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A cell complex in number theory ☆

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ABSTRACT

Let Δ_n be the simplicial complex of squarefree positive integers less than or equal to n ordered by divisibility. It is known that the asymptotic rate of growth of its Euler characteristic (the Mertens function) is closely related to deep properties of the prime number system.

In this paper we study the asymptotic behavior of the individual Betti numbers $\beta_k(\Delta_n)$ and of their sum. We show that Δ_n has the homotopy type of a wedge of spheres, and that as $n \to \infty$

$$\sum \beta_k(\Delta_n) = \frac{2n}{\pi^2} + O\left(n^\theta\right), \quad \text{for all } \theta > \frac{17}{54}.$$

Furthermore, for fixed k,

$$\beta_k(\Delta_n) \sim \frac{n}{2\log n}\frac{(\log\log n)^k}{k!}.$$

As a number-theoretic byproduct we obtain inequalities

$$\partial_k\left(\sigma_{k+1}^{odd}(n)\right) \leqslant \sigma_k^{odd}(n/2),$$

where $\sigma_k^{odd}(n)$ denotes the number of odd squarefree integers $\leqslant n$ with k prime factors, and ∂_k is a certain combinatorial shadow function.

We also study a CW complex $\widetilde{\Delta}_n$ that extends the previous simplicial complex. In $\widetilde{\Delta}_n$ *all* numbers $\leqslant n$ correspond to cells and its Euler characteristic is the summatory Liouville function. This

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cell complex $\widetilde{\Delta}_n$ is shown to be homotopy equivalent to a wedge of spheres, and as $n \to \infty$

$$\sum \beta_k(\widetilde{\Delta}_n) = \frac{n}{3} + O\left(n^{\theta}\right), \quad \text{for all } \theta > \frac{22}{27}.$$

1. Introduction

Let $M(n) \stackrel{\text{def}}{=} \sum_{k=1}^{n} \mu(k)$, where $\mu(k)$ is the number-theoretic Möbius function. The rate of growth of the function $M(n)$ is of great interest and importance in number theory, as is clear from the following facts:

$$\text{Prime Number Theorem} \iff \left|M(n)\right| \leqslant \varepsilon n, \quad \text{for all } \varepsilon > 0 \text{ and all sufficiently large } n.$$
$$\text{Riemann Hypothesis} \iff \left|M(n)\right| \leqslant n^{1/2+\varepsilon}, \quad \text{for all } \varepsilon > 0 \text{ and all sufficiently large } n.$$

The key questions concerning the growth of $M(n)$ are subtle. For instance, Mertens conjectured in 1897 that $|M(n)| \leqslant n^{1/2}$ for all sufficiently large n. This conjecture was disproved in 1985 by Odlyzko and te Riele. See e.g. the books by Hardy and Wright [5] and Ivić [6, especially §1.9] for more information about these matters.

This paper has its genesis in the observation that the Mertens function $M(n)$ can be interpreted as the Euler characteristic of a simplicial complex. Namely, for each positive squarefree integer k, let $P(k)$ be the set of its prime factors. For instance, $P(165) = \{3, 5, 11\}$. Then the set family

$$\Delta_n \stackrel{\text{def}}{=} \left\{P(k)\colon\ k \text{ is squarefree and } k \leqslant n\right\}$$

is closed under taking subsets. In other words, it is an abstract simplicial complex. (Remark: The integer 1 is squarefree, so we include $P(1) = \emptyset$ in Δ_n.)

Since, by definition,

$$\mu(k) = \begin{cases} (-1)^{|P(k)|}, & \text{if } k \text{ is squarefree,} \\ 0, & \text{otherwise} \end{cases}$$

it follows that

$$M(n) = -\chi(\Delta_n), \tag{1.1}$$

where $\chi(\Delta_n)$ is the reduced Euler characteristic (the ordinary Euler characteristic minus one). Let $\beta_k(\Delta_n)$ denote the k-th Betti numbers of reduced simplicial homology, i.e., $\beta_k(\Delta_n) \stackrel{\text{def}}{=} \operatorname{rank} \widetilde{H}_k(\Delta_n, \mathbb{Z})$. Then, from the Euler–Poincaré formula and Eq. (1.1) we have

$$M(n) = \sum_{k \geqslant 0} (-1)^{k-1} \beta_k(\Delta_n). \tag{1.2}$$

Thus, important number-theoretic propositions, such as the Prime Number Theorem and the Riemann Hypothesis, are equivalent to statements about the asymptotic rate of growth of the Euler characteristic $\chi(\Delta_n)$. It therefore seems reasonable to inquire about the Betti numbers of the complex Δ_n and their asymptotics.

We prove the following estimates: As $n \to \infty$

$$\sum_{k \geqslant 0} \beta_k(\Delta_n) = \frac{2n}{\pi^2} + O\left(n^\theta\right), \quad \text{for all } \theta > \frac{17}{54},$$

$$\sum_{k \text{ even}} \beta_k(\Delta_n) \sim \frac{n}{\pi^2},$$

$$\sum_{k \text{ odd}} \beta_k(\Delta_n) \sim \frac{n}{\pi^2},$$

$$\text{for fixed } k\text{:} \quad \beta_k(\Delta_n) \sim \frac{n}{2\log n} \frac{(\log\log n)^k}{k!}.$$

Unfortunately these results fall short of shedding new light on the rate of growth of the Euler characteristic $M(n)$. Perhaps a study of deeper topological invariants of Δ_n could add something of value.

