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ScienceDirect
Topology and its Applications 155 (2008) 965–971
Topology and its Applications
www.elsevier.com/locate/topol

$\mathbb{Z}_2$ actions on complexes with three non-trivial cells

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Received 7 September 2007; received in revised form 18 December 2007; accepted 18 December 2007

Abstract

In this paper, we study $\mathbb{Z}_2$ actions on a cell complex X having its cohomology ring isomorphic to that of the wedge sum $P^2(n) \vee S^{3n}$ or $S^n \vee S^{2n} \vee S^{3n}$. We determine the possible fixed point sets depending on whether or not X is totally non-homologous to zero in $X_{\mathbb{Z}_2}$ and give examples realizing all possible cases.

MSC: primary 55S17; secondary 55R20

Keywords: Cohomology ring; Fibration; Group action; Join; Totally non-homologous to zero; Wedge sum

1. Introduction

Toda [9] studied the cohomology ring of a space X having only non-trivial cohomology groups $H^{in}(X;\mathbb{Z}) = \mathbb{Z}$ for $i = 0, 1, 2$ and 3, where n is a fixed positive integer. Let $u_i \in H^{in}(X;\mathbb{Z})$ be a generator for $i = 1, 2$ and 3. Then the ring structure of $H^*(X;\mathbb{Z})$ is completely determined by the integers a and b such that

$$u_1^2 = au_2 \quad \text{and} \quad u_1u_2 = bu_3.$$

Such a space is said to be of type (a,b). Note that, when n is odd, we must have $a = 0$ [9, Theorem 1].

Let p be a prime. One can see that for a space X of type (a,b) there exists always a cell complex $K = S^n \cup e^{2n} \cup e^{3n}$ with three non-trivial cells such that $H^*(X;\mathbb{F}_p) \cong H^*(K;\mathbb{F}_p)$. We shall write $X \simeq_p Y$ if there is an abstract isomorphism of graded rings $H^*(X;\mathbb{F}_p) \xrightarrow{\cong} H^*(Y;\mathbb{F}_p)$ (not necessarily induced by a continuous map $Y \to X$). Similarly, we use the notation $X \simeq_p P^h(n)$ to mean that $H^*(X;\mathbb{F}_p) \cong \mathbb{F}_p[z]/z^{h+1}$, where z is a homogeneous element of degree n.

Given spaces X_i with chosen base points $x_i \in X_i$ for $i = 1, 2, \ldots, n$, their wedge sum $\bigvee_{i=1}^n X_i$ is the quotient of the disjoint union $\coprod_{i=1}^n X_i$ obtained by identifying the points $x_1, x_2, \ldots, x_n$ to a single point called the wedge point.

One can see that a space X of type (a,b) is determined by the integers a and b in terms of the familiar spaces as follows.

If $b \not\equiv 0 \mod p$, then

$$X \simeq_p S^n \times S^{2n} \quad \text{for } a \equiv 0 \mod p$$

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.12.008

and

$$X \simeq_p P^3(n) \quad \text{for } a \not\equiv 0 \mod p.$$

And, if $b \equiv 0 \mod p$, then

$$X \simeq_p S^n \vee S^{2n} \vee S^{3n} \quad \text{for } a \equiv 0 \mod p$$

and

$$X \simeq_p P^2(n) \vee S^{3n} \quad \text{for } a \not\equiv 0 \mod p.$$

Let the cyclic group $G = \mathbb{Z}_p$ act on a space X of type (a, b). This gives a fibration $X \hookrightarrow X_G \to B_G$, where $X_G = (X \times E_G)/G$ is the orbit space of the diagonal action on $X \times E_G$ and is called the Borel construction on X (see [2, Chapter IV]) and B_G is the base space of the universal principal G-bundle $G \hookrightarrow E_G \to B_G$ called the classifying space of the group G. We say that X is totally non-homologous to zero in X_G if the inclusion of a typical fiber $X \hookrightarrow X_G$ induces a surjection in the cohomology $H^*(X_G; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p)$. This condition is equivalent to a nice relation between the cohomology of the space and the fixed point set (Proposition 2).

