JOURNAL OF NUMBER THEORY 55, 28–45 (1995)

# On a Generalization of the $3x+1$ Problem

FILIPPO MIGNOSI*

*Dept. Mat. et Appl., Via Archirafi 34, 90143 Palermo, Italia; L.I.T.P.-Université Paris 7, 2 Place Jussieu, 75251 Paris Cedex 05, France*

*Communicated by R. L. Graham*

Received June 25, 1991; revised September, 1993

We consider the following analogue of the $3x+1$ function,

$$T_\beta(n) = \begin{cases} \lceil \beta n \rceil & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even,} \end{cases}$$

where $\beta > 1$ is real, and $\lceil\ \rceil$ is the ceiling function (next largest integer). The case $\beta = \frac{3}{2}$ is just the $3x+1$ function. We prove that for almost all $\beta$, $T_\beta$ decreases iterates on average when $1 < \beta < 2$ and increases iterates on average when $\beta > 2$. We find certain values of $\beta$ where the analogue of the $3x+1$ conjecture has an affirmative answer and other values where it has a negative answer. 

## INTRODUCTION

The $3x+1$ function $T: \mathbb{Z} \to \mathbb{Z}$ is defined as follows

$$T(n) = \begin{cases} \dfrac{3n+1}{2} & \text{if } n \text{ is odd} \\ \dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The $3x+1$ Conjecture asserts that for every integer $n \geqslant 1$, there exists an integer $k \geqslant 1$, such that $T^{(k)}(n) = 1$. An account of known results on this conjecture is given in Lagarias [6].

This paper studies a generalization of the $3x+1$-problem. More precisely, for any real number $\beta > 1$, define the function $T_\beta: \mathbb{Z} \to \mathbb{Z}$ by the rule

$$T_\beta(n) = \begin{cases} \lceil \beta n \rceil & \text{if } n \text{ is odd,} \\ \dfrac{n}{2} & \text{if } n \text{ is even,} \end{cases}$$

* This work was supported by a C.N.R. (Consiglio Nazionale delle Ricerche) fellowship.



0022-314X/95 $12.00

The function $T_{3/2}$ is the $3x+1$ function $T$. Note that $T(\mathbb{Z}^+) \subseteq \mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the positive integers.

*Conjecture* $C_\beta$. The set of periodic points

$$L_\beta := \{n \in Z^+, n \geqslant 1 : T_\beta^{(k)}(n) = n \text{ for some } k \geqslant 1\}$$

of $T_\beta$ is finite, and for every integer $n \geqslant 1$, there exists some iterate $T_\beta^{(k)}(n)$ that is a periodic point.

In the case $\beta = 3/2$, with $L_{3/2} = \{1, 2\}$, Conjecture $C_\beta$ is equivalent to the $3x+1$ Conjecture.

We prove that if $\beta$ is a transcendental number or a rational number with an even denominator (in lowest terms) then, for any real number $z \geqslant 1$, under iterations of $T_\beta$ almost all natural numbers become smaller than $1/z$-times themselves when $1 < \beta < 2$, and become greater than $z$ times themselves when $\beta > 2$.

It is easy to see that if for some $z \geqslant 1$, all sufficiently large $n$ under iterations of $T_\beta$ becomes smaller than $1/z$ times itself, then Conjecture $C_\beta$ is true. Since the set of transcendental numbers $\beta$, $1 < \beta < 2$ has measure 1, the result above supports the truth of Conjecture $C_\beta$ for almost all such $\beta$ having $1 < \beta < 2$.

We also give examples of numbers $\beta$ for which Conjecture $C_\beta$ is true and examples of numbers $\beta$ for which Conjecture $C_\beta$ is false. This is, to our knowledge, the first generalization of the $3x+1$ problem for which it is possible to settle the generalized conjecture in some cases (cf. the Existence Conjecture in Section 3 of [6]).

The results of this paper easily extend to functions $T_{\beta,\alpha}$ defined for any real number $\alpha$ by the rule

$$T_\beta(n) = \begin{cases} \lceil \beta n + \alpha \rceil & \text{if } n \text{ is odd}, \\ \dfrac{n}{2} & \text{if } n \text{ is even}, \end{cases}$$

Several results of this paper were previously obtained in [7].

## 1.1. *Preliminaries*

Given a positive integer $n$, define the sequence of 0–1 valued quantities $x_{\beta,i}(n)$ by

$$T_\beta^{(i)}(n) \equiv x_{\beta,i}(n) \pmod 2$$

where $T_\beta^{(0)}(n) = n$.

Call

$$\mathbf{v}_{\beta,k}(n) := (x_{\beta,0}(n), x_{\beta,1}(n), ..., x_{\beta,k-1}(n))$$

the *parity vector* of length $k$ of $n$.

It is easy to prove by induction on $k$ that, for $k \geqslant 1$,

$$T_\beta^{(k)}(n) = 2^{-k}(2\beta)^{\sum_{i=0}^{k-1} x_{\beta,i}(n)}\, n + \rho_{\beta,k}(n) \tag{1.1}$$

where $\rho_{\beta,k}(n)$ is bounded by $(\beta^k - 1)/(\beta - 1)$.

We let $\lambda_{\beta,k}(n)$ denote the coefficient $2^{-k}(2\beta)^{\sum_{i=0}^{k-1} x_{\beta,i}(n)}$ of $n$ in (1.1).

More generally, given a 0–1 vector $\mathbf{v} = (x_0, x_1, ..., x_{k-1})$ of length $k$, set

$$\lambda_\beta(\mathbf{v}) := 2^{-k}(2\beta)^{\sum_{i=0}^{k-1} x_i} \tag{1.2}$$

and, consequently, for any $n$ and for any $\beta$ we have that $\lambda_{\beta,k}(n) = \lambda_\beta(\mathbf{v}_{\beta,k}(n))$.

We define for any $z \geqslant 1$ and for any $n \in \mathbb{Z}^+$

$$\sigma_{\beta,z}(n) := \min\left\{\sum_{i=0}^{k-1} x_{\beta,i}(n) \mid k \text{ is such that } T_\beta^{(k)}(n) < n/z\right\}$$

if the above minimum exists, otherwise $\sigma_{\beta,z}(n) := +\infty$. Also set

$$\omega_{\beta,z}(n) := \min\left\{\sum_{i=0}^{k-1} x_{\beta,i}(n) \mid k \text{ is such that } \lambda_{\beta,k}(n) < 1/z\right\}$$

if the above minimum exists, otherwise $\omega_{\beta,z}(n) := +\infty$.

For a fixed $\beta$ we call, similarly to [6] and to [8], $\sigma_{\beta,z}(n)$ the *z-stopping time* of $n$ and $\omega_{\beta,z}(n)$ the *z-coefficient stopping time* of $n$.

We also define analogously

$$\sigma^*_{\beta,z}(n) := \min\left\{\sum_{i=0}^{k-1} x_{\beta,i}(n) \mid k \text{ is such that } T_\beta^{(k)}(n) > zn\right\}$$

if the above minimum exists, otherwise $\sigma^*_{\beta,z}(n) := +\infty$. Also set

$$\omega^*_{\beta,z}(n) := \min\left\{\sum_{i=0}^{k-1} x_{\beta,i}(n) \mid k \text{ is such that } \lambda_{\beta,k}(n) > z\right\}$$

if the above minimum exists, otherwise $\omega^*_{\beta,z}(n) := +\infty$.

