form $w_k = \frac{1}{\beta_k} p_k$, with $p_k \in (N_0)_{\phi_0}$ and $v_{\beta_k}^k \in N_0$ satisfying $\phi_0(v_{\beta_k}^k X) = \beta_k \phi(X v_{\beta_k}^k) \forall X \in N_0$, for some scalars $\beta_k$.

It follows that it is sufficient to prove (a) for $y_{1,2}$ of the form $\bigotimes_{k \in J} w_k^k$, with $J \subset K$ finite and $w_k$ as above. But for each such element $y_i$ there exists a non-zero scalar $c_i$ such that $\phi(X' y_i) = c_i \phi(y_i X') \forall X' \in N$. Thus $\phi(y_1 \sigma_k(x), y_2) = c_2 \phi(y_2 \sigma_k(x))$. By approximating $x$ in the $s^*$-topology with elements in the algebraic tensor product $\bigotimes_k (N_0, \phi_0)_k$ and by applying again Cauchy–Schwartz inequality, it follows that in calculating a limit of the form $\lim_{n \to \infty} \phi(y_2 \sigma_n(x))$, it is sufficient to take $x \in \bigotimes_k (N_0, \phi_0)_k$. But for such $x, y_1, y_2$ we clearly have $\lim_{n \to \infty} \phi(y_2 \sigma_n(x)) = \phi(y_2 y_1) \phi(x) = c_2^{-1} \phi(y_1 y_2) \phi(x)$. Altogether, it follows that $\lim_{n \to \infty} \phi(y_1 \sigma_n(x) y_2) = \phi(y_1 y_2) \phi(x)$.

The proof that if $G \cap K$ satisfies (2.4.3'') then $\sigma$ satisfies (b) is similar. $\square$
then both the weak mixing condition (2.4.3') and the proper outerness condition (2.2.4') are satisfied.

2.4.4. Lemma. Free Bernoulli actions of infinite groups are strongly mixing.

Proof. This is trivial by the definition of free shifts (see e.g. [S]). □

2.5. Product-type actions

Let \((N_0, \phi_0)\) be an ITPF1 factor. Let \(\alpha_0 : G^0 \hookrightarrow \text{Aut}(N_0, \phi_0)\) be a subgroup of automorphisms with \(\alpha_{0,v} \neq id \quad \forall v \neq e\), and \(\phi_0 \circ \alpha_{0,v} = \phi_0 \quad \forall v \in G^0\). \(G^0\) will be considered with the topology inherited from \(\text{Aut}(N_0, \phi_0)\). Let \(K\) be an infinite set and denote \((N, \phi) = \otimes_k (N_0, \phi_0)_k\). We let \(\alpha : G^0 \to \text{Aut}(N, \phi)\) be defined by \(\alpha_v(\otimes b_k) = \otimes_k b'_k\), where \(b'_k = \alpha_{0,v}(b_k)\forall k \in K\).

It is easy to see that \(\alpha\) is a continuous, properly outer action of \(G^0\) on \((N, \phi)\). In particular, since \(\alpha\) preservers \(\phi\), it leaves the centralizer \(N = N_\phi\) of \(\phi\) invariant, thus implementing on it an action. Moreover, if \(N_0\) is of type \(I_k\), for some \(2 \leq k \leq \infty\), then the action \(\alpha\) has fixed point algebra \(N''\) satisfying \(N'' \cap N = C1 [W]\).

Note that if a group \(G\) acts on \(K\) by \(G \cap K\) then the corresponding \((N_0, \phi_0)\)-Bernoulli \((G \cap K)\)-action \(\sigma\) on \((N, \phi)\) commutes with \(\alpha\). Thus, \(\alpha|_N\) commutes with the associated CS Bernoulli action \(\sigma|_N\) as well.

A particular case of product-type automorphism is the \(\text{flip}\) automorphism on \(N \otimes N\), that takes \(x \otimes y\) into \(y \otimes x\). This automorphism will play an important role in Section 4. In fact, we will need the existence of some continuous path of automorphisms that relate the identity to the flip automorphism.

2.5.1. Lemma. Let \((N_0, \phi_0)\) be a type \(I_k\) factor, \(2 \leq k \leq \infty\). Then there exists a continuous unitary representation \(\alpha_0\) of \(\mathbb{R}\) in the centralizer algebra \((N_0 \otimes N_0)_{\phi_0} \otimes \phi_0 \subset N_0 \otimes N_0\) such that:

(i) \(\alpha_0(1) (N_0 \otimes C) \alpha_0(1)^* = C \otimes N_0\).

If in addition we assume \((N_0, \phi_0) = \otimes_{i=1}^N \text{M}_{2\times 2}(\mathbb{C}), \phi_0, \phi_0\), for some \(1 \leq n \leq \infty\), then \(\alpha_0\) can be constructed so that there exists a self-adjoint unitary element \(\beta_0\) in \(C \otimes (N_0)_{\phi_0}\) with the property:

(ii) \(\beta_0 \alpha_0(t) \beta_0^{-1} = \alpha_0(-t)\forall t\).

Proof. Consider first the case \(N_0 = \text{M}_{2\times 2}(\mathbb{C})\). We let \(\{e_{ij}\}_{i,j=1,2}\) be a matrix unit for \(N_0\) such that the state \(\phi_0\) on \(N_0\) is given by a diagonal operator \(\Sigma_i \lambda_i e_{ii}\).

We define the unitaries \(\alpha_0(t) \in N_0 \otimes N_0\) by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \pi t/2 & \sin \pi t/2 & 0 \\
0 & -\sin \pi t/2 & \cos \pi t/2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
This clearly defines a continuous unitary representation of \( \mathbb{R} \). Then we define \( \beta_0 \in \mathbb{C} \otimes \mathcal{N}_0 \) to be the unitary element:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

Note that the unitaries \( x_0(t), t \in \mathbb{R} \), and \( \beta_0 \) belong to the centralizer of the state \( \varphi_0 \otimes \varphi_0 \) on \( \mathcal{N}_0 \otimes \mathcal{N}_0 \). An easy calculation shows that \( \beta_0 x_0(t) \beta_0^{-1} = x_0(-t) \) and that \( \text{Ad}(x_0(1))(\mathcal{N}_0 \otimes \mathbb{C}) = \mathbb{C} \otimes \mathcal{N}_0 \). Thus, \( x_0, \beta_0 \) satisfy all the required conditions.

Note that by taking tensor products of the above construction, this also proves the case \((\mathcal{N}_0, \varphi_0) = \bigotimes_{i=1}^n (M_{2 \times 2}(\mathbb{C}), \varphi_{0,i})\). For the general case when \( \mathcal{N}_0 = M_{k \times k}(\mathbb{C}) \), let \( \{e_{ij}\}_{i,j} \) be a matrix unit with \( \varphi_0 \) given by a diagonal operator in \( \text{Alg}\{e_{ii}\}_i \), as before. Let also

\[
x_0(t) = \sum_i e_{ii} \otimes e_{ii} + \sum_{i<j} (\cos \pi t/2(e_{ii} \otimes e_{jj} + e_{jj} \otimes e_{ii}) + \sin \pi t/2(e_{ij} \otimes e_{ji} - e_{ji} \otimes e_{ij})).
\]

It is easy to check that (i) is then satisfied. □

2.5.2. Corollary. Let \((\mathcal{N}_0, \varphi_0)\) be either of the form \((\mathcal{N}_0, \varphi_0) = \bigotimes_i (M_{2 \times 2}(\mathbb{C}), \varphi_{0,i})\), with \( \varphi_{0,i} \) faithful states on \( M_{2 \times 2}(\mathbb{C}) \), or of the form \((\mathcal{N}_0, \varphi_0) = \bigotimes_j (\mathcal{N}_{0,j}, \varphi_{0,j})\), where each \((\mathcal{N}_{0,j}, \varphi_{0,j})\) is either the hyperfinite type II\(_1\) factor with its trace, or the hyperfinite type III\(_\lambda\) factor with its generalized trace, for some \( 0 < \lambda_j < 1 \), with the indices \( i, j = 1, 2, \ldots \) over a finite or infinite sequence. Then there exist a continuous action \( x_0 \) of \( \mathbb{R} \) and a period 2 automorphism \( \beta_0 \) on \((\mathcal{N}_0 \otimes \mathcal{N}_0, \varphi_0 \otimes \varphi_0)\) such that:

(i) \( x_{0,1}(\mathcal{N} \otimes \mathbb{C}) = \mathbb{C} \otimes \mathcal{N} \).
(ii) \( \mathcal{N}_0 \otimes \mathbb{C} 1 \subset (\mathcal{N}_0 \otimes \mathcal{N}_0)^{\beta_0} \).
(iii) \( \beta_0 x_{0,t} \beta_0^{-1} = x_{0,-t} \forall t \).

Proof. This is immediate by 2.5.1. Indeed, by Connes’ classification of hyperfinite II\(_1\) and III\(_\lambda\) factors, in each of the cases considered the factors \((\mathcal{N}_{0,j}, \varphi_{0,j})\) are tensor powers of some \((M_{2 \times 2}(\mathbb{C}), \phi)\), i.e., of the form \( \bigotimes_n (M_{2 \times 2}(\mathbb{C}), \phi)_n \). Thus, by taking product-type actions, it is sufficient to prove the result in the case \( \mathcal{N}_0 = M_{2 \times 2}(\mathbb{C}) \). But this case is trivial by 2.5.1, taking \( x_{0,t} \) and \( \beta_0 \) to be the automorphism groups implemented by the corresponding unitary representations. □

2.5.3. Corollary. Let \((\mathcal{N}_0, \varphi_0)\) be as in 2.5.2. Let \( G \) be a discrete group and \( G \acts K \) an action of \( G \) on a set \( K \). Let \( \sigma \) be the \((\mathcal{N}_0, \varphi_0)\)-Bernoulli \((G \acts K)\)-action of \( G \) on
Let \( (N, \varnothing) = \overline{\otimes}_k (N_0, \varnothing_0)_k \). Then there exist a continuous action \( \alpha \) of \( \mathbb{R} \) and a period 2 automorphism \( \beta \) on \( (N \overline{\otimes} N, \varnothing \otimes \varnothing) \) such that:

1. \( \alpha, \beta \) commute with the \( (N_0, \overline{\otimes} N_0, \varnothing_0 \otimes \varnothing_0) \)-Bernoulli \((G \blacktriangleright K)\)-action \( \sigma \otimes \sigma \) of \( G \) on \( (N \overline{\otimes} N, \varnothing \otimes \varnothing) \).
2. \( \alpha \) is cocycle conjugate to \( \beta \).
3. \( N \otimes C1 \subset (N \overline{\otimes} N) \).
4. \( \beta \varphi \beta^{-1} = \varnothing \forall \varnothing, t \).

**Proof.** Let \( \alpha_0, \beta_0 \) be as in 2.5.2 and define \( \alpha_t = \otimes_k (\alpha_{0,t})_k \) and \( \beta = \otimes_k (\beta_0)_k \). Then both \( \alpha \) and \( \beta \) leave the product state \( \varnothing \otimes \varnothing = \otimes_k (\varnothing_0 \otimes \varnothing_0)_k \) invariant. Moreover, by the way \( \alpha, \beta \) are defined and by properties (i)-(iii) of 2.5.2, the above conditions (i)-(iv) are trivially satisfied. \( \square \)

### 3. Examples of cocycles

In this section we consider a method for constructing cocycle actions, from a given action. We also construct a class of examples of non-trivial 1-cocycles for actions on type II\(_1\) factors that come from a discrete decomposition.

Thus, let \( (N, \varnothing) \) be a von Neumann factor with centralizer \( N = N_\varnothing \). We assume \( N \) to be a factor itself. We consider properly outer cocycle actions \( (\sigma, \varnothing) \) of the discrete group \( G \) on \( (N, \varnothing) \), as defined in Section 1.2. Note that \( N \) is invariant to any such \( \sigma \).

Let \( p \in \mathcal{P}(N) \) be a non-zero projection. Since \( N \) is a factor and \( \sigma_g \) are trace preserving on \( N \), for each \( g \in G \) the projections \( p \) and \( \sigma_g(p) \) are equivalent in \( N \). Choose partial isometries \( w_g \in N_\varnothing = N \) of left support \( p \) such that \( w_g \sigma_g(p) w_g^* = p \) and \( w_e = p \).

#### 3.1. Proposition. Let \( \sigma^p : G \to \text{Aut}(N_p, \varnothing_p) \) be defined by \( \sigma^p_g(pxp) = w_g \sigma_g(pxp) w_g^* \), for \( x \in N \), where \( \varnothing_p(pxp) = \phi(pxp)/\varnothing(p) \forall x \in N \). Denote \( v_{g,h}^p = w_g \sigma_g(w_h) \varnothing_{g,h} v_{g,h}^p \), \( g, h \in G \), which we regard as an element in \( (N_\varnothing)_p = pN_\varnothing p \). Then \( (\sigma^p, v^p) \) is a properly outer cocycle action of \( G \) on \( (N_p, \varnothing_p) \). Moreover, up to cocycle conjugacy in \( \text{Aut}(N_p, \varnothing_p) \), the cocycle action \( (\sigma^p, v^p) \) does not depend on the choice in \( \sigma^p \) of the partial isometries \( w_g \). Also, if \( (\sigma', v') \) is another cocycle action of \( G \) on a von Neumann factor \( (N_\varnothing', \varnothing') \) and \( (\sigma', v') \) is cocycle conjugate (respectively, outer conjugate) to \( (\sigma, v) \), in the sense of 1.5, and \( p' \in N_\varnothing' \) is a projection with \( \varnothing'(p') = \varnothing(p) \), then \( (\sigma'^{p'}, v'^{p'}) \) is cocycle conjugate (respectively, outer conjugate) to \( (\sigma^p, v^p) \).

**Proof.** A straightforward calculation shows that \( v^p \) checks conditions (1.2.1), (1.2.2) with respect to \( \sigma^p \). The last part is trivial by the definitions. \( \square \)

Assume now \( N \) is the type II\(_1\) factor \( N \). Since \( (pNp, \sigma^p) \simeq (qNq, \sigma^q) \) if \( p, q \in \mathcal{P}(N) \) have same trace, from Proposition 3.1 we see that the isomorphism class of \( (N_p, \sigma^p, v^p) \), up to cocycle conjugacy, only depends on \( \tau(p) \). Thus, we can denote it by \( (N^t, \sigma^t) \), for each \( 0 \leq t \leq 1 \). More generally, we consider the following:
3.1.1. Definition. If \( t > 0 \) then we denote by \( (N', \sigma', v') \) the reduced of \( (M_{n \times n}(N), \text{id} \otimes \sigma, 1 \otimes v) \) by a projection \( p \) with normalized trace \( \tau(p) \) satisfying \( n \tau(p) = t \) (for \( n \) suitably large). Again, by Proposition 3.1 this cocycle action only depends on \( t \), up to cocycle conjugacy. We call \( (N', \sigma', v') \) the amplification of \( (N, \sigma, v) \) by \( t \).

3.1.2. Definition. We denote by \( \mathcal{F}(\sigma) \) the set of all \( t > 0 \) with the property that \( \sigma' \) is outer conjugate to \( \sigma \) (as defined in Section 1.5). We call \( \mathcal{F}(\sigma) \) the fundamental group of \( \sigma \). Similarly, we denote by \( \mathcal{F}(\sigma) \) the set of \( t > 0 \) for which \( (\sigma', v') \) is cocycle conjugate to \( (\sigma, v) \) (see 1.5). By the definition and Proposition 3.1, \( \mathcal{F}(\sigma) \) is clearly an outer conjugacy invariant for \( \sigma \). Thus, it is a cocycle conjugacy invariant as well. \( \mathcal{F}(\sigma) \) is cocycle conjugacy invariant only. Also, \( \mathcal{F}(\sigma) \subset \mathcal{F}(\sigma) \).

Note that by Section 1.5, \( \mathcal{F}(\sigma) \) coincides with the fundamental group of the inclusion \( (N \subset M) \) where \( M = N \rtimes_{\sigma,v} G \), i.e., with \( \mathcal{F}(N \subset M) = \{ t > 0 \mid (N \subset M) \} \). Note also that if in 3.1 we take \( \sigma \) to be a genuine action (so \( v = 1 \)) and take the projection \( p \) in the fixed point algebra \( N^\sigma \) then the 2-cocycle \( v^p \) is vanishing, in other words the cocycle action \( (\sigma^p, v^p) \) can be perturbed to an actual action. Indeed, because we can take \( w_s = p\forall g \). Thus, if \( N^\sigma \) contains projections of any trace, then all 2-cocycles as in 3.1, vanish. If in addition \( N^\sigma \subset N \) splits off the hyperfinite type II\(_1\) factor, then \( \mathcal{F}(\sigma) = \mathcal{F}(\sigma) = \mathbb{R}^*_+ \).

We now consider actions \( \sigma \) on type II\(_1\) factors \( N \) with the property that \( N \) is the core of a discrete decomposition of some factor \( (N', \varphi) \) (possibly of type III), in a way that the action \( \sigma \) itself be the restriction to \( N \) of an action on \( (N', \varphi) \). Examples of actions satisfying this property are provided by the CS Bernoulli actions defined in 2.2.4. The next result provides a class of examples of (generalized, weak) non-trivial 1-cocycles for such actions. Along the lines, we show that all the scalars in the spectrum \( S(N', \varphi) \) of the discrete decomposition \( (N', \varphi) \) belong to \( \mathcal{F}(\sigma) \).

To simplify notations, for each scalar 2-cocycle \( \mu \) of \( G \) (see 1.2) we denote by \( \mathcal{F}_\mu(G) \) (or simply \( \mathcal{F}_\mu \), when no confusion is possible) the set of finite dimensional, unitary, projective representations of \( G \) with scalar 2-cocycle \( \mu \), i.e., \( \pi \) belongs to \( \mathcal{F}_\mu \) if \( \pi : G \to \mathcal{U}(n) \), for some \( n \geq 0 \), and if it satisfies \( \pi(\mu) = \mu \forall g, h \in G \). The 0 representation is contained in \( \mathcal{F}_0 \). Note that if \( \mu = 1 \) then \( \mathcal{F}_1 \) is just the set of finite-dimensional representations \( \mathcal{F} \) of \( G \). This set contains the trivial representation of the group \( G \) and in fact any finite multiple of this representation.

Let \( (N, \varphi) \) be a von Neumann factor with a factorial discrete decomposition, as in 2.2.0. Let \( N = N_{\varphi} \) and denote by \( S = S(N', \varphi) \), which as noticed in 2.2.0 is a group \((\subset \mathbb{R}^*_+) \). More precisely, \( S \) is the multiplicative subgroup of \( \mathbb{R}^*_+ \) such that \( (N \subset N_{\varphi}) \otimes \mathcal{B}(\ell^2(\mathbb{N})) = (N \otimes \mathcal{B}(\ell^2(\mathbb{N}))) \rtimes S \). Let \( S_1 = \{ \beta \in S \mid 0 < \beta < 1 \} \). We denote by \( \mathcal{F}_\mu^{S_1} = \mathcal{F}_\mu^{S_1} \) the set of families \( \{ \pi_\beta \}_{\beta \in S_1} \) of representations \( \pi_\beta \) in \( \mathcal{F}_\mu \) indexed by \( \beta \in S_1 \) such that \( \Sigma_{\beta \mu} \beta \beta \leq 1 \), where \( \beta \mu = \dim \pi_\beta \). Note that some of the representations \( \pi_\beta \) may be equal to 0. In fact, if \( \pi_1 \neq 0 \) then this forces \( \pi_1 \) to be a one-dimensional representation, thus a character of \( G \), and all other \( \pi_\beta, \beta < 1 \), must be zero.