The fixed point sets of $\mathbb{Z}_p$ actions for the case $b \not\equiv 0 \mod p$ have been investigated in detail by Bredon [1] and Su [7,8] for all primes p. And the fixed point sets of $\mathbb{Z}_p$ actions for the case $b \equiv 0 \mod p$ have been completely determined by Dotzel and Singh [3,4] for odd primes p. In this paper, we settle the remaining case when $p = 2$ and obtain the following results:

Theorem 1. *Let $G = \mathbb{Z}_2$ act on a space X of type $(a, 0) \mod 2$ with trivial action on $H^*(X; \mathbb{Q})$ and fixed point set F. Suppose X is totally non-homologous to zero in X_G, then F has at most four components satisfying the following*:

(1) *If F has four components, then each is acyclic, n is even and $a \equiv 0 \mod 2$.*
(2) *If F has three components, then n is even and*

$$F \simeq_2 S^r \sqcup \{point_1\} \sqcup \{point_2\} \quad \text{for some even integer } 2 \leqslant r \leqslant 3n.$$

(3) *If F has two components, then either*

$$F \simeq_2 S^r \sqcup S^s \quad \text{or} \quad (S^r \vee S^s) \sqcup \{point\} \quad \text{for some integers } 1 \leqslant r, s \leqslant 3n$$

or

$$F \simeq_2 P^2(r) \sqcup \{point\} \quad \text{for some even integer } 2 \leqslant r \leqslant n.$$

(4) *If F has one component, then either*

$$F \simeq_2 S^r \vee S^s \vee S^t \quad \text{for some integers } 1 \leqslant r, s, t \leqslant 3n$$

or

$$F \simeq_2 S^s \vee P^2(r) \quad \text{for some integers } 1 \leqslant r \leqslant n \text{ and } 1 \leqslant s \leqslant 3n.$$

Further, if n is even, then X is always totally non-homologous to zero in X_G.

Theorem 2. *Let $G = \mathbb{Z}_2$ act on a space X of type $(a, 0) \mod 2$ with trivial action on $H^*(X; \mathbb{Q})$ and fixed point set F. Suppose X is not totally non-homologous to zero in X_G, then either $F = \phi$ or $F \simeq_2 S^r$, where $1 \leqslant r \leqslant 3n$ is an odd integer.*

We shall prove Theorem 1 in Section 3 and Theorem 2 in Section 4. We include examples in the proofs to show that all the cases are realizable.

2. Preliminaries

Our methods will be standard and for details we refer to Bredon [1]. As the spaces of concern in this paper are finite cell complexes, the cohomology used will be the cellular cohomology with coefficients in the field $\mathbb{F}_2$ of two elements unless otherwise stated. Recall that, $X \simeq_2 P^h(n)$ means that the mod 2 cohomology ring of X is isomorphic to $\mathbb{F}_2[z]/z^{h+1}$, where z is a homogeneous element of degree n. The following result is well known.

Proposition 1. *If X is a finite cell complex such that $X \simeq_2 P^h(n)$, then*

$$n = 1, 2, 4 \quad \textit{for } h \geqslant 2$$

and

$$n = 8 \quad \textit{for } h = 2.$$

See [6, *Chapter* I, 4.5].

The following facts about $\mathbb{Z}_2$ actions can be easily deduced.

Proposition 2. *Let $G = \mathbb{Z}_2$ act on a finite cell complex X with fixed point set F. Then X is totally non-homologous to zero in X_G if and only if*

$$\sum_{i \geqslant 0} rk\, H^i(F) = \sum_{i \geqslant 0} rk\, H^i(X).$$

See [1, *Chapter* VII, 1.6].

Proposition 3. *Let $G = \mathbb{Z}_2$ act on a finite cell complex X with fixed point set F. Then*

$$\sum_{i \geqslant 0} rk\, H^i(F) \leqslant \sum_{i \geqslant 0} rk\, H^i(X).$$

See [1, *Chapter* III, 7.9].