## 2. The Set $U$ and Density Results

For any real $\beta > 1$ and $\mathbf{v} \in \{0, 1\}^k$ define

$$S_\beta(\mathbf{v}) := \{n \in \mathbb{Z}^+ \mid \mathbf{v}_{\beta, k}(n) = \mathbf{v}\}.$$

Next define the *uniform distribution set*

$$U := \{\beta \in \mathbb{R} \mid \beta > 1 \text{ and } \forall \mathbf{v} \in \{0, 1\}^k, \mathrm{d}(S_\beta(\mathbf{v})) = 2^{-k} \text{ for all } k \geqslant 1\},$$

where $\mathrm{d}(S_\beta(\mathbf{v})) = \lim_{n \to \infty} (1/n) \operatorname{Card}\{m \in S_\beta(\mathbf{v}) \mid m \leqslant n\}$ is the usual asymptotic density of $S_\beta(\mathbf{v})$ on $\mathbb{Z}^+$.

This definition says that if $\beta \in U$ then there is a sort of uniform distribution of the positive integers with respect to the parity vectors generated by $T_\beta$; indeed techniques of uniform distribution of sequences will be useful in order to analyse the structure of $U$. In Section 3 we show that almost all real numbers belong to $U$.

Now we show that membership in $U$ guarantees for $1 < \beta < 2$ that almost all integers have a finite $z$-stopping time. The proof technique generalizes that given for $\beta = \frac{3}{2}$ and $z = 1$ by Terras [8], who also proved that $\beta = \frac{3}{2}$ belongs to $U$; the extension to $\beta = \frac{3}{2}$ and any real $z \geqslant 1$ was proved in [7] and [9].

Theorem A. (a) *If $\beta \in U$ and $1 < \beta < 2$, then for any $z \geqslant 1$ the set of positive integers having some $T_\beta^{(i)}(n) < n/z$ has density one.*

(b) *If $\beta \in U$ and $\beta > 2$, then for any $z \geqslant 1$ the set of positive integers having some $T_\beta^{(i)}(n) > zn$ has density one.*

We first establish some intermediate results.

For any $i < k - 1$, $i \geqslant 0$, $\mathbf{v}_i = (x_0, ..., x_i)$ is called a *proper prefix* of the vector $\mathbf{v} = (x_0, x_1, ..., x_{k-1})$; we also define $\mathbf{V}_{\beta, z}(l) := \{\mathbf{v} = (x_0, x_1, ..., x_{k-1}), x_i \in \{0, 1\} \mid \sum_{i=0}^{k-1} x_i = l, x_0 = 1$ and $\lambda_\beta(\mathbf{v}) < 1/z$ and for any proper prefix $\mathbf{v}_i$ of $\mathbf{v}$, $\lambda_\beta(\mathbf{v}_i) \geqslant 1/z\}$.

We show below that $\mathbf{V}_{\beta, z}(l)$ is finite because it is a set of 0–1 vectors of the fixed length $k = \lfloor \log_2(2^{l+1}\beta^l z) \rfloor$.

Since $\lambda_{\beta, k}(n)$ is completely determined by the parity vector $\mathbf{v}_{\beta, k}(n)$, then, by definition, $\omega_{\beta, z}(n)$ is determined by properties concerning parity vectors; indeed this is the gist of what follows.

Proposition 2.1. *For any real $\beta$ satisfying $1 < \beta < 2$, for any real $z \geqslant 1$ and for any integer $l \geqslant 1$,*

$$\{n \text{ odd} \mid \omega_{\beta, z}(n) = l\} = \{n \in \mathbb{Z}^+ \mid \mathbf{v}_{\beta, k}(n) \in \mathbf{V}_{\beta, z}(l)\} = \bigcup_{\mathbf{v} \in \mathbf{V}_{\beta, z}(l)} S_\beta(\mathbf{v}).$$

*Moreover any vector in $\mathbf{V}_{\beta, z}(l)$ has length $k = \lfloor \log_2(2^{l+1}\beta^l z) \rfloor$.*

*Proof.* The second equality of the proposition derives directly from the definitions.

If $m$ is such that $\mathbf{v}_{\beta,k}(m) \in \mathbf{V}_{\beta,z}(l)$ for some $k$, then it is easy to see by the definitions that $m \in \{n \text{ odd} \mid \omega_{\beta,z}(n) = l\}$.

Conversely, if $n$ is such that $\omega_{\beta,z}(n) = l$ then there exists a natural $h$ such that $\lambda_{\beta,h}(n) < 1/z$; let $k$ be the smallest of these $h$.

By the minimality it follows that for any $i$, $0 < i < k-1$, $\lambda_{\beta,i}(n) \geqslant 1/z$; in particular $\lambda_{\beta,k}(n) < 1/z \leqslant \lambda_{\beta,k-1}(n)$; hence $x_{\beta,k-1}(n) = 0$ and, consequently, by the definition of $\omega_{\beta,z}(n)$, $\sum_{i=0}^{k-1} x_{\beta,i}(n) = l = \sum_{i=0}^{k-2} x_{\beta,i}(n)$.

By the definition of $\lambda_{\beta,k}(n)$ and of $\lambda_{\beta,k-1}(n)$ it follows that $2^{-k}(2\beta)^l < 1/z \leqslant 2^{-k+1}(2\beta)^l$, which implies $\log_2(2^l\beta^l z) < k \leqslant \log_2(2^{l+1}\beta^l z)$ that is to say $k = \lfloor \log_2(2^{l+1}\beta^l z) \rfloor$.

If $n$ is odd $x_{\beta,0}(n) = 1$ and, consequently, $\mathbf{v}_{\beta,k}(n) \in \mathbf{V}_{\beta,z}(l)$. ∎

*Remark.* The fact that $n$ is odd is equivalent to $x_{\beta,0}(n) = 1$; thus it is possible to prove that

$$\{n \mid \omega_{\beta,z}(n) = l\} = \{n \in N \mid \mathbf{v}_{\beta,k}(n) \in \mathbf{V}'_{\beta,z}(l)\},$$

where $\mathbf{V}'_{\beta,z}(l)$ is defined as $\mathbf{V}_{\beta,z}(l)$ without the requirement that the first component of the vectors belonging to it be one.

COROLLARY 2.1a. *If* $\beta \in U$ *then* $d(\{n \text{ odd} \mid \omega_{\beta,z}(n) = l\}) = \mathrm{Card}(\mathbf{V}_{\beta,z}(l))\, 2^{-k}$ *where* $k$ *is the length of the vectors in* $\mathbf{V}_{\beta,z}(l)$.

*Proof.* If $\mathbf{v}_1$ and $\mathbf{v}_2$ are two distinct 0–1 vectors of the same length then the intersection $S_\beta(\mathbf{v}_1) \cap S_\beta(\mathbf{v}_2)$ is empty because $T_\beta$ is a function; the result now follows from the above propostion and the finite additivity of the asymptotic density. ∎

PROPOSITION 2.2. *For any real* $\beta$ *satisfying* $1 < \beta < 2$, *for any real* $z \geqslant 1$ *and for any integer* $l \geqslant 1$, *the set*

$$\{n \text{ odd} \mid \omega_{\beta,z}(n) = l\}$$

*differs by a finite set from*

$$\{n \text{ odd} \mid \sigma_{\beta,z}(n) = l\}.$$

*Proof.* If $\sigma_{\beta,z}(n) = l$ then $\omega_{\beta,z}(n) \leqslant l$.

Let us define

$$D_\beta(l) := \max\{\lambda_\beta(\mathbf{v}) \mid \lambda_\beta(\mathbf{v}) < 1/z,\ \mathbf{v} \in \{0,1\}^h,\ 1 \leqslant h \leqslant \lfloor \log_2(2^{l+1}\beta^l z) \rfloor\}.$$

If $\omega_{\beta,z}(n) \leqslant l$ there exists a smallest natural $h$ such that $\lambda_{\beta,h}(n) < 1/z$ and $\sum_{i=0}^{h-1} x_{\beta,i}(n) \leqslant l$; from the remark after Proposition 2.1, $h \leqslant \lfloor \log_2(2^{l+1}\beta^l z) \rfloor$. By the above definition $\lambda_{\beta,h}(n) \leqslant D_\beta(l)$.