Let $\sigma_k^{odd}(n)$ be the number of odd squarefree integers $\leqslant n$ with k prime factors. It turns out that the homology information of Δ_n can be expressed in terms of these numbers, which, in turn, leads to some purely number-theoretic consequences. Namely, for certain "shadow functions" ∂_k and ∂^k, well known in extremal combinatorics and defined in Section 4, we obtain inequalities

(1) $\partial_k(\sigma_{k+1}^{odd}(n)) \leqslant \sigma_k^{odd}(n/2)$,
(2) $\partial^k(\sigma_{2k+2}^{odd}(n) + \sigma_{2k+1}^{odd}(n)) \leqslant \sigma_{2k}^{odd}(n/2) + \sigma_{2k-1}^{odd}(n/2)$.

In Section 5 we study a CW complex $\widetilde{\Delta}_n$ that extends the previous simplicial complex. In $\widetilde{\Delta}_n$ *all* numbers $\leqslant n$ correspond to cells and its Euler characteristic is the summatory Liouville function. We get a sequence of embeddings

$$\widetilde{\Delta}_2 \hookrightarrow \widetilde{\Delta}_3 \hookrightarrow \cdots \hookrightarrow \widetilde{\Delta}_n \hookrightarrow \cdots$$

that can be seen as a filtration of the join of denumerably many copies of certain infinite-dimensional spaces such as S^∞ or $\mathbb{R}P^\infty$. The cell complex $\widetilde{\Delta}_n$ is shown to be homotopy equivalent to a wedge of spheres, and as $n \to \infty$

$$\sum \beta_k(\widetilde{\Delta}_n) = \frac{n}{3} + O\left(n^\theta\right), \quad \text{for all } \theta > \frac{22}{27}.$$

The construction of $\widetilde{\Delta}_n$ is based on a more general construction for multicomplexes given in [3]. In the last section we recall and expand on the details of the construction from [3].

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2. Preliminaries

Later on we need to refer to a few results from combinatorial topology and number theory. Here this information is recalled.

A simplicial complex Δ on a linearly ordered vertex set $x_1, x_2, \ldots, x_t$ is said to be *shifted* if for every $F \in \Delta$:

$$i < j, \quad x_i \notin F \quad \text{and} \quad x_j \in F \quad \Longrightarrow \quad F \setminus \{x_j\} \cup \{x_i\} \in \Delta.$$

Theorem 2.1. [2, p. 292] *Suppose that* Δ *is a shifted complex on the vertex set* $x_1, x_2, \ldots, x_t$. *Then* Δ *has the homotopy type of a wedge of spheres. Its Betti numbers are*

$$\beta_k(\Delta) = \text{number of } F \in \Delta \quad \text{such that} \quad F \cup \{x_1\} \notin \Delta \text{ and } |F| = k+1.$$

The number $\Omega(n)$ of prime factors of a positive integer n is called its *weight*. I.e., by definition,

$$\Omega\left(p_{i_1}^{e_1} p_{i_2}^{e_2} \cdots p_{i_k}^{e_k}\right) = e_1 + e_2 + \cdots + e_k.$$

It is convenient to define the following integer-valued functions for all positive real numbers x (not only for integers).

Definition 1. For $x \in \mathbb{R}^+$, let

(1) $\sigma(x) \stackrel{\text{def}}{=}$ the number of squarefree integers in $(0, x]$,
(2) $\sigma^{odd}(x) \stackrel{\text{def}}{=}$ the number of odd squarefree integers in $(0, x]$,
(3) $\sigma_k(x) \stackrel{\text{def}}{=}$ the number of squarefree integers in $(0, x]$ of weight k,
(4) $\sigma_k^{odd}(x) \stackrel{\text{def}}{=}$ the number of odd squarefree integers in $(0, x]$ of weight k,
(5) and analogously $\sigma^{even}(x)$ and $\sigma_k^{even}(x)$.

Note that $\sigma(x) = 0$ for $x < 1$ and similarly for the other functions.

The following estimates belong to the classics of number theory.

Theorem 2.2. *As* $x \to \infty$ *we have that*

(a) (*Gegenbauer, 1885*; *Jia, 1993*; *see* [5, pp. 355, 359].)

$$\sigma(x) = \frac{6x}{\pi^2} + O\left(x^\theta\right), \quad \text{for all } \theta > \frac{17}{54},$$

(b) (*Landau, 1900*; *see* [5, p. 491].) *For fixed* k,

$$\sigma_k(x) \sim \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}.$$

Remark 2.3. (i) Gegenbauer's estimate of the error term was $O(\sqrt{n})$. The sharper exponent cited here is due to Jia.

(ii) Landau's asymptotic formula for $\sigma_k(x)$ was conjectured by Gauss. Note that the $k = 1$ case is the Prime Number Theorem. Estimates of the error term exist but will not be used in this paper.

3. The number-theoretic simplicial complex Δ_n

Some of the basic concepts of elementary number theory have direct geometric meaning for the complex Δ_n. In fact, one can read parts of the book by Hardy and Wright [5] as describing the size and shape of this complex.

For instance, the dimension of a simplex $P(k)$ is the weight of k minus one:

$$\dim P(k) = \Omega(k) - 1$$

and the typical dimension of a simplex in Δ_n is circa $\lfloor \log\log n \rfloor$. For integers $k \geqslant 1$, let $k\#!$ denote the *primorial number* $p_1 p_2 \cdots p_k$, that is, the product of the first k prime numbers. Then

$$\dim \Delta_n = \max\{k\colon\ k\#! \leqslant n\} - 1.$$

The *round numbers* [5, p. 476] less than n (roughly) correspond to the high-dimensional simplices in Δ_n.

