



Sine-Gordon expectation values of exponential fields with variational perturbation theory

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Abstract

In this Letter, expectation values of exponential fields in the 2-dimensional Euclidean sine-Gordon field theory are calculated with variational perturbation approach up to the second order. Our numerical analysis indicates that for not large values of the exponential-field parameter a , our results agree very well with the exact formula conjectured by Lukyanov and Zamolodchikov [Nucl. Phys. B 493 (1997) 571].

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1. Introduction

This Letter briefly reports our investigation on vacuum expectation value (VEV) of the exponential field $e^{ia\phi(\vec{r})}$ in the sine-Gordon (sG) field theory, G_a , with variational perturbation theory (VPT). Here, a is a parameter, and $\phi(\vec{r})$ is the field operator at the 2-dimensional Euclidean space (2DES) point $\vec{r} = (x, \tau)$ (τ is the Euclidean time).

In 1997, starting from the exact expressions for the three special cases: the coupling $\beta \rightarrow 0$ (semi-classical limit), $\beta = \frac{1}{2}$ and $a = \beta$, Lukyanov and Zamolodchikov guessed an exact formula for the VEV of $e^{ia\phi(\vec{r})}$ in the sG field theory at any $\beta^2 < 1$ and $|\text{Re}(a)| < 1/(2\beta)$ [1]. Then, defining “fully connected” one-point functions, σ_{2n} (n is any natural number), from the VEVs of even-power fields ϕ^{2n} , they showed that σ_2 and σ_4 from the above-mentioned exact formula agree with those from perturbation theory for the sG field theory up to β^4 and that σ_2 agrees with

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the corresponding one-point function from perturbation theory up to g the coupling in the massive Thirring model, which is the fermion version of the sG field theory [2]. Furthermore, based on the reflection relations with Liouville reflection amplitude [3], some extra arguments for supporting the conjectured exact formula were presented by their collaborators with them in the subsequent papers [4]. Slightly later, in 2000, checks from perturbation theories in both an angular and a radial quantization approaches for the massive Thirring model indicated that the perturbation result up to g exactly coincides with the corresponding result obtained by expanding the exact formula according to the coupling g [5]. Additionally, a numerical study for the model in finite volume also provides evidence for the conjectured exact formula [6]. Thus, these investigations have given the conjectured exact formula for the VEV of $e^{ia\phi(\vec{r})}$ a complete check for the case of $\beta \rightarrow \frac{1}{2}$ ($g \rightarrow 0$ is equivalent to $\beta^2 \rightarrow \frac{1}{2}$, from Eq. (8) in Ref. [1]), and some indirect evidences for its validity. Obviously, a direct check for the cases of $\beta^2 \neq \frac{1}{2}$ (Ref. [1] has provided a partial check for the case of $\beta \rightarrow 0$) is still needed.

In fact, VPT [7] may provide such a check. It is some kind of expansion theory similar to the perturbation theory. However, because it properly “grafts” the variational approach onto the perturbation theory, the VPT produces non-perturbative results which are valid for any coupling strength, including both weak and strong couplings. Moreover, because the principle of minimal sensitivity (PMS) [8] is used to determine a man-made parameter, the VPT is believed to be a convergent theory, and its approximate accuracy can be systematically controlled and improved to tend towards the exact [7]. The primitive idea of VPT is not a new one and can date back to 1955 at least [9]. Now, it has developed with many equivalent practical schemes, and has been applied to quantum field theory (QFT), condensed matter physics, statistical mechanics, chemical physics, and so on [7]. In this Letter, based on the variational perturbation scheme in one of our former joint papers [10] (the scheme in Ref. [10] was stemmed from the Okopinska’s optimized expansion [11], and proposed by Stancu and Stevenson [12]), we will develop a variational perturbation scheme for the purpose of the present Letter and calculate approximately G_a up to the second order. No explicit divergences exist in the resultant expression owing to the adoption of the Coleman’s normal-ordering renormalization prescription [2,10,13]. One will see that our investigation strongly support the conjectured exact formula.

In next section, we will develop the VPT to calculate the VEV of $e^{ia\phi(\vec{r})}$ and give the VEV of $e^{ia\phi(\vec{r})}$ up to the second order in VPT. In Section 3, we will report our numerical results and make a numerical comparison between our results and the exact ones. A brief conclusion will be made in Section 4.