If we take $\underline{n}$ such that

$$\underline{n} > \frac{\beta^k - 1}{\beta - 1}(1/z - D_\beta(l))^{-1},$$

where $k = \lfloor \log_2(2^{l+1}\beta^l z) \rfloor$, and if $n$ is odd and $n \geqslant \underline{n}$ then

$$\omega_{\beta,z}(n) \leqslant l \Rightarrow \omega_{\beta,z}(n) = \sigma_{\beta,z}(n);$$

indeed

$$T_\beta^h(n) = \lambda_{\beta,h}(n)\, n + \rho_{\beta,h}(n) \leqslant D_\beta(l)n + \frac{\beta^k - 1}{\beta - 1}$$

$$< D_\beta(l)n + n(1/z - D_\beta(l)) = \frac{1}{z}n.$$

In particular if $n$ is odd and $n \geqslant \underline{n}$ then

$$\omega_{\beta,z}(n) = l \Leftrightarrow \sigma_{\beta,z}(n) = l. \quad \blacksquare$$

COROLLARY 2.2a. *If $b \in U$ then*

$$d(\{n \text{ odd} \mid \sigma_{\beta,z}(n) = l\}) = \operatorname{Card}(\mathbf{V}_{\beta,z}(l))\, 2^{-k}$$

*where $k$ is the length of the vectors in $\mathbf{V}_{\beta,z}(l)$.*

*Proof.* If $\beta \in U$ then, from Corollary 2.1a, $d(\{n \text{ odd} \mid \omega_{\beta,z}(n) = l\}) = \operatorname{Card}(\mathbf{V}_{\beta,z}(l))2^{-k}$. By Proposition 2.2 $d(\{n \text{ odd} \mid \sigma_{\beta,z}(n) = l\})$ exists and has the same value. $\blacksquare$

PROPOSITION 2.3. *For any real $\beta$ satisfying $1 < \beta < 2$ and for any real $z \geqslant 1$,*

(a) $\operatorname{Card}(\mathbf{V}_{\beta,z}(1))\, 2^{-k} = 2^{-k}$, *where $k = \lfloor \log_2(2^2\beta z) \rfloor$ is the length of the vectors in $\mathbf{V}_{\beta,z}(1)$.*

(b) *Let $l > 1$.* $\operatorname{Card}(\mathbf{V}_{\beta,z}(l))\, 2^{-k_0} = \sum_{i=1}^{\lfloor \log_2(2\beta z) \rfloor} \operatorname{Card}(\mathbf{V}_{\beta,2^{-i+1}\beta z}(l-1))\, 2^{-i-k_i}$ *where $k_0 = \lfloor \log_2(2^{l+1}\beta^l z) \rfloor$ is the length of the vectors in $\mathbf{V}_{\beta,z}(l)$ and $k_i = \lfloor \log_2(2^l\beta^{l-1}2^{-i+1}\beta z) \rfloor = k_0 - i$ are the lengths of the vectors in $\mathbf{V}_{\beta,2^{-i+1}\beta z}(l-1)$.*

*Proof.* Since $\operatorname{Card}(\mathbf{V}_{\beta,z}(1)) = 1$ then (a) is trivial. In order to prove (b) we want to prove that

$$\operatorname{Card}(\mathbf{V}_{\beta,z}(l)) = \sum_{i=1}^{\lfloor \log_2(2\beta z) \rfloor} \operatorname{Card}(\mathbf{V}_{\beta,2^{-i+1}\beta z}(l-1));$$

by dividing both sides of this equality by $k_0$ we obtain (b).

Let $\mathbf{v} \in \mathbf{V}_{\beta, z}(l)$, $\mathbf{v} = (x_0, x_1, ..., x_{k-1})$ and let $i$ be such that $x_i = 1$ while $x_p = 0$, $1 \leqslant p < i$; it follows by definition that $\mathbf{v}_i = (x_i, ..., x_{k-1}) \in \mathbf{V}_{\beta, 2^{-i+1}\beta z}(l-1)$.

Conversely, if $\mathbf{v}_i = (x_i, ..., x_{k-1}) \in \mathbf{V}_{\beta, 2^{-i+1}\beta z}(l-1)$ then it follows by definition that $\mathbf{v} = (x_0, ..., x_i, ..., x_{k-1}) \in \mathbf{V}_{\beta, z}(l)$, where $x_0 := 1$ and $x_p := 0$, $1 \leqslant p < i$. ∎

We define the set $H_{\beta, z}(l)$ as, for a fixed $\beta$, the set of the odd numbers with $z$-stopping time smaller than or equal to $l$.

Since $H_{\beta, z}(l) = \bigcup_{i=1}^{l} \{n \text{ odd} \mid \sigma_{\beta, z}(n) = i\}$ and since this union is finite and composed of disjoint sets, then, from Corollary 2.2a, $\mathrm{d}(H_{\beta, z}(l))$ exists and it is equal to $\sum_{i=1}^{l} \mathrm{d}(\{n \text{ odd} \mid \sigma_{\beta, z}(n) = i\})$.

Now define the density function of odd numbers

$$g_{\beta, l}(z) := \mathrm{d}(H_{\beta, z}(l)) = \sum_{i=1}^{l} \mathrm{d}(\{n \text{ odd} \mid \sigma_{\beta, z}(n) = i\}).$$

PROPOSITION 2.4. *For any real $\beta$ satisfying $1 < \beta < 2$ and for any real $z \geqslant 1$*

(a) $g_{\beta, 1}(z) = 2^{-k}$, *where* $k = \lfloor \log_2(2^2 \beta z) \rfloor$.

(b) *For* $l \geqslant 2$ $g_{\beta, l}(z) = g_{\beta, 1}(z) + \sum_{i=1}^{\lfloor \log_2(2\beta z) \rfloor} g_{\beta, l-1}(2^{-i+1}\beta z)\, 2^{-i}$.

*Proof.* The statement of the proposition follows from the definition of $g_{\beta, l}(z)$; from the Corollary 2.2a and from Proposition 2.3. ∎

We derive part (a) of Theorem A as a consequence of the following stronger result

THEOREM B. *Suppose $\beta \in U$ and $1 < \beta < 2$ for any real $z \geqslant 1$, let $f_{\beta, l}(z)$ denote the density of integers having $z$-stopping time less than or equal to* 1. *Then there are constants $c(\beta) > 0$ and $h(\beta) < 1$ such that*

$$f_{\beta, l}(z) \geqslant 1 - 2z^{c(\beta)} h(\beta)^l \tag{2.1}$$

*for all integers $l \geqslant 1$. In fact one may take $h(\beta) = w \log_2(2\beta)/(2w - 1)$, and $c(\beta) = \log_2(w)$ where $w = \frac{1}{2} + 1/2\log_2(\beta)$ satisfies $w > 1$.*

*Proof.* We first prove a density bound in the odd integers only, namely

$$g_{\beta, l}(z) \geqslant \tfrac{1}{2} - z^{c(\beta)} h(\beta)^l, \tag{2.2}$$

and then we derive (2.1) from (2.2). Indeed we prove the stronger inequality, for any real $r > 1$

$$\tfrac{1}{2} - g_{\beta, l}(z) \leqslant y^l z^{\log_2(r)} \tag{2.3}$$

where $y = r^{\log_2(2\beta)}/(2r-1)$. If we choose $r$ in order to minimize $y$ and we call $w$ this value of $r$, we obtain $w = \frac{1}{2} + 1/2\log_2(\beta)$. Call $h(\beta)$ the value that $y$ assumes for $r = w$. Since $1 < \beta < 2$, we find $1 < w < +\infty$, so $\frac{1}{2} < h(\beta) < 1$, and (2.2) follows.