The following lemma is crucial for our results.

Lemma 4. *Let $G = \mathbb{Z}_2$ act on a finite cell complex X with trivial action on the rational cohomology $H^*(X; \mathbb{Q})$, then*

$$\chi(X) = \chi(F).$$

Proof. By Theorem 7.2 of Bredon [1, Chapter III], we have

$$\pi^i : H^i(X/G; \mathbb{Q}) \xrightarrow{\cong} H^i(X; \mathbb{Q})^G \quad \text{for all } i \geqslant 0,$$

where $\pi : X \to X/G$ is the orbit map. Since G acts trivially on the cohomology, the fixed point set $H^i(X; \mathbb{Q})^G = H^i(X; \mathbb{Q})$ for all $i \geqslant 0$. This gives $H^i(X/G; \mathbb{Q}) \cong H^i(X; \mathbb{Q})$ for all $i \geqslant 0$ and hence $\chi(X) = \chi(X/G)$. By Theorem 7.10 of Bredon [1, Chapter III], we have

$$\chi(X) + \chi(F) = 2\chi(X/G)$$

and hence $\chi(X) = \chi(F)$. $\square$

Remark. The results quoted above are true for a general class of spaces called finitistic spaces using the Čech cohomology with coefficients in the field $\mathbb{F}_2$ (which is the same as the cellular cohomology on cell complexes). Recall that, a paracompact Hausdorff space is said to be finitistic if its every open covering has a finite dimensional open refinement, where the dimension of a covering is one less than the maximum number of members of the covering which intersect non-trivially [1, Chapter III]. Clearly a compact space is finitistic. Hence a space X of type $(a, 0)$ mod 2 is finitistic being compact.

Now we consider a $\mathbb{Z}_2$ action on the unit sphere $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$ that we shall use in constructing examples in the following sections. For $0 \leqslant r \leqslant n$, $S^r \subseteq S^n$, where $S^r = \{(x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_{r+2} = x_{r+3} = \cdots = x_{n+1} = 0\}$. The $\mathbb{Z}_2$ action on S^n given by

$$(x_1, x_2, \ldots, x_{n+1}) \mapsto (x_1, x_2, \ldots, x_{r+1}, -x_{r+2}, -x_{r+3}, \ldots, -x_{n+1})$$

has S^r as its fixed point set. Given any point $x \in S^n$, we consider $\{x, -x\}$ as $S^0 \subset S^n$. Then the above action on S^n, for $r = 0$, has $\{x, -x\}$ as its fixed point set.

We shall also use the join $X \star Y$ of two spaces X and Y, which is defined as the quotient of $X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$, where I is the unit interval. That is, we are collapsing the subspace $X \times Y \times \{0\}$ to X and $X \times Y \times \{1\}$ to Y. Note that, if a group G acts on both X and Y with fixed point sets F_1 and F_2, respectively, then the induced action of G on the join $X \star Y$ has $F_1 \star F_2$ as its fixed point set.

3. Proof of Theorem 1

Let X be totally non-homologous to zero in X_G. Then by Proposition 2,

$$\sum_{i \geqslant 0} rk\, H^i(F) = \sum_{i \geqslant 0} rk\, H^i(X) = 4.$$

It follows that F has at most four components.

Case 1. Suppose F has four components, then it is clear that each is acyclic. Let $\bar{u}_i$ denote the reductions of $u_i \mod 2$. If $a \not\equiv 0 \mod 2$, then $\bar{u}_1^2 = \bar{u}_2 \neq 0$ and hence $H^n(F) \neq 0$ [1, Chapter VII, 7.3] showing that F has a non-acyclic component. Therefore, in this case $a \equiv 0 \mod 2$. By Lemma 4, we have $\chi(X) = \chi(F) = 4$ and hence n must be even.