For example, $\dim(\Delta_{10^7}) = 7$ and a typical simplex of Δ_{10^7} is 2-dimensional, whereas $\dim(\Delta_{10^{80}}) = 44$ and a typical simplex is of dimension 4 or 5. Compare this to the discussion in [5, p. 477], where in passing it is mentioned that 10^{80} is roughly the number of protons in the universe.

Theorem 3.1. *The complex Δ_n is shifted. Its Betti numbers are*

$$\beta_k(\Delta_n) = \sigma^{odd}_{k+1}(n) - \sigma^{odd}_{k+1}(n/2). \tag{3.1}$$

Proof. The vertices of Δ_n are the prime numbers $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$. If for a squarefree number $k \leqslant n$ one of its prime factors is replaced by a smaller prime number, then the new number k' so obtained satisfies $k' < k$, and in particular $k' < n$. Hence Δ_n is shifted.

According to Theorem 2.1 the Betti numbers are

$$\beta_k(\Delta_n) = \#\bigl\{\text{odd squarefree integers } b \leqslant n \text{ such that } 2b \nleqslant n \text{ and } \Omega(b) = k+1\bigr\},$$

which agrees with the stated expression. □

The plan from here on is to get rid of the “odd” condition in formula (3.1). We seek to instead express the Betti numbers in terms of the standard number theoretic functions $\sigma(x)$ and $\sigma_k(x)$, so that we can benefit from the known estimates for these functions.

Lemma 3.2.

$$\sigma^{even}_k(x) = \sigma^{odd}_{k-1}(x/2).$$

Proof. Multiplication by 2 gives a bijective map

$$\bigl\{\text{odd squarefree numbers in } (0, x/2] \text{ of weight } k-1\bigr\}$$
$$\leftrightarrow \bigl\{\text{even squarefree numbers in } (0, x] \text{ of weight } k\bigr\}. \qquad \square$$

Lemma 3.3.

$$\sigma^{odd}_k(x) = \sigma_k(x) - \sigma_{k-1}(x/2) + \sigma_{k-2}(x/4) - \sigma_{k-3}(x/8) + \cdots, \tag{3.2}$$

$$\sigma^{odd}(x) = \sigma(x) - \sigma(x/2) + \sigma(x/4) - \sigma(x/8) + \cdots. \tag{3.3}$$

Proof. Lemma 3.2 implies that

$$\sigma_k(x) = \sigma^{odd}_k(x) + \sigma^{odd}_{k-1}(x/2),$$

from which Eq. (3.2) follows. Eq. (3.3) is then obtained by summation over all k. □

Theorem 3.4. *As $n \to \infty$ we have that*

$$\sum_{k \geqslant 0} \beta_k(\Delta_n) = \frac{2n}{\pi^2} + O\left(n^\theta\right), \quad \textit{for all } \theta > \frac{17}{54}.$$

Proof. Let $N \overset{\text{def}}{=} \lceil \log_2(n) \rceil$. We then have that $n \leqslant 2^N < 2n$. Using Theorem 2.2 and Lemma 3.3 we compute

$$\begin{aligned}
\sigma^{odd}(n) &= \sum_{i=0}^{N} (-1)^i \sigma\left(\frac{n}{2^i}\right) \\
&= \sum_{i=0}^{N} (-1)^i \left[\frac{6}{\pi^2}\frac{n}{2^i} + O\left(\left(\frac{n}{2^i}\right)^\theta\right)\right] \\
&= \frac{6n}{\pi^2} \sum_{i=0}^{N} \left(-\frac{1}{2}\right)^i + \sum_{i=0}^{N} O\left(\left(\frac{n}{2^i}\right)^\theta\right) \\
&= \frac{6n}{\pi^2}\frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^{N+1}\right) + n^\theta\, O\left(\sum_{i=0}^{N} \frac{1}{2^{i\theta}}\right) \\
&= \frac{4n}{\pi^2} + O(1) + n^\theta\, O\left(1 - \left(\frac{1}{2^\theta}\right)^{N+1}\right) \\
&= \frac{4n}{\pi^2} + O\left(n^\theta\right).
\end{aligned}$$

Hence, by Eq. (3.1)

$$\sum_{k \geqslant 0} \beta_k(\Delta_n) = \sigma^{odd}(n) - \sigma^{odd}(n/2) = \frac{4n}{\pi^2} + O\left(n^\theta\right) - \frac{2n}{\pi^2} - O\left((n/2)^\theta\right). \qquad \square$$

Theorem 3.5. *As $n \to \infty$ we have that*

$$\sum_{k \text{ even}} \beta_k(\Delta_n) \sim \frac{n}{\pi^2} \quad \textit{and} \quad \sum_{k \text{ odd}} \beta_k(\Delta_n) \sim \frac{n}{\pi^2}.$$

Proof. Let $a(n)$ denote the first sum and $b(n)$ the second. Theorem 3.4 and the Prime Number Theorem show, respectively, that

$$\frac{a(n) + b(n)}{n} \to \frac{2}{\pi^2} \quad \text{and} \quad \frac{a(n) - b(n)}{n} = \frac{M(n)}{n} \to 0$$

as $n \to \infty$. Hence,

$$\frac{2a(n)}{n} = \frac{a(n) + b(n)}{n} + \frac{a(n) - b(n)}{n} \to \frac{2}{\pi^2} + 0$$

and similarly for $b(n)$. $\square$

Theorem 3.6. *For fixed k and $n \to \infty$,*

$$\beta_k(\Delta_n) \sim \frac{n}{2\log n}\frac{(\log\log n)^k}{k!}.$$

Proof. We begin with a small auxiliary computation. Using Theorem 2.2 we have that, as $n \to \infty$,

$$\begin{aligned}\frac{\sigma_{k-i}(n/2^j)}{\sigma_k(n)} &\sim \frac{\frac{n/2^j}{\log(n/2^j)}}{\frac{n}{\log n}} \cdot \frac{\frac{(\log\log(n/2^j))^{k-i-1}}{(k-i-1)!}}{\frac{(\log\log n)^{k-1}}{(k-1)!}} \\ &= \frac{1}{2^j} \cdot \frac{\log n}{\log(n/2^j)} \cdot \left(\frac{\log\log(n/2^j)}{\log\log n}\right)^{k-i-1} \cdot \frac{(k-1)(k-2)\cdots(k-i)}{(\log\log n)^i} \\ &\to \frac{1}{2^j} \cdot 1 \cdot 1 \cdot \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}\end{aligned}$$