Let us prove (2.3) by induction on $l$. From (a) of Proposition 2.4, for any $z, g_{\beta,1}(z) \leqslant \frac{1}{4}$. Since $yr^{\log_2(z)} = r^{\log_2(2\beta z)}/(2r-1) \geqslant r/(2r-1) \geqslant \frac{1}{2}$, the statement is the theorem is true in the case $l = 1$.

Now let $l > 1$. From (b) of Proposition 2.4,

$$
\begin{aligned}
\tfrac{1}{2} - g_{\beta,l}(z) &= \tfrac{1}{2} - g_{\beta,1}(z) - \sum_{i=1}^{\lfloor \log_2(2\beta z)\rfloor} g_{\beta,l-1}(2^{-i+1}\beta z)\,2^{-i} \\
&= 2^{-\lfloor \log_2(2\beta z)\rfloor} - 2^{-\lfloor \log_2(2\beta z)\rfloor} \\
&\quad + \sum_{i=1}^{\lfloor \log_2(2\beta z)\rfloor} (2^{-i-1} - g_{\beta,l-1}(2^{-i+1}\beta z)\,2^{-i}) \\
&= \sum_{i=1}^{\lfloor \log_2(2\beta z)\rfloor} 2^{-i}(\tfrac{1}{2} - g_{\beta,l-1}(2^{-i+1}\beta z)) \\
&\leqslant \sum_{i=1}^{\lfloor \log_2(2\beta z)\rfloor} 2^{-i}(y^{l-1} r^{\log_2(2\beta z)} r^{-i}) \\
&\leqslant y^{l-1} r^{\log_2(2\beta z)} 2^{-1} r^{-1} \sum_{i=0}^{\infty} 2^{-i} r^{-i} \\
&= y^{l-1} r^{\log_2(2\beta z)} 2^{-1} r^{-1} \frac{2r}{2r-1} = y^l r^{\log_2(z)},
\end{aligned}
$$

completing the induction step.

We now derive (2.1) from (2.2).

It is not difficult to see by the definitions that if $g_{\beta,l}(z)$ exists then $f_{\beta,l}(z)$ also exists and

$$
f_{\beta,l}(z) = 2^{-\lfloor \log_2(z)\rfloor + 1} + \sum_{i=0}^{\lfloor \log_2(z)\rfloor - 1} g_{\beta,l}(2^{-i}z)\,2^{-i}.
$$

By (2.2)

$$
\begin{aligned}
f_{\beta,1}(z) &\geqslant 1 - \sum_{i=0}^{\lfloor \log_2(z)\rfloor - 1} (2^{-i}z)^{c(\beta)}\, h(\beta)^l\, 2^{-i} \\
&= 1 - z^{c(\beta)} h(\beta)^l \sum_{i=0}^{\lfloor \log_2(z)\rfloor - 1} 2^{-ic(\beta)} 2^{-i}.
\end{aligned}
$$

Since $c(\beta) > 0$, $2^{-ic(\beta)} < 1$ and, consequently $\sum_{i=0}^{\lfloor \log_2(z)\rfloor - 1} 2^{-ic(\beta)} 2^{-i} < 2$; inequality (2.1) is now straightforward. ∎

We remark that for a fixed $\beta$ the value $h(\beta)$ that $y$ assumes when $r = w$ cannot be improved; for any $\varepsilon > 0$

$$f_{\beta, l}(z) \geqslant 1 - z^{c(\beta)}(h(\beta) - \varepsilon)^l$$

holds for all sufficiently large $l$ depending on $\varepsilon$, $\beta$, $z$ (see the remark to Theorem D of [6]).

*Proof of Theorem* A. To prove part (a); since $h(\beta)$ is always smaller than one, if $\beta \in U$ and $1 < \beta < 2$, then $f_{\beta, l}(z)$ approaches 1 at an exponential rate as $l \to \infty$; but $f_{\beta, l}(z)$ is the density of the set of integers having $z$-stopping time less than or equal to 1. Hence if $\beta \in U$ and $1 < \beta < 2$, for any $z \geqslant 1$, the set of positive integers having some $T_\beta^{(i)}(n) < n/z$ contains a sequence of sets whose densities tend to 1, and (a) follows.

The proof of part (b) is similar to (a). We use instead of the $\mathbf{V}_{\beta, z}(l)$ the quantities $\mathbf{V}^*_{\beta, z}(l) = \{\mathbf{v} \mid \sum_{i=0}^{k-1} x_i = l,\ x_0 = 1$ and $\lambda_\beta(\mathbf{v}) > z$ and for any proper prefix $\mathbf{v}_i$ of $\mathbf{v}$, $\lambda_\beta(\mathbf{v}_i) \leqslant z\}$, and $\sigma^*_{\beta, z}(n)$ and $\omega^*_{\beta, z}(n)$ replace $\sigma_{\beta, z}(n)$ and $\omega_{\beta, z}(n)$. ∎

*Remark.* Theorem B tells us only that, for any $\delta > 0$ and $x$ large enough, the percentage of numbers smaller than $x$ and having a $z$-stopping time greater than $l$ is smaller than $2z^{c(\beta)}h(\beta)^l + \delta$, but we do not know how large $x$ must be as a function of $\delta$. This is why, if $\beta \in U$, it would be of interest to know bounds for the function $f(x, l, \beta, z)$ defined by

$$f(x, l, \beta, z) := \mathrm{d}(H_{\beta, z}(l)) - \frac{1}{x}\,\mathrm{Card}\{m \in H_{\beta, z}(l) \mid m \leqslant x\}.$$

It is possible to prove results analogous to Theorem Q of [6] (see [4] and also [1]), if a bound is given on how rapidly $f(x, l, \beta, z)$ converges to 0.

## 3. The Structure of the Set $U$

Recall the definition:

$$U := \{\beta \in \mathbb{R} \mid \beta > 1 \text{ and } \forall \mathbf{v} \in \{0, 1\}^k,\ \mathrm{d}(S_\beta(\mathbf{v})) = 2^{-k}, \text{ for all } k \geqslant 1\},$$

where $S_\beta(\mathbf{v}) := \{n \in N \mid \mathbf{v}_{\beta, k}(n) = \mathbf{v}\}$.

Our main results about the structure of $U$ are the following:

THEOREM C. *If $\beta$ is a rational number and $t/m$ is its expression in lowest terms, then $\beta$ belongs to $U$ if and only if $m$ is an even number. In particular $\frac{3}{2}$ belongs to $U$.*

THEOREM D. *If $\beta$ is a transcendental number then $\beta$ belongs to $U$.*

We do not settle membership in $U$ for algebraic numbers of degree $\geqslant 2$. In Section 4 we give examples of algebraic numbers not in $U$.

In order to prove that $\beta$ belongs to $U$ it suffices to prove $\mathrm{d}(S_\beta(\mathbf{v}))=2^{-k}$ for all vectors $\mathbf{v}=(x_0, x_1, ..., x_{k-1})$ such that $x_0=1$ and $k\geqslant 2$.

Indeed if $\mathbf{v}=(x_0)$, $x_0=1$ or 0, then $\mathrm{d}(S_\beta(\mathbf{v}))=\frac{1}{2}$ is trivial, while if $\mathbf{v}=(x_0, x_1, ..., x_{k-1})$ is such that $x_0=x_1=\cdots=x_{i-1}=0$, then $S_\beta(\mathbf{v})=2^{-i}S_\beta((x_i, x_{i+1}, ..., x_{k-1}))$.

### 3.1. *The Rational Case*

Theorem C is an immediate consequence of the definition of $U$ and of the following result.