For $a \equiv 0 \mod 2$, we can take $X = S^n \vee S^{2n} \vee S^{3n}$. Consider the $\mathbb{Z}_2$ actions on the spheres S^n, S^{2n} and S^{3n} with exactly two fixed points each and then take their wedge sum at some fixed points. This gives a $\mathbb{Z}_2$ action on X with the disjoint union of four points as its fixed point set.

Case 2. Suppose that F has three components, then

$$F \simeq_2 S^r \sqcup \{point_1\} \sqcup \{point_2\} \quad \text{for some integer } 1 \leqslant r \leqslant 3n.$$

Note that $\chi(F) = 2$ or 4 according as r is odd or even. As $\chi(X) = \chi(F)$, both n and r are even.

For $a \equiv 0 \mod 2$ and even integers r and n such that $2 \leqslant r \leqslant 3n$, we take $X = S^n \vee S^{2n} \vee S^{3n}$. Consider the $\mathbb{Z}_2$ actions on the spheres S^n and S^{2n} with exactly two fixed points each and the action on S^{3n} with S^r as its fixed point set. Taking their wedge sum at some fixed points gives a $\mathbb{Z}_2$ action on X with $F = S^r \sqcup \{point_1\} \sqcup \{point_2\}$.

For $a \not\equiv 0 \mod 2$, we know that $X \simeq_2 P^2(n) \vee S^{3n}$.

If Y is a space such that $H^*(Y; \mathbb{F}_2) = \mathbb{F}_2[z]/z^{h+1}$, where z is of degree n, then by Proposition 1, we have $n = 2$, 4 or 8 for $h = 2$. Therefore, we can take $Y = \mathbb{C}P^2$ the complex projective 2-space, $\mathbb{H}P^2$ the quaternionic projective 2-space or $\mathbb{O}P^2$ the Cayley projective plane, according as $n = 2$, 4 or 8, respectively.

For $n = 2$, let $S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid \sum_{i=1}^3 |z_i|^2 = 1\}$. Consider the $\mathbb{Z}_2$ action on S^5 given by $(z_1, z_2, z_3) \mapsto (z_1, z_2, -z_3)$. This action commutes with the usual S^1 action on S^5 and hence descends to an action on $\mathbb{C}P^2$. As $S^3 \subset S^5$ is fixed under the $\mathbb{Z}_2$ action on S^5, it is easy to see that

$$S^2 \sqcup \{point\}$$

is the fixed point set of the $\mathbb{Z}_2$ action on $\mathbb{C}P^2$.

Similarly, for $n = 4$, let $\mathbb{H}$ be the normed division algebra of quaternions and $S^{11} = \{(w_1, w_2, w_3) \in \mathbb{H}^3 \mid \sum_{i=1}^3 |w_i|^2 = 1\}$ and consider the $\mathbb{Z}_2$ action on S^{11} given by $(w_1, w_2, w_3) \mapsto (w_1, w_2, -w_3)$. This action commutes with the usual S^3 action on S^{11}. As above, one can see that

$$S^4 \sqcup \{point\}$$

is the fixed point set of the induced action of $\mathbb{Z}_2$ on $\mathbb{H}P^2$.

For $n = 8$, Bredon [1, Chapter VII] has constructed a $\mathbb{Z}_2$ action on $\mathbb{O}P^2$ with

$$S^8 \sqcup \{point\}$$

as its fixed point set.

Now, consider the $\mathbb{Z}_2$ action on S^{3n} with exactly two fixed points. Taking $X = Y \vee S^{3n}$, where the wedge sum is taken at the isolated fixed point of Y and a fixed point of S^{3n}, we get a $\mathbb{Z}_2$ action on X with the fixed point set $F = S^r \sqcup \{point_1\} \sqcup \{point_2\}$ for some even integer $2 \leqslant r \leqslant 3n$.