Hence, for fixed $i, j \geqslant 0$

$$\lim_{n\to\infty} \frac{\sigma_{k-i}(n/2^j)}{\sigma_k(n)} = \begin{cases} 1/2^j, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases} \tag{3.4}$$

Eqs. (3.1) and (3.2) imply

$$\begin{aligned}\beta_{k-1}(\Delta_n) &= \sigma_k^{odd}(n) - \sigma_k^{odd}(n/2) \\ &= \sum_{j=0}^{k} (-1)^j \big[\sigma_{k-j}\big(n/2^j\big) - \sigma_{k-j}\big(n/2^{j+1}\big)\big].\end{aligned}$$

Using (3.4) we have that

$$\begin{aligned}\frac{\beta_{k-1}(\Delta_n)}{\sigma_k(n)} &= 1 - \frac{\sigma_k(n/2)}{\sigma_k(n)} + \sum_{j=1}^{k}(-1)^j\left[\frac{\sigma_{k-j}(n/2^j)}{\sigma_k(n)} - \frac{\sigma_{k-j}(n/2^{j+1})}{\sigma_k(n)}\right] \\ &\to 1 - \frac{1}{2} + 0 = \frac{1}{2}.\end{aligned}$$

Hence,

$$\beta_{k-1}(\Delta_n) \sim \frac{1}{2}\sigma_k(n) \sim \frac{n}{2\log n}\frac{(\log\log n)^{k-1}}{(k-1)!}. \qquad \square$$

4. Some number-theoretic consequences

The fact that the Betti numbers of the complex Δ_n are determined by counting certain odd square-free integers has some purely number-theoretic implications. Namely, knowing the number of such integers of weight $k+1$ in the interval $(0, n]$ one obtains a lower bound for the number of such integers of weight k in the interval $(0, n/2]$.

We need to recall the following definitions. Two number-theoretic functions $\partial_k(n)$ and $\partial^k(n)$ are defined in the following way. For $n, k \geqslant 1$ the integer n can in a unique way be expressed in the following form

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i},$$

where $a_k > a_{k-1} > \cdots > a_i \geqslant i \geqslant 1$. Then let:

$$\partial_{k-1}(n) \stackrel{\text{def}}{=} \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_i}{i-1},$$

and

$$\partial^{k-1}(n) \stackrel{\text{def}}{=} \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-1}.$$

Also, we let $\partial_{k-1}(0) = \partial_{k-1}(0) = 0$.

Theorem 4.1. *For all $k \geqslant 1$ we have that*

(1) $\partial_k(\sigma_{k+1}^{odd}(n)) \leqslant \sigma_k^{odd}(n/2)$,
(2) $\partial^k(\sigma_{2k+2}^{odd}(n) + \sigma_{2k+1}^{odd}(n)) \leqslant \sigma_{2k}^{odd}(n/2) + \sigma_{2k-1}^{odd}(n/2)$.

Proof. For an arbitrary finite simplicial complex with f-vector $(f_0, f_1, \ldots)$ and Betti numbers $\beta_0, \beta_1, \ldots$, let for $k \geqslant 0$

$$\chi_{k-1} \stackrel{\text{def}}{=} \sum_{j \geqslant k} (-1)^{j-k} (f_j - \beta_j).$$

The following relations appear as Theorem 1.1 in [2] and Theorem 3.3 in [3], respectively. For all $k \geqslant 1$,

$$\partial_k(\chi_k + \beta_k) \leqslant \chi_{k-1}, \tag{4.1}$$

$$\partial^k(f_{2k+1} + \beta_{2k}) \leqslant f_{2k-1} - \beta_{2k-1}. \tag{4.2}$$

Let us now see what these relations mean for the particular complex Δ_n:

$$\begin{aligned} \chi_{k-1} &= \sum_{j \geqslant k} (-1)^{j-k} \big(\sigma_{j+1}(n) - \beta_j(\Delta_n)\big) = \sum_{j \geqslant k} (-1)^{j-k} \big(\sigma_{j+1}^{even}(n) + \sigma_{j+1}^{odd}(n/2)\big) \\ &= \sum_{j \geqslant k} (-1)^{j-k} \big(\sigma_j^{odd}(n/2) + \sigma_{j+1}^{odd}(n/2)\big) = \sigma_k^{odd}(n/2). \end{aligned}$$

Hence,

$$\chi_k + \beta_k = \sigma_{k+1}^{odd}(n/2) + \big(\sigma_{k+1}^{odd}(n) - \sigma_{k+1}^{odd}(n/2)\big) = \sigma_{k+1}^{odd}(n),$$

$$\sigma_{2k+2}(n) + \beta_{2k}(\Delta_n) = \sigma_{2k+2}(n) + \big(\sigma_{2k+1}^{odd}(n) - \sigma_{2k+1}^{odd}(n/2)\big) = \sigma_{2k+2}^{odd}(n) + \sigma_{2k+1}^{odd}(n),$$

$$\sigma_{2k}(n) - \beta_{2k-1}(\Delta_n) = \sigma_{2k}(n) - \big(\sigma_{2k}^{odd}(n) - \sigma_{2k}^{odd}(n/2)\big) = \sigma_{2k-1}^{odd}(n/2) + \sigma_{2k}^{odd}(n/2).$$