PROPOSITION 3.1. (a) *For any rational number $\beta=t/m$, where $t$ and $m$ are relatively prime and $m$ is even, for any vector $\mathbf{v}=(x_0, x_1, ..., x_{k-1})$ such that $x_0=1$ and $k\geqslant 2$, there exists a positive integer $q$ such that $S_\beta(\mathbf{v})$ is a disjoint union of $q$ arithmetical progressions of ratio $q^{2^k}$; hence* $\mathrm{d}(S_\beta(\mathbf{v}))=2^{-k}$.

(b) *If $\beta=t/m$, where $t$ and $m$ are relatively prime and $m$ is odd, then there exists an integer $r$ such that $S_\beta((1,0))$ is a disjoint union of $r$ arithmetical progressions of ratio $2m$; hence $d(S_\beta((1,0)))=r/2m\neq\frac{1}{4}$.*

The proof of Proposition 3.1 is based on the following two lemmas. Their proofs are elementary; similar proofs can be found in almost all papers on generalizations of the $3x+1$ problem (see [6] for references and [7] for detailed proofs).

If a set $X$ is a disjoint union of $q$ arithmetical progressions of ratio $s$ we say that $X$ is *s-periodic*; clearly the asymptotic density $d(X)$ of $X$ exists and $d(X)=q/s$. ∎

Also let $\mathbf{O}$ denote the set of the odd natural numbers, and for $X\subset\mathbb{Z}^+$ define:

$$P_{\beta,k}(X):=\{n\in\mathbf{O}\mid \mathbf{v}_{b,k}(n)=(1, x_1, ..., x_{k-1}), x_i=0, 0<i<k, T_\beta^{(i)}(n)\in X\}.$$

LEMMA 3.1.1. *Let $X\subset\mathbf{O}$ be $2p$-periodic, let $\beta=t/m$, and suppose $(t, pm)=1$. Then, for any $k\geqslant 1$, $P_{\beta,k}(X)$ is $2^kpm$-periodic and* $\mathrm{d}(P_{\beta,k}(X))=2^{-k}d(X)$.

*Proof.* By definition $\ulcorner\beta P_{\beta,k}(X)\urcorner=2^{k-1}X\cap\ulcorner\beta\mathbf{O}\urcorner$. $2^{k-1}X$ is $2^kp$-periodic and $\mathrm{d}(2^{k-1}X)=2^{-k+1}\mathrm{d}(X)$. $\ulcorner\beta\mathbf{O}\urcorner$ is $2t$-periodic and $\mathrm{d}(\ulcorner\beta\mathbf{O}\urcorner)=m/2t$. Since $t$ and $p$ are relatively prime and since $X\subset\mathbf{O}$, their intersection (i.e., $\ulcorner\beta P_{\beta,k}(X)\urcorner$) is $2^kpt$-periodic and $\mathrm{d}(\ulcorner\beta P_{\beta,k}(X)\urcorner)=m/2^kt\ \mathrm{d}(X)$ and, consequently, $P_{\beta,k}(X)$ is $2^kpm$-periodic and $\mathrm{d}(P_{\beta,k}(X))=2^{-k}d(X)$. ∎

LEMMA 3.1.2. *If* $\beta = t/m$, *where* $t$ *and* $m$ *are relatively prime and* $m$ *is even, then* $P_{\beta,k}(\mathbb{Z}^+)$ *is* $2^k m$*-periodic and* $\mathrm{d}(P_{\beta,k}(\mathbb{Z}^+)) = 2^{-k}$. *If* $\beta = t/m$, *where* $t$ *and* $m$ *are relatively prime and* $m$ *is odd, then* $P_{\beta,2}(\mathbb{Z}^+)$ *is* $2m$*-periodic.*

*Proof.* Similar to the proof of Lemma 3.1.1. ▮

*Proof of Proposition* 3.1. By definition, for any vector $\mathbf{v} = (x_0, x_1, ..., x_{k-1})$ such that $x_0 = 1$ and $k \geq 2$, $S_\beta(\mathbf{v})$ is equal to $P_{\beta,k_1}(P_{\beta,k_2}(\cdots(P_{\beta,k_d}(\mathbb{Z}^+))))$ for a suitable sequence $k_1, k_2, ..., k_d$. The proof of the first part of Proposition 3.1 is by induction on $d$; the case $d = 1$ is the first part of Lemma 3.1.2. The case $d + 1$ follows by the inductive hypothesis and Lemma 3.1.1.

The proof of the second part of Proposition 3.1 follows by the second part of Lemma 3.1.2. ▮

### 3.2. *The Transcendental Case*

We prove that if $\beta$ is a transcendental number then $\beta$ belongs to $U$.

We will prove that $\mathrm{d}(S_\beta(\mathbf{v})) = 2^{-k}$ for the vectors $\mathbf{v} = (x_0, x_1, ..., x_{k-1})$ such that $x_0 = 1$ and $k \geq 2$, by using the fact that if $\beta$ is trancendental then for any natural $l$ and for any rational numbers $a_i$, $1 \leq i \leq l$, the numbers $1, a_1\beta, a_2\beta^2, a_3\beta^3, ..., a_l\beta^l$, are linearly independent over the rationals and, consequently (see [5] ch. 1, example 6.1), the sequence $n(a_1\beta, a_2\beta^2, a_3\beta^3, ..., a_l\beta^l)$, $n = 1, 2, ...$ is uniformly distributed (mod. 1) in $\mathbb{R}^l$. We assume basic notations of the theory of uniform distribution of sequences, as in [5].

DEFINITIONS. Let $i$ be a natural number greater than zero; for any real number $r$ we define

$$[r]_i := i\left\lfloor \frac{r}{i} \right\rfloor$$

$$\{r\}_i := r - [r]_i.$$

If $i = 1$ then $[r]_i$ is the usual integer part of $r$ and $\{r\}_i$ is the usual fractional part of $r$.

We also define

$$\{(r_1, r_2, ..., r_k)\}_i := (\{r_1\}_i, \{r_2\}_i, ..., \{r_k\}_i)$$

For a set $I$ of real intervals, let $\mu(I)$ denote its total length. By an *half-open interval* we mean one of the form $[a, b)$.

PROPOSITION 3.2. *Let $n$ be odd, and let* $\mathbf{v} = 10^{s}$ *or* $10^{s-1}1$. *Then, for* $h \geqslant s$

$$n \in S_\beta(\mathbf{v}) \Leftrightarrow \{\beta n\}_{2^h} \in I(\mathbf{v}),$$

where $I(\mathbf{v})$ is a finite union of half-open subintervals of $[0, 2^h)$ having measure $\mu(I(\mathbf{v})) = 2^{h-s}$.

*Proof.* The proof is by induction on $h - s$. It is easy to see by definition that if $n$ is odd, $\mathbf{v}_1 = 10^s$ and $\mathbf{v}_2 = 10^{s-1}1$ then

(a) $n \in S_\beta(\mathbf{v}_1) \Leftrightarrow 2^s$ divides $T_\beta(n) \Leftrightarrow 2^s - 1 \leqslant \{\beta n\}_{2^s} < 2^s$.

(b) $n \in S_\beta(\mathbf{v}_2) \Leftrightarrow 2^{s-1}$ divides $T_\beta(n)$ and $2^s$ does not divide $T_\beta(n) \Leftrightarrow 2^{s-1} - 1 \leqslant \{\beta n\}_{2^s} \leqslant 2^{s-1}$.

Hence the case $h - s = 0$ is true.

Now if $I$ is a subset of $[0, 2^s)$, then $\{r\}_{2^s} \in I \Leftrightarrow \{r\}_{2^{s+1}} \in I \cup (2^s + I)$, where $2^s + I$ is the set $\{2^s + i \mid i \in I\}$.