Case 3. Suppose F has two components, then

$$F \simeq_2 S^r \sqcup S^s, \quad \left(S^r \vee S^s\right) \sqcup \{point\} \quad \text{or} \quad P^2(r) \sqcup \{point\} \quad \text{for some } r \text{ and } s.$$

By Lemma 4, $\chi(X) = \chi(F)$. If n is odd, $\chi(F) = 0$ and hence

$$F \simeq_2 S^r \sqcup S^s \quad \text{or} \quad \left(S^r \vee S^s\right) \sqcup \{point\} \quad \text{for odd integers } 1 \leqslant r, s \leqslant 3n.$$

And if n is even, $\chi(F) = 4$ and hence

$$F \simeq_2 S^r \sqcup S^s \quad \text{or} \quad \left(S^r \vee S^s\right) \sqcup \{point\} \quad \text{for even integers } 2 \leqslant r, s \leqslant 3n$$

or

$$F \simeq_2 P^2(r) \sqcup \{point\} \quad \text{for some even integer } 2 \leqslant r \leqslant n.$$

For $a \equiv 0 \mod 2$, let $Y = S^{n-1} \star P^2(n)$. Consider a free $\mathbb{Z}_2$ action on S^{n-1} and that action on $P^2(n)$ which has the fixed point set $S^r \sqcup \{point\}$ for some r (which we constructed in Case 2). Let $\mathbb{Z}_2$ act on S^n with its fixed point set S^s for some s. Take $X = S^n \vee Y$, where the wedge sum is taken at the isolated fixed point of Y and some point of S^s. Then $X \simeq_2 S^n \vee S^{2n} \vee S^{3n}$ and has a $\mathbb{Z}_2$ action with the fixed point set $F \simeq_2 S^r \sqcup S^s$.

If we take the wedge sum at some point of S^r and some point of S^s, then X has a $\mathbb{Z}_2$ action with the fixed point set $F \simeq_2 (S^r \vee S^s) \sqcup \{point\}$.

Further, if we consider a free $\mathbb{Z}_2$ action on S^{n-1}, the trivial action on $P^2(n)$ and the action on S^n with exactly two fixed points, then $X = S^n \vee Y$, where the wedge is taken at some point of $P^2(n)$ and some fixed point of S^n, has a $\mathbb{Z}_2$ action with the fixed point set $F \simeq_2 P^2(n) \sqcup \{point\}$.

For $a \not\equiv 0 \mod 2$, take $X = P^2(n) \vee S^{3n}$. Consider the $\mathbb{Z}_2$ action on $P^2(n)$ with $S^r \sqcup \{point\}$ as its fixed point set and the action on S^{3n} with S^s as its fixed point set. By taking the wedge sum at suitable points, we get a $\mathbb{Z}_2$ action on X with $F \simeq_2 S^r \sqcup S^s$ or $(S^r \vee S^s) \sqcup \{point\}$. Similarly, suitable actions on $P^2(n)$ and S^{3n} gives an action on X with $F \simeq_2 P^2(r) \sqcup \{point\}$.

Case 4. Suppose F has one component, then either

$$F \simeq_2 S^r \vee S^s \vee S^t \quad \text{for some integers } 1 \leqslant r, s, t \leqslant 3n$$

or

$$F \simeq_2 S^s \vee P^2(r) \quad \text{for some integers } 1 \leqslant r \leqslant n \text{ and } 1 \leqslant s \leqslant 3n.$$

As $\chi(F) = \chi(X)$, for $F \simeq_2 S^r \vee S^s \vee S^t$ we must have either r, s and t all are even or exactly one of them is even. Similarly, for $F \simeq_2 S^s \vee P^2(r)$ we must have either s and r both even or both odd.

For $a \equiv 0 \mod 2$, take $X = S^n \vee S^{2n} \vee S^{3n}$. Consider the $\mathbb{Z}_2$ actions on S^n, S^{2n} and S^{3n} with S^r, S^s and S^t respectively as their fixed point sets. This gives an action on X with $S^r \vee S^s \vee S^t$ as its fixed point set, where the wedge is taken at some fixed points on the subspheres.