Inserting these evaluations into relations (4.1) and (4.2) one obtains the theorem. □

5. The number-theoretic cell complex $\widetilde{\Delta}_n$

The system of squarefree numbers less than or equal to n and ordered by divisibility presents itself immediately as a simplicial complex. The same is not true for the larger system of *all* such numbers. However, a construction is known [3] which produces a CW complex of a similar nature. The CW structure is no longer uniquely defined. However, if one demands that its closed i-dimensional cells are $(i-1)$-connected for all i then the complex is uniquely determined up to homotopy type.

The construction from [3] of cellular realizations of multicomplexes is reviewed in Section 6. We take it here for known.

The system of positive integers less than or equal to n, ordered by divisibility, is isomorphic to a multicomplex of monomials. Namely, associate an indeterminate x_i with each prime number $p_i \leqslant n$ and then extend this to a bijection

$$p_{i_1}^{e_1} p_{i_2}^{e_2} \cdots p_{i_k}^{e_k} \leftrightarrow x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_k}^{e_k}, \quad i_1 < i_2 < \cdots < i_k.$$

Therefore the construction of a cellular realization Γ, reviewed in Section 6, is applicable.

Definition 2. Let $\widetilde{\Delta}_n$ denote the CW complex $\Gamma(M)$, where M is the multicomplex of positive integers less than or equal to n, and the construction of Γ is based on the choice of a well-connected CW string.

Here is a summary of the main features of $\widetilde{\Delta}_n$, see Section 6 for further details.

(1) The positive integers $k \leqslant n$ are in bijection with the closed cells $c(k)$ of $\widetilde{\Delta}_n$.
(2) $\dim c(k) = \Omega(k) - 1$.
(3) The integer k_1 divides k_2 if and only if $c(k_1) \subseteq c(k_2)$.
(4) The cell $c(k)$ has the following homotopy type:

$$c(k) \simeq \begin{cases} S^{d-1}, & \text{if } k \text{ is a full square and } \Omega(k) = d, \\ \text{a point}, & \text{otherwise.} \end{cases}$$

(5) In case we are using the S^∞ string even more is true. If $\Omega(k) = d$, then the cell $c(k)$ has the following *homeomorphy* type (sphere or ball):

$$c(k) \cong \begin{cases} S^{d-1}, & \text{if } k \text{ is a full square,} \\ B^{d-1}, & \text{otherwise.} \end{cases}$$

(6) The Euler characteristic is

$$\chi(\widetilde{\Delta}_n) = \#\{k \mid k \leqslant n,\ \Omega(k) \text{ is odd}\} - \#\{k \mid k \leqslant n,\ \Omega(k) \text{ is even}\}.$$

The Euler characteristic of $\widetilde{\Delta}_n$ is equivalent to one of the classic functions of number theory. The *summatory Liouville function* $L(n)$ is defined as follows:

$$L(n) = \sum_{k \leqslant n} (-1)^{\Omega(k)}.$$

See [4] for information about this function. So,

$$\chi(\widetilde{\Delta}_n) = -L(n).$$

Thus, the summatory Liouville function $L(n)$ plays for the cell complex $\widetilde{\Delta}_n$ the same role that the Mertens function $M(n)$ plays for the simplicial complex Δ_n. The two functions are related as follows.

Proposition 5.1.

$$L(n) = \sum_{r=1}^{\lfloor\sqrt{n}\rfloor} M\left(\left\lfloor \frac{n}{r^2} \right\rfloor\right) \quad \textit{and} \quad M(n) = \sum_{r=1}^{\lfloor\sqrt{n}\rfloor} \mu(r) L\left(\left\lfloor \frac{n}{r^2} \right\rfloor\right).$$

Proof. Let $[n] \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$ and $[n]_{\text{sqf}} \stackrel{\text{def}}{=} \{k \in [n] \mid k \text{ is squarefree}\}$. Every positive integer k can be uniquely factored as a product of a full square r^2 and a squarefree integer s: $k = r^2 s$. This implies the following set-theoretic disjoint union decomposition

$$[n] = \biguplus_{r=1}^{\lfloor\sqrt{n}\rfloor} r^2 \cdot \left[\left\lfloor \frac{n}{r^2} \right\rfloor\right]_{\text{sqf}}.$$

Since $\Omega(k) = \Omega(r^2 s) \equiv \Omega(s) \pmod 2$, the first formula follows. Then the second one is obtained by Möbius inversion of the form presented in [5, p. 307]. □

The rates of growth of the two functions $L(n)$ and $M(n)$ are essentially identical, as the following proposition shows.

Proposition 5.2. *For every $\varepsilon > 0$, as $n \to \infty$,*

$$M(n) = o\left(n^{1/2+\varepsilon}\right) \quad \textit{if and only if} \quad L(n) = o\left(n^{1/2+\varepsilon}\right).$$

Proof. In what follows all numbers that are not integers are to be rounded down to the closest smaller integer. So, $\sqrt{n}$ is to be read $\lfloor\sqrt{n}\rfloor$, etc.