2. VPT for VEVs and approximate G_a up to the second order

We consider the 2-dimensional Euclidean sG field theory with the following Lagrangian density

$$\mathcal{L}_{\text{sG}} = \frac{1}{2} \nabla_{\vec{r}} \phi_{\vec{r}} \nabla_{\vec{r}} \phi_{\vec{r}} - 2\Omega \cos(\sqrt{8\pi} \beta \phi_{\vec{r}}). \quad (1)$$

In this Letter, the subscript \vec{r} represents the coordinate argument, for example, $\phi_{\vec{r}} \equiv \phi(\vec{r})$, and $\nabla_{\vec{r}}$ is the gradient in the 2DES. The Lagrangian density Eq. (1) is nothing but Eq. (5) in Ref. [1] if one makes the transform $\phi \rightarrow \frac{\phi}{\sqrt{8\pi}}$ (hereafter, we will use $e^{i\sqrt{8\pi}a\phi(\vec{r})}$ instead of $e^{ia\phi(\vec{r})}$ and consequently the parameter a and the coupling β in this Letter are identical to those in Ref. [1], respectively). If taking $\sqrt{8\pi}\beta \rightarrow \beta$ and $2\Omega = m^2/\beta^2$ and adding the term m^2/β^2 in the Lagrangian density, one will get the Euclidean version of the sG Lagrangian density which discussed in Ref. [10]. In Eq. (1), the dimensionless β is the coupling parameter and Ω is another parameter with the dimension [length] $^{-2}$ in the natural unit system. It is always viable to have $\beta \geq 0$ without loss of generality. The classical potential $V(\phi_{\vec{r}}) = -2\Omega \cos(\sqrt{8\pi}\beta\phi_{\vec{r}})$ is invariant under the transform $\phi \rightarrow \phi + \frac{2\pi n}{\sqrt{8\pi}\beta}$ with any integer n , and so the classical vacua are infinitely degenerate. So do the quantum vacua of the sG field theory according to Ref. [10]. Here, as did in Ref. [1], we choose to consider the symmetry vacuum with the expectation value of the sG field operator $\phi_{\vec{r}}$ vanishing instead of those spontaneous symmetry broken vacua.

The VEV of the exponential field $e^{i\sqrt{8\pi}a\phi(\vec{r})}$ is defined as follows [1]

$$G_a \equiv \langle e^{i\sqrt{8\pi}a\phi(0)} \rangle \equiv \frac{\int \mathcal{D}\phi \exp\{i\sqrt{8\pi}a\phi(0)\} \exp\{-\int d^2\vec{r} \mathcal{L}_{sG}\}}{\int \mathcal{D}\phi \exp\{-\int d^2\vec{r} \mathcal{L}_{sG}\}}. \quad (2)$$

For simplicity, the exponential field in Eq. (2) is taken at $r = 0$. It is evident that the numerator and denominator in the right-hand side of Eq. (2) can easily be got from the following sG generating functional

$$Z[J] = \int \mathcal{D}\phi \exp\left\{-\int d^2\vec{r} [\mathcal{L}_{sG} - J_{\vec{r}}\phi_{\vec{r}}]\right\}, \quad (3)$$

by taking $J_{\vec{r}} = i\sqrt{8\pi}a\delta(\vec{r})$ ($\delta(\vec{r}) \equiv \delta(x)\delta(\tau)$) and $J = 0$, respectively. In Eq. (3), $J_{\vec{r}}$ is an external source at \vec{r} . For renormalizing G_a , we will use its normal-ordering form [10,14].