If we take $X = S^n \vee Y$, where $Y = S^{n-1} \star P^2(n)$ and consider the $\mathbb{Z}_2$ action on S^n with S^s as its fixed point set for some s and the action on Y with $P^2(r)$ as its fixed point set for some r, then we get a $\mathbb{Z}_2$ action on X with its fixed point set $F \simeq_2 S^s \vee P^2(r)$.

For $a \not\equiv 0 \mod 2$, taking a suitable $\mathbb{Z}_2$ action on $X = P^2(n) \vee S^{3n}$ gives $F \simeq_2 S^s \vee P^2(r)$ for some integers r and s. Note that in this case the fixed point set cannot be a wedge of three spheres.

Finally, suppose that n is even and X is not totally non-homologous to zero in X_G. Then by Proposition 2,

$$\sum_{i\geqslant 0} rk\, H^i(F) \neq \sum_{i\geqslant 0} rk\, H^i(X) = 4.$$

And by Proposition 3,

$$\sum_{i\geqslant 0} rk\, H^i(F) \leqslant 3.$$

This gives $\chi(F) = -1$, 0, 1, 2 or 3. But, $\chi(F) = \chi(X) = 4$, a contradiction. This completes the proof of the theorem. □

4. Proof of Theorem 2

Let X be not totally non-homologous to zero in X_G. Then n is odd and hence $\chi(X) = 0$. By Lemma 4, we have $\chi(F) = 0$.

As above $\sum_{i\geqslant 0} rk\, H^i(F) \leqslant 3$.

Observe that

$$\text{if } \sum_{i\geqslant 0} rk\, H^i(F) = 1, \quad \text{then } \chi(F) = 1$$

and

$$\text{if } \sum_{i\geqslant 0} rk\, H^i(F) = 3, \quad \text{then } \chi(F) = 1, -1 \text{ or } 3.$$

Therefore, these cases do not arise. Further,

$$\text{if } \sum_{i\geqslant 0} rk\, H^i(F) = 0, \quad \text{then } F = \phi$$

and

$$\text{if } \sum_{i\geqslant 0} rk\, H^i(F) = 2, \quad \text{then } \chi(F) = 0 \text{ or } 2.$$

But, $\chi(F) = 0$ and hence $F \simeq_2 S^r$ for some odd integer $1 \leqslant r \leqslant 3n$.

Recall that, when n is odd $a \equiv 0 \mod 2$ (see [9]). Let $h : S^3 \to S^2$ be the Hopf map and Y be the union of mapping cylinders of the sphere bundle maps

$$S^2 \times S^n \xleftarrow{h\times 1} S^3 \times S^n \xrightarrow{\text{projection}} S^3.$$

Then $H^*(Y;\mathbb{Z}) = H^*(S^2 \times S^{n+2};\mathbb{Z})$ and Y is a manifold (see [5]). Let $\mathbb{Z}_2$ act freely on S^n and trivially on both S^2 and S^3, then it act on Y with the fixed point set homeomorphic to S^3. Remove a fixed point from Y to obtain a space $Z \simeq_2 S^2 \vee S^{n+2}$ with a $\mathbb{Z}_2$ action and contractible fixed point set. With $\mathbb{Z}_2$ acting trivially on S^{n-3}, consider the induced action on the join $W = S^{n-3} \star Z$ which is homotopically equivalent to $S^n \vee S^{2n}$. This action on W has a contractible fixed point set. For a given odd integer $1 \leqslant r \leqslant 3n$, consider the $\mathbb{Z}_2$ action on S^{3n} with S^r as the fixed point set. Then the wedge sum of W and S^{3n} at some fixed points is a space $X \simeq_2 S^n \vee S^{2n} \vee S^{3n}$ and has a $\mathbb{Z}_2$ action with its fixed point set $F \simeq_2 S^r$. It is clear that every $\mathbb{Z}_2$ action on $X = S^n \vee S^{2n} \vee S^{3n}$ has a non-empty fixed point set. □

Acknowledgement

The author would like to thank the referee for many useful comments and suggestions which made the paper more readable.

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