Let \( G \) be an infinite group and \( \sigma \) an action of \( G \) on \( (N', \varphi) \), with its restriction to \( N = N_{\varphi} \) still denoted by \( \sigma \).
3.2. Theorem. Let \( \{\pi_\beta\}_\beta \in \mathcal{F}_\mu^{S_1} \). For each \( \beta \) let \( v_i^\beta, 1 \leq i \leq n_\beta \), be isometries in \( V_\beta \) such that \( \{v_i^\beta\}_{\beta,i} \) have mutually orthogonal left supports. Let \( (\pi_\beta(g))_{ij} \) denote the coefficients of the representation \( \pi_\beta \) with respect to some orthonormal basis. Let \( w : G \to N \) be defined by

\[
w_g = \sum_{i,j} (\pi_\beta(g))_{ij} v_i^\beta \sigma_g (v_j^\beta^*), \quad g \in G.
\]

(1) \( w_g \) are partial isometries in \( N = N_\phi \) and in fact \( w \) is a generalized weak 1-cocycle for \( \sigma = \sigma|_N \) with support \( p = \sum_{\beta,i} v_i^\beta v_i^{\beta*} \) and scalar 2-cocycle \( \mu \).

(2) Let \( \sigma'_g = \text{Ad}(w_g) \circ \sigma_g \) and denote by \( B \) the von Neumann subalgebra of \( pNp \) generated by the matrix unit \( \{v_i^\beta v_i'^{\beta*} \mid 1 \leq i \leq n_\beta, 1 \leq i' \leq n_{\beta'}, \beta, \beta' \in S_1\} \).

Let also \( B \) be the von Neumann subalgebra of \( B \) generated by

\[
\{v_i^\beta v_i'^{\beta*} \mid 1 \leq i, i' \leq n_\beta, \beta \in S_1\}.
\]

Then \( B = B \cap N = (B)_\phi = E_N^{\phi}(B) \) and \( \sigma'_g(B) = B, \sigma'_g(B) = B, \) with \( \sigma'_gB \simeq \text{Ad}(\pi(g))\psi_g \in G \), where \( \pi = \oplus_{\beta} \tau_\beta \) is viewed as a representation on the Hilbert space \( \oplus_{\beta} \ell^2(n_\beta) \), with the inclusion \( B \subset B \) being identified with the inclusion

\[
\oplus_{\beta} B(\ell^2(n_\beta)) \subset B(\oplus_{\beta} \ell^2(n_\beta)).
\]

(3) Assume in addition that \( \sigma \) is weakly mixing on \( N \). Then \( (pNp)^{\sigma'} = B^{\sigma'} = \pi(G)' \cap B, (pNp)^{\sigma'} = B^{\sigma'} = \pi(G)' \cap B \). Moreover, any finite-dimensional vector subspace of \( pNp \) (resp. \( pNp \)) which is invariant to \( \sigma' \) is contained in \( B \) (resp. \( B \)) and in fact \( B \) (resp. \( B \)) is the closure of the span of finite-dimensional \( \sigma' \)-invariant subspaces of \( pNp \) (resp. \( pNp \)).

(4) Let \( w, \) respectively \( w' \), be cocycles constructed out of \( \{\pi_\beta\}_\beta \in \mathcal{F}_\mu^{S_1} \) and \( \{v_i^\beta\}_{\beta,i} \), respectively \( \{\pi'_\beta\}_\beta \in \mathcal{F}_\mu^{S_1} \) and \( \{v_i'^\beta\}_{\beta,i} \), as in (3.2.1). If \( \mu = \mu' \) and \( \pi_\beta \sim \pi'_\beta \forall \beta \in S_1 \) then \( w \) and \( w' \) are equivalent. If in addition we assume \( \sigma \) to be weakly mixing then, conversely, \( w \sim w' \) implies \( \mu = \mu' \) and \( \pi_\beta \sim \pi'_\beta \forall \beta \).

(5) \( S \subset \mathcal{F}^c(\sigma) \subset \mathcal{F}(\sigma) \). More precisely, if \( \beta \in S \) and \( v \in V_\beta, p = vv^* \), then \( \text{Adv} \) implements an isomorphism of \( (N \subset N \rtimes_{\sigma} G) \) onto \( p(N \subset N \rtimes_{\sigma} G)p \) which conjugates the action \( \sigma_g, g \in G \), on \( N \) to the action \( \text{Ad}(v\sigma_g(v^*)) \circ \sigma_g|_{pNp}, g \in G \), on \( pNp \).

Proof. (1) The statement is clearly true when all \( \pi_\beta \) are equal to 0.
An easy calculation shows that if we denote \( p_\beta = \sum_i v^\beta_i v_i^\beta * \) then \( p_\beta \) is fixed by \( \sigma'_g \forall g \). Note also that \( p_\beta \) this way defined is in the center of \( B \). Thus, by replacing \( w_g \) by \( p_\beta w_g \), we see that in order to prove (1) in its full generality it is sufficient to prove it in the case all but one of the representations \( \pi_\beta, \beta \in S_1 \), are equal to zero, i.e., for \( \pi = \pi_\beta \neq 0 \), for some \( \beta \in S_1 \). Thus, we may assume \( w_g \) is of the form

\[
\Sigma_{i,j}(\pi(g))_{ij} v^\beta_i \sigma_g(v^\beta_j *).
\]

For simplicity, denote \( v_i = v^\beta_i \) and \( c^g_{ij} = (\pi(g))_{ij} \). Taking into account that

\[
\Sigma_{j}c^g_{ij}c^h_{jl} = \mu_{g,h}c^g_{il}
\]

and that \( v_i \) have mutually orthogonal left supports, we get:

\[
w_g \sigma_g(w_h) = (\Sigma_{i,j}c^g_{ij} v_i \sigma_g(v_j *))\sigma_g(\Sigma_{k,l}c^h_{kl} v_k \sigma_h(v_l *))
= \Sigma_{i,j}(\Sigma_{j}c^g_{ij}c^h_{jl}) v_i \sigma_g(v_j *) = \mu_{g,h} \Sigma_{i,j} c^g_{ij} v_i \sigma_g(v_i *) = \mu_{g,h}w_{gh}.
\]

(2) Since \( E_N(\phi)(v^\beta v^*) = 0 \) for \( v \in V_\beta \) and \( v' \in V_{\beta'} \) with \( \beta \neq \beta' \), we have \( E_N^\phi(B) = B \).

To check that \( \sigma'_g(B) = B \), we need to show that \( \sigma'_g(v^\beta_i v^\beta'_j) \in B \forall i, \beta, \beta' \). We have

\[
\sigma'_g(v^\beta_i v^\beta'_j) = w_g \sigma_g(v^\beta_i v^\beta'_j) w^*_g
= (\Sigma_{k,l}(\pi_\beta(g))_{kl} v^\beta_k \sigma_g(v^\beta_* l)) \sigma_g(v^\beta_i v^\beta'_j) (\Sigma_{k',l'}(\pi_{\beta'}(g))_{k'l'} v^\beta'_{k'} \sigma_g(v^\beta'_j *))
= \Sigma_{k,l}(\pi_\beta(g))_{kl} v^\beta_k \sigma_g(v^\beta_i v^\beta'_j) \sigma_g(v^\beta_i v^\beta'_j)
= \Sigma_{k'} v^\beta_i v^\beta'_j.
\]

Since this latter element lies in \( B \), we are done. The above computation also shows that if we take \( \beta = \beta' \) then \( \sigma'_g(v^\beta_i v^\beta'_j) \) lies in \( \text{sp}\{v^\beta_k v^\beta'_k \}_{k,k'} \), thus in \( B \).

Moreover, it shows that if we identify the von Neumann algebra \( B \) generated by the matrix unit \( \{v^\beta_i v^\beta'_j \mid i, i', \beta, \beta' \} \) with \( B(\oplus \beta \ell^2(n_\beta)) \), then for each \( x = v^\beta_i v^\beta'_j \in B \) we have \( \sigma'_g(x) = \pi(g) x \pi(g)^* \).

(3) Recall that

\[
w_g = \Sigma_{k,l}(\pi_\beta(g))_{kl} v^\beta_k \sigma_g(v^\beta_* l).
\]

Hence

\[
w_g^* = \Sigma_{k',l'}(\pi_{\beta'}(g))_{k'l'} v^\beta'_{k'} \sigma_g(v^\beta'_j *).
\]
Any element $x \in pNp$ can be uniquely expressed in the form

$$x = \Sigma_{\beta, \beta'} \Sigma_{i, i'} v_{i}^{\beta} x_{i i'}^{\beta \beta'} v_{i'}^{\beta' *}$$

for some $x_{i i'}^{\beta \beta'} \in N$. For such $x$ we have

$$\sigma_{g}(x) = \Sigma_{\beta, \beta'} \Sigma_{i, i'} \sigma_{g}(v_{i}^{\beta}) \sigma_{g}(x_{i i'}^{\beta \beta'}) \sigma_{g}(v_{i'}^{\beta' *}).$$

By taking into account that $v_{i}^{\beta} v_{i}^{\beta} = \delta_{x} \delta_{x} \delta_{i}$ and $v_{i}^{\beta' *} v_{i'}^{\beta'} = \delta_{x} \delta_{x} \delta_{i'}$ it follows that $\sigma_{g}(v_{i}^{\beta}) \sigma_{g}(v_{i}^{\beta}) = \delta_{x} \delta_{x} \delta_{i}$ and $\sigma_{g}(v_{i}^{\beta'}) \sigma_{g}(v_{i'}^{\beta'}) = \delta_{x} \delta_{x} \delta_{i'}$ as well. Thus we have

$$w_{g} \sigma_{g}(x) w_{g}^{*} = \Sigma_{\beta, \beta'} \Sigma_{k, k'} \Sigma_{i, i'} (\pi_{\beta}(g))_{k i} (\pi_{\beta'}(g))_{k' i'} v_{k}^{\beta} \sigma_{g}(x_{i i'}^{\beta \beta'}) v_{k'}^{\beta' *}.$$  \hspace{1cm} (3.2.2)

The equation $w_{g} \sigma_{g}(x) w_{g}^{*} = x$ is then equivalent to the set of equations

$$\Sigma_{i, i'} (\pi_{\beta}(g))_{k i} (\pi_{\beta'}(g))_{k' i'} \sigma_{g}(x_{i i'}^{\beta \beta'}) = x_{i i'}^{\beta \beta'} \forall i, i', \beta, \beta'.$$  \hspace{1cm} (3.2.3)

Letting $y_{i i'}^{\beta \beta'} = \sigma_{g}(x_{i i'}^{\beta \beta'})$ and $x_{i i'}^{\beta \beta'} = \sigma_{g-1}(y_{i i'}^{\beta \beta'})$, this shows in particular that the finite-dimensional space $sp\{y_{i i'}^{\beta \beta'} | 1 \leq i \leq n_{\beta}, 1 \leq i' \leq n_{\beta'}\}$ is invariant to $\sigma_{g-1} \forall g \in G$, thus to $\sigma_{h} \forall h \in G$. Since $\sigma$ is weakly mixing on $N$, by 2.4.2 this implies $y_{i i'}^{\beta \beta'}$ are all scalars. Thus, $x_{i i'}^{\beta \beta'}$ are scalars as well implying that $x$ lies in $B$. By part (2) it follows that $x \in B' = \pi(G)' \cap B$. Restricting to elements $x \in pNp$, we also get $(pNp)^{\sigma'} = B' = \pi(G)' \cap B$.

Let now $H_{0} \subset pNp$ be a finite-dimensional vector subspace invariant to $\sigma'$. Since the projection $p_{\beta} = \Sigma_{i} v_{i}^{\beta} v_{i}^{\beta *}$ are fixed by $\sigma' \forall \beta$, it follows that for any $\beta, \beta'$ the vector space

$$X = p_{\beta} H_{0} p_{\beta'} = \Sigma_{i} v_{i}^{\beta} x_{i i'}^{\beta \beta'} v_{i'}^{\beta' *},$$

is invariant to $\sigma'$ as well, where $X_{i i'}^{\beta \beta'} = v_{i}^{\beta *} H_{0} v_{i'}^{\beta'}$. But calculation (3.2.2) above shows that

$$w_{g} \sigma_{g}(X) w_{g}^{*} = \Sigma_{k, k'} \Sigma_{i, i'} (\pi_{\beta}(g))_{k i} (\pi_{\beta'}(g))_{k' i'} v_{k}^{\beta} \sigma_{g}(X_{i i'}^{\beta \beta'}) v_{k'}^{\beta' *}.$$  \hspace{1cm} (3.2.5)
Since \( \sigma' \circ \sigma(\mathcal{X}) = \mathcal{X} \), from (3.2.4) and (3.2.5) we get

\[
\Sigma_{ii'} v_i^\beta \mathcal{X}_{ii'}^{i'} v_i^{\beta*} = \Sigma_{k,k'} \Sigma_{i,i'} (\pi_\beta(g))_{k'i'}(\pi_\beta(g))_{k'i'} v_k^\beta \sigma_g(\mathcal{X}_{ii'}^{i'}) v_{k'}^{\beta*}.
\]  

(3.2.6)

Fixing \( l, l' \) and multiplying both sides of (3.2.6) by \( v_l^{\beta*} \cdot v_{l'}^{\beta} \) we thus get

\[
\mathcal{X}_{ll'}^{\beta'^*} = \Sigma_{i,i'} (\pi_\beta(g))_{ii'}(\pi_\beta(g))_{i'i'} v_i^\beta \sigma_g(\mathcal{X}_{ll'}^{\beta'^*}).
\]  

(3.2.7)

Letting \( \mathcal{Y}_{ll'}^{\beta'^*} = \sigma_g(\mathcal{X}_{ll'}^{\beta'^*}) \), \( \mathcal{X}_{ll'}^{\beta'^*} = \sigma_{g^{-1}}(\mathcal{Y}_{ll'}^{\beta'^*}) \) in (3.2.7), it follows that the finite-dimensional vector space spanned by \( \cup_{l,l'} \mathcal{Y}_{ll'}^{\beta'^*} \) is invariant to \( \sigma_g \forall g \in G \), thus to \( \sigma_h \forall h \in G \). By 2.4.2 it follows again that all the spaces \( \mathcal{Y}_{ll'}^{\beta'^*} \) are equal to \( \mathbb{C} \). Thus \( \mathcal{X}_{ll'}^{\beta'^*} = \mathbb{C} \) as well, implying that \( \mathcal{X} = \Sigma_{i,i'} \mathbb{C} v_i^{\beta'^*} \). This proves that \( \mathcal{H}_0 \subset \mathcal{B} \).

Since \( p_\beta B p_\beta' \) are all invariant subspaces and they span \( \mathcal{B} \), the last part of the statement follows as well.

(4) Note first that the equivalence class of the cocycle \( w \) does not depend on the choice of the isometries \( v_i^\beta \), once the family of representations \( \pi_\beta \) is fixed.

If the two given families of representations satisfy \( \pi_\beta \sim \pi'_\beta \forall \beta \in S_1 \), then in particular \( \dim(\pi_\beta) = \dim(\pi'_\beta) \forall \beta \). It follows that in the construction of \( w \) and \( w' \) we can take the same set of isometries \( \{v_i^\beta\}_{\beta,i} \). Thus, both \( \pi_\beta, \pi'_\beta \) “live” in \( p_\beta B p_\beta' \), where \( p_\beta = \Sigma_i v_i^\beta v_i^{\beta*} \) as usual. If for each \( \beta \in S_1 \) with \( n_\beta \neq 0 \) we take \( u_\beta \in p_\beta B p_\beta' \) to be the unitary element that implements the equivalence of \( \pi_\beta, \pi'_\beta \) then an immediate calculation shows that \( u = \oplus_\beta u_\beta \) also implements the equivalence of \( w \) with \( w' \).

Note that this same computation shows that the converse also holds true, provided \( \dim(\pi_\beta) = \dim(\pi'_\beta) \). But this equality does hold true if we assume \( \sigma \) is weakly mixing, by the last part of (3). Indeed, because the fixed point algebras \( B \), respectively \( B' \), of the actions \( \text{Ad}(w) \circ \sigma \) and, respectively, \( \text{Ad}(w') \circ \sigma \), are conjugate in \( pNp \). Since the traces of the minimal projections in \( p_\beta B p_\beta \) and \( p'_\beta B' p'_\beta \) must both be equal to \( \beta \), thus being different for distinct \( \beta \), it follows that the unitary element \( u \in pNp \) that satisfies \( u B u^* = B' \) must carry projections of trace \( \beta \) onto projections of trace \( \beta \), implying that \( u p_\beta B p_\beta u^* = p'_\beta B' p'_\beta \) and thus \( n_\beta = n'_\beta \forall \beta \).

(5) This follows from the fact that \( v(\sigma_g(v^*xv)v^*) = \text{Ad}(v \sigma_g(v^*)) \sigma(x) \) for \( x \in pNp \) and \( g \in G \).

An immediate consequence of Theorem 3.2 is that in case \( (\mathcal{N}, \varphi) \) is of type III, any cocycle action constructed by amplification of a CS Bernoulli action \( \sigma \) on the core type II_1 factor \( \mathcal{N} = \mathcal{N}_\varphi \) can in fact be perturbed to an action. We will later see that if \( \mathcal{N} \) is of type II_1, then this is no longer the case.

### 3.3 Corollary

With \( \sigma \) as in 3.2 above, assume in addition that \( S \neq \{1\} \) (equivalently, the factor \( (\mathcal{N}, \varphi) \) is of type III). For each \( t > 0 \), let \( \sigma' \) be a choice of a cocycle action
on \( N^t \) obtained by amplifying the action \( \sigma \) on \( N = N_\varphi \) by \( t \), as in 3.1. Then \( \sigma^t \) can be perturbed to an action.

**Proof.** Let \( m \) be an integer such that \( m \geq t \). Since \( S_1 \neq \{1\} \), it contains some \( \beta < 1 \). Thus, \( \beta^m \in S_1 \forall n \geq 1 \) as well. But then there exist some non-negative integers \( k_n \) such that \( \sum_k k_n \beta^m = t/m \) and we can apply part (1) of Theorem 3.2 to the action \( \sigma \otimes id_m \) of \( G \) on \( N \otimes M_{m \times m}(\mathbb{C}) \) (which is the core of the discrete decomposition of \( (N \otimes M_{m \times m}(\mathbb{C}), \varphi \otimes tr) \)), to provide a generalized 1-cocycle \( w \) for \( \sigma \) with support \( p \), where \( p \in N \otimes M_{m \times m}(\mathbb{C}) \) is a projection of trace \( t/m \). \( \square \)

Our next result gives an abstract characterization of the cocycles \( w \) constructed in Theorem 3.2. Namely, we show that \( w \) is locally of form (3.2.1) iff the projective representation \( \xi \mapsto w_\varphi \sigma_g(\xi) \) has finite dimensional invariant subspaces. This observation will be needed in Section 4.

**3.4. Proposition.** With the same notations and hypothesis as in 3.2, assume the action \( \sigma : G \to \text{Aut}(N, \varphi) \) is weakly mixing. Let \( w \) be a generalized weak 1-cocycle for \( \sigma \) with support \( p \in \mathcal{P}(N) \) and scalar 2-cocycle \( \mu \).

(1) For \( g \in G \) and \( \xi \in L^2(pN, \varphi) \) denote \( \sigma^w_g(\xi) = w_\varphi \sigma_g(\xi) \). Then \( \sigma^w \) is a projective unitary representation of \( G \) on \( L^2(pN, \varphi) \) with scalar 2-cocycle \( \mu \).