Let $\varepsilon > 0$ and assume that $M(n) = o(n^{1/2+\varepsilon})$ as $n \to \infty$. Choose any $\delta > 0$, and let N be such that $\sum_{r=N+1}^{\infty} \frac{1}{r^{1+2\varepsilon}} < \delta$ and $1/N \leqslant \delta$. Then, if n is large enough that $\sqrt{n} > N$ and

$$\frac{|M(\frac{n}{r^2})|}{(\frac{n}{r^2})^{1/2+\varepsilon}} \leqslant 1/N^2, \quad \text{for all } r \in \{1, 2, \dots, N\},$$

we have that

$$\begin{aligned}
\frac{|L(n)|}{n^{1/2+\varepsilon}} &\leqslant \sum_{r=1}^{\sqrt{n}} \frac{1}{r^{1+2\varepsilon}} \frac{|M(\frac{n}{r^2})|}{(\frac{n}{r^2})^{1/2+\varepsilon}} \\
&\leqslant \sum_{r=1}^{N} \frac{1}{r^{1+2\varepsilon}} \frac{|M(\frac{n}{r^2})|}{(\frac{n}{r^2})^{1/2+\varepsilon}} + \sum_{r=N+1}^{\infty} \frac{1}{r^{1+2\varepsilon}} \\
&\leqslant N \cdot \frac{1}{N^2} + \delta \\
&\leqslant 2\delta.
\end{aligned}$$

The same computation, exchanging the roles of $M(n)$ and $L(n)$, gives by Proposition 5.1 the opposite implication. □

By way of the $\varepsilon = 1/2$ case of Proposition 5.2, the Prime Number Theorem implies that

$$L(n) = o(n), \quad \text{as } n \to \infty. \tag{5.1}$$

Also, one can conclude from Proposition 5.2 that the Riemann Hypothesis is equivalent to

$$L(n) = O\left(n^{1/2+\varepsilon}\right), \quad \text{as } n \to \infty, \text{ for every } \varepsilon > 0. \tag{5.2}$$

The preceding shows that the Euler characteristic of the cell complex $\widetilde{\Delta}_n$ has the same number-theoretic relevance as that of the simplicial complex Δ_n. This motivates seeking information about the Betti numbers of $\widetilde{\Delta}_n$, as we did in Section 3 for Δ_n.

Theorem 5.3. *We have the following homotopy equivalence*

$$\widetilde{\Delta}_n \simeq \bigvee_{r^2 \leqslant n} \mathrm{susp}^{2\Omega(r)}\left(\Delta_{\lfloor n/r^2 \rfloor}\right).$$

Proof. This is a direct consequence of Theorem 6.4. □

Corollary 5.4. $\widetilde{\Delta}_n$ *has the homotopy type of a wedge of spheres, and*

$$\beta_k(\widetilde{\Delta}_n) = \sum_{r=1}^{\lfloor \sqrt{n} \rfloor} \beta_{k-2\Omega(r)}\left(\Delta_{\lfloor n/r^2 \rfloor}\right).$$

Proof. Combine with Theorem 5.3 the information coming from Theorems 3.1 and 6.5. □

Theorem 5.5. *As $n \to \infty$ we have that*

$$\sum_{k \geqslant 0} \beta_k(\widetilde{\Delta}_n) = \frac{n}{3} + O\left(n^{\theta}\right), \quad \textit{for all } \theta > \frac{22}{27}.$$

Proof. Let $\psi > \frac{17}{54}$. Using Theorem 3.4 and Corollary 5.4 we get:

$$\begin{aligned}
\sum_{k \geqslant 0} \beta_k(\widetilde{\Delta}_n) &= \sum_{r=1}^{\lfloor \sqrt{n} \rfloor} \sum_{k} \beta_k\left(\Delta_{\lfloor n/r^2 \rfloor}\right) \\
&= \sum_{r=1}^{\lfloor \sqrt{n} \rfloor} \left(\frac{2(n/r^2)}{\pi^2} + O\left(\left(\frac{n}{r^2}\right)^{\psi}\right)\right) \\
&= \frac{2n}{\pi^2} \sum_{r=1}^{\lfloor \sqrt{n} \rfloor} \frac{1}{r^2} + \sqrt{n} \cdot O\left(n^{\psi}\right) \\
&= \frac{2n}{\pi^2} \frac{\pi^2}{6} + O\left(n \sum_{r=\lfloor \sqrt{n} \rfloor}^{\infty} \frac{1}{r^2}\right) + O\left(n^{1/2+\psi}\right) \\
&= \frac{n}{3} + O\left(n^{1/2}\right) + O\left(n^{1/2+\psi}\right) \\
&= \frac{n}{3} + O\left(n^{\theta}\right), \quad \text{where } \theta = \frac{1}{2} + \psi > \frac{1}{2} + \frac{17}{54} = \frac{22}{27}.
\end{aligned}$$

□

Theorem 5.6. *As $n \to \infty$ we have that*

$$\sum_{k \text{ even}} \beta_k(\widetilde{\Delta}_n) \sim \frac{n}{6} \quad \textit{and} \quad \sum_{k \text{ odd}} \beta_k(\widetilde{\Delta}_n) \sim \frac{n}{6}.$$

Proof. Let $a(n)$ denote the first sum and $b(n)$ the second. Theorem 5.5 and Eq. (5.1) show, respectively, that

$$\frac{a(n)+b(n)}{n} \to \frac{1}{3} \quad \text{and} \quad \frac{a(n)-b(n)}{n} = \frac{L(n)}{n} \to 0$$

as $n \to \infty$. Hence, the proof can be completed as that of Theorem 3.5. □

6. Cellular realization of multicomplexes

In this section we review the construction of CW complexes from [3]. We expand on some details and present a new result concerning their homotopy type.