To perform a variational perturbation expansion on G_a , now we modify $Z[J]$ in Eq. (3) by following Ref. [10] only without shifting the field $\phi_{\vec{r}}$ (this is not necessary here because we choose to consider the symmetrical vacuum as aforementioned). That is, first introduce a parameter μ by adding a vanishing term $\int d^2\vec{r} \frac{1}{2}\phi_{\vec{r}}(\mu^2 - \mu^2)\phi_{\vec{r}}$ into the exponent of the functional integral in Eq. (3), then rearrange the exponent into a free-field part plus a new interacting part, and finally insert a formal expansion factor ϵ before the interacting part. Consequently, one has

$$Z[J] \rightarrow Z[J; \epsilon] = \exp\left\{-\int d^2\vec{r} \left[\frac{1}{2}I_{(0)}(\mu^2) - \frac{1}{2}I_{(0)}(\mathcal{M}^2) + \frac{1}{2}\mathcal{M}^2 I_{(1)}(\mathcal{M}^2)\right]\right\} \\ \times \exp\left\{-\epsilon \int d^2\vec{r} \mathcal{H}_I\left(\frac{\delta}{\delta J_{\vec{r}}}, \mu\right)\right\} \exp\left\{\frac{1}{2}Jf^{-1}J\right\} \quad (4)$$

with

$$\mathcal{H}_I(\phi_{\vec{r}}, \mu) = -\frac{1}{2}\mu^2\phi_{\vec{r}}^2 - 2\Omega \cos(\sqrt{8\pi}\beta\phi_{\vec{r}}) \exp\{4\pi\beta^2 I_{(1)}(\mathcal{M}^2)\}. \quad (5)$$

Here, \mathcal{M} is a normal-ordering mass,

$$I_{(n)}(Q^2) \equiv \begin{cases} \int \frac{d^2\vec{p}}{(2\pi)^2} \frac{1}{(p^2+Q^2)^n}, & \text{for } n \neq 0, \\ \int \frac{d^2p}{(2\pi)^2} \ln(p^2 + Q^2), & \text{for } n = 0 \end{cases}$$

with \vec{p} a Euclidean momentum in 2DES, and $Jf^{-1}J \equiv \int d^2\vec{r}' d^2\vec{r}'' J_{\vec{r}'} f_{\vec{r}'\vec{r}''}^{-1} J_{\vec{r}''}$ with

$$f_{\vec{r}'\vec{r}''}^{-1} = \int \frac{d^2\vec{p}}{(2\pi)^2} \frac{1}{p^2 + \mu^2} e^{i\vec{p}\cdot(\vec{r}'' - \vec{r}')} = \frac{1}{2\pi} K_0(\mu|\vec{r}'' - \vec{r}'|). \quad (6)$$

In Eq. (6), $K_n(z)$ is the n th-order modified Bessel function of the second kind. Note that Eq. (4) in the extrapolating case of $\epsilon = 1$ is only the normal-ordering expression of Eq. (3).

Expanding $\exp\{-\epsilon \int d^2\vec{r} \mathcal{H}_I(\frac{\delta}{\delta J_{\vec{r}}}, \mu)\}$ in Eq. (4) with Taylor series of the exponential, one can write normal-ordered G_a as

$$G_a = e^{4\pi a^2 I_{(1)}(\mathcal{M}^2)} \frac{\left[\sum_{n=0}^{\infty} \epsilon^n \frac{(-1)^n}{n!} \int \prod_{k=1}^n d^2\vec{r}_k \mathcal{H}_I\left(\frac{\delta}{\delta J_{\vec{r}_k}}, \mu\right) \exp\left\{\frac{1}{2}Jf^{-1}J\right\}\right]_{J_{\vec{r}}=i\sqrt{8\pi}a\delta(\vec{r})}}{\left[\sum_{n=0}^{\infty} \epsilon^n \frac{(-1)^n}{n!} \int \prod_{k=1}^n d^2\vec{r}_k \mathcal{H}_I\left(\frac{\delta}{\delta J_{\vec{r}_k}}, \mu\right) \exp\left\{\frac{1}{2}Jf^{-1}J\right\}\right]_{J=0}} \Bigg|_{\epsilon=1}. \quad (7)$$

Thus, according to the formula (0.313) on p. 14 in Ref. [15], G_a can be expanded as the following series of ϵ

$$G_a = [G_a^{(0)} + \epsilon^1 G_a^{(1)} + \epsilon^2 G_a^{(2)} + \dots + \epsilon^n G_a^{(n)} + \dots]_{\epsilon=1}. \quad (8)$$

This series with $\epsilon = 1$ is independent of the man-made parameter μ , but to get its closed form is beyond our ability. Hence one can truncate it at any order of ϵ to approximate it and then the truncated results will be

dependent upon μ . This arbitrary parameter μ should be determined according to the PMS [8] as mentioned in the introduction. That is, under the PMS, μ will be chosen from roots which make the first (or second) derivative of the truncated result with respect to μ vanish [8,10,12]. Thus, μ will depend on the truncated order. It is believed that it is this dependence that makes the truncated result approach the exact G_a order by order [7]. Thus the above procedure provides an approximate method of calculating G_a which can systematically control its approximation accuracy. It is evident that this procedure has no limits to the model coupling and is a non-perturbative method.