(2) Let \( \mathcal{H} \) be the Hilbert space of Hilbert–Schmidt operators on \( L^2(pN, \varphi) \). For each \( X \in \mathcal{H} \) let \( \tilde{\sigma}^w_g(X) = \sigma^w_g X \sigma^w_g^* \). Then \( \tilde{\sigma}^w \) is a unitary representation of \( G \) on \( \mathcal{H} \).

(3) The following conditions are equivalent:
   
   (i) \( \tilde{\sigma}^w \) contains a copy of the trivial representation.
   
   (ii) \( \sigma^w \) has a non-trivial, invariant, finite-dimensional subspace \( \mathcal{H}_0 \subset L^2(pN, \varphi) \).
   
   (iii) There exist some \( \beta \in S_1 \) and a finite set of isometries \( v_1, v_2, \ldots, v_n \in \mathcal{V}_\beta \) with mutually orthogonal left supports such that \( q = \sum_i v_i v_i^* \leq p \) is fixed by the action \( \sigma' = Ad w \circ \sigma \) and \( \mathcal{H}_0 = \sum_i \mathcal{C} v_i \) is invariant to \( \sigma^w \).
   
   (iv) There exists a non-zero projection \( q \in pNp \) fixed by \( \sigma' = Ad(w) \circ \sigma \) such that \( qw_\varphi = \sum_{i,j} (\mathcal{P}_i(g))_{i,j} v_i \sigma_g(v_j^*) \), \( g \in G \), for some \( \mathcal{P}_i(\mathcal{F}_\mu) \) and some isometries \( v_i, 1 \leq i \leq \text{dim}(\mathcal{P}_0) \), lying all in some \( \mathcal{V}_\beta \) and having mutually orthogonal left supports.
   
   (v) There exists a non-zero projection \( q_0 \in pNp \) fixed by \( \sigma' = Ad w \circ \sigma \) such that \( q_0 w \) is of form (3.2.1), for some family \( \{\mathcal{P}_\beta\}_{\beta} \in \mathcal{F}_\mu^{S_1} \), and such that there are no non-zero projections \( q \leq (p - q_0) \), \( q \in B_0 \) with the property that \( qw \) is of form (3.2.1).

**Proof.** (1) By the definitions, \( \sigma^w_g \) are clearly unitary operators acting on \( L^2(pN, \varphi) \). Also, we have

\[
\sigma^w_g(\sigma^w_{h}(\xi)) = \sigma^w_g(w_h \sigma_h(\xi)) = w_\varphi \sigma_g(w_h \sigma_h(\xi)) = w_\varphi \sigma_g (w_h) \sigma_{gh}(\xi) = \mu_{g,h} w_{gh} \sigma_{gh}(\xi),
\]

proving (1).
(2) By (1) it follows that

$$\tilde{\sigma}_g^w(\tilde{\sigma}_h^w(X)) = \sigma_g^w \sigma_h^w X \sigma_h^{w*} \sigma_g^{w*} = \mu_{g,h} \tilde{\sigma}_g^w X \sigma_h^{w*} = \tilde{\sigma}_{gh}^w(X).$$

(3) Noticing that $\tilde{\sigma}_g^w$ extends from the ideal of Hilbert–Schmidt operators to all $B(L^2(pN, \varphi))$, it follows that if $X \in \mathcal{H}$ is a fixed point for $\tilde{\sigma}_g^w$ then the trace class operator $XX^*$ is also a fixed point for $\tilde{\sigma}_g^w$. Thus all spectral projections $P$ of $XX^*$, are fixed points for $\tilde{\sigma}_g^w$ as well. Since $P$ have finite trace, they are finite dimensional. Let $\mathcal{H}_0 \subset L^2(pN, \varphi)$ be a non-zero finite-dimensional space corresponding to some spectral projection $P \neq 0$. By the definitions, $\tilde{\sigma}_g^w(P) = P$, it follows that $\sigma_g^w(\mathcal{H}_0) = \mathcal{H}_0$. This proves (i) $\Rightarrow$ (ii).

To prove (ii) $\Rightarrow$ (iii), note first that if $v \in \mathcal{V}$ then each of the subspaces $L^2(pv^*N, \varphi)$ and $L^2(pNV, \varphi)$ of $L^2(pN^*, \varphi)$ is invariant to $\sigma^w$. Thus, if $\sigma^w$ has a non-trivial finite-dimensional invariant subspace $\mathcal{H}_0$ then by compressing it to one of these spaces and taking into account that they span all $L^2(pN, \varphi)$ it follows that we may assume $\mathcal{H}_0$ is contained either in some $L^2(pv^*N, \varphi)$ or in some $L^2(pNV, \varphi)$.

In either case, denote by $\{\xi_i\}_i$ an orthonormal basis of $\mathcal{H}_0$. By the definition of $\sigma^w$ it follows that $\Sigma_i \xi_i^* \xi_i \in L^1(\mathcal{N}, \varphi)$ is fixed by $\sigma$. Indeed, this is because

$$\sigma_g(\Sigma_i \xi_i^* \xi_i) = \Sigma_i \sigma_g(\xi_i^*) \sigma_g(\xi_i) = \Sigma_i (\sigma_g(\xi_i)^*w_g^*)(w_g \sigma_g(\xi_i))$$

and because $\Sigma_i \eta_i^* \eta_j = \Sigma_i \xi_i^* \xi_j$, for any other orthonormal bases $\{\eta_i\}_i$ of $\mathcal{H}_0$. Since $\sigma$ is weakly mixing, it is ergodic. Thus $\Sigma_i \xi_i^* \xi_i \in C1$ implying that $\xi_i$ are actually in $pN$.

Similarly, the finite-dimensional vector space $\mathcal{H}'_0 = sp\mathcal{H}_0^* \mathcal{H}_0 \subset \mathcal{N}$ is invariant to $\sigma$. It follows that $\mathcal{H}'_0 \subset C$ implying that $\xi_i^* \xi_i \in C1 \forall i$. This in turn implies that each $\xi_i$ is a scalar multiple of an isometry $v_i$ and that $A = sp\{v_i v_j^* | 1 \leq i, j \leq n\}$ is a finite-dimensional factor. Thus, by replacing if necessary the elements $\{v_i\}_i$ by some elements $\{a_{ij} v_j\}_i$ for some appropriate partial isometries $a_j \in A$, we may assume the isometries $v_i$ have mutually orthogonal left supports, yet still generate $\mathcal{H}_0$. Since all elements in $L^2(pv^*N, \varphi)$ have right supports $\neq 1$ while the right support of $\xi_i = v_i$ is 1, it follows that $\mathcal{H}_0$ cannot be a subspace of some $L^2(pv^*N, \varphi)$, forcing it to be a subspace of some $L^2(pNv, \varphi), v \in \mathcal{V}$. This implies $v_i \in \mathcal{V}_\beta \forall i$, where $\beta = \varphi(vv^*) \in S_1$.

Moreover, since $sp\mathcal{H}_0 \mathcal{H}_0^*$ is invariant to

$$pNp \ni x \mapsto w_g \sigma_g(x)w_g^* = \sigma_g^w(x)$$

and the support projection $q$ of the elements in this vector space is $\Sigma_i v_i^* v_i$, it follows that $

\sigma_g^w(q) = q \forall g \in G.$

Assuming (iii) holds true, let $\beta \in S_1$ and $v_1, v_2, \ldots, v_n \in \mathcal{V}_\beta$ be such that $\mathcal{H}_0 = \Sigma_i C v_i$ is invariant to $\sigma^w$. It follows that for each $1 \leq i \leq n$ and $g \in G$ there exist some scalars $\{c_{ij}^g\}_j$ such that $w_g \sigma_g(v_i) = \Sigma_j c_{ij}^g v_j$. To prove (iv) it is sufficient to prove that $c_{ij}^g$ are the coefficients of a projective unitary representation of $G$ with the same scalar
2-cocycle $\mu$ as $w$. Since $w_g \sigma_g(v_i) = \Sigma_j b_{ij} v_j \forall i$, we have $q w_g = \Sigma_i, j c_{ij}^g v_j \sigma_g(v_i^*)$. Similarly $q w_h = \Sigma_i, j c_{ij}^h v_j \sigma_h(v_i^*)$. It follows that

$$q w_g \sigma_g(w_h) = (\Sigma_i, j c_{ij}^g v_j \sigma_g(v_i^*)) (\Sigma_k, l c_{lk}^h \sigma_g(v_l) \sigma_{gh}(v_k^*)) = \Sigma_j, k (\Sigma_i c_{ji}^g c_{ik}^h) v_j \sigma_{gh}(v_k^*).$$

Similarly, we have

$$q w_g h = \Sigma_j, k c_{jk}^{gh} v_j \sigma_{gh}(v_k^*).$$

Replacing the above in the equation $q w_g \sigma_g(w_h) = \mu_{g, h} q w_g h$ and multiplying to the left by $v_j v_j^*$ and to the right by $\sigma_{gh}(v_k v_k^*)$, we get

$$\Sigma_i c_{ij}^g c_{jk}^{gh} = \mu_{g, h} c_{jk}^{gh} \forall j, k, g, h.$$

But this means $c_{ij}^g$ are the coefficients of a projective representation $\pi_0$ of $G$ on $\ell^2(n)$ with 2-cocycle $\mu$.

To prove $(iv) \iff (v)$ we use a maximality argument. Thus, we let $P$ denote the set of families of mutually orthogonal projections $(q_i)_{i \in pNp}$ such that $q_i \in (pNp)^{\sigma'}$ and $q_i^* w$ is of form (3.2.1). We endow $P$ with the obvious order $\leq$ given by inclusion. $(P, \leq)$ this way defined is clearly inductively ordered. Let $(q_0^i)_{i \in pNp}$ be a maximal element, which by $(iv)$ we may suppose non-zero. Let $q_0 = \Sigma_i q_0^i \in (pNp)^{\sigma'}$. Then $q_0$ clearly satisfies the required conditions, or else the maximality of $(q_0^i)_{i \in pNp}$ would be contradicted.

The implication $(v) \implies (iv)$ is trivial. Finally, $(iv) \implies (i)$ follows by taking $P \in \mathcal{H}$ to be the orthogonal projection of $L^2(pN, \phi)$ onto $\mathcal{H}_0 = \Sigma_i \mathbb{C} v_i$. Indeed, because by hypothesis we have

$$w_g \sigma_g(v_i) = \Sigma_j (\pi_0(g))_{ij} v_j$$

showing that $\sigma_g^w(\mathcal{H}_0) = \mathcal{H}_0$, equivalently $\sigma_g^w(P) = P \forall g \in G$. \qed

4. Cohomology of CS Bernoulli actions

In this section we prove the main technical result of this paper. Thus, we consider a CS Bernoulli action $\sigma$ of a group $G$ on the hyperfinite $\Pi_1$ factor and calculate the complete cohomology picture of its restrictions $\sigma|_H$ to subgroups $H \subset G$ with the relative property (T) of Kazhdan–Margulis [Ma]; see A.1 for the definition. Note that if the subgroup $H$ is finite, then it is already known that any 1-cocycle is trivial on $H$ [Su], so we only have to consider infinite subgroups with the relative property (T). In fact, the result works for more general type of actions, that we define below:
4.0. Definition. An action \( \sigma : G \to \text{Aut}(N, \tau) \) of a discrete group \( G \) on a finite von Neumann algebra \( (N, \tau) \) is called malleable if there exists a von Neumann algebra with factorial discrete decomposition \( (N', \phi) \) and an action \( \tilde{\sigma} : G \to \text{Aut}(N' \bar{\otimes} N', \phi \otimes \phi) \) with the following properties:

\[
(4.0.1) \quad \tilde{\sigma}(N' \otimes 1) = N' \otimes 1, \quad N = N'_{\phi} \quad \text{and if one regards } N \text{ as the subalgebra } N \otimes 1 = N'_{\phi} \otimes 1 \text{ of } N' \bar{\otimes} N \text{ then } \tilde{\sigma}|_N = \sigma.
\]

(4.0.2) There exists a continuous action \( \alpha \) of \( \mathbb{R} \) on \( (N' \bar{\otimes} N', \phi \otimes \phi) \) such that:

(i) \( \alpha \) commutes with the action \( (\tilde{\sigma}, G) \).
(ii) \( \alpha_{\pm 1}(N' \otimes \mathbb{C}) = \mathbb{C} \otimes N' \).

A discrete decomposition \( (N', \phi) \) with an action \( (\tilde{\sigma}, G) \) on \( (N' \bar{\otimes} N', \phi \otimes \phi) \) satisfying conditions (4.0.1), (4.0.2) is called a gauged extension for the malleable action \( \sigma \). The action \( \sigma \) is called \( s \)-malleable if it has a gauged extension \( (\tilde{\sigma}; (N', \phi)) \) which in addition to conditions (4.0.1), (4.0.2) satisfies the following property:

\[
(4.0.3) \quad \text{There exists a period } 2 \text{ automorphism } \beta \text{ of } (N' \bar{\otimes} N', \phi \otimes \phi) \text{ such that:}
\]

(i) \( \beta \) commutes with the action \( (\tilde{\sigma}, G) \).
(ii) \( N' \otimes \mathbb{C} \subset (N' \bar{\otimes} N')\beta \).
(iii) \( \beta \sigma_1 \beta^{-1} = \alpha_{-1} \beta_1 \).

By 2.2.4 and 2.5.2 if \( G \curvearrowright K \) is an action of a group \( G \) on a set \( K \) and \( (N_0, \phi_0) \) is an ITPF1 factor \( \neq \mathbb{C} \), then the \( (N_0, \phi_0) \)-CS Bernoulli \( (G \curvearrowright K) \)-action \( \sigma \) is malleable. If in addition \( (N_0, \phi_0) \) satisfies the conditions in the hypothesis of 2.5.3, then \( \sigma \) is \( s \)-malleable. Also, if \( 2.2.4' \) is satisfied then \( \sigma \) is properly outer and if a subgroup \( H \subset G \) is so that \( H \curvearrowright K \) satisfies condition \( 2.4.3' \) then \( \sigma|_H \) is weakly mixing.

4.1. Theorem. Let \( \sigma : G \to \text{Aut}(N, \tau) \) be a properly outer action of a discrete group \( G \) on a \( \Pi_1 \) factor \( N \). Assume \( \sigma \) is malleable and let \( (\tilde{\sigma}; (N', \phi)) \) be a gauged extension for \( \sigma \), as defined in 4.0. We still denote by \( \sigma \) the restriction of \( \tilde{\sigma} \) to \( N' = N \otimes 1 \). Let \( H \subset G \) be a subgroup with the relative property \( (T) \), such that \( \sigma|_H \) is weakly mixing. Let \( w \in N \) be a generalized weak 1-cocycle for \( \sigma \), with support \( p \in \mathcal{P}(N) \) and scalar 2-cocycle \( \mu \in H^2(G) \). Let \( B_0 \) be the fixed point algebra of the action \( \sigma'_g = \text{Ad}(w_g) \circ \sigma_g, g \in H \), of the group \( H \) on the algebra \( pNp \). With the notations \( \tilde{V}, \tilde{V}_\beta, S_1 \) in 2.2.0 for \( (N, \phi) \) and the notations \( \mathcal{F}_\mu = \mathcal{F}_{\mu}(H), \mathcal{F}_{\mu}^{S_1} = \mathcal{F}_{\mu}^{S_1}(H) \) considered in Theorem 3.2, we have:

(1) If \( z \in B_0 \) is the maximal central projection of \( B_0 \) such that \( B_0z \) is atomic, then there exist a family of finite dimensional, projective, unitary representations \( \{\pi_\beta \mid \beta \in S_1\} \subset \mathcal{F}_{\mu}^{S_1} \) of the group \( H \) and some isometries \( \{v^\beta_i \mid 1 \leq i \leq \dim \pi_\beta\} \subset \mathcal{V}_\beta, \beta \in S_1 \), such that \( \Sigma_{\beta,i}v^\beta_i v^\beta_i^* = z \) and such that the restriction to \( H \) of the generalized weak 1-cocycle \( zw \) is given by the formula:

\[
zw_h = \Sigma_{\beta}(\Sigma_{i,j}(\pi_\beta(h))_{ij}v^\beta_i \sigma_h(v^\beta_j^*) \quad \text{for } h \in H.
\]
(2) If in addition \( \sigma \) is \( s \)-malleable and \( (\tilde{\sigma}; (N, \varphi)) \) is a graded gauged extension for \( \sigma \) then \( B_0 \) follows atomic; i.e., \( z = 1_{B_0} = p \), and the above formula (4.1.1) holds true for the given cocycle \( w|_H (= pw|_H) \) itself.

To prove the theorem we first need some lemmas. To state them, recall some notations from 2.2.0–2.2.2. Thus, we denote by \( \tilde{N} \) the centralizer of \( \varphi \otimes \varphi \) on \( N \otimes N \). We denote by \( \theta \) the restriction of \( \sigma \otimes \sigma \) to \( \tilde{N} \), i.e., \( \theta_g = (\sigma_g \otimes \sigma_g)|_{\tilde{N}}, \quad g \in G \).

Note that the actions \( \alpha \) and \( \beta \) on \( (N \otimes N, \varphi \otimes \varphi) \) given by (4.0.2) and, respectively, (4.0.3) implement actions on \( \tilde{N} \) that we still denote by \( \alpha, \beta \). By (4.0.2), \( \alpha \) commutes with \( \theta \). Thus, \( \theta \) and \( \alpha \) implement an action \( \theta \times \alpha \) of the group \( G \times \mathbb{R} \) on \( \tilde{N} \), by 

\[
\theta_g \times \alpha_t = \theta_g \alpha_t, \quad g \in G, \quad t \in \mathbb{R}.
\]

For the next two lemmas, the discrete groups \( H \subset G \) can be arbitrary. Also, we will not need the existence of the period 2 automorphism \( \beta \) satisfying (4.0.3) until Lemma 4.9. Other than that, we are under the general assumptions and notations of (1) of Theorem 4.1.

### 4.2. Lemma

Let \( \tilde{M} \) denote the type \( \Pi_1 \) factor obtained by taking the cross product of \( \tilde{N} \) by the action \( \theta \times \alpha \) of the group \( \mathcal{G} = G \times \mathbb{R} \) on it, in which \( \mathbb{R} \) is regarded as a discrete group. Let \( (U_h)_{h \in \mathcal{G}} \subset M \) be the canonical unitaries implementing this action. Let \( \sigma_g' = \text{Ad}(w_g) \circ \sigma_g, U_g' = w_g U_g, \quad g \in G = G \times \{0\} \). Let \( P = \{U_h'\}_{h \in H} \subset pMp \) be the von Neumann subalgebra with support \( 1_P = p \) generated by the unitaries \( \{U_h'| h \in H\} \) in \( p\tilde{M}p \) and denote \( B_0 = (pNp)\sigma'|_{\mathcal{H}} = P' \cap pNp \). Fix \( x \in N = N \otimes \mathbb{C} \). For each \( t \in \mathbb{R} \) put \( x_t \overset{\text{def}}{=} E_{P' \cap pMp}(pxU_t)pU_t^* \). Then we have:

1. \( x_t \) belongs to \( \tilde{N} \) being in fact the unique element of minimal norm-2 in the weakly closed convex subset of \( \tilde{N} \)

\[
K'_t(x) = \overline{co}^W \{U_h' x U_h U_h'^* U_t^* \}_{h \in H} = \overline{co}^W \{w_h \sigma_h(x) x_t (w_h^*) \}_{h \in H}.
\]

2. \( x_t \) satisfies the equivalent conditions

\[
U_h' x_t = x_t \text{Ad} U_t (U_h') \quad \forall h \in H, \tag{a}
\]

\[
w_h \theta_h (x_t) = x_t \sigma_t (w_h) \quad \forall h \in H. \tag{b}
\]

Also, \( x_t \) is the unique element in \( K'_t(x) \) that satisfies these equivalent conditions.