This completes the induction step. ∎

LEMMA 3.3. *Let $n$ be odd and suppose that* $\mathbf{v} = (1, x_1, ..., x_{k-1})$ *is of the form* $\mathbf{v} = 10^{s_1-1}10^{s_2-1} \cdots 10^{s_\lambda-1}10^{s_{\lambda+1}-1}1$ *or* $\mathbf{v} = 10^{s_1-1}10^{s_2-1} \cdots 10^{s_\lambda-1}10^{s_{\lambda+1}}$ *for some* $\lambda \geqslant 0$ *with all* $s_i \geqslant 1$. *Then, for* $h \geqslant k-1$, *there are sets* $I_j$, $0 \leqslant j \leqslant \lambda$, *each a finite union of half-open subintervals of* $[0, 2^h)$, *such that*:

$$n \in S_\beta(\mathbf{v}) \Leftrightarrow \{\beta n\}_{2^h} \in I_0 \quad \text{and} \quad \{\beta T_\beta^{(s_1+s_2+\cdots+s_j)}(n)\}_{2^h} \in I_j \quad \text{for } 0 \leqslant j \leqslant \lambda.$$

*Furthermore the measure* $\mu(I_j) = 2^{h-s_j}$.

*Proof.* The proof is by induction on $\lambda$, using Proposition 3.2. ∎

The following proposition is a multidimensional analogue of Proposition 3.2.

PROPOSITION 3.4. *Let $n$ be odd and suppose* $\mathbf{v} = (1, x_1, ..., x_{k-1})$ *is of the form* $\mathbf{v} = 10^{s_1-1}10^{s_2-1} \cdots 10^{s_\lambda-1}10^{s_{\lambda+1}-1}1$ *or* $\mathbf{v} = 10^{s_1-1}10^{s_2-1} \cdots 10^{s_\lambda-1}10^{s_\lambda+1}$ *for some* $\lambda \geqslant 0$, *with all* $s_i \geqslant 1$. *Then, for* $h \geqslant k-1$

$$n \in S_\beta(\mathbf{v}) \Leftrightarrow n \text{ is odd and } \{(\beta n, 2^{-s_1+1}\beta^2 n, ..., 2^{-s_1-s_2\cdots-s_\lambda+\lambda}\beta^\lambda n)\}_{2^h} \in C(\mathbf{v})$$

*where* $C(\mathbf{v})$ *is a Jordan measurable subset of* $[0, 2^h)^{\lambda+1}$ *with measure* $\mu(C(\mathbf{v})) = 2^{h(\lambda+1)-(k-1)}$

*Proof.* The proof is by induction on $\lambda$. The case $\lambda = 0$ is true by Proposition 3.2.

Let us suppose $\lambda > 0$. From Lemma 3.3 and by the inductive hypothesis we have that $n \in S_\beta(\mathbf{v}) \Leftrightarrow n$ is odd and $\{(\beta n, 2^{-s_1+1}\beta^2 n, ..., 2^{-s_1-s_2-\cdots-s_{\lambda-1}+\lambda-1}\beta^{\lambda-1}n)\}_{2^h} \in C(\mathbf{w})$ and $\{\beta T_\beta^{(s_1+s_2+\cdots+s_\lambda)}(n)\}_{2^h} \in I_{\lambda+1}$, where $\mathbf{w}$ is the prefix of $\mathbf{v}$ of length $1+s_1+s_2+\cdots+s_\lambda$, $C(\mathbf{w})$ is a subset of $[0, 2^h)^\lambda$ measurable for the Jordan measure with measure $\mu(C(\mathbf{w}))=2^{h\lambda-(s_1+s_2+\cdots+s_\lambda+s_\lambda)}$ and where $I_{\lambda+1}$ is a finite union of half-open subintervals of $[0, 2^h)$, such that $\mu(I_{\lambda+1})=2^{h-s_{\lambda+1}}$.

Hence, in order to prove Proposition 3.4, we have to prove that there exists a set $C(\mathbf{v})$, subset of $[0, 2^h)^{\lambda+1}$, measurable for the Jordan measure with $\mu(C(\mathbf{v}))=2^{h(\lambda+1)-(k-1)}$ (i.e., $\mu(C(\mathbf{v}))=\mu(C(\mathbf{w}))\,\mu(I_{\lambda+1})$) and which is such that:

$$\{(\beta n, 2^{-s_1+1}\beta^2 n, ..., 2^{-s_1-s_2-\cdots-s_{\lambda-1}+\lambda-1}\beta^{\lambda-1}n)\}_{2^h}\in C(\mathbf{w})$$

and

$$\{\beta T_\beta^{s_1+s_2+\cdots+s_\lambda}(n)\}_{2^h}\in I_{\lambda+1}$$
$$\Leftrightarrow \{(\beta n, 2^{-s_1+1}\beta^2 n, ..., 2^{-s_1-s_2-\cdots-s_\lambda+\lambda}\beta^\lambda n)\}_{2^h}\in C(\mathbf{v}).$$

Notice that

$$\{\beta T_\beta^{(s_1)}(n)\}_{2^h}=\{(\beta n+1-\{\beta n\}_1)\,2^{-s_1+1}\beta\}_{2^h}=g_2(\beta n, 2^{-s_1+1}\beta^2 n),$$

where $g_2$ is the piecewise linear real function

$$g_2\colon \mathbb{R}^2\to[0, 2^h)$$

defined by

$$g_2(y_1, y_2):=\{y_2+2^{-s_1+1}\beta-\{y_1\}_1\,2^{-s_1+1}\beta\}_{2^h}.$$

Observe that there exists a finite partition of $[0, 2^h)\times[0, 2^h)$ composed of Jordan measurable sets, such that, on each element of this partition, the function $g_2$ is linear for both variables with nonzero partial derivatives. More precisely, since the $g_2$ is of the form $\{y_2+Y\}_{2^h}$ with $Y$ not containing the second variable $y_2$, each element of this partition is a "vertical stripe"; indeed, if the couple $(r_1, r_2)$ belongs to an element of the partition then $(r_1, s)$, $0\leqslant s<2^h$ belongs to same element.

In the same way (more formally by induction on $\lambda$) it is possible to prove that there exists a piecewise linear real function $g_{\lambda+1}$, $g_{\lambda+1}$: $R^{\lambda+1}\to[0, 2^h)$ such that

$$\{\beta T_\beta^{(s_1+s_2+\cdots+s_\lambda)}(n)\}_{2^h}=g_{\lambda+1}((\beta n, 2^{-s_1+1}\beta^2 n, ..., 2^{-s_1-s_2-\cdots-s_\lambda+\lambda}\beta^\lambda n)),$$

and that there exists a finite partition of $[0, 2^h)^\lambda$ composed of Jordan measurable sets, such that, on each element of this partition, the function $g_{\lambda+1}$ is linear for all variables all partial derivatives of which are nonzero.

More precisely, since the $g_{\lambda+1}$ is of the form $\{y_{\lambda+1}+Y\}_{2^h}$ with $Y$ non containing the last variable $y_{\lambda+1}$, each element of this partition is a "vertical stripe"; indeed, if $(r_1, r_2, ..., r_{\lambda+1})$ belongs to an element of the partition then $(r_1, r_2, ..., s)$, $0 \leqslant s < 2^h$ belongs to same element.