Executing the above procedure, we have truncated the series in Eq. (8) at the second order of ϵ , and the three coefficients are

$$G_a^{(0)} = \exp\{4\pi a^2 I_{(1)}(\mathcal{M}^2)\} \left[\exp\left\{\frac{1}{2} J f^{-1} J\right\} \right]_{J_{\vec{r}}=i\sqrt{8\pi}a\delta(\vec{r})}, \tag{9}$$

$$G_a^{(1)} = -\exp\{4\pi a^2 I_{(1)}(\mathcal{M}^2)\} \left[\int d^2\vec{r} \mathcal{H}_I\left(\frac{\delta}{\delta J_r}, \mu\right) \exp\left\{\frac{1}{2} J f^{-1} J\right\} \right]_{J_{\vec{r}}=i\sqrt{8\pi}a\delta(\vec{r})} \\ + G_a^{(0)} \left[\int d^2\vec{r} \mathcal{H}_I\left(\frac{\delta}{\delta J_r}, \mu\right) \exp\left\{\frac{1}{2} J f^{-1} J\right\} \right]_{J_{\vec{r}}=0}, \tag{10}$$

and

$$G_a^{(2)} = \frac{1}{2!} e^{4\pi a^2 I_{(1)}(\mathcal{M}^2)} \left[\int d^2\vec{r}_1 d^2\vec{r}_2 \mathcal{H}_I\left(\frac{\delta}{\delta J_{\vec{r}_1}}, \mu\right) \mathcal{H}_I\left(\frac{\delta}{\delta J_{\vec{r}_2}}, \mu\right) \exp\left\{\frac{1}{2} J f^{-1} J\right\} \right]_{J_{\vec{r}}=i\sqrt{8\pi}a\delta(\vec{r})} \\ - \frac{1}{2!} G_a^{(0)} \left[\int d^2\vec{r}_1 d^2\vec{r}_2 \mathcal{H}_I\left(\frac{\delta}{\delta J_{\vec{r}_1}}, \mu\right) \mathcal{H}_I\left(\frac{\delta}{\delta J_{\vec{r}_2}}, \mu\right) \exp\left\{\frac{1}{2} J f^{-1} J\right\} \right]_{J_{\vec{r}}=0} \\ + G_a^{(1)} \left[\int d^2\vec{r} \mathcal{H}_I\left(\frac{\delta}{\delta J_r}, \mu\right) \exp\left\{\frac{1}{2} J f^{-1} J\right\} \right]_{J_{\vec{r}}=0}, \tag{11}$$

respectively.

Performing carefully operations in Eqs. (9)–(11), we obtain the following expression of G_a approximated up to the second order of ϵ , $G_a^{\text{II}} = [G_a^{(0)} + \epsilon G_a^{(1)} + \epsilon^2 G_a^{(2)}]_{\epsilon=1}$,