3. \( x_t x_t^* \in B_0 \otimes \mathbb{C} \) and \( x_t^* x_t \in \pi_t(B_0 \otimes \mathbb{C}) \).

4. \( p_{r^*} = \pi_{-r} (p_{r^*})^* \) and \( bx_t = b(x_t) \forall b \in B_0 \).

**Proof.** Since the set \( \{U_h'\}_{h \in H} \) is total in \( P, E_{P' \cap p\tilde{M}p}(pxU_t)p \) is the unique element of minimal norm-2 in \( K_t(x) = \overline{co}^W \{U_h' x U_t U_h'^* \}_{h \in H} \) (see e.g., [Po4]). But \( U_h' x U_t U_h'^* = w_h \sigma_h(x) U_t w_h^* = w_h \sigma_h(x) x_t (w_h^*) U_t \), implying that \( K'_t(x) = K_t(x) U_t^* \) and that \( x_t = E_{P' \cap p\tilde{M}p}(x U_t) U_t^* \) is the unique element of minimal norm-2 in the set \( K'_t(x) \). This proves (1) of Lemma 4.2.
The commutation relation $U'_h(x_t U_t) = x_t U_t U'_h$, which holds true for all $h \in H$, is equivalent to the condition $U'_h x_t = x_t A_d U_t (U'_h)$ (by multiplying the former to the right by $U'_h$). This shows that (a) in (2) of Lemma 4.2 holds true. But condition (a) amounts to $w_h U_t x_t = x_t U_t w_h U_t U'_h$, which multiplied from the right by $U'_h$ gives condition (b) (after appropriate simplifications).

Noticing that by the uniqueness of the trace preserving conditional expectation $\mathrm{E}_{p' \cap \tilde{M}_p}$ of $p \tilde{M}_p$ onto $p' \cap \tilde{M}_p$ we have $u^* \mathrm{E}_{p' \cap \tilde{M}_p} (u \cdot u^*) u = \mathrm{E}_{p' \cap \tilde{M}_p} (\cdot)$, it follows that for all $u$ in $\mathcal{U}(P)$ we have

$$\mathrm{E}_{p' \cap \tilde{M}_p} (u(p x U_t) u^*) = u^* \mathrm{E}_{p' \cap \tilde{M}_p} (u(p x U_t) u^*) u = \mathrm{E}_{p' \cap \tilde{M}_p} (p x U_t),$$

it follows that $\mathrm{E}_{p' \cap \tilde{M}_p} (y) = \mathrm{E}_{p' \cap \tilde{M}_p} (p x U_t) \forall y \in K_t(x)$, so that $x_t U_t$ is the unique element in $K_t(x) = K'_t(x) U_t$ that commutes with all $U'_h, h \in H$. By the equivalence between the commutation relation and the conditions (a) and (b), this proves the uniqueness in (2) of Lemma 4.2.

Since by the way it is defined the element $x_t U_t$ commutes with the $^*$-algebra $P$ and since $p \tilde{N} p \cap (p \tilde{N} p \otimes \mathbb{C}) = p \tilde{N} p \otimes \mathbb{C}$, 2.4.2 implies that

$$x_t x_t^* = (x_t U_t)(x_t U_t)^* \in p' \cap p \tilde{N} p$$

$$= (p \tilde{N} p)^{\theta'} = (p \tilde{N} p \otimes \mathbb{N})^{\sigma' \otimes \sigma} \cap p \tilde{N} p = ((p \tilde{N} p)^{\sigma'} \otimes \mathbb{C}) \cap p \tilde{N} p$$

$$= ((p \tilde{N} p)^{\sigma'} \otimes \mathbb{C}) \cap (p \tilde{N} p \otimes \mathbb{C}) = (p \tilde{N} p)^{\sigma'} \otimes \mathbb{C} = B_0 \otimes \mathbb{C},$$

where $\theta'$ is the action of $H$ on $p \tilde{N} p$ given by $\theta'_h = \sigma'_h \otimes \sigma_h, h \in H$.

Similarly, we get

$$\pi_t(x_t^* x_t) = (x_t U_t)^* (x_t U_t) \in p' \cap p \tilde{N} p = B_0 \otimes \mathbb{C},$$

implying that $x_t^* x_t \in \pi_t(B_0 \otimes \mathbb{C})$. This proves (3) of Lemma 4.2.

Taking $-t$ for $t$, by the definitions we get $p_{-t} U_{-t} = \mathrm{E}_{p' \cap \tilde{M}_p} (p U_{-t})$. Thus

$$U_{-t} p_{-t} U_t^* = \mathrm{E}_{p' \cap \tilde{M}_p} (p U_t) U_{-t} = p_t,$$

showing that $\pi_t(p_{-t})^* = p_t$, which by taking adjoints and applying $(\pi_t)^{-1} = \pi_{-t}$ gives the first part of (4) of Lemma 4.2. The second part is trivial by the definitions.

4.3. Lemma. Let $u(t) = (p_t p_t^*)^{-1/2} p_t \in \tilde{N}$ denote the partial isometry in the polar decomposition of $p_t$ and denote by $l(t) = u(t) u(t)^*$, $r(t) = u(t)^* u(t)$ its left support and right support. Then we have:

1. $U'_h u(t) = u(t) U_t U'_h U'_t \forall h \in H$.
2. $u(-t) = \pi_{-t}(u(t))^*$.
3. $l(t) \in B_0, r(t) \in \pi_t(B_0)$ and $u(t)^* B_0 u(t) = r(t) \pi_t(B_0) r(t)$.
**Proof.** By multiplying both sides of Eq. (a) in (2) of Lemma 4.2 by \((pt, p_t^*)^{-1/2} \in B_0\) from the left, we get (1) of Lemma 4.3. Since by (4) of Lemma 4.2 we have \(p_{-t} = \pi_{-t}(u(t))^*\), by the definitions of \(u(t)\) and \(u(-t)\) we immediately get \(u(-t) = \pi_{-t}(u(t))^*\). This proves (2) of Lemma 4.3.

Taking \((pt, p_t^*)^{-1/2}\) to be the element \(b\) in (4) of Lemma 4.2, by (3) of Lemma 4.2 and the definitions we get 
\[
l(t) = u(t)u(t)^* = bt^*b_t \in B_0\text{ and } r(t) = u(t)^*u(t) = b_t^*b_t \in \pi_{t}(B_0).
\]
Similarly, if we let \(q\) be an arbitrary projection in \(l(t)B_0l(t)\) and take this time \(q(pt, p_t^*)^{-1/2}\) to be the element \(b\) in (4) of Lemma 4.2, then we get \(u(t)^*qu(t) = b_t^*b_t \in \pi_{t}(B_0)\). Since any element in \(B_0\) can be approximated in uniform norm by a linear combination of projections, it follows that 
\[
u(t)^*B_0u(t) \subset r(-t)\pi_{-t}(B_0)r(-t).
\]
Applying \(\pi_{t}\) on both sides we get
\[
u(-t)^*B_0u(-t) \subset r(-t)\pi_{-t}(B_0)r(-t).
\]
Applying \(\pi_{t}\) on both sides we get
\[
u_{t}(u(-t)^*)\pi_{t}(B_0)\pi_{t}(u(-t)) \subset \pi_{t}(r(-t))B_0\pi_{t}(r(-t)).
\]
But by (2) of Lemma 4.3 we have \(\pi_{t}(u(-t)) = u(t)^*\) and \(l(t) = l(\pi_{t}(u(-t))^*) = \pi_{t}(r(-t))\), so the above implies
\[
u(t)\pi_{t}(B_0)\nu(t)^* \subset l(t)B_0l(t).
\]
Equivalently,
\[
u(t)\pi_{t}(B_0)\nu(t) \subset u(t)^*B_0u(t).
\]
This proves the opposite inclusion in (3) of Lemma 4.3. □

For the next three lemmas we will assume that the subgroup \(H \subset G\) has the relative property (T), as in the hypothesis of 4.1. All other assumptions and notations are as in 4.1.1, 4.2, 4.3.

**4.4. Lemma.** For each \(b \in B_0\), the function \(t \mapsto b_t\) from \(\mathbb{R}\) into the unit ball of \(\tilde{N} \subset \tilde{M}\) is continuous at \(t = 0\), with respect to the norm-2 topology. Moreover, 
\[
lim_{t \to 0} \|u(t) - p\|_2 = 0.
\]

**Proof.** By the second part of (4) of Lemma 4.2, to prove the continuity at \(t = 0\) of the function \(t \mapsto b_t\) it is sufficient to prove it for the function \(t \mapsto p_t\).

Let \(\varepsilon > 0\). By the relative property (T) of \(H \subset G\) (see A.1), there exist \(F \subset G\) and \(\delta > 0\) such that if \(\pi\) is a unitary representation of the group \(G\) on a Hilbert space \(\mathcal{H}\), with a unit vector \(\xi \in \mathcal{H}\) satisfying \(\|\pi(g)\xi - \xi\| \leq \delta\forall g \in F\), then \(\|\pi(h)\xi - \xi\| \leq \varepsilon/2\) \(\forall h \in H\).
Since $t \mapsto z_t$ is pointwise continuous in the norm-2 topology and $z_0(p) = p$, $z_0(w_g) = w_g$, $g \in F$, it follows that there exists $\delta_1 > 0$ such that if $t \in \mathbb{R}$ satisfies $|t| \leq \delta_1$ then $\|z_t(p) - p\|_2 \leq (\varepsilon/2)\|p\|_2$ and

$$\|w_g z_t(w_g)^* - p z_t(p)\|_2 \leq \delta \|p z_t(p)\|_2 \quad \forall g \in F. \tag{4.4.1}$$

Consider the representation $\pi$ of $G$ on $L^2(p\tilde{M}p)$ given by $\pi(g)z = U_g'z U_g'^*$. For each $t \in \mathbb{R}$, if we take $\zeta' = pU_t p$ we then have $\|\zeta'\|_2 = \|p z_t(p)\|_2$ and

$$\|\pi(g)\zeta' - \zeta'\| = \|U_g'U_tU_g'^* - pU_t p\|_2$$

$$= \|w_g U_t w_g^* - pU_t p\|_2 = \|w_g z_t(w_g)^* - p z_t(p)\|_2 \quad \forall g \in G.$$

Applying this to $g \in F$, by (4.4.1) we get

$$\|\pi(g)\zeta' - \zeta'\| = \|w_g z_t(w_g)^* - p z_t(p)\|_2 \leq \delta \|p z_t(p)\|_2.$$

It thus follows that

$$\|w_h z_t(w_h)^* - p z_t(p)\|_2 \leq (\varepsilon/2)\|p z_t(p)\|_2 \quad \forall h \in H. \tag{4.4.2}$$

Since by its definition (see (1) of Lemma 4.2) $p_t$ is in $\overline{\text{co}}^W\{w_h z_t(w_h)^*\}_{h \in H}$, the above implies that

$$\|p_t - p z_t(p)\|_2 \leq (\varepsilon/2)\|p z_t(p)\|_2$$

for all $t \in \mathbb{R}$ with $|t| \leq \delta_1$. Thus,

$$\|p_t - p\|_2 \leq \|p_t - p z_t(p)\|_2 + \|p z_t(p) - p\|_2 \leq \varepsilon \|p\|_2 \forall |t| \leq \delta_1.$$

This shows the continuity at $t = 0$ of the function $p_t$. The continuity of $u(t)$ at $t = 0$ follows now trivially from the first part and 4.3. \(\square\)

**4.5. Lemma.** Let $z_0 \in B_0$ be a central projection of $B_0$ such that $B_0z_0$ is a finite-dimensional algebra. Then there exists $\varepsilon > 0$ such that if $|t| < \varepsilon$ then $z_0 \leq l(t) = u(t)u(t)^*$ and $u(t)^*B_0z_0u(t) = z_t(B_0z_0)$.

**Proof.** By Lemma 4.4 we have $\lim_{t \to 0} u(t)u(t)^* = 1_{B_0} = p$. Also, by (3) of Lemma 4.3 we have $l(t) = u(t)u(t)^* \in B_0$. Since $z_0$ is central in $B_0$ and $B_0z_0$ is a finite-dimensional von Neumann algebra, it follows that for $|t|$ small enough we have $z_0 \leq u(t)u(t)^*$.

Let $c > 0$ be the minimal trace of a non-zero projection in $B_0z_0$. Note that for $|t|$ small enough, $z_t(z_0)$ is close to $z_0$. By 4.4, $u(t)^*z_0u(t)$ is also close to $z_0$. Thus, by (3)
of Lemma 4.3 it follows that \( z_1(z_0) \) and \( u(t)^*z_0u(t) \) are projections of \( z_t(B_0) \) which are close one to another for \(|t| \) small. Moreover, \( z_t(z_0) \) is central in \( z_t(B_0) \).

Since the minimal trace of a non-zero projection in both

\[
u(t)^*(B_0z_0)u(t) = u(t)^*z_0u(t)z_t(B_0)u(t)^*z_0u(t)
\]

and in

\[
z_t(B_0z_0) = z_t(B_0)z_t(z_0)
\]

is \( c \), it follows that if \(|u(t)^*z_0u(t) - z_t(z_0)|_2^2 < c\) then \( u(t)^*z_0u(t) = z_t(z_0)\). Thus, for \(|t| \) small enough we have \( u(t)^*B_0z_0u(t) = z_t(B_0z_0) \). □

4.6. **Lemma.** Let \( z \in B_0 \) be the maximal atomic projection of \( B_0 \). Then there exists a partial isometry \( u \in \tilde{N} \) such that \( uu^* = z \otimes 1 \in B_0 \), \( u^*u = 1 \otimes z \in z_1(B_0) \) and \((w_h \otimes 1)UHu = u(1 \otimes w_h)U_h \forall h \in H\).

**Proof.** Let \( \{z_i\}_i \) be a partition of \( z \) with central projections in \( B_0 \) such that each \( B_0z_i \) is finite dimensional. It is sufficient to prove that for each \( z_i \) there exists a partial isometry \( u_i \in \tilde{N} \) such that \( z_i = z_i \otimes 1 = u_iu_i^* \), \( u_i^*u_i = z_1(z_i) = 1 \otimes z_i \) and \( (w_h \otimes 1)Uhu_i = u_i(1 \otimes w_h)U_h \forall h \in H \). Indeed, because then we can just define \( u = \oplus_i u_i \) and use the fact that for this \( u \) we have \( z_i(w_h \otimes 1)UHu = (w_h \otimes 1)U_uu_i \). Summing up this gives \((w_h \otimes 1)UHu = u(1 \otimes w_h)U_h \).

This shows that by replacing \( u \) by \( z_iu \) and \( B_0 \) by \( B_0z_i \) we may suppose \( B_0 \) is finite dimensional. Denote \( z_0 = 1_{B_0} \). By (3) of Lemma 4.3 and 4.5 it follows that there exists \( n \geq 1 \) such that \( u(t_n)u(t_n)^* = z_0 \) and \( u(t_n)^*z_0u(t_n) = z_{t_n}(z_0) \), where \( t_n = 1/n \).

By applying recursively \( (z_{t_n})^k \) to this equation, for \( k = 1, 2, \ldots, n - 1 \), it follows that the element

\[
u = u(t_n)(z_{t_n}(u(t_n)))(z_{t_n}^2(u(t_n))) \cdots (z_{t_n}^{n-1}(u(t_n)))
\]

is a partial isometry in \( \tilde{N} \) which satisfies \( u^*z_0u = z_n(z_0) = z_{1}(z_0) \) and \( uu^* = z_0 \). Moreover, by multiplying the relation

\[
U_hu(t_n) = u(t_n)U_{t_n}U_h^*U_{t_n}^* \quad \forall h \in H
\]

(which holds true by (1) of Lemma 4.3) from the right by \( z_n(u(t_n)) \), it follows that

\[
U_h^*(u(t_n)z_n(u(t_n))) = u(t_n)U_{t_n}U_h^*U_{t_n}^*z_n(u(t_n))
\]

\[
= u(t_n)U_{t_n}U_h^*U_{t_n}^*(U_{t_n}u(t_n)U_{t_n}^*) = u(t_n)U_{t_n}(U_h^*u(t_n))U_{t_n}^*
\]

\[
= u(t_n)U_{t_n}(u(t_n)U_{t_n}U_h^*U_{t_n}^*)U_{t_n}^*
\]

\[
= (u(t_n)z_n(u(t_n)))(U_{t_n}^2U_h^2U_{t_n}^2) \quad \forall h \in H.
\]
Recursively, we obtain that $u = \prod_{k=0}^{n-1} x_k^{\gamma_k}(u(t_n))$ satisfies $U_n^* u = u(U_n^*) \overline{U_n^*} (U_n^*)^*$, which by taking into account that $U_n^* = U_1$ gives $U_n^* u = uU_1 U_n^*. Since \( U_n^* = (w_h \otimes 1)U_h \) and $U_1(w_h \otimes 1)U_n^* = 1 \otimes w_h$, from this last relation we get $(w_h \otimes 1)U_h u = u(1 \otimes w_h)U_h \forall h \in H$, as desired. □

4.7. Corollary. Let $z \in B_0 \subset \mathcal{N}$ and $u \in \tilde{\mathcal{N}} \subset \mathcal{N} \otimes \mathcal{N}$ be as in Lemma 4.6. Let $v^w$ be the representation of $H$ on the Hilbert space $L^2(p\mathcal{N}, \varphi)$, as defined in (2) of Proposition 3.4. If we identify $\mathcal{H}$ in the usual way with the Hilbert space $L^2(p\mathcal{N}, \varphi) \overline{\otimes} L^2(\mathcal{N} \cap \mathcal{N}, \varphi)$ and regard $u \in p\mathcal{N} \overline{\otimes} \mathcal{N} \cap \mathcal{N}$ as an element in $\mathcal{H}$, then $\tilde{\sigma}^w_{\mathcal{H}}(u) = u \forall h \in H$.

Proof. Under the identification of $\mathcal{H}$ with the space $L^2(p\mathcal{N}, \varphi) \overline{\otimes} L^2(\mathcal{N} \cap \mathcal{N}, \varphi)$, the representation $\tilde{\sigma}^w$ becomes

$$\tilde{\sigma}^w(h)(x' \otimes x'') = (w_h \sigma_h(x')) \otimes (\sigma_h(x'') w_h^*),$$

$$= (w_h \otimes 1)(\text{Ad}U_h(x' \otimes x''))(1 \otimes w_h^*) \forall h \in H, x', x'' \in p\mathcal{N}.$$ 

On the other hand, the left support $z$ of $u$ belongs to $B_0 = B_0 \otimes \mathcal{C} \subset \mathcal{N} \otimes \mathcal{C}$ and the right support $z_1(z) = 1 \otimes z$ to $\mathcal{C} \otimes \mathcal{N}$. Thus, $u \in p\mathcal{N} \overline{\otimes} \mathcal{N} \cap \mathcal{N}$. By Lemma 4.6 we have

$$(w_h \otimes 1)U_h u = u(1 \otimes w_h)U_h \forall h \in H.$$ 

But the latter is equivalent to $\tilde{\sigma}^w(g)(u) = u$. □

Proof of (1) of Theorem 4.1. By (3)(v) of Proposition 3.4 there exists a projection $q_0 \in B_0$ such that $q_0 w^i$ is of form (4.1.1) and such that there are no non-zero projections $q_1 \leq (p - q_0), q_1 \in B_0$ with $q_1 w$ of form (4.1.1). Note right away that $q_0 \leq z$, where $z$ is the maximal atomic projection of $B_0$, i.e., maximal projection of $Z(B_0)$ with $B_0z$ atomic. Indeed, this is because $q_0 w^i$ is a generalized weak 1-cocycle with $\text{Ad}(q_0 w^i) \circ \sigma$ having atomic fixed point algebra (cf. (3) of Theorem 3.2).