By a *CW string* we shall mean an infinite sequence of embeddings

$$c^0 \hookrightarrow c^1 \hookrightarrow \cdots \hookrightarrow c^j \hookrightarrow \cdots$$

such that

(1) $\{c^j\}_{j \geqslant 0}$ are the closed cells of a CW decomposition of the colimit $\bigcup c^j$.
(2) $\dim c^j = j$, for all $j \geqslant 0$.

The CW string is said to be *well connected* if

(3) c^j is $(j-1)$-connected, for all $j \geqslant 0$.

Lemma 6.1. *A CW string is well connected if and only if c^j is contractible for all even j and has the homotopy type of the j-sphere for all odd j.*

Proof. It is known that a $(j-1)$-connected and j-dimensional space has the homotopy type of a wedge of j-dimensional spheres. The number of spheres is given by the Euler characteristic. Since the string has exactly one cell in each dimension, the reduced Euler characteristic of c^j is 0 for all even j and 1 for all odd j. □

Here are three constructions of CW strings. The first one was suggested to us by Michael J. Falk (private communication). The other two appear in [3].

1. The S^∞ string. This is a CW decomposition of the infinite-dimensional sphere $S^\infty = \{(x_1, x_2, \ldots) \mid x_n = 0$ for sufficiently large n, and $x_1^2 + x_2^2 + \cdots = 1\}$. The cells are

$$c^j = \{(x_1, x_2, \ldots) \subseteq S^\infty \mid x_k = 0 \text{ for all } k > j+1\}, \quad \text{if } j \text{ is odd,}$$
$$c^j = \{(x_1, x_2, \ldots) \subseteq S^\infty \mid x_{j+1} \geqslant 0,\ x_k = 0 \text{ for all } k > j+1\}, \quad \text{if } j \text{ is even.}$$

It is clear, when j is odd, how to attach a $(j+1)$-ball onto the sphere c^j in order to obtain the ball c^{j+1}. When j is even the attachment map has the following description. Note that c^j is here the upper hemisphere of S^j, the j-dimensional sphere. Consider the $(j+1)$-ball

$$B^{j+1} = \{(x_1, x_2, \ldots) \subseteq S^\infty \mid x_{j+2} \geqslant 0,\ x_k = 0 \text{ for all } k > j+2\},$$

i.e., the upper hemisphere of S^{j+1}. By switching to polar coordinates for the last two coordinates, the points of S^j can be coordinatized

$$S^j = \{(x_1, x_2, \ldots, x_{j-1}, r\cos\theta, r\sin\theta, 0, 0, \ldots)\} \subseteq S^\infty.$$

In terms of these coordinates the attachment map $\alpha : \mathrm{bd}(B^{j+1}) = S^j \to c^j$ sends $(x_1, x_2, \ldots, x_{j-1}, r\cos\theta, r\sin\theta)$ to $(x_1, x_2, \ldots, x_{j-1}, r\cos 2\theta, r\sin 2\theta)$.

Thus we have a CW string which is well connected in a strong sense, namely its cells are *homeomorphic* to balls and spheres of the appropriate dimensions.

2. The D^∞ string. The *j-dimensional dunce hat* D^j is obtained from the j-dimensional simplex $(a_0, a_1, \ldots, a_j)$ by identifying all of its $(j-1)$-dimensional faces, with their vertices in the induced order. Note that D^2 is the usual dunce hat. It is shown in [1] that D^j is contractible, but not collapsible, for all even $j > 0$, and that D^j is a homotopy sphere for all odd j. Thus, with the obvious attaching maps we have a well-connected CW string

$$D^0 \hookrightarrow D^1 \hookrightarrow \cdots \hookrightarrow D^j \hookrightarrow \cdots$$

giving a CW decomposition of its colimit D^∞, the infinite-dimensional dunce hat.

3. The $\mathbb{R}P^\infty$ string. The standard CW decomposition of infinite-dimensional real projective space, with one cell in each dimension, provides a well-known example of a CW string

$$\mathbb{R}P^0 \hookrightarrow \mathbb{R}P^1 \hookrightarrow \cdots \hookrightarrow \mathbb{R}P^j \hookrightarrow \cdots.$$

This string is not well connected. However, on the level of rational homology $H(\cdot, \mathbb{Q})$ this string behaves in a way that for algebraic purposes parallels that of well-connected strings, see Remark 6.6.

Let M be a multicomplex. By this we mean a finite collection of monomials in indeterminates $x_1, x_2, \ldots$ closed under divisibility. A CW complex $\Gamma(M)$ is constructed as follows. It depends on a choice of CW string $\{c^j\}_{j\geqslant 0}$, which once chosen remains fixed and will not be included in the notation.

For each indeterminate x_i take a copy $\{c_i^j\}_{j\geqslant 0}$ of the string. Then, to each monomial $m = x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_k}^{e_k} \in M$ associate the space

$$c(m) \stackrel{\mathrm{def}}{=} c_{i_1}^{e_1-1} * c_{i_2}^{e_2-1} * \cdots * c_{i_k}^{e_k-1}$$

where "$*$" denotes the join of topological spaces. Note that if m is squarefree then $c(m)$ is a $(k-1)$-dimensional simplex, since c^0 is a single vertex (for every choice of CW string).

We say that m is a *full square* if $m = r^2$ for some monomial r.

Lemma 6.2. [3, Prop. 2.2] *The space $c(m)$ has the following homotopy type*:

$$c(m) \simeq \begin{cases} S^{d-1}, & \text{if } m \text{ is a full square of degree } d, \\ \text{a point}, & \text{otherwise}. \end{cases}$$

We remark that in case we are using the S^∞ string even more is true: $c(m)$ is *homeomorphic* to the $(d-1)$-sphere or the $(d-1)$-ball, respectively.