$$G_a^{\text{II}} = \left(\frac{\mu^2}{\mathcal{M}^2}\right)^{a^2} (1 - 2a^2 K_{02}) + 4\pi \frac{\Omega}{\mathcal{M}^2} \left(\frac{\mu^2}{\mathcal{M}^2}\right)^{a^2+\beta^2-1} K_{0c} - \frac{1}{2\pi^2} a^2 \left(\frac{\mu^2}{\mathcal{M}^2}\right)^{a^2} K_{0111} \\ + 2a^4 \left(\frac{\mu^2}{\mathcal{M}^2}\right)^{a^2} (K_{02})^2 - 8\pi \frac{\Omega}{\mathcal{M}^2} a^2 \left(\frac{\mu^2}{\mathcal{M}^2}\right)^{a^2+\beta^2-1} K_{02} K_{0c} - \frac{2}{\pi} \frac{\Omega}{\mathcal{M}^2} \beta^2 \left(\frac{\mu^2}{\mathcal{M}^2}\right)^{a^2+\beta^2-1} K_{02c} \\ + \frac{4}{\pi} \frac{\Omega}{\mathcal{M}^2} a\beta \left(\frac{\mu^2}{\mathcal{M}^2}\right)^{a^2+\beta^2-1} K_{011s} + 8\pi^2 \left(\frac{\Omega}{\mathcal{M}^2}\right)^2 \left(\frac{\mu^2}{\mathcal{M}^2}\right)^{a^2+2\beta^2-2} (K_{0c})^2 \\ + \left(\frac{\Omega}{\mathcal{M}^2}\right)^2 \left(\frac{\mu^2}{\mathcal{M}^2}\right)^{a^2+2\beta^2-2} K_{0ec11} + \left(\frac{\Omega}{\mathcal{M}^2}\right)^2 \left(\frac{\mu^2}{\mathcal{M}^2}\right)^{a^2+2\beta^2-2} K_{0ee11}, \tag{12}$$

where

$$K_{02} \equiv \int_0^\infty dx x K_0^2(x), \quad K_{0c} \equiv \int_0^\infty dx x [\cosh(4a\beta K_0(x)) - 1], \\ K_{0111} = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^\infty d\rho_1 \int_0^\infty d\rho_2 \rho_1 \rho_2 K_0(R) K_0(\rho_1) K_0(\rho_2),$$

$$K_{02c} = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^\infty d\rho_1 \int_0^\infty d\rho_2 \rho_1 \rho_2 (K_0(R))^2 [\cosh(4a\beta K_0(\rho_2)) - 1],$$

$$K_{011s} = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^\infty d\rho_1 \int_0^\infty d\rho_2 \rho_1 \rho_2 K_0(R) K_0(\rho_1) \sinh(4a\beta K_0(\rho_2)),$$

$$K_{0ec11} = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^\infty d\rho_1 \int_0^\infty d\rho_2 \rho_1 \rho_2 [\exp\{-4\beta^2 K_0(R)\} - 1] [\cosh(4a\beta (K_0(\rho_1) + K_0(\rho_2))) - 1],$$

$$K_{0ee11} = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^\infty d\rho_1 \int_0^\infty d\rho_2 \rho_1 \rho_2 [\exp\{4\beta^2 K_0(R)\} - 1] [\exp\{-4a\beta (K_0(\rho_1) - K_0(\rho_2))\} - 1]$$

with $R = \sqrt{\rho_1^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2) + \rho_2^2}$.

In the right-hand side of Eq. (12), the first two terms are the expression of G_a approximated up to the first order of ϵ , $G_a^I = [G_a^{(0)} + \epsilon G_a^{(1)}]_{\epsilon=1}$. To determine the arbitrary parameter μ for G_a^I , as stated in the above, we can require that $\frac{dG_a^I}{d(\mu^2)} = 0$ according to the PMS, and have

$$\frac{\mu^2}{\mathcal{M}^2} = \left(4\pi \frac{\Omega}{\mathcal{M}^2} K_{0c} \frac{1 - a^2 - \beta^2}{a^2(1 - 2a^2 K_{02})} \right)^{1/(1-\beta^2)}. \tag{13}$$

Thus, substituting last equation into the expression of G_a^I gives approximately the result of G_a up to the first order in VPT. Note that \mathcal{M} can be taken as any positive value, and usually it can be referred to as unit when one renders various quantities dimensionless for numerical calculations.

As for μ at the second order, following the next section, one can numerically check that the condition $\frac{dG_a^{II}}{d(\mu^2)} = 0$ does not produce a real μ^2 for a real value of a . However, for a real a , G_a should be real because EVEs of odd-power fields in the sG field theory vanish (see Eq. (2)). Hence we have to resort to $\frac{d^2 G_a^{II}}{d(\mu^2)^2} = 0$ for determining μ at this order (note that μ enters the scheme in its squared power μ^2). Fortunately, μ can explicitly be obtained with a long expression from this condition, and then the approximate G_a up to the second order in VPT can be concretely given from Eq. (12).

In the same way, one can approximately give G_a up to higher orders. Here we do not continue to consider it, and next section we will carry out a numerical calculation on the first and the second order results.