We have to prove that $q_0 = z$, i.e., $(z - q_0) \neq 0$. We will proceed by contradiction, assuming that $(z - q_0) \neq 0$. By replacing if necessary $w$ by $(z - q_0) w$, which is still a generalized weak 1-cocycle for $\sigma$, with the same scalar 2-cocycle as $w$, we may assume that the algebra $B_0 = (p \mathfrak{N})^\sigma$ is atomic and, furthermore, that there exists no non-zero projection $q_1 \leq p = 1_{B_0}, q_1 \in B_0$, such that $q_1 w$ is of form (4.1.1).

But by Proposition 3.4 the latter condition is equivalent to the fact that the representation $\tilde{\sigma}^w$ of $H$ on the Hilbert space $\mathcal{H}$ of Hilbert–Schmidt operators on $L^2(p\mathcal{N}, \varphi)$ does not contain the trivial representation. By Corollary 4.7, this is a contradiction. □

In order to prove (2) of Theorem 4.1 (and thus end the proof of 4.1) we only need to show that if also assume the existence of the “grading” $\beta$ satisfying (4.0.3) (i.e. that $\sigma$ is s-malleable with $(\mathcal{N}, \varphi); \tilde{\sigma}$ a graded gauged extension for $\sigma$), then the fixed point algebra $B_0$ must be atomic. We do this in the next two lemmas.
4.8. Lemma. Assume $\alpha, \beta$ are as in (4.0.2), (4.0.3). Assume there exists a diffuse weakly closed $\ast$-subalgebra $B^0 \subset N = N \otimes \mathbb{C}$ and a partial isometry $v_0 \in \tilde{N} = (N' \overline{\otimes} N)_{\varphi \otimes \varphi}$ such that $v_0^* v_0 = 1_{B^0}$ and $v_0 B^0 v_0^* \subset \alpha_{1/2^n}(N \otimes \mathbb{C})$, for some $n \geq 1$. Then $v_0 = 0$.

Proof. We will construct by induction over $k \geq 0$ some partial isometries $v_k \in \tilde{N}$ and diffuse weakly closed von Neumann subalgebras $B_k \subset N \otimes \mathbb{C}$ such that

$$\tau(v_k v_k^*) = \tau(v v^*), v_k^* v_k = 1_{B_k}, v_k B_k v_k^* \subset \alpha_{1/2^n-k}(N \otimes \mathbb{C}). \quad (4.8.1)$$

Letting $B_0 = B^0$, $v_0 = v_0$, we see that the relation holds true for $k = 0$. Assume we have constructed $v_j, B_j$ for $j = 0, 1, \ldots, k$. By applying the automorphism $\beta$ of (4.0.3) to the inclusion in (4.8.1) and using the properties of $\beta$, it follows that

$$\beta(v_k) B_k \beta(v_k)^* \subset \alpha_{-1/2^n-k}(N \otimes \mathbb{C}). \quad (4.8.2)$$

By further applying $\alpha_{1/2^n-k}$ to this latter inclusion it follows that

$$B_{k+1} \overset{\text{def}}{=} \alpha_{1/2^n-k}(\beta(v_k)) \alpha_{1/2^n-k}(B_k) \alpha_{1/2^n-k}(\beta(v_k)^*) \subset N \otimes \mathbb{C}. \quad (4.8.3)$$

By conjugating (4.8.3) with $\alpha_{1/2^n-k}(\beta(v_k)^*)$ we thus obtain

$$\alpha_{1/2^n-k}(\beta(v_k)) B_{k+1} \alpha_{1/2^n-k}(\beta(v_k)^*) = \alpha_{1/2^n-k}(B_k) \subset \alpha_{1/2^n-k}(N \otimes \mathbb{C}). \quad (4.8.4)$$

On the other hand, by applying $\alpha_{1/2^n-k}$ to (4.8.1) we also have

$$\alpha_{1/2^n-k}(v_k) \alpha_{1/2^n-k}(B_k) \alpha_{1/2^n-k}(v_k^*) \subset \alpha_{1/2^n-k-1}(N \otimes \mathbb{C}). \quad (4.8.5)$$

Altogether, it follows that if we let $v_{k+1} = \alpha_{1/2^n-k}(v_k) \alpha_{1/2^n-k}(\beta(v_k^*))$ then by (4.8.4) and (4.8.5) we obtain

$$v_{k+1} B_{k+1} v_{k+1} \subset \alpha_{1/2^n-k-1}(N \otimes \mathbb{C}).$$

Moreover, since $\beta(v_k^* v_k) = v_k^* v_k$, we also have $v_{k+1} v_{k+1}^* = \alpha_{1/2^n-k}(v_k v_k^*)$, so that $\tau(v_{k+1} v_{k+1}^*) = \tau(v_k v_k^*)$. This ends the induction argument. Taking $k = n$, it follows that $v_n B_n v_n^* \subset \alpha_1(N \otimes \mathbb{C}) = \mathbb{C} \otimes N$ (since $\alpha_1$ is the flip automorphism). But by (Lemma 4.3 in [Po3], or 2.6 in [Po2]), this implies that $v_n = 0$. Since $\tau(v_n v_n^*) = \tau(v v^*)$, it follows that $v = 0$. \hfill \square

4.9. Corollary. Under the assumptions of (2) of Theorem 4.1, the algebra $B_0 = N^{\sigma'}$ follows atomic.
Proof. By 4.3 and 4.4 it follows that if \( n \geq 1 \) is sufficiently large then the partial isometry \( v' = u(1/2^n)^* \) is non-zero, has right support in \( B_0 \) and satisfies \( v'B_0v'^* \subset \mathcal{A}_1/2^n(N \otimes \mathbb{C}) \). But if we assume \( B_0v'^*v' \) has a non-zero diffuse part \( B_0 \) and we let \( v_0 = v'1_{B_0} \), then 4.8 applies to infer that \( v_0 = 0 \), a contradiction. Thus, \( v'^*v' = u(1/2^n)u(1/2^n)^* \) tends to \( p \) as \( n \) tends to \( \infty \), it follows that \( B_0 \) is atomic. □

Proof of (2) of Theorem 4.1. By Corollary 4.9 the algebra \( B_0 \) follows atomic. But then (1) of Theorem 4.1 applies to obtain that \( w \) is necessarily of form (3.2.1). □

4.10. Corollary. Let \( (N_0, \varphi_0) \) be an ITPF1 factor, as in 2.2.2. Let \( H \subset G \) be an inclusion of discrete groups with the relative property (T). Let \( G \rtimes K \) be an action of \( G \) on a set \( K \), which we assume to satisfy the proper outerness condition (2.2.4) and be weak mixing when restricted to \( H \) (i.e. \( H \rtimes K \) satisfies (2.4.3)). Let \( \sigma \) denote the \( (N_0, \varphi_0)-\text{Bernoulli} \) \((G \rtimes K)\)-action of \( G \) on \( (N, \varphi) = \mathcal{O}_K(N_0, \varphi_0)_K \), as in 2.2.3. We still denote by \( \sigma \) the associated \( (N_0, \varphi_0)\)-CS Bernoulli \((G \rtimes K)\)-action on the hyperfinite II_1 factor \( N \), obtained by restricting \( \sigma_g \) to \( N = N_\varphi \), \( g \in G \), as in 2.2.4. If \( w \) is a generalized weak 1-cocycle for the CS Bernoulli action \( (\sigma, G) \) on \( N \), with support \( p \in \mathcal{P}(N) \) and scalar 2-cocycle \( \mu \in H^1(G) \), then the restriction \( w|_H \) of \( w \) to the subgroup \( H \) is of the form

\[
w_h = \Sigma_\beta \Sigma_{i,j} (\pi_\beta(h))_{ij} v_i^\beta \sigma_h(v_j^{\beta^*}), \quad h \in H
\]

(4.10.1)

for some family of finite-dimensional projective unitary representations \( \{\pi_\beta \mid \beta \in S_1\} \subset \mathcal{F}_\mu^1 \) and some isometries \( \{v_i^\beta \mid 1 \leq i \leq \dim \pi_\beta\} \subset \mathcal{V}_\beta(N, \varphi) \) satisfying \( \Sigma_{\beta,i} v_i^\beta v_i^{\beta^*} = p \).

Proof. By 2.5.3, if \( (N_0, \varphi_0) \) is a tensor product of type III_{\lambda_i} \ ITPF1 factors, \( 0 < \lambda_i < 1 \), then conditions (4.0.2), (4.0.3) are satisfied. Thus, part (2) of Theorem 4.1 applies to get the conclusion.

For the general case, let \( (N_0, \varphi_0) = \mathcal{O}_f(N_0,j, \varphi_0,j) \), where each \( (N_0,j, \varphi_0,j) \) is a type I_j factor with a faithful normal state \( \varphi_0,j \) given by the weights \( t_{1,j}^i \geq t_{2,j}^i \geq \cdots \). Let \( \lambda_i^n = t_{n,j}^i/t_{1,j}^i \) and \( (N_{0,n}^i, \varphi_0^n) = \mathcal{O}_\mathcal{F}(N_{0,n}^i, \varphi_0^n) \), where each \( (N_{0,n}^i, \varphi_0^n) \) is the hyperfinite type III_{\lambda_i^n} factor with its generalized trace. Thus, \( S(N_{0,n}^i, \varphi_0^n) \) is equal to the multiplicative group generated by \( \{\lambda_i^n\}_n \), which coincides with the multiplicative group generated by \( \{t_{n,j}^i/t_{1,j}^i\}_n \).

We claim that there exists a von Neumann subalgebra \( N_{1,j} \subset N_{0,n}^i \) such that \( (N_{1,j}, \varphi_{0,n}N_{1,j}) \simeq (N_0,j, \varphi_0,j) \) and \( (N_{1,j}) \varphi_{0,n}N_{1,j} \subset (N_{0,n}^i) \varphi_0^n \). Indeed, if for each \( n \geq 1 \) we take \( v_n^i \in \mathcal{V}_{t_{n,j}^i/t_{1,j}^i}(N_{0,n}^i, \varphi_0^n) \), with \( v_1^i = 1 \), and choose \( q \in (N_{0,n}^i) \varphi_0^n \) of trace \( t_{1,j}^i \), then \( \varphi_0^n(v_q q v_q^*) = t_{n,j}^i \) and so, since \( \Sigma_n t_{n,j}^i = 1 \), we can choose the isometries \( \{v_n^i\}_n \) such that \( \Sigma_n v_n^i q v_n^* = 1 \). Thus, if we put \( e_n^m = v_q q v_q^* \) and let \( N_{1,j} = \mathcal{O}_\mathcal{F}(e_n^m)_{n,m} \) then all conditions are satisfied.
Let now \((N^0, \phi^0) = \bigotimes_j (N^0_j, \phi^0_j)\). Let also \((M, \phi) = \bigotimes_{k \in K} (N^0_k, \phi^0_k)\). Note that the embeddings \(N_{1,j} \subset N^0_j, j \in J\), implement an embedding of \(N = N_\phi \) into \(M = M_\phi\). Also, if we denote by \(\rho\) the \((N^0, \phi^0)\)-Bernoulli \((G \rtimes K)\)-action of \(G\) on \((M, \phi)\), then \(\rho(N) = N\) and \(\rho|_N = \sigma|_N\). Thus, if \(w : G \to N\) is a generalized weak 1-cocycle for the action \(\sigma\) of \(G\) on \(N\), then \(w\) is also a generalized weak 1-cocycle for the \((N^0, \phi^0)\) CS Bernoulli \((G \rtimes K)\)-action of \(G\) on \(M = M_\phi\). By (2) of Theorem 4.1 the fixed point algebra of the action \(\text{Ad}w_h \circ \rho_h, h \in H\), of \(H\) on \(M\) is atomic. Thus, the fixed point algebra \(B_0\) of the action \(\text{Ad}w_h \circ \sigma_h, h \in H\), of \(H\) on \(N\) is atomic as well. But by 2.5.3, the action \(\sigma_h, h \in H\), of \(H\) on \((N, \phi)\) satisfies the conditions in (1) of Theorem 4.1. Since \(B_0\) is atomic, we get that in fact \(w|_H\) is of form (4.1.1) with \(z = p\). □

5. \(I\)-Invariants and their calculation

Motivated by the results and considerations in the previous sections, we introduce here the outer conjugacy invariants \(I(\sigma)\) and \(I_s(\sigma)\) for cocycle actions \(\sigma\) of arbitrary discrete groups \(G\) on type II\(1\) factors. These invariants account for the reals \(t > 0\) for which \(\sigma^t\) satisfies certain vanishing cohomology properties: plain vanishing cohomology in the case of \(I\) and vanishing cohomology with weak mixing in the case of \(I_s\). More precisely, we set:

5.1. Definition. Let \(\sigma\) be a cocycle action of a discrete group \(G\) on a type II\(1\) factor \(N\). We denote by \(I(\sigma)\) the set of all \(t > 0\) for which there exists an amplification \(\sigma^t\) of \(\sigma\) such that \(\sigma^t\) is a genuine action of \(G\) on \(N^t\). We denote by \(I_s(\sigma)\) the set of those \(t \in I(\sigma)\) for which \(\sigma^t\) can be chosen a weakly mixing action. We call \(I(\sigma)\) the spectrum of \(\sigma\) and \(I_s(\sigma)\) the strong spectrum of \(\sigma\). Note that \(I_s(\sigma) \subset I(\sigma) \subset \mathbb{R}^*_+\) could a priori be empty sets (see 5.3 below).

The next proposition lists some basic properties of \(I(\sigma), I_s(\sigma)\).

5.2. Proposition. (1) \(I(\sigma), I_s(\sigma)\) are outer conjugacy invariants for \(\sigma\) (so, in particular, cocycle conjugacy invariants as well).

(2) \(I(\sigma)I(\sigma) = I(\sigma), I(\sigma)I_s(\sigma) = I_s(\sigma)\) and \(\mathbb{Z}^*_+I(\sigma) = I(\sigma)\).

(3) \(I(\sigma^t) = I(\sigma)/t, I_s(\sigma^t) = I_s(\sigma)/t \forall t > 0\).

(4) \(I(\sigma)I(\sigma^t) \subset I(\sigma \otimes \sigma^t), I(\sigma)I_s(\sigma^t) \subset I_s(\sigma \otimes \sigma^t)\) and \(I_s(\sigma)I_s(\sigma^t) \subset I_s(\sigma \otimes \sigma^t)\).

Proof. (1) The definition of \(I(\sigma), I_s(\sigma)\) ignores the cocycles associated with \(\sigma\) and \(\sigma^t\). On the other hand, we see from Proposition 3.1 that \(\sigma^t\) depends only on the class of \(\sigma\) in \(\text{Aut} N/\text{Int} N\).

(2) If \(\sigma^t\) is outer conjugate to \(\sigma\) and \(\sigma^t\) is a genuine action (respectively, a weakly mixing action), then \((\sigma^t)^{t'} \sim_o \sigma^{t'}\) can be chosen an action too (resp. weakly mixing action), so \(tt'\) lies in \(I(\sigma)\) (resp. \(I_s(\sigma)\)). This proves the first two equalities. Also, if \(\sigma^t\) is an action and \(n \geq 1\) then \(\sigma^{nt} \sim_o \sigma^t \otimes id_n\) is also an action.
(3) These equalities follow from the fact that \((\sigma^t)^{t'}\) can be chosen to be an action (resp. weakly mixing action) iff \(\sigma^{tt'}\) can be chosen to be an action (resp. weakly mixing action).

(4) This is trivial by taking into account that \((\sigma \otimes \sigma')^{tt'} \sim_c \sigma^t \otimes \sigma'^{t'}\). □

The next result summarizes some well known results in [Oc,CJ,Po1,Po4], using this new terminology.

5.3. Theorem. Let \(\sigma\) be a cocycle action of an infinite, countable group \(G\) on a separable type \(\text{II}_1\) factor \(N\).

(1) If \(G\) is amenable then \(\mathcal{S}(\sigma) = \mathbb{R}_+^\ast\). If in addition \(N \cong R\) then \(\mathcal{F}(\sigma) = \mathcal{F}^c(\sigma) = \mathcal{S}_s(\sigma) = \mathbb{R}_+^\ast\) as well.

(2) If \(G\) has the property (T) and has finite radical (= subgroup of all elements having finite conjugacy class) then \(\mathcal{S}_s(\sigma)\) is countable. If in addition \(\mathcal{S}_s \neq \emptyset\) then \(\mathcal{F}^c(\sigma)\) is also countable.

(3) Assume \(G\) has an infinite subgroup with the relative property (T) (see A.1 for the definition). If \(\sigma\) is the Connes–Jones cocycle action [CJ] (see also A.2), then \(\mathcal{S}(\sigma) = \emptyset\).

Proof. (1) By [Po1] any cocycle action of the amenable group \(G\) (including \(\sigma^t \forall t > 0\)) can be perturbed to an action. Moreover, by Ocneanu [Oc], if \(N\) is the hyperfinite type \(\text{II}_1\) factor then any cocycle action of \(G\) (including the \(\sigma^t\)) can be perturbed to a Bernoulli action, which is weakly mixing by 2.4.3. Also, any two cocycle actions are cocycle conjugate, so in particular \(\sigma^t \sim_c \sigma \forall t\).

(2) Let \(M = N \rtimes_\sigma G\) and \(\{u_g\}_g \subseteq M\) be the canonical unitaries implementing \(\sigma\) on \(N\). Let \(t \in \mathcal{S}_s(\sigma), t \leq 1\), and let \(w : G \to N\) be such that \(w_tw_t^* = p\) and \(w_tw_g = \sigma_g(p)\forall g \in G\), for some projection \(p \in N\) with \(\tau(p) = t\), and such that \(\sigma_g = \text{Ad}(w_g) \circ \sigma, g \in G\), is a weak mixing action of \(G\) on \(pNp\). Denote \(u_g' = w_gu_g^* \in pMp, g \in G\), and \(P = \langle u_g' \rangle''\). Then \(P \simeq L\mu(G)\) has finite-dimensional center (because \(G\) has finite radical, see [Di]). More than that, \(P' \cap pM^p = P' \cap P\) is finite dimensional.

But by Popa [Po4] the number of property (T) von Neumann subalgebras \(P \subseteq M\) with \(P' \cap 1_p M^1_p\) finite dimensional is countable, up to conjugacy by unitary elements in \(M\). In particular, the set of traces of the supports of such factors \(P, \tau(1_p)\), is countable. Thus, the set of \(1 \geq t > 0\) for which there exists \(P \simeq L\mu(G)\) in \(M\), with support \(1_p\) of trace \(t\) and with \(P' \cap 1_p M^1_p\) finite dimensional is countable. This proves that \(\mathcal{S}_s(\sigma)\) is countable.

If in addition \(\sigma^t\) can be chosen a weak mixing action then it follows that \(M^t = N^t \rtimes_{\sigma^t} G\) contains a von Neumann subalgebra \(P\) isomorphic to \(L\mu(G)\), for some scalar 2-cocycle \(\mu\). Since \(G\) has finite radical, \(P\) has finite-dimensional center and in fact \(P' \cap M^t\) is finite dimensional. Also, since \(G\) has the property T, each direct summand of \(P\) has the property T [CJ,B-N]. Thus \(\mathcal{F}(M) = \mathcal{F}(M^t)\) follows countable by Popa [Po4], so even more so \(\mathcal{F}^c(\sigma) = \mathcal{F}(N \subseteq M) \subset \mathcal{F}(M)\) is countable.