We know that $I_{\lambda+1} = [\gamma_1, \delta_1) \cup [\gamma_2, \delta_2) \cup \cdots \cup [\gamma_z, \delta_z)$, where $\gamma_i, \delta_i \in [0, 2^h)$; let us define the sets $E_i = \{x \in [0, 2^h)^{\lambda+1} \mid \gamma_i \leqslant g_{\lambda+1}(x) < \delta_i\}$. Clearly

$$\{\beta T_\beta^{(s_1+s_2+\cdots+s_\lambda)}(n)\}_{2^h} \in I_{\lambda+1}$$
$$\Leftrightarrow g_{\lambda+1}((\beta n, 2^{-s_1+1}\beta^2 n, ..., 2^{-s_1-s_2-\cdots-s_\lambda+\lambda}\beta^\lambda n)) \in I_{\lambda+1}$$
$$\Leftrightarrow (\beta n, 2^{-s_1+1}\beta^2 n, ..., 2^{-s_1-s_2-\cdots-s_\lambda+\lambda}\beta^\lambda n) \in E := \bigcup_{i=1}^{z} E_i.$$

Let us define $C^*(\mathbf{w})$ as the set of all vectors of $[0, 2^h)^{\lambda+1}$ having their prefix of length $\lambda$ which belongs to $C(\mathbf{w})$; $C^*(\mathbf{w})$ is composed of "vertical stripes" and $\mu(C^*(\mathbf{w})) = \mu(C(\mathbf{w}))\, 2^h$. Finally let us define $C(\mathbf{v}) := C^*(\mathbf{w}) \cap E$.

Since there exists a partition of $[0, 2^h)^{\lambda+1}$ composed of measurable sets such that $g_{\lambda+1}$ is linear on all the variables on each element of this partition and since the partial derivative of $g_{\lambda+1}$ with respect to the last variable is not equal to zero on each element of this partition, it is easy to see that each $E_i$, and, consequently $E$ is a Jordan-measurable set with $\mu(E_i) = 2^{h\lambda}(\delta_i - \gamma_i)$ and $\mu(E) = 2^{h\lambda}\mu(I_{\lambda+1})$. It follows that $C(\mathbf{v})$ is Jordan-measurable and by standard techniques it also follows that $\mu(C(\mathbf{v})) = \mu(C(\mathbf{w}))\,\mu(I_{\lambda+1})$. ∎

*Proof of Theorem* D. Let $\mathbf{v} = (1, x_1, ..., x_{k-1})$, $x_i \in \{0, 1\}$, $k \geqslant 2$, let $h \geqslant k-1$ and let $C(\mathbf{v})$ be as in the statement of Proposition 3.4.

If $\beta$ is a transcendental number then for any natural $\lambda$ the numbers $1, \beta, 2^{-s_1+1}\beta^2, 2^{-s_2+2}\beta^3, ..., 2^{-(s_1+s_2+\cdots+s_\lambda)+\lambda}\beta^\lambda$ are linearly independent over the rationals. From [5, Chap. 1, ex, 6.1] the sequence $(\beta, 2^{-s_1+1}\beta^2, 2^{-s_2+2}\beta^3, ..., 2^{-(s_1+s_2+\cdots+s_\lambda)+\lambda}\beta^\lambda)(2m+1)$, $m = 1, 2, ...$, is uniformly distributed (mod. $2^h$) in $R^{\lambda+1}$, for any fixed $h$.

By Proposition 3.4, $C(\mathbf{v})$ is measurable (for the Jordan measure) and, consequently,

$$\begin{aligned} &\mathrm{d}(\{m \in \mathbb{Z}^+ \mid (\beta, 2^{-s_1+1}\beta^2, 2^{-s_2+2}\beta^3, ..., 2^{-(s_1+s_2+\cdots+s_\lambda)+\lambda}\beta^\lambda) \\ &\quad \times (2m+1) \in C(\mathbf{v})\}) \\ &= \mu(C(\mathbf{v}))\,(\mu([0, 2^h)^{\lambda+1}))^{-1} \\ &= 2^{-(k-1)}. \end{aligned}$$

But, by Proposition 3.4 once again, $S_\beta(\mathbf{v}) = 2\{m \in \mathbb{Z}^+ \mid (\beta, 2^{-s_1+1}\beta^2, 2^{-s_2+2}\beta^3, ..., 2^{-(s_1+s_2+\cdots+s_\lambda)+\lambda}\beta^\lambda)(2m+1) \in C(\mathbf{v})\} + 1$; this implies that $d(S_\beta(\mathbf{v})) = \frac{1}{2} d(\{m \in \mathbb{Z}^+ \mid (\beta, 2^{-s_1+1}\beta^2, 2^{-s_2+2}\beta^3, ..., 2^{-(s_1+s_2+\cdots+s_\lambda)+\lambda}\beta^\lambda)(2m+1) \in C(\mathbf{v})\}) = 2^{-k}$. ▮

## 4. Examples

Theorem A provides evidence for the truth of Conjecture $C_\beta$ when $\beta \in U$ and $1 < \beta < 2$. However we are unable to prove Conjecture $C_\beta$ for any $\beta \in U$. In this section we explicitly produce values $1 < \beta < 2$ for which Conjecture $C_\beta$ is true and other values for which it is false. None of these values belongs to $U$ and, furthermore, $S_\beta(\mathbf{v})$ is empty for numerous vectors $\mathbf{v}$ in both cases.

The fact that all transcendental numbers belong to $U$ gives us the following information: if there exist real numbers leading to easier problems they must be algebraic.

Theorem E. *For $\beta = \sqrt{2}$, Conjecture $C_\beta$ is true. In this case $\{1, 2\}$ is a cycle for $T_\beta$, and for any natural number $n$ there exist a $k$ such that $T_\beta^{(k)}(n) = 1$, with $k \leqslant O(\log^2(n))$.*

Before proving Theorem E we establish some intermediate results.

Proposition 4.1. *Suppose that $\beta = \sqrt{2}$.*

(a) *If for $0 \leqslant i \leqslant 2q$, $T_\beta^i(n)$ is odd, then $T_\beta^{(2q)}(n) = 2^q(n+1) - 1$.*

(b) *If for $0 \leqslant i \leqslant 2p-1$, $T_\beta^{(i)}(n)$ is odd and $T_\beta^{(2p)}(n)$ is even, then $T_\beta^{(2p)}(n) = 2^p(n+1)$.*

*Proof.* We claim that if $n$ is odd and $T_\beta(n)$ is odd then $T_\beta^{(2)}(n) = 2n+1$ or $T_\beta^{(2)}(n) = 2n+2$. Indeed we know that $T_\beta(n) = \sqrt{2}\, n + \rho_1$, where $0 < \rho_1 < 1$ and $T_\beta^{(2)}(n) = 2n + \sqrt{2}\,\rho_1 + \rho_2$, where $0 < \rho_2 < 1$; the claim follows from the fact that $0 < \sqrt{2}\,\rho_1 + \rho_2 < 3$.

Using this claim it is easy to prove (a) by induction on $q$.

To prove (b) take $q = p-1$; we have by (a) that $T_\beta^{(2q)}(n) = 2^{(q)}(n+1) - 1$ and that $T_\beta^{(2q+1)}(n)$ is odd. It follows from the claim that $T_\beta^{(2p)}(n) = T_\beta^{(2q+2)}(n) = 2(2^q(n+1) - 1) + 2 = 2^q(n+1)$. ▮

The following lemma is a special case of Liouville's Theorem.

Lemma 4.1. *For any integer $p$ and for any natural number $q$, $q > 0$, $|\sqrt{2}/2 - p/q| > 1/4q^2$.*

*Proof.* This follows from Theorem 191 of [3]. ▮

PROPOSITION 4.2. *For $\beta=\sqrt{2}$ and any n, one of the first $\lfloor 2\log_2(n+1)\rfloor+5$ iterates $T_\beta^{(i)}(n)$ is even.*

*Proof.* Suppose not, and let $2p$ be the largest even number less than $2\log_2(n+1)+5$ so that $T_\beta^i(n)$ is odd for $0\leqslant i\leqslant 2p+1$.

We show that the denominator $q=n+1$ violates Lemma 4.1.