In passing, $G_a^{(n)}$ in Eq. (8) is only the sum of all n th-order connected Feynman diagrams consisting of the external vertices from $\exp\{i\sqrt{8\pi}a\phi(0)\}$ and n vertices from $\mathcal{H}_I(\phi_{\vec{r}}, \mu)$. Hence, one can also obtain the above results Eq. (12) by borrowing Feynman diagrammatic technique with the propagator of Eq. (6).

3. Numerical calculations and comparisons

To perform numerical calculations, we take $\mathcal{M} = 1$ and all physical quantities in Eqs. (12) and (13) are dimensionless. Simultaneously, one can check that this treatment (taking $\mathcal{M} = 1$) amounts to taking the same normalization conditions as Eqs. (6) and (16) in Ref. [1]. Thus, our results can be compared with the exact formula.

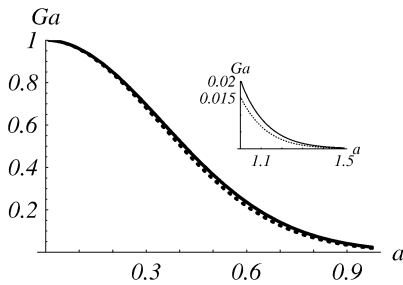


Fig. 1. Comparison between the first-order (G_a^I , solid curve) and conjectured exact (G_a^{exact} , dashed curve) sG expectation values of exponential fields at $\beta = 0.2$.

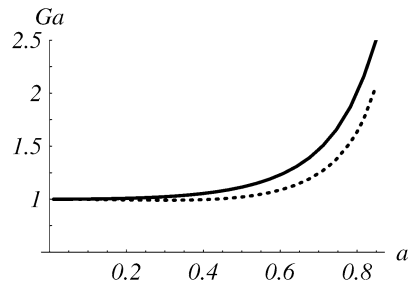


Fig. 2. Similar to Fig. 1 but at $\beta = 0.5$.

Table 1

Comparisons: G_a^{II} , G_a^I and G_a^{exact} for $\beta = 0.5$

a	G_a^{exact}	G_a^I	G_a^{II}	Δ^I	Δ^{II}
0.01	0.999981	1.00001758	1.00001187	0.003661	0.003090
0.1	0.998193	1.001847531	1.00137859	0.36542	0.3185
0.2	0.993954	1.00850388	1.00678750	1.4639	1.2912
0.3	0.990960	1.02358276	1.01961512	3.2920	2.8917
0.4	0.995987	1.05423835	1.04769147	5.8486	5.1913
0.5	1.020682	1.11390679	1.10415979	9.1336	8.1786
0.6	1.086765	1.22971602	1.21627955	13.1538	11.9174
0.7	1.242866	1.46517988	1.44784716	17.8872	16.4926
0.8	1.634342	2.00162788	1.99288037	22.4730	21.9378

From Eq. (12) in Ref. [1], the dimensionless Ω in Eqs. (12) and (13) can be written as

$$\Omega = \frac{\Gamma(\beta^2)}{\pi \Gamma(1 - \beta^2)} \left[\frac{\sqrt{\pi} \Gamma(\frac{1}{2} + \frac{\xi}{2})}{2\Gamma(\frac{\xi}{2})} \right]^{2-2\beta^2} \tag{14}$$

with $\xi = \frac{\beta^2}{1-\beta^2}$. In last equation, we took the soliton mass as unit to make Ω dimensionless. For comparisons, we also do so for the conjectured exact formula Eq. (20) in Ref. [1], G_a^{exact} , when it is employed.