(3) The argument is the same as in [CJ]: Let \(M = L(\mathbb{F}_n) = L(\mathbb{F}_\infty) \rtimes_{\sigma} G\). Then \(M^s = (L(\mathbb{F}_n))^s\) still has Haagerup’s compact approximation property [H] \(\forall s > 0\). If the
Connes–Jones cocycle action $\sigma$ could be perturbed to an actual action, then there would exist a scalar 2-cocycle $\mu$ for $G$ such that $M^s = L(\mathbb{F}_n)^s$ would contain a von Neumann algebra $B_0 \subset M^s$ isomorphic to $L_\mu(G)$. By the rigidity of the inclusion $H \subset G$, this would imply that for any $\epsilon > 0$ there exists a compact, unital, completely positive map $\Phi : M^s \to M^s$ such that $\|\Phi(x) - x\|_2 \leq \epsilon \forall x \in B$, $\|x\| \leq 1$, where $B \simeq L_\mu(H)$ denotes the von Neumann subalgebra of $B_0$ corresponding to $L_\mu(H) \subset L_\mu(G)$. Since $H$ is an infinite group, $B$ contains the sequence of unitary elements $\{u_g\}_{g \in H}$ which tends weakly to 0. Thus, since $\Phi$ is compact, $\lim_{g \to \infty} \|\Phi(u_g)\|_2 = 0$, contradicting $\|\Phi(u_g) - u_g\|_2 \leq \epsilon \forall g \in H$, whenever $\epsilon < 1$. □

5.4. Definition. A discrete group $G$ is called weakly rigid ($w$-rigid) if it has an infinite normal subgroup with the relative property (T) of Kazhdan–Margulis [Ma]. Note that any infinite property (T) group is $w$-rigid. Any group $G$ of the form $\mathbb{Z}^2 \rtimes \Gamma$, with $\Gamma \subset SL(2, \mathbb{Z})$ a non-amenable group, is $w$-rigid [K, Ma, Bu], and so are all groups of the form $\mathbb{Z}^d \rtimes \Gamma$ constructed in [Va]. Also, note that if $G$ is $w$-rigid and $K$ is arbitrary then $G \times K$ is $w$-rigid.

5.5. Theorem. Let $G$ be a countable, discrete, $w$-rigid group. Let $G \curvearrowright K$ be an action of $G$ on a set $K$ such that $\forall g \neq e, |\{k \in K \mid gk \neq k\}| = \infty$. We also assume $G$ has an infinite normal subgroup $H \subset G$ with the relative property (T) such that $H \curvearrowright K$ is weak mixing (i.e. it satisfies (2.4.3')). Let $(N_0, \varphi_0)$ be an ITPF1 factor and let $\sigma$ denote the $(N_0, \varphi_0)$ CS Bernoulli $(G \curvearrowright K)$-action of $G$ on the hyperfinite factor (see 2.2.4). Let $S$ be the multiplicative group generated by the spectrum of $(N_0, \varphi_0)$ (see 2.2.0). (Thus, if $\sigma$ is the CS Bernoulli $(G \curvearrowright K)$-action of weights $\{t_j\}_j$, then $S$ is the multiplicative group generated by $\{t_i/t_j\}_{i,j}$. Then we have:

1. $\mathcal{F}^e(\sigma) = \mathcal{F}(\sigma) = \mathcal{S}_s(\sigma) = S$. Thus, $\mathcal{S}_s(\sigma') = S/t \forall t > 0$.
2. If $S \neq \{1\}$ then $\mathcal{S}(\sigma) = \mathbb{R}^*_+$. If $S = \{1\}$ then $\mathcal{S}(\sigma) = \mathbb{Z}^*_+$, so that $\mathcal{S}(\sigma') = \mathbb{Z}^*_+/t \forall t > 0$.

Proof. (1) By (5) of Theorem 3.2 we have $S \subset \mathcal{F}^e(\sigma) \subset \mathcal{F}(\sigma)$. Since $\sigma$ is a weakly mixing action itself, we have $1 \in \mathcal{S}_s(\sigma)$. By (2) of Proposition 5.2, we thus have $\mathcal{F}(\sigma) \subset \mathcal{S}_s(\sigma)$. We still need to prove that $\mathcal{S}_s(\sigma') \subset S$. To this end, assume first that $S \neq \{1\}$. Since $S \subset \mathcal{F}(\sigma)$, by (5) of Proposition 5.2 we have $S\mathcal{S}_s(\sigma) = \mathcal{S}_s(\sigma)$. Since $S$ is a multiplicative group and $\sigma' \neq \{1\}$, it follows that if we can prove that the unit ball of $\mathcal{S}_s(\sigma)$ is contained in $S$ then all $\mathcal{S}_s(\sigma)$ follows contained in $S$.

Thus, in order to finish the proof of (1) in the case $S \neq \{1\}$, we only need to prove that if $w$ is a generalized weak 1-cocycle for $\sigma$ such that $\text{Ad}(w_g) \circ \sigma_g, g \in G$, is a weakly mixing action then the support $p$ of $w$ has trace $\tau(p)$ lying in $S$.

To prove this fact note first that by 4.10 the generalized weak 1-cocycle $w_h, h \in H$, is of the form (4.10.1). Thus, by 3.2, the action $\sigma'_h = \text{Ad}(w_h) \circ \sigma_h |_{pNp}, h \in H$, has atomic fixed point algebra $B_0 = \{x \in pNp \mid \sigma'_h(x) = x \forall h \in H\}$ and there exists an atomic von Neumann subalgebra $B \subset pNp$ such that $\sigma'_h(B) = B \forall h \in H$, and such that any other $H$-invariant atomic von Neumann subalgebra (in particular $B_0$) is contained in $B$. Moreover, the algebras $B, B_0$ satisfy $B_0 = \{b \in B \mid \sigma'_h(b) = b \forall h \in H\}$. Since
$H$ is normal in $G$, by the maximality of $B$ it follows that $\sigma'_g(B) = B \forall g \in G$. But the action $\sigma'_g, g \in G$, is weakly mixing, so it has no finite dimensional invariant subspaces, and thus no atomic invariant subalgebras either, other than $C_1$. Thus $B = C$.

This shows that $\sigma'_h, h \in H$, must be weakly mixing itself. Like before, by 4.10 this forces $w_h = \lambda_h v \sigma_h(v^*), h \in H$, for some $\beta \in S_1$ and $v \in \mathcal{V}_\beta, \lambda_g \in C_1$. Thus $\beta = \tau(vv^*) = \tau(p)$, finishing the proof of this case as well.

Assume now that $S = \{1\}$. We need to prove that $\mathcal{J}_s(\sigma)$ is equal to $\{1\}$. Assume by contradiction that there exists $t \in \mathcal{J}_s(\sigma)$ with $t \neq 1$.

Let $\tilde{\lambda}_0 \in \mathbb{R}_+$ be such that $\tilde{\lambda}_0 > t, \tilde{\lambda}_0 \neq 1$, and such that $t$ does not lie in the multiplicative group $S_0$ generated by $\tilde{\lambda}_0$, equivalently $\tilde{\lambda}_0 \notin \{1^{1/n} | n \in \mathbb{Z}\}$. Let $\mathcal{O}_0$ be the CS Bernoulli $(G \curvearrowright K)$-action with weights $\{1 - t_0, t_0\}$, where $t_0 = (1 + \tilde{\lambda}_0)^{-1}$. Note that the multiplicative group $S_0$ generated by the weights of $\mathcal{O}_0$ is equal to $\mathbb{Z}$. By the remark at the end of Section 2.2.4, $\sigma_1 = \sigma \otimes \mathcal{O}_0$ is still a CS Bernoulli $(G \curvearrowright K)$-action and its ratio group $S_0$ is equal to $S_0 = \mathbb{Z}$. By the first part, we have $\mathcal{J}_s(\sigma_0) = \mathcal{J}_s(\sigma \otimes \mathcal{O}_0) = \mathbb{Z}$. Thus, $1/\tilde{\lambda}_0 \in \mathcal{J}_s(\sigma_0)$, so by (4) of Proposition 5.2 it follows that $t/\tilde{\lambda}_0 \in \mathcal{J}_s(\sigma \otimes \mathcal{O}_0) = \mathbb{Z}$. This is in contradiction with $t \notin \mathbb{Z}$.

(2) If $S \neq \{1\}$ then, by 3.3, $\sigma'$ can be perturbed to an action for any $t > 0$.

If $S = \{1\}$ and we assume $\sigma'$ is an action, then let $B_1 \subset N^t$ be the fixed point algebra of $\sigma'$, $B_1 = (N^t)^{\sigma'}$. Assume there exists a non-zero projection $q \in B_1$ of trace $\tau(q) < 1/t$. Then $\sigma'_g(q \cdot q), g \in G$, is an action of $G$ on $qN^tq \simeq N^t$, where $t' = \tau t(q) < 1$. Thus, $\sigma$ would have a generalized 1-cocycle of support $\neq 1$. But since $S = \{1\}$ and $\mathcal{V}(N) = \mathcal{U}(N)$, by 4.10 the restriction to $H$ of any generalized weak 1-cocycle for $\sigma$ must have support 1, a contradiction.

In particular, this shows that $B_1$ is finite dimensional and $t \geq 1$. To prove that $t$ is an integer, it is sufficient to prove that any minimal projection $q$ in $B_1 = (N^t)^{\sigma'}$ satisfies $t \tau(q) \in \mathbb{Z}$. By reducing with such projections, this amounts to proving that if $\sigma'$ is an ergodic action of $G$ on $N^t$ then $t$ is an integer.

To prove this we use the same trick as in the proof of (1) of Theorem 5.5. Thus, we proceed by contradiction, assuming $t$ is not an integer, and let $\lambda_0 > 0, \lambda_0 \neq 1$, be such that $t$ cannot be expressed as an integer multiple of some $\lambda_0^n, n \in \mathbb{Z}$, i.e., $t \notin \mathbb{Z}_+ \lambda_0^\mathbb{Z}$ (such $\lambda_0$ clearly exist because $t \notin \mathbb{Z}_+^\mathbb{Z}$). We take $\mathcal{O}_0$ to be the CS Bernoulli $(G \curvearrowright K)$-action of weights $\{t_0, 1 - t_0\}$, where $t_0/(1 - t_0) = \lambda_0$. The multiplicative group $S_0$ generated by $\lambda_0$ is thus $\mathbb{Z}_+^\mathbb{Z}$. Then $\sigma_1 = \sigma \otimes \mathcal{O}_0$ is still a CS Bernoulli $(G \curvearrowright K)$-action, and its corresponding ratio group $S_1$ is equal to $S_0 = \lambda_0^\mathbb{Z}$. Since $\sigma'$ is ergodic and since $\sigma'_0$ can be chosen a weakly mixing action for any $s \in \lambda_0^\mathbb{Z}$ (cf. 3.2), it follows that for any $t' \in t\lambda_0^\mathbb{Z}$ there exists a $t'$-amplification of $\sigma'_{def} = \sigma \otimes \mathcal{O}_0$ which is an ergodic action of the group $G$ on the hyperfinite type II$_1$ factor $R$.

In particular, if we take $t' \in t\lambda_0^\mathbb{Z}$, with $t' \leq 1$, this implies that there exists a generalized weak 1-cocycle $w'$ for $\sigma'$ of support $p'$ of trace $\tau(p') = t'$, such that $\sigma'_g = Adw_g' \circ \sigma'_{g \circ p' \cdot R}g, g \in G$, is an ergodic action of $G$. But since $\sigma'$ is a CS Bernoulli action of the type considered in 4.10, from that statement it follows that $B_0 = \{x \in p' \cdot R | \sigma'_h(x) = x \ \forall h \in H\}$ is atomic. By the normality of $H$ in $G$, it follows that $\sigma''_g(B_0) = B_0 \ \forall g \in G$. Since $B''_0 = \mathbb{C}$, this implies that the minimal projections of $B_0$ have equal traces. But if so is the case, then by formula (4.1.1) for $w'_{|H}$ it follows that
there can be only one $\beta$ with the property $\pi_\beta \neq 0$ in that summation. Thus, the support $p'$ of $w'$ has trace $t'$ equal to \( \dim \pi_\beta \beta \), with $\beta \in \lambda_0^\mathbb{Z}$. This implies $t \lambda_0^\mathbb{Z} \cap Z_4^+ \lambda_0^\mathbb{Z} \neq \emptyset$, a contradiction. \( \square \)

5.6. Corollary. Let $G$ be a $w$-rigid group and $\sigma, \sigma'$ be CS Bernoulli actions of $G$ on the hyperfinite $\text{II}_1$ factor, satisfying the hypothesis of Theorem 5.5. Let $S, S'$ denote the ratio groups of $\sigma$, respectively, $\sigma'$. Let $t, t' > 0$ and $\sigma', \sigma''$ be some choices of cocycle actions, defined as in 3.1.1. If $\sigma'$ is outer conjugate to $\sigma''$, then $S = S'$ and $t/t' \in S$. If in addition $\sigma', \sigma''$, are genuine actions and $\sigma'$ is assumed weakly mixing, then $t, t' \in S = S'$ and $\sigma, \sigma', \sigma''$, $\sigma'$ are all conjugate.

Proof. By 5.5 we have $\mathcal{S}_S(\sigma) = S/t, \mathcal{S}_S(\sigma) = S'/t'$. Since the strong spectrum is an outer conjugacy invariant for cocycle actions, $\sigma' \sim_\sigma \sigma''$ implies $S/t = S'/t'$. Thus $(t/t')S = S$ is a group so $t/t' \in S'$, implying also $S = S'$.

To prove the last part, note that by (5) of Theorem 5.2 we may assume $t \leq 1$. By 4.10, if $\sigma'_H$ does not have finite dimensional invariant subspaces (this is equivalent to $\sigma'_H$ being weakly mixing, by 2.4.2), then $t \in S$. By 3.2, this implies $\sigma'$, $\sigma$ are conjugate, as actions of $G$, i.e., $\sigma' \sim \sigma$. By the first part, this entails $t' \in S' = S$, and a similar argument implies $\sigma'' \sim \sigma'$. Since by 4.10 and 3.2 any weak 1-cocycle $w$ for $\sigma$ with the property that $\text{Ad}(w_g) \circ \sigma, g \in H$, is weakly mixing is weakly trivial on $H$, it follows that $\sigma \sim \sigma'$, as actions of $G$, we need the following:

5.7. Lemma. Let $G$ be a discrete group and $H \subset G$ an infinite normal subgroup of $G$. Let $\sigma$ be a properly outer action of $G$ on a type $\text{II}_1$ factor $N$ and $w : G \rightarrow \mathcal{U}(N)$ a weak 1-cocycle for $\sigma$. Assume $\sigma'_H$ is weakly mixing and $w_H$ is weakly trivial, i.e., there exists a unitary element $v \in N$ such that $vw_g \sigma_g(v^*) \in \mathbb{C} \forall g \in H$. Then $w$ is weakly trivial on all $G$, more precisely $vw_g \sigma_g(v^*) \in \mathbb{C} \forall g \in G$.

Proof. By replacing the weak 1-cocycle $w$ by $w'_g = vw_g \sigma_g(v^*)$, $g \in G$, one gets a weak 1-cocycle $w'$ for the action $\sigma$ of $G$ on $N$ which is scalar valued when restricted to $H$. Let $g \in G$. Since $\sigma'_g = \text{Ad}w'_g \circ \sigma_g, g \in G$, is an action, it follows that $\sigma'_g$ normalizes $\sigma'(H) = \sigma(H)$, i.e.,

$$\left(\text{Ad}(w'_g) \circ \sigma_g\right)\sigma_h(\sigma'^{-1}_g \circ \text{Ad}(w'_g)) \in \sigma(H) \quad \forall h \in H.$$

Thus, if we take $h \in H$ of the form $g^{-1}hg$, we get

$$\text{Ad}(w'_g \sigma_h(w'_g^*)) \sigma_h = \text{Ad}w'_g \sigma_h \text{Ad}w'_g^* \in \sigma(H) \quad \forall h \in H$$

implying that

$$w'_g \sigma_h(w'_g^*) \in \mathbb{C} \quad \forall h \in H.$$
Thus, $\sigma_h(w'_g) \in \mathbb{C}w'_g \ \forall h \in H$. But since $\sigma|_H$ is weakly mixing, the only finite-dimensional vector subspace of $L^2(N, \tau)$ invariant to $\sigma|_H$ is $\mathbb{C}1$ (cf. 2.4.3). Thus, $w'_g \in \mathbb{C}1$. Since $g \in G$ was taken arbitrary, it follows that $w$ is weakly trivial on all $G$, i.e., $w_g = v^* \sigma_g(v) \mod \mathbb{C} \ \forall g \in G$. □

**Proof of 5.6 (End).** Since $\sigma \sim_o \sigma'$ and $\sigma|_H \sim \sigma'|_H$, it follows that we may assume there exists a weak 1-cocycle $w' : G \to \mathcal{U}(N)$ for $\sigma$ such that $\sigma'_g = \text{Ad}(w'_g) \circ \sigma_g \ \forall g \in G$, and $w'_H$ is scalar valued. But by Lemma 5.7 this implies $w'$ is scalar valued on all $G$, i.e., $\sigma \sim \sigma'$. □

**5.8. Corollary.** Let $G$ be a $w$-rigid group and $\sigma_i$ be left $M_{n_i \times n_i}(\mathbb{C})$-Bernoulli $G$-actions, $i = 1, 2$, on the hyperfinite $\text{II}_1$ factor, as in 2.1. Let $t_1, t_2 > 0$. If $\sigma_1^H$ is outer conjugate to $\sigma_2^H$ then $t_1 = t_2$ and $\sigma_1$ is conjugate to $\sigma_2$. In particular, taking $n_1 = n_2 = n$, $\sigma_1 = \sigma_2 = \sigma$, $t_1 = 1$ and $t_2 = m$ it follows that the genuine actions $\{\text{id}_m \otimes \sigma\}_m$ are mutually non-outner conjugate.

**Proof.** This is just a particular case of 5.6. □

**5.9. Corollary.** Let $G$ be a $w$-rigid group and $G \rtimes K$ an action of $G$ on a set $K$, which is assumed to satisfy (2.2.4') and to be weak mixing on some infinite normal subgroup of $G$ with the relative property (T). Let $\sigma$ be a $(N_0, \tau_0)$-Bernoulli $(G \rtimes K)$-action, where $(N_0, \tau_0) = (M_{n \times n}(\mathbb{C}), \text{tr})$, for some $n \geq 2$, or $N_0 = R$ with $\tau_0$ the unique normalized trace. Then any weak 1-cocycle $w$ for $\sigma$ is coboundary modulo the scalars, i.e. if $w : G \to \mathcal{U}(N)$ is so that $\text{Ad}(w_g) \circ \sigma_g$ is an action of $G$ on $N$, then there exists a unitary element $v \in \mathcal{U}(N)$ such that $w_g = v^* \sigma_g(v) \mod \mathbb{C} \ \forall g \in G$. Moreover, the set of 1-cocycles for $\sigma$, modulo equivalence, coincides with the set of characters of $G$. In particular, if $G$ has no characters, e.g., if $G = \text{SL}(n, \mathbb{Z})$, $n \geq 3$, then any 1-cocycle for $\sigma$ is coboundary.

