NEUTRALIZED VALUES. II
NEUTRALIZED VALUES WITH MORE THAN ONE NEUTRIX

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1. Special neutrix calculus with two or three neutrices

Let us first consider the case that the structure consists of two neutrices \( L \) and \( M \) with the structure relation \( L \rightarrow M \) given by the mapping \( \zeta \rightarrow \xi(\zeta) \). Let \( f(\zeta, \xi) \) be a function defined for each element \( \zeta \) of \( \Delta L \) and for \( \xi = \xi(\zeta) \) which can be written for each element \( \zeta \) of \( \Delta L \) in the form

\[
f(\zeta, \xi(\zeta)) = \mu(\xi(\zeta)) \lambda(\zeta) \alpha,
\]

where \( \alpha \) denotes an element of the fundamental set \( \Gamma \) independent of \( \zeta \), where \( \lambda(\zeta) \) is a fundamental operator belonging to \( L \) and where \( \mu(\xi) \) is a fundamental operator belonging to \( M \). Notice that \( f(\zeta, \xi) \) is not necessarily defined for each element \( \zeta \) of \( \Delta L \) and each element \( \xi \) of \( \Delta M \).

Since \( M \) is a homomorphic image of \( L \) with mapping \( \zeta \rightarrow \xi(\zeta) \), the operator \( \mu(\xi(\zeta)) \) is by definition a fundamental operator belonging to \( L \). Consequently the product \( \mu(\xi(\zeta)) \lambda(\zeta) \) is also a fundamental operator belonging to \( L \). It follows therefore from (1) that the constant \( \alpha \) is uniquely defined and is equal to

\[
\alpha = f(L, \xi(L)).
\]

In this way \( \alpha \) is represented by an expression involving one neutrix, namely \( L \). This notation involves also the function \( \xi(\zeta) \) occurring in the mapping \( \zeta \rightarrow \xi(\zeta) \). On the other hand, if the structure is given, then this function is known, so that it is better to have a notation which does not involve this function. For this reason \( \alpha \) will also be represented by the symbol \( f(L, M) \), so that

\[
\alpha = f(L, M).
\]
In a structure with two neutrices $L$ and $M$ we admit either the structure relation $L \rightarrow M$ or the structure relation $M \rightarrow L$ which is obtained from the first one by interchanging $L$ and $M$.

In a structure formed by three neutrices $L$, $M$ and $N$ we admit the structure relations $L \sim M$ and $M \sim L$ which is obtained from the first one by permutation of the neutrices $L$, $M$ and $N$.

The structure relations $L \rightarrow M \rightarrow N$ mean that $M$ is a given homomorphic image of $L$ with a given mapping $\zeta \rightarrow \xi(\zeta)$ and that $N$ is a given homomorphic image of $M$ with a given mapping $\xi \rightarrow \eta(\xi)$. The structure relations $L \triangleleft M \triangleleft N$ mean that $M$ is a given homomorphic image of $L$ with a given mapping $\zeta \rightarrow \xi(\zeta)$ and that $N$ is a given homomorphic image of $L$ with a given mapping $\xi \rightarrow \eta(\xi)$.

In the structure with structure relation $L \rightarrow M \rightarrow N$ the neutrix $N$ is according to the theorem in § 15 the homomorphic image of $L$ with the mapping $\zeta \rightarrow \eta(\xi(\zeta))$. We assume that $f(\zeta, \xi(\zeta), \eta(\xi(\zeta)))$ is defined for each element $\zeta$ of $\Delta L$ and can be written in the form

$$f(\zeta, \xi(\zeta), \eta(\xi(\zeta))) = v(\eta(\xi(\zeta))) \mu(\xi(\zeta)) \lambda(\zeta) \alpha,$$

where $\alpha$ is an element of $\Gamma$ independent of $\zeta$, and where $\lambda(\zeta), \mu(\xi)$ and $v(\eta)$ denote fundamental operators belonging respectively to $L$, $M$ and $N$. By hypothesis $\lambda(\zeta), \mu(\xi(\zeta))$ and $v(\eta(\xi(\zeta)))$ are fundamental operators belonging to $L$, so that

$$\alpha = f(L, \xi(L), \eta(\xi(L))).$$

Then $\alpha$ will also be represented by

$$\alpha = f(L, M, N).$$

In the structure with structure relation $L \triangleleft M \triangleleft N$ we assume that $f(\zeta, \xi(\zeta), \eta(\zeta)))$ is defined for each element $\zeta$ of $\Delta L$ and can be written in the form

$$f(\zeta, \xi(\zeta), \eta(\zeta)) = v(\eta(\xi(\zeta))) \mu(\xi(\zeta)) \lambda(\xi) \alpha,$$

where $\alpha$ is an element of $\Gamma$ independent of $\zeta$ and where $\lambda(\zeta), \mu(\xi)$ and $v(\eta)$ are fundamental operators belonging respectively to $L$, $M$ and $N$. By hypothesis $\lambda(\zeta), \mu(\xi(\zeta))$ and $v(\eta(\xi(\zeta)))$ are fundamental operators belonging to $L$, so that also their product is a fundamental operator belonging to $L$. Consequently it follows from (4) that

$$\alpha = f(L, \xi(L), \eta(L)),$$

but for this element of $\Gamma$ also the notation

$$\alpha = f(L, M, N)$$

is used.
The neutrices $L$, $M$ and $N$ with the structure $L \to M \to N$ are called compatible. Similarly the neutrices $L$, $M$ and $N$ with the structure $L_\alpha^M \overset{\alpha}{\to} N_\alpha^M$ are compatible.

When does $f(M, N)$ have a meaning in the special neutrix calculus with structure $L_\alpha^M \overset{\alpha}{\to} N_\alpha^M$? In that case $f(\xi, \eta) = f(\xi, \eta)$ is independent of $\xi$. Furthermore $f(M, N)$ is equal to an element $\alpha$ of the fundamental set if and only if $f(\xi(\xi), \eta(\xi))$ can be written in the form

$$f(\xi(\xi), \eta(\xi)) = \nu(\eta(\xi)) \mu(\xi(\xi)) \lambda(\xi) \alpha,$$

where $\nu(\eta), \mu(\xi)$ and $\lambda(\xi)$ denote fundamental operators belonging respectively to $N$, $M$ and $L$.

Above the definition of $f(L, M)$ in the special neutrix calculus has been given for the structure relation $L \to M$ and also for the structure relation $M \to L$. Here we have the definition of $f(M, N)$ in the special neutrix calculus in a completely different case, namely for the structure $L_\alpha^M \overset{\alpha}{\to} N_\alpha^M$.

2. Special neutrix calculus with an arbitrary number of neutrices

Let us first consider a structure formed by four neutrices $