Thus, by (a) of Proposition 4.1, $T_\beta^{(2i+1)}(n)=\lfloor\sqrt{2}\,T_\beta^{2i}(n)\rfloor+1=\lfloor\sqrt{2}(2^i(n+1)-1)\rfloor+1$ is odd, which is equivalent to $0\leqslant\{\sqrt{2}(2^i(n+1)-1)\}_2<1$. This, in turn, is equivalent to saying that for any $i$, $0\leqslant i\leqslant 2p$, either

$$2-(2-\sqrt{2})\leqslant\{\sqrt{2}\,2^i(n+1)\}_2<2 \quad\text{or}\quad 0\leqslant\{\sqrt{2}\,2^i(n+1)\}_2<\sqrt{2}-1.$$

Now it can be proved by induction on $p$ that either

$$2-\frac{(2-\sqrt{2})}{2^p}\leqslant\{\sqrt{2}(n+1)\}_2<2 \quad\text{or}\quad 0\leqslant\{\sqrt{2}(n+1)\}_2<\frac{\sqrt{2}-1}{2^p}.$$

In either case there is an integer $m$ with

$$\left|\frac{\sqrt{2}}{2}-\frac{m}{n+1}\right|<\frac{1}{(n+1)\,2^{p+1}}.$$

Since the bound on $p$ implies $2^p>2(n+1)$, this contradicts Lemma 4.1. ▮

*Proof of Theorem* E. We claim that for any natural number $n$ there exists a number $h<3(\log_2(n+1)+3)$ such that $T_\beta^{(h)}(n)<(\sqrt{2}/2)(n+1)$. Without loss of generality, $n$ is odd, for otherwise the statement of the claim is trivially verified.

Let $i$ be the smallest integer such that $T_\beta^{(i)}(n)$ is even, and by Proposition 4.2 $i\leqslant 2\log_2(n+1)+5$. If $i$ is even set $p=i/2$ and if $i$ is odd set $p=(i-1)/2$.

In the case $p=0$ the claim is easily verified. If $p>0$, then from Proposition 4.1, $T_\beta^{(i)}(n)=2^p(n+1)$ or $T_\beta^{(i)}(n)=2^p(T_\beta(n)+1)$ and, consequently $T_\beta^{(i+1+p)}(n)=(n+1)/2$ or $T_\beta^{(i+1+p)}(n)=(T_\beta(n)+1)/2$. In both cases

$$T_\beta^{(i+1+p)}(n)<\frac{\sqrt{2}}{2}(n+1),$$

and clearly $h:=i+p+1<3(\log_2(n+1)+3)$.

Hence the claim is proved.

By the claim it follows that $T_\beta^{(h)}(n)<n$ whenever $n>4$. Hence some iterate $T_\beta^{(i)}(n)=1$ or 2 or 3, and since $\{1,2\}$ is a cycle and $T_\beta^{(5)}(3)=1$, the first part of Theorem follows. The claim also shows that $k \leqslant O(\log^2(n))$ if $k$ is the smallest natural such that $T_\beta^{(k)}(n) = 1$. ∎

The idea upon which Theorem E is based is that if many of the digits of $\mathbf{v}_{\beta,k}(n)$ are equal to one (which means that $\lambda_{\beta,k}(n) = \lambda_\beta(\mathbf{v}_{\beta,k}(n))$ is large) then $(\sqrt{2}/2)(n+1)$ must be close to an integer; this allows us to use Liouville's Theorem.

With the same techniques we can show that Conjecture $C_\beta$ is true for all the $\beta$ of the form $\sqrt[n]{2}$ and for some other fractional powers of 2 among which $\sqrt[3]{4}$. We have not been able to prove Conjecture $C_\beta$ for a dense set of real numbers in $[1, 2]$, however.

Brocco [2] proved that Conjecture $C_\beta$ is false for all Pisot numbers (Pisot numbers are those real algebraic integers $\beta > 1$, such that all their conjugates are of modulus strictly smaller than one). Here we give an elementary proof for $\beta = (\sqrt{5}+1)/2$, the "golden number".

THEOREM F (Brocco). *Let* $\beta = (\sqrt{5}+1)/2$. *Then* $\lim_{n\to\infty} T_\beta^{(k)}(3) = \infty$.

*Proof.* We claim that $T_\beta^{(2)}(x)$ is odd when $x$ and $T_\beta(x)$ are odd. Let $x$ be odd; definition $T_\beta(x) = \beta x + \rho$ where $0 < \rho < 1$ and

$$T_\beta^{(2)}(x) = \lfloor \beta^2 x + \beta\rho \rfloor + 1 = \lfloor \beta x + x + \beta\rho \rfloor + 1 = x + \lfloor \beta x + \beta\rho \rfloor + 1.$$

Since $0 < \rho < 1$ and $1 < \beta < 2$, $\rho < \beta\rho < \rho + 1$, which implies that $T_\beta(x) < \beta x + \beta\rho < T_\beta(x) + 1$, i.e., $\lfloor \beta x + \beta\rho \rfloor = T_\beta(x)$. Thus $T_\beta^{(2)}(x) = x + T_\beta(x) + 1$; since $x$ and $T_\beta(x)$ are odd then $x + T_\beta(x)$ is even and $T_x^{(2)}(x)$ is odd.

Since $T_\beta(3) = 5$, by iterating the above claim we obtain that for any $k \geqslant 0$, $T_\beta^{(k)}(3)$ is odd, and, consequently, $T_\beta^{(k+1)}(3) > T_\beta^{(k)}(3)$. More precisely the sequence $T_\beta^{(k)}(3)$ is completely defined by the inductive rule: $T_\beta^{(0)}(3) = 3$, $T_\beta^{(1)}(3) = 5$, $T_\beta^{(k+2)}(3) = T_\beta^{(k+1)}(3) + T_\beta^{(k)}(3) + 1$. ∎

We remark that the proof shows that there are no natural numbers $x$ such that $x$ and $T_\beta(x)$ are odd and $T_\beta^2(x)$ is even. In consequence $S_\beta((1, 1, 0))$ is empty, so $d(S_\beta((1, 1, 0))) = 0$ and, consequently, $\beta = (\sqrt{5}+1)/2$ does not belong to $U$.

## ACKNOWLEDGMENTS

I thank J. P. Allouche, S. Brocco, Professor A. Restivo, and especially the referee for their suggestions and advice.

## REFERENCES

1. J.-P. ALLOUCHE, "Sur la conjecture de Syracuse-Kakutani-Collatz", Séminaire de Théorie de Nombres, 1978–1979," Exp. No. 9, 15 pp., CNRS, Talence, France, 1979.

2. S. BROCCO, A Note on Mignosi's generalization of the $3x+1$ problem, *J. Number Theory* **52** (1995), 173–178.
3. G. H. HARDY AND E. M. WRIGHT, "An Introduction to the Theory of Numbers," 5th ed., Oxford Univ. Press, Oxford, 1983.
4. E. HEPPNER, Eine Bemerkung zum Hasse-Syracuse Algorithmus, *Archiv. Math.* **31** (1978), 317–320.
5. KUIPERS AND NIEDERREITER, "Uniform Distribution of Sequences," Wiley–Interscience New York, 1974.
6. J. C. LAGARIAS, The $3x+1$ problem and its generalizations, *Amer. Math. Monthly* **92**, No. 1 (1985), 3–23.
7. F. MIGNOSI, "Sulla congettura di Collatz-Thwaites," Tesi di laurea, Università degli Studi di Palermo (Italia), luglio 1987.
8. R. TERRAS, A stopping time problem on the positive integers, *Acta Arith.* **30** (1976), 241–252.
9. G. VENTURINI, On the $3x+1$ problem, *Adv. Appl. Math.* **10** (1989) 344–347.