**Proof.** Let $H \subset G$ be an infinite, normal, subgroup with the relative property (T). By 4.10, any weak 1-cocycle $w$ for the action $\sigma$ of the group $G$ is of the form $w_h = v^* \sigma_h(v) \mod \mathbb{C} \ \forall h \in H$, for some unitary element $v \in N$. By Lemma 5.7, this implies $w$ is weakly trivial on all $G$, i.e. $w_g = \lambda_g v \sigma_g(v^*) \ \forall g \in G$, for some $\lambda_g \in \mathbb{T}$, $g \in G$. This implies $\lambda_e = 1$ and $\lambda_{g, h} = \lambda_g \lambda_h \lambda_{g, h} \ \forall g, h \in H$.

Assume further that $w$ is a 1-cocycle. Then $\mu = 1$, so that $\lambda$ follows a character. Moreover, if $\lambda$ is a character and $v$ is a unitary element in $N$ such that $\lambda_g = v^* \sigma_g(v) \ \forall g \in G$ then $\sigma_g(v) \in \mathbb{C}v \ \forall g \in G$. Since $\sigma$ is weakly mixing, by 2.4.2 this implies $v \in \mathbb{C}$ itself. This shows that different characters give non-equivalent 1-cocycles. □

The next result solves a problem posed by Connes and Jones [CJ], on whether there exist cocycle actions on the hyperfinite $\text{II}_1$ factor that cannot be perturbed to genuine actions.
5.10. Corollary. Let \((\sigma, G)\) be as in 5.9. For each non-integer \(t > 0\), let \(\sigma'\) be a choice of a cocycle action on \(N^t \simeq R\) obtained by amplifying the action \(\sigma\), as in 3.1.1. Then the cocycle action \(\sigma'\) of \(G\) on \(R\) cannot be perturbed by inner automorphisms to a genuine action.

**Proof.** This is just a reformulation of (2) of Theorem 5.5. \(\square\)

5.11. Corollary. Given any countable subgroup \(S\) of \(\mathbb{R}_+^*\) and any w-rigid group \(G\) there exist CS Bernoulli \(G\)-actions such that \(\mathcal{F}(\sigma) = S\). Moreover, one can choose \(\sigma\) to be a \((M_n \times \mathbb{R}(\mathbb{C}, \varphi_0))\) CS Bernoulli \(G\)-action, for some \(2 \leq k \leq \infty\).

**Proof.** The case \(S = \{1\}\) is clear by 5.5. If \(S \neq \{1\}\) then let \(\{s_n \mid n \geq 1\} = \{s \in S \mid s \leq 1/2\}\). Clearly, the multiplicative group generated by \(\{s_n\}_n\) is equal to \(S\). In fact, for any \(k \geq 1\) the group generated by \(\{s_n \mid n \geq k\}\) is equal to \(S\). By Theorem 5.5 it is sufficient to prove that there exists a sequence of positive numbers \(\{t_n\}_n\) such that \(\Sigma_n t_n = 1\) and such that \(\{s_n \mid n \geq 2\} = \{t_{n+1}/t_n \mid n \geq 1\} \cup \{1\}\).

First define \(t'_n = s_1 s_2 \ldots s_n, n = 1, 2, 3, \ldots\) Note that \(t'_{n+1} \leq t_n/2\) and \(\Sigma_n t'_n \leq \Sigma_n 2^{-n} \leq 1\). But then by repeating each \(t'_n\) conveniently many times one can get some multiplicities \(k_n \geq 1\) such that \(\Sigma_n k_n t'_n = 1\). Indeed, one defines \(k_j, j \geq 1\), recursively by requiring \(k_n\) to be the largest integer such that

\[
1 - \sum_{j=1}^n k_j t'_j \geq \sum_{i=n+1}^{\infty} t'_i.
\]

We then define \(\{t_n\}_n\) to be the \(t'_n\) taken with these multiplicities.

Note that another construction can be made by taking \(\sigma\) to be a (left) \((\mathcal{N}_0, \varphi_0)\) CS Bernoulli \(G\)-action with \((\mathcal{N}_0, \varphi_0) = \overline{\otimes} J(\mathcal{N}_0, j, \varphi_0, j)\), where \((\mathcal{N}_0, j, \varphi_0, j)\) is the hyperfinite type \(III_{\lambda_j}\) factor with its generalized trace, \(\{\lambda_j\}_j\) being a set of generators for \(S\). \(\square\)

5.12. Notation. If \(\sigma\) is a cocycle action of a group \(G\) on a type \(II_1\) factor \(N\) and we put \(M = N \rtimes_{\sigma} G\) then define \(\mathcal{F}(N \subset M) \overset{\text{def}}{=} \mathcal{F}(\sigma), \mathcal{F}_s(N \subset M) \overset{\text{def}}{=} \mathcal{F}_s(\sigma)\). Note that if \(\{u_g\}_g \subset M\) are the canonical unitaries implementing \(\sigma\), then \(t \in \mathcal{F}(N \subset M)\) iff there exist partial isometries \(\{w_g\}_g \subset N\) such that \(w_g w^*_g = p, \forall g,\) with \(\tau(p) = t,\) and such that \(\{w_g u_g\}_g\) gives a copy of the left regular representation of \(G\). Similarly, \(t \in \mathcal{F}_s(N \subset M)\) if in addition \(w_g\) can be chosen so that \(\Ad w_g \circ \sigma_g, g \in G,\) is weakly mixing on \(p N p\).

Denote also \(\mathcal{F}^0(N \subset M) = \mathcal{F}(\sigma)\) and \(\mathcal{F}_s^0(N \subset M) = \mathcal{F}_s(\sigma)\). Moreover, let \(\mathcal{F}(N \subset M)\) denote the fundamental group of the inclusion \(N \subset M\), i.e., the set \(\{t > 0 \mid (N \subset M)^t \simeq (N \subset M)\}\). Note that \(\mathcal{F}(N \subset M) = \mathcal{F}(\sigma)\). Put \(\mathcal{F}^0(N \subset M) = \mathcal{F}(\sigma)\). We then have:

5.13. Corollary. (1) If \(N \subset N \rtimes_{\sigma} G = M\) are as above, then

\(\mathcal{F}(N \subset M), \mathcal{F}_s(N \subset M), \mathcal{F}^0(N \subset M), \mathcal{F}_s^0(N \subset M), \mathcal{F}(N \subset M), \mathcal{F}^0(N \subset M)\)

are all isomorphism invariants for the inclusion \(N \subset M\).
(2) Each of the invariants $\mathcal{S}^*_\ast$ satisfies

$$\mathcal{S}^*_\ast((N \subset M)^t) = \mathcal{S}^*_\ast(N \subset M)/t \quad \forall t > 0.$$  

(3) Let $G$ be a w-rigid group. Let $\sigma$ be a CS Bernoulli action of $G$ on the hyperfinite factor $N$, as in the hypothesis of 5.5, with $S$ denoting its ratio group. Let $M = N \rtimes_\sigma G$ as above. Then

$$\mathcal{S}_\ast(N \subset M) = \mathcal{S}^0_\ast(N \subset M) = \mathcal{F}(N \subset M) = \mathcal{F}^0(N \subset M) = S.$$  

Moreover, if $S \neq \{1\}$ then $\mathcal{S}(N \subset M) = \mathcal{F}^0(N \subset M) = \mathbb{R}^*_+$, while if $S = \{1\}$ then $\mathcal{S}(N \subset M) = \mathcal{F}^0(N \subset M) = \mathbb{Z}^*_+$.  

**Proof.** Part (1) is trivial, since $\mathcal{S}(\sigma), \mathcal{S}_\ast(\sigma)$ are outer conjugacy invariants (thus cocycle conjugacy invariants as well) and $\mathcal{S}^c(\sigma), \mathcal{S}^c_\ast(\sigma)$ are cocycle conjugacy invariants.

Part (2) is immediate by the similar properties for $\mathcal{S}^*_\ast(\sigma)$.

To prove part (3), note that by 5.5 and the definitions it is sufficient to show that $\mathcal{S}^c_\ast(\sigma) = S$ and $\mathcal{S}^c(\sigma)$ equals $\mathbb{R}^*_+$ when $S \neq \{1\}$ and it equals $\mathbb{Z}^*_+$ when $S = \{1\}$. But this is trivial by 3.3 and the inclusions $S \subset \mathcal{S}^c_\ast(\sigma) \subset \mathcal{S}_\ast(\sigma), \mathbb{Z}^*_+ \subset \mathcal{S}^c(\sigma) \subset \mathcal{S}(\sigma)$. 

**5.14. Remarks.** (1) Part (2) of Corollary 5.13 above shows that the $\mathcal{S}$-type invariants for cross-product inclusions $N \subset M = N \rtimes_\sigma G$ satisfy “scaling” properties similar to the ones satisfied by the $\ell^2$-Betti numbers defined in [G] for inclusions of the form $A \subset M$, where $A \subset M$ is a Cartan subalgebra (e.g., of the form $L^\infty(X, \mu) \subset L^\infty(X, \mu) \rtimes G$). Note however that the calculations of our invariants does not depend on the group $G$, but rather on the action $\sigma$, while the Betti numbers for a Cartan subalgebra inclusion $A \subset A \rtimes G$ only depend on the group $G$.

(2) Some of the results in 5.5–5.10 may in fact be true for a larger class of non-amenable groups. Related to this, the following question is of interest:

*What is the class $\mathcal{G}(\mathcal{R})$ of all groups $G$ for which any cocycle action $\sigma$ on the hyperfinite type II$_1$ factor $R$ can be perturbed to a genuine action?*

Note that this is the same as requiring that $G$ satisfies $\mathcal{S}(\sigma) = \mathbb{R}^*_+$ $\forall \sigma$ cocyclic action of $G$ on $R$. Besides the amenable groups (cf. [Oc]), the class $\mathcal{G}(\mathcal{R})$ of such groups contains the free groups $F_n$ (which have no 2-cocycles) and more generally any group $G$ of the form $G = G_1 \ast H G_2 \ast H G_3 \ldots$, with $G_i, i \geq 1$, amenable and $H \subset G_i$ a common finite subgroup (cf. [Su,J1]). In turn, note that not all amalgamated free products of amenable groups belong to $\mathcal{G}(\mathcal{R})$. Indeed, by 5.10 we have that $G = SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$ does not belong to $\mathcal{G}(\mathcal{R})$, and yet $G = G_1 \ast H G_2$, with $H$ the dihedral group $\mathbb{Z}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ and $[G_1 : H] = 2, [G_2 : H] = 3$ (thus both $G_1, G_2$ being amenable).

(3) Choda has constructed in [Ch] a continuous family of mutually non-conjugate ergodic actions of the property (T) groups $G = SL(n, \mathbb{Z})$ on $R$, for $n \geq 3$. Moreover, these actions were proved to be mutually non-cocycle conjugate, modulo countable sets, in [Ka]. The argument in [Ka] relies on first proving that the set of ergodic 1-cocycles
of an ergodic action $\sigma$ of a property (T) group $G$ is countable, then using [Ch]. In this respect, our Corollary 5.9 gives an actual list of all 1-cocycles (not just the “ergodic” ones), in the case when $\sigma$ is a $M_{n \times n}(\mathbb{C})$-Bernoulli $G$-action, for $G$ arbitrary property (T) group (even w-rigid).

The actions of $G = SL(n, \mathbb{Z})$ on $R$ considered in [Ch,Ka] are not cocycle conjugate to Bernoulli shifts. Indeed, those actions are so that the corresponding crossed products $R \rtimes G$ have the property (T) as von Neumann algebras (in the sense of [CJ]), while the crossed product algebras associated to Bernoulli shift actions can never have the property (T) (cf. [Ch]). This is because the automorphism group of such a cross product is large (for instance, Aut $R \rtimes G/\text{Int} R \rtimes G$ contains a copy of $\mathbb{T}$, see 2.5.1).

Although the focus of interest in this paper is on actions of groups on factors, let us mention that the vanishing cohomology for Bernoulli actions on the hyperfinite II$_1$ factor in Corollary 5.9 readily entails the vanishing cohomology for all $(N_0, \tau_0)$-Bernoulli actions of w-rigid groups, with $(N_0, \tau_0)$ an arbitrary separable, finite, approximately finite-dimensional (AFD) von Neumann algebra (equivalently injective, by Connes [C1]):

**5.15. Corollary.** Let $G$ be a w-rigid group and $G \rtimes K$ an action of $G$ on a set $K$, which we assume to be weak mixing when restricted to some infinite normal subgroup $H \subset G$ with the relative property (T) (i.e. $H \rtimes K$ satisfies (2.4.3')) and to satisfy the outerness condition (2.2.4'). Let $(N_0, \tau_0)$ be an arbitrary separable finite, AFD von Neumann algebra and let $\sigma$ be the $(N_0, \tau_0)$-Bernoulli $(G \rtimes K)$-action, i.e. the action of $G$ on $(N, \tau) = \boxtimes_k (N_0, \tau_0)_k$ given by $\sigma_g(\boxtimes_k x_k) = \boxtimes_k x'_k$, where $x'_k = x_k^{-1} x_k \forall k \in K, g \in G$. Then any 1-cocycle $w$ for $\sigma$ is cohomologous to a character of $G$, i.e. $w : G \to \mathcal{U}(N)$ is so that $w_g \sigma_g(w_h) = w_{gh} \forall g, h \in G$ then there exists $v \in \mathcal{U}(N)$ and $\gamma \in \text{Char}(G) = \text{Hom}(G, \mathbb{T})$ such that $w_g = \gamma(g)v^* \sigma_g(v) \forall g \in G$.

**Proof.** Let $C$ be the class of finite, AFD von Neumann algebras $(N_0, \tau_0)$ for which the statement holds true. We first show that if $(N_0, \tau_0)$ can be embedded into a hyperfinite II$_1$ factor $R_0$ such that there exists a sequence of von Neumann subalgebras $N^j_0 \subset R_0, j \geq 1$, with the property that $\cap_j N^j_0 = N_0$ and $(N^j_0, \tau) \in C \forall j$, then $(N_0, \tau_0) \in C$. To this end, denote by $\sigma_0$ the $(R_0, \tau)$-Bernoulli $(G \rtimes K)$-action and by $\sigma_j$ (resp. $\sigma$) its restriction to $N^j_0$ (resp. $N_0$), $j \geq 1$. Thus, we can view $w$ as a 1-cocycle for $\sigma_j \forall j \geq 0$.

Since $R_0, N^j_0 \in C$, for each $j \geq 0$ there exists $v_j \in \mathcal{U}(N^j_0)$ such that $w_g \in v_j^* \sigma_j(g)(v_j) \forall g \in G$. Thus $\sigma_0(v_i^*) \in \mathbb{T}, v_j^* \forall i, j \geq 0$. Since $\sigma_0$ is weakly mixing (because $G \rtimes K$ is weakly mixing), it follows that $v_i = v_j$ modulo multiplication by scalars $\forall i, j \geq 0$.

In particular, $v_0 \in \cap_j N^j_0 = N_0$, showing that $N_0 \in C$.

Secondly, notice that if $A$ denotes the separable diffuse abelian von Neumann algebra then $\overline{A \boxtimes N} \subset C$ whenever $N = R$ or $N = M_{n \times n}(\mathbb{C}), n \geq 1$. Indeed, because if we let $S_\infty$ act on $A$ by left Bernoulli shifts and denote by $S_\infty$ the subgroup of all finite permutations of $\mathbb{N}$ that leave $1, 2, \ldots, n$ fixed, then $R_n = (A \rtimes S_\infty) \boxtimes N \cong R \forall n$, and $\cap_n R_n = A \boxtimes \mathbb{N}$, so the first part applies.

Notice thirdly that if $(p_n)_{n \geq 0} \subset \mathcal{P}(A)$ is a partition of 1 with projections and we denote $A_\infty = Ap_0 + \sum_{n \geq 1} \mathbb{C}p_n$ then there exists a sequence of diffuse...
subalgebras $A_m \subset A$ such that $A_\infty = \cap_mA_m$. Indeed, just let $A = \bigotimes_{k \geq 1} L^{\infty}(\mathbb{T}, \lambda)$, $B_m = \bigotimes_{k \geq m} L^{\infty}(\mathbb{T}, \lambda) \subset A$ and $A_m = Ap_0 + \sum_{n \geq 1} B_m p_n$. Thus, by the remarks above, we have $A_\infty \bigotimes N \in C$.

Finally, notice that by Connes [C1] any separable, finite AFD von Neumann algebra $(N_0, \tau_0)$ is of the form

$$N_0 = \sum_{n \geq 1} R_n q_n + A \bigotimes R_0 q_0 + \sum_{j \geq 1} A \bigotimes P_j p_{0j} + \sum_{i,k \geq 1} P_k p_{ik},$$

with $\{q_n\}_{n \geq 0} \cup \{p_{ij}\}_{i,j \geq 0}$ a partition of 1 with projections in $Z(N_0)$, $R_n \simeq R$, $P_k = M_{k \times k}(\mathbb{C})$. By applying the remarks above one gets that $\Sigma_{n \geq 0} Q_n q_n + \sum_{i,j \geq 0} Q_{ij} p_{ij} \in C$, with all $Q_{ij} \simeq R$, $Q_n \simeq R$. Then applying the third remark above we get $N_0 \in C$, thus finishing the proof. □

6. Calculation of $\mathcal{S}$-invariants for free Bernoulli actions

In the previous section we only needed the good deformation properties (namely the malleability property 4.0) of Bernoulli actions $\sigma$ of w-rigid groups $G$, in order to calculate the complete “cohomology picture” of $\sigma$. In this section we consider another example of actions satisfying “good deformation” properties, namely $G$-actions on $L(\mathbb{F}_\infty) = L(\mathbb{F}_G)$ by “free shifts”, showing that if $G$ is w-rigid then results similar to 5.5–5.10 hold true.

Thus, let $\{a_g \mid g \in G\}$ denote the generators of the free group with infinitely (but countably) many generators $\mathbb{F}_\infty = \mathbb{F}_G$. Denote by $\sigma$ the (left) action of the group $G$ on $L(\mathbb{F}_G)$, determined by

$$\sigma_g(a_h) = a_{g^{-1}h} \quad \forall g, h \in G$$

and call it the action of $G$ on $L(\mathbb{F}_G)$ by (left) free shifts.

6.1 Theorem. With $G$ a w-rigid group and $L(\mathbb{F}_G), \sigma$ as above, we have:

1. $\mathcal{S}(\sigma') = \mathbb{Z}_+^*/t$ and $\mathcal{S}_x(\sigma') = \{1/t\} \forall t > 0$. Thus, if $t > 0$ is not an integer then $\sigma'$ cannot be perturbed by inner automorphisms to a genuine action. Also, $\sigma', t > 0$, are all mutually non-outer conjugate.

2. Any weak 1-cocycle $w$ for $\sigma$ is weakly trivial and any 1-cocycle is equivalent to a character.

Proof. For simplicity, we only consider the case when $G$ has the property (T) itself. Also, we only prove the equalities hold when intersecting with the interval $(0, 1]$, leaving as an exercise the case $t > 1$, which can be done by a suitable adaptation of the similar case for Bernoulli actions in 5.5.

Denote $N = L(\mathbb{F}_G)$. We need to prove that if $w$ is a generalized weak 1-cocycle with support $p$ then $p = 1$ and there exists $v \in \mathcal{U}(N)$ such that $vw_8 \sigma_8(v)^* \in C \forall g \in G$. 
Indeed, because then the arguments in the proofs of 5.5–5.10 apply identically to get the above (1), (2).