Now we first report numerical results up to the first order. Mathematica5.0 program gives that $K_{02} = 0.5$ and K_{0c} is finite for the case of $a\beta \leq 0.426925$ which is involved in the range of $a\beta < \frac{1}{2}$ for the conjectured exact formula. Taking $\beta = 0.2$ and 0.5 as examples, we compared our results G_a^I with the conjectured exact results of Ref. [1] in Figs. 1 and 2, respectively. In Figs. 1 and 2, the solid curves are G_a^I , and the dashed curves are G_a^{exact} . In Fig. 1, G_a decreases from 1 to 0 with the increase of a . This tendency exists in the case of $\beta < 0.4$ or so. When $a < 1$ and $a^2 + \beta^2 > 1$, Eq. (13) will produce a complex μ^2 and one should use the second derivative for determining μ . For sufficiently small β , one need not do so because G_a tends to zero with the increase of a before μ^2 approaches complex values. In Fig. 2, G_a increases almost from 1 with the increase of a . This tendency appears for other larger values of β . These figures indicate that for the case of $a < 0.2$ or so, the first-order results almost completely agree with the conjectured exact results, and for larger a , our results differ from the exact results with about ten percents or so (at most with 20 more percents when a approaches values with $a\beta = 0.426925$ satisfied, see Table 1).

For calculating G_a^{II} , we need to calculate integrals in Eq. (12) which are involved in the zeroth order modified Bessel functions of the second kind. Noting Gegenbauer's addition formula for the zeroth-order modified Bessel functions of the second kind [16] $K_0(R) = I_0(\rho_1)K_0(\rho_2) + 2 \sum_{n=1}^{\infty} \cos[n(\theta_1 - \theta_2)]I_n(\rho_1)K_n(\rho_2)$ with $\rho_1 < \rho_2$

($I_n(\rho_1)$ is the n th-order modified Bessel functions of the first kind), one can finish those integrals by dividing the plane $\{\rho_1, \rho_2\}$ into two parts: one part with $\rho_1 < \rho_2$ and the other with $\rho_1 > \rho_2$. In performing the calculations, because of the oscillatory property of $\cos[n(\theta_1 - \theta_2)]$, it is enough to truncate the series in the Gegenbauer's addition formula at some n . Finishing those integrals for $\beta = 0.5$ with long-time calculations of Mathematica5.0 system, we obtained the results of G_a^{II} and accordingly compared them with G_a^{I} and the conjectured exact results in Table 1. In Table 1, $\Delta^{\text{I}} = \frac{G_a^{\text{I}} - G_a^{\text{exact}}}{G_a^{\text{exact}}} \times 100$ and $\Delta^{\text{II}} = \frac{G_a^{\text{II}} - G_a^{\text{exact}}}{G_a^{\text{exact}}} \times 100$. This table indicates that G_a^{II} has an about one percent smaller difference from the conjectured exact result than G_a^{I} at $\beta = 0.5$. We also checked the cases of smaller and larger β ($\beta^2 < 0.426925$), and found the same conclusion.

4. Conclusion

In this Letter, in order to check the conjectured exact formula in Ref. [1], we have calculated the sG expectation value of the exponential field $e^{ia\phi(\vec{r})}$ up to the second order with variational perturbation approach. According to our numerical results in last section, for not large a , the exact formula conjectured by Lukyanov and Zamolodchikov in Ref. [1] is correct for all values of β in the range of $\beta^2 < 1$, and for larger a ($|\text{Re}(a)| < 1/(2\beta)$), our numerical results can be believed to support the conjectured exact formula since our method is an approximate one. Thus, from the existed reports in Refs. [1,4,5] and our report here, we believe that the conjectured exact formula in Ref. [1] is completely convincible.

As ending the present Letter, we mention some interesting problems. The normal-ordering prescription amounts to a renormalization procedure on the mass parameter and makes G_a^{I} finite for the range of $a\beta < \frac{1}{2}$. If introducing additional renormalization scheme on the coupling β and the exponential-field parameter a , it will be possible to obtain a finite G_a for the range of $a\beta > \frac{1}{2}$. Furthermore, it will be also interesting to calculate the sG expectation values of the exponential fields on the asymmetrical vacua. On the other hand, because it was an important progress in calculating VEVs of local fields, the Lukyanov–Zamolodchikov conjecture in Ref. [1] has stimulated generalizations or conjectures on the VEVs of local fields in some field theories [17], and hence the method in the present Letter and its generalization to finite temperature case can be used to check or confirm those generalizations or conjectures in Ref. [4,17]. Besides, although our results up to the second order improved the results up to the first order, Table 1 indicates that the improvement is small. Since the Lukyanov–Zamolodchikov conjecture can be believed to be correct, comparing it with higher-order contributions in VPT for the sG expectation values of the exponential fields will provide the first check in QFT on the convergency of VPT.

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