The CW complex $\Gamma(M)$ associated with M is constructed as follows. With each monomial $m \in M$ associate the space $c(m)$, and then glue these together to form $\Gamma(M)$. The attaching maps are everywhere the ones coming from how a j-cell is attached to c_i^{j-1} to obtain c_i^j, and the joins of these maps.

Proposition 6.3. *(See [3, Sect. 2].) The space $\Gamma(M)$ is a CW complex with closed cells $c(m)$, $m \in M$. A cell $c(m)$ is contained in another cell $c(m')$ if and only if m divides m'.*

We think of $\Gamma(M)$ as the geometric realization of the multicomplex M. Note that if all monomials in M are squarefree then the construction reduces to the usual geometric realization of a simplicial complex.

Every monomial can be uniquely factorized as a product of a full square monomial and a square-free monomial: $m = r^2 s$. For each full square monomial r^2 in M we define

$$M_{r^2} = \{ s \in M \mid r^2 s \in M, s \text{ squarefree} \}.$$

Notice that each M_{r^2} is a simplicial complex, possibly empty.

Let $|r|$ denote the degree of a monomial r.

Theorem 6.4. *Suppose that a well-connected CW string has been used for the construction of $\Gamma(M)$. Then*

$$\Gamma(M) \simeq \bigvee_{r^2 \in M} \operatorname{susp}^{2|r|} \Gamma(M_{r^2}).$$

Proof. We are going to use the theory of homotopy colimits of diagrams of spaces. The tools from this theory that we need are summarized in an accessible way in [7]. We refer the reader to that paper for all explanations of terminology and basic facts used in the sequel. We frequently use the fact that the join operation of topological spaces commutes with the needed operations, up to homotopy.

The multicomplex M, ordered by divisibility is a poset. The functor

$$\mathcal{D} : m \mapsto c(m) \quad \text{and} \quad \mathcal{D} : m|m' \mapsto \big(c(m) \hookrightarrow c(m')\big)$$

gives us a diagram of spaces $\mathcal{D} : M \to \mathit{Top}$. By the "Projection Lemma" [7, Prop. 3.1] we have that

$$\Gamma(M) \simeq \operatorname{hocolim} \mathcal{D}. \tag{6.1}$$

Let $\mathcal{E}$ be a diagram over M with constant maps. Since $\dim(\bigcup_{m'<m} c(m')) = \dim c(m) - 1$ and $c(m)$ is $(\dim c(m) - 1)$-connected, it follows that any two maps from $\bigcup_{m'<m} c(m')$ to $c(m)$ are homotopic. In particular, the inclusion map is homotopic to any constant map. Therefore, using the "Homotopy Lemma" [7, Prop. 3.7] and the homotopy extension property, we get

$$\operatorname{hocolim} \mathcal{D} \simeq \operatorname{hocolim} \mathcal{E}. \tag{6.2}$$

Finally, the "Wedge Lemma" [7, Lemma 4.9] implies that

$$\operatorname{hocolim} \mathcal{E} \simeq \bigvee_{m \in M} \big(c(m) * \Delta\big(M^{>m}\big)\big), \tag{6.3}$$

where $\Delta(M^{>m})$ denotes the order complex of the poset of monomials in M that are strictly above m in the partial order, i.e., that are divisible by m.

Many of the spaces appearing in the wedge are contractible and can therefore be removed without affecting the homotopy type. Namely, we have that

$$c(m) * \Delta\big(M^{>m}\big) \simeq \begin{cases} \operatorname{susp}^{2|r|}(\Delta(M^{>m})), & \text{if } m = r^2 \in M, \\ \text{a point}, & \text{otherwise.} \end{cases}$$

This follows from Lemma 6.2 and the fact that taking join with the $(d-1)$-dimensional sphere is, up to homeomorphism, equivalent to taking d-fold suspension. Combining Eqs. (6.1), (6.2) and (6.3) we obtain the homotopy equivalence

$$\Gamma(M) \simeq \bigvee_{r^2 \in M} \operatorname{susp}^{2|r|} \Delta\big(M^{>r^2}\big). \tag{6.4}$$

Thus, the following homotopy equivalences complete the proof:

$$\Delta\big(M^{>r^2}\big) \simeq \Delta(M_{r^2}) \simeq \Gamma(M_{r^2}).$$

The first equivalence is implied by [3, Prop. 4.1]. The second follows from the fact that the homeomorphism type of a simplicial complex is invariant under barycentric subdivision. □

Corollary 6.5. [3, p. 55]

$$\widetilde{H}_k\big(\Gamma(M), \mathbb{Z}\big) \cong \bigoplus_{r^2 \in M} \widetilde{H}_{k-2|r|}\big(\Gamma(M_{r^2}), \mathbb{Z}\big).$$

Remark 6.6. The $\mathbb{R}P^\infty$ string is not well connected, since its cells c^j are real projective spaces with plenty of mod 2 torsion. However, the string is "rationally well connected", since

$$\widetilde{H}_k\big(c^j, \mathbb{Q}\big) \cong \begin{cases} \mathbb{Q}, & \text{if } j \text{ is odd and } k = j, \\ 0, & \text{in all other cases.} \end{cases}$$

Thus, in terms of rational homology the $\mathbb{R}P^\infty$ string behaves like the well-connected strings. By replacing the diagrammatic tools used in the proof of Theorem 6.4 by their homological counterparts one can derive a splitting formula for $\widetilde{H}_k(\Gamma(M), \mathbb{Q})$ analogous to Corollary 6.5. The rational Betti numbers produced for the number-theoretic cell complexes using joins of real projective spaces as cells will therefore be the same as those computed in Sections 3 and 5.

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