The proof of the vanishing of the generalized weak 1-cocycles is basically the same as the proof of 4.1, except that the role of $N \overline{\otimes} N$ and $(\sigma \otimes \sigma, G)$ will this time be played by $N \ast N$ and $(\sigma \ast \sigma, G)$. Due to this, it is convenient to keep the notation $\{a_g\}_g$ for the generators of $N \ast \mathbb{C}$ and denote these same generators of $\mathbb{C} \ast N$ by $\{b_g\}_g$. We split the argument into a number of “steps”.

**Step 1:** We first show that there exist a continuous action $\alpha : \mathbb{R} \to \text{Aut}(N \ast N)$ and a period 2 automorphism $\beta \in \text{Aut}(N \ast N)$ such that

(a) $\alpha_1(N \ast \mathbb{C}) = N_0$ where $N_0$ is free with respect to $N = N \ast \mathbb{C}$ and $N, N_0$ generate $N \ast N$.

(b) $N \ast \mathbb{C} \subset (N \ast N)^\beta$.

(c) $\beta \alpha_t \beta = \pi_{-t}$ $\forall t$.

(d) $\alpha$ and $\beta$ commute with the free shift $\theta = \sigma \ast \sigma$.

This construction plays the role of Lemma 2.5.1 in the sequel. To construct these automorphisms, let $h_g \in \mathbb{C} \ast N$ be self-adjoint elements with spectrum in $[0, 2]$ such that $b_g = \exp(\pi i t g) \forall g$. We then put $\alpha_t(a_g) = \exp(\pi i t g) a_g$ and $\alpha_t(b_g) = b_g \forall g \in G, t \in \mathbb{R}$. By Voiculescu [VI] $\exp(\pi i t g) a_g$ and $b_g$ are mutually free and they clearly generate the same von Neumann algebra as $a_g, b_g$. Thus, $\alpha_t$ defines an automorphism of $N \ast \mathbb{C} \forall t$. In particular, $b_g a_g$ is free with respect to $b_g$ and they jointly generate the same algebra as $a_g, b_g$ do. Thus, (a) is satisfied.

Moreover, by the definition we clearly have $\alpha_t \alpha_s = \alpha_{t+s} \forall t, s \in \mathbb{R}$. Thus, $\alpha$ is a continuous action and it is immediate to see that it commutes with $\theta = \sigma \ast \sigma$. Also, define $\beta(a_g) = a_g$ and $\beta(b_g) = b_g^* \forall g \in G$. This clearly defines a period 2 automorphism of $N \ast N$ that commutes with $\theta$. Moreover, $\alpha_1(N \ast \mathbb{C}) = N_0$, with $N, N_0$ generating $N \ast N$ and $N \ast \mathbb{C} \subset (N \ast N)^\beta$. Furthermore, we have

$$\beta(\alpha_t(\beta(a_g))) = \beta(\alpha_t(a_g)) = \beta(\exp(\pi i t g) a_g)$$

$$= \exp(\pi i t g)^* a_g = \exp(-\pi i t g) a_g = \pi_{-t}(a_g).$$

Similarly, we get

$$\beta(\alpha_t(\beta(b_g))) = b_g = \pi_{-t}(b_g),$$

showing that all conditions (a)–(d) are satisfied.

**Step 2:** Denote by $\hat{\sigma}^w$ the representation of $G$ on $L^2(pN, \tau) \overline{\otimes} L^2(Np, \tau)$, as in 3.4. Apply (3) of Proposition 3.4 for $N = N$ and $S_1 = \{1\}$ to deduce that if $\hat{\sigma}^w$ contains the trivial representation then $p = 1$ and there exists a unitary element $v \in N$ such that $w_{\sigma} \sigma_g(v)^* \in \mathbb{C} \forall g \in G$. Thus, we are left to prove the Theorem 6.1 when $\hat{\sigma}^w$ does not contain the trivial representation. More precisely, we want to show that this assumption leads to a contradiction.

**Step 3:** Denote $\sigma' = \text{Ad}(w) \circ \sigma$ and $B_0 = (pNp)^{\sigma'}$. Consider the action of $G \oplus \mathbb{R}$ implemented by the commuting actions $\theta = \sigma \ast \sigma$ and $\alpha$ on $N \ast N$. Then define
Thus, \( u(t) \) is an isometry in \( N \ast N \) such that if we put \( l(i) = u(t)u(i)^{\ast} \), \( r(i) = u(i)^{\ast}u(t) \) then conditions (1)–(3) of Lemma 4.3 are satisfied, where \( B_0 \) is identified with \( B_0 \ast \mathbb{C} \).

Step 4: The function \( t \mapsto p_t \) follows continuous with respect to the norm-2 topology in exactly the same way as in Lemma 4.4, by using the property (T) of \( G \).

Step 5: Lemma 4.8 holds in this “free product” context as well: Indeed, we just replace \( \tilde{N} \) by \( N \ast N \) and \( N \otimes \mathbb{C} \) by \( N \ast \mathbb{C} \) throughout the proof of that lemma to deduce that if there exists a diffuse von Neumann subalgebra \( B^0 \subset N \ast \mathbb{C} \) and a partial isometry \( v_0 \in N \ast N \) satisfying \( v_0^{\ast}v_0 = 1_{B^0} \) and \( v_0B_0^{0}v_0^{*} \subset \mathcal{z}_{1/2^{n}}(N \ast \mathbb{C}) \) for some \( n \geq 1 \) then there exists a partial isometry \( v_n \in N \ast N \) and a diffuse algebra \( B_n \subset N \ast \mathbb{C} \) such that \( \tau(v_nv_n^{*}) = \tau(v_0v_0^{*}) \), \( v_n^{*}v_n = 1_{B_n} \) and \( v_nB_nv_n^{*} \subset N_0 \), where \( N_0 = \mathcal{z}_1(N \ast \mathbb{C}) \) as denoted in Step 1.

Since \( N = N \ast \mathbb{C} \) and \( N_0 \) are mutually free and they generate \( N \ast N \), by (4.3 in [Po6]) it follows that \( v_n = 0 \), thus \( v_0 = 0 \) and \( B^0 = 0 \).

Step 6: One applies the results in Steps 4 and 5 to \( B_0, u(1/2^n) \) the same way as in the proof of Lemma 4.5, to first conclude that \( B_0l(1/2^n) \) is atomic, then pursue with the same argument as in that proof to conclude that all \( B_0 \) follows atomic, and that on each minimal central projection of \( B_0 \) the automorphism \( \text{Ad}(u(1/2^n)) \) implements \( \mathcal{z}_{1/2^{n}} \), for \( n \) sufficiently large.

Step 7: We conclude, by using exactly the same construction as in the proof of Lemma 4.6, that there exists a partial isometry \( u \in N \ast N \) such that \( uu^{*} = 1_{B_0} \) and

\[
w_gU_gu = u\mathcal{z}_1(w_g)U_g \quad \forall g \in G,
\]

where \( U_g \) are the canonical unitaries in \((N \ast N) \rtimes G\) implementing the action \( \theta \) of \( G \) on \( N \ast N \).

Step 8: To end the proof of Theorem 6.1 we will prove that the relation in Step 7 entails \( u = 0 \), giving us the desired contradiction. To show this, we prove that \( \tau(uX) = 0 \) \( \forall X \in N \ast N \). We use the fact that \( N \ast N \) is generated by \( N = N \ast \mathbb{C} \) and \( N_0 = \mathcal{z}_1(N \ast \mathbb{C}) \), with \( N, N_0 \) mutually free. We denote by \( b_g' = \mathcal{z}_1(a_g) \). Thus, it is sufficient to prove that \( \tau(uX) = 0 \) for \( X \) of the form \( X_1Y_1X_2Y_2 \ldots \) or of the form \( Y_0X_1Y_1 \ldots \), where \( X_i \) are words in \( a_g \) and \( Y_j \) are words in \( b_g' \).

By the relation in Step 7 we have

\[
\tau(uX) = \tau(puXp) = \tau(w_gU_g(uX)U_g^{*}w_g^{*}) = \tau(aw_1(w_g)U_gXU_g^{*}w_g^{*}) = \tau(aw_1(w_g)U_gXw_g^{*}).
\]

Thus, if \( X \) is a word of length at least 3 in \( X_i, Y_i \), with all “letters” \( X_i, Y_i \) of trace 0, then the “middle” letters have only \( \theta_g \) acting on them, being un-altered by the right multiplication by \( w_g^{*} \in N \ast \mathbb{C} \) and by the left multiplication by \( \mathcal{z}_1(w_g) \in N_0 \). Since \( u \) can be approximated arbitrarily well by an element in the algebra generated by \( \{a_g, b_g' \mid g, h \in S\} \), for some finite subset \( S \subset G \), it follows that we have

\[
\lim_{g \to \infty} \tau(aw_1(w_g)\theta_g(X)w_g^{*}) = 0.
\]
for any such \(X\). From the above equations, this implies \(\tau(uX) = 0\) for any \(X\) with word length at least 3. A similar argument shows that if \(X\) is of the form \(X_1Y_1\) with \(\tau(X_1) = \tau(Y_1) = 0\) then \(\tau(uX) = 0\).

Thus, we are left to check the equality \(\tau(uX) = 0\) for \(X = Y_0X_1\). But this brings us to a situation equivalent to the hypothesis of Lemma 4.7. By using Step 2 and Lemma 4.7 in the same way as in the Proof of (1) of Theorem 4.1, it follows that \(\tau(uX) = 0\) in this case as well. □

6.2. Remark. (1) Note that Theorem 6.1 provides a different class of cocycle actions of property (T) groups \(G\) on \(L(F_\infty)\) that cannot be perturbed to genuine actions, than the cocycle action of Connes and Jones [CJ] (see also A.2). This is because each of our cocycle actions \(\sigma^t\) on the factor \((L(F_\infty))^t\) (which is isomorphic to \(L(F_\infty)\) \(\forall t\), by Voiculescu [V2], Radulescu [Ra]), obtained by reducing a free Bernoulli shift \(\sigma\) by a projection of trace \(t\), has the property that its \(t^{-1}\) amplification can be perturbed to an action. While none of the amplifications of the cocycle action in [CJ] can be perturbed to an action. Indeed, the argument is the same: \((L(F_n))^t\) still has Haagerup’s compact approximation property [H] \(\forall s > 0\), so \(L(G)\) cannot be embedded in it \(\forall t\) a scalar 2-cocycle of \(G\).

(2) Related to 6.1, the examples in [CJ] and problem (2) of Remark 5.14, the following question is in place:

Find the class \(\mathbb{G}\) of all groups \(G\) with the property that any cocycle action of \(G\) on any \(\Pi_1\) factor \(N\) can be perturbed to a genuine action.

Note that, like for cocycle actions on the hyperfinite type \(\Pi_1\) factor, the class \(\mathbb{G}\) contains all amenable groups [Po1] and, more generally, all the free products of amenable groups with amalgamation over finite subgroups (this follows trivially by applying [Su,J1,Po1]). A good candidate for \(\mathbb{G}\) seems to be the family of all amalgamated free products of amenable groups over common finite subgroups.

(3) The Connes–Jones cocycles seem to be “universally bad”, so the class of groups \(G\) (with some specified presentation \(F_n \to G \to 1\)) for which such cocycles vanish (or do not vanish) is an interesting test example to study for Problem (2) of Remark 6.2. The obvious obstruction for vanishing cohomology to look at in this case would be the non-embeddability of the group algebra \(L(\mu(G))\) into \(L(F_n)\), like in the argument in [CJ].

(4) Related to the result in [Ch] showing that there exist actions of \(SL(n, \mathbb{Z})\), \(n \geq 3\), on the hyperfinite \(\Pi_1\) factor such that the associated crossed product factor has the property (T) (in the sense of [CJ]), it would be interesting to know whether there exists a (cocycle) action \(\sigma\) of a property (T) group \(G\) on \(L(F_\infty)\) such that the corresponding crossed product \(\Pi_1\) factor \(L(F_\infty) \rtimes \sigma G\) has the property (T) of Connes–Jones [CJ]. Same for \(L(F_n), 2 \leq n < \infty\).

(5). Related to the proofs of Theorem 6.1 above and [Po4, Lemma 4.3.2], the following question on the group of automorphisms of the free group factors is of interest

Is \(\text{Aut}(L(F_n)), 2 \leq n \leq \infty\), path-wise connected?

One should note that \(\text{Aut} R\) is path-wise connected. Indeed, this follows trivially by the classification of single automorphisms in [C2,C3], which reduces the problem to some model automorphisms and ultimately to aperiodic such automorphisms, as
follows: Any automorphism \( \theta \in \text{Aut} R \) is conjugate to an inner perturbation of a model automorphism \( \rho_0 \). But \( \text{Ad}(u) \circ \rho_0 \) is path-wise connected to \( \rho_0 \) and \( \rho_0 \sim \rho_0 \otimes \text{id}_R \). Taking the “model” aperiodic automorphism to be an irrational rotation \( \rho_1 \), it follows that \( \rho_1 \) is path-wise connected to \( \text{id}_R \). Thus, \( \rho_0 \) follows path-wise connected to \( \rho_0 \otimes \rho_1 \). But the latter is aperiodic, so it is connected to \( \rho_1 \), thus to \( \text{id}_R \).

Acknowledgments

I am very grateful to Dima Shlyakhtenko, Masamichi Takesaki and Antony Wassermann, for pointing out to me several pertinent references related to this work. I am also grateful to Roman Sasyk, for drawing my attention to a redundancy in the initial proof of 4.1.1, and to Paul Jolissaint, for his remarks in A.1.

Appendix

A.1. Relative property \((T)\) for pairs of groups

Let \( G \) be a discrete group. A subgroup \( H \subset G \) has the relative property \((T)\) of Kazhdan–Margulis [Ma,dHV] if there exist a finite set of elements \( F_0 \subset G \) and \( \varepsilon > 0 \) such that if \(( \pi, \mathcal{H} )\) is a unitary representation of \( G \) on the Hilbert space \( \mathcal{H} \) with a unit vector \( \zeta \in \mathcal{H} \) satisfying \( \| \pi(h) \zeta - \zeta \| < \varepsilon \ \forall h \in F_0 \), then \( \mathcal{H} \) has a non-zero vector fixed by \( \pi|_H \).

If \( H = G \) this property amounts to \( G \) having the property \((T)\) of Kazhdan (see [K,Deki]). In fact, if a group \( H \) has the property \((T)\) of Kazhdan, then \( H \subset G \) has the relative property \((T)\) for any group \( G \) that contains \( H \). Also, if \( G \) has the property \((T)\) then \( H \subset G \) has the relative property \((T)\) for any subgroup \( H \subset G \). The classical non-trivial example of an inclusion of groups having the relative property \((T)\), for which neither the ambient group nor the subgroup are rigid, is \( \mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}) \) (cf. [K]). Moreover, by Burger [Bu], given any non-amenable group \( \Gamma \subset SL(2, \mathbb{Z}) \) the inclusions \( \mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma \) has the relative property \((T)\). Valette has recently proved that for any arithmetic lattice \( \Gamma \) in \( SO(n, 1) \) or \( SU(n, 1), n \geq 2 \), there exist some \( m \geq 1 \) and an action of \( \Gamma \) on \( \mathbb{Z}^m \) such that \( \mathbb{Z}^m \subset \mathbb{Z}^m \rtimes \Gamma \) has the relative property \((T)\).

In order to use the hypothesis of (relative) rigidity for a (sub)group, one typically needs to relate the distance \( \varepsilon \) from the \( \delta \)-almost invariant vector to the \( H \)-invariant one, showing that \( \varepsilon \to 0 \) as \( \delta \to 0 \). Such a continuity result, with an estimate \( \delta = O(\varepsilon) \), has already been obtained in [DeKi,K] in the case \( H = G \) has itself the property \((T)\). That argument is easily seen to work equally well for normal subgroups \( H \subset G \) with the relative property \((T)\), with the same estimate. In the case \( H \) is not necessarily normal, the appropriate continuity result was proved by Jolissaint [Jo]:

**Theorem** (Jolissaint [Jo]). Let \( G \) be a countable, discrete group and \( H \subset G \) a subgroup with the relative property \((T)\). Given any \( \varepsilon > 0 \), there exist a finite subset \( F \subset G \)
and $\delta > 0$ such that if $(\pi, \mathcal{H})$ is a unitary representation of $G$ on the Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ is a unit vector with

$$\|\pi(g)\xi - \xi\| \leq \delta \quad \forall g \in F,$$

then

$$\|\pi(h)\xi - \xi\| \leq \epsilon \quad \forall h \in H.$$

### A.2. The Connes–Jones cocycles

An important class of cocycle actions can be obtained from inclusions of group von Neumann algebras, as follows:

Let $1 \to H \hookrightarrow K \to G \to 1$ be an exact sequence of discrete groups and note that one has $L(H)' \cap L(K) = C$ if and only if $K$ has infinite conjugacy classes relative to $H$, i.e., \{hgh^{-1} \mid h \in H\} is an infinite set $\forall g \in G$, $g \neq e$. Indeed, the proof is identical to the classical case for single groups [MvN,Di]. For each $g \in G$ let $k(g) \in K$ be a lifting of $g$ in $K$ and denote by $\gamma_g \in \text{Aut} L(H)$ the automorphism $\gamma_g(x) = u_{k(g)}xu_{k(g)}^*$, $x \in L(H)$. Also for each $g_1, g_2 \in G$ denote by $v_{g_1, g_2}$ the unitary element $u_h \in L(H) \subset L(K)$, where $h = k(g_1)k(g_2)k(g_1g_2)^{-1} \in H$. Then $(\gamma, v)$ is clearly a properly outer cocycle action and $L(H) \subset L(K)$ can be viewed as the crossed product inclusion $N \subset N \rtimes \gamma, vG$, where $N = L(H)$ (see [NT]).

Along these lines, the Connes–Jones cocycle are defined as follows [CJ]: Let $G$ be an arbitrary countable discrete group with generators $g_1, g_2, \ldots, g_n$, where $1 \leq n \leq \infty$, but with $G \neq F_n$. Let $F_n \to G \to 1$ be the corresponding presentation of $(G; g_1, \ldots, g_n)$. Let $H = \ker(F_n \to G)$ and note that $H \simeq F_{nm}$, where $m = |G|$. Then $F_n$ has infinite conjugacy class relative to its normal subgroup $H$. Thus $(L(H) \subset L(F_n)) = (L(H) \subset L(H) \rtimes \gamma, vG)$ for an appropriate properly outer cocycle action $(\gamma, v)$ of $G$ on $L(H)$, with $H \simeq F_{\infty}$ whenever $G$ is an infinite group with generators $g_1, \ldots, g_n$ not all mutually free.

The starting point for the argument in [CJ] is the observation that if such a cocycle is (weakly) vanishing then $L(F_n)$ must contain a copy of a von Neumann algebra of the form $L_{\mu}(G)$, with $\mu$ a scalar 2-cocycle for $G$. The next key observation in [CJ] is that if $G$ contains an infinite subgroup $H$ such that $H \subset G$ has the relative property (T) then $L_{\mu}(G)$ cannot be embedded in $L(F_n)$, due to Haagerup’s compact approximation property of $L(F_n)$ [H]. Indeed, if one takes the approximation of the identity on $L(F_n)$ by compact, unital, trace preserving completely positive maps then the identity on $L_{\mu}(G)$ follows close to a compact operator, uniformly on all the unitaries in $L_{\mu}(G)$, a contradiction (see e.g. [Po4] for a complete argument along these lines).

### References

[H] U. Haagerup, An example of non-nuclear C*-algebra which has the metric approximation property, Invent. Math. 50 (1979) 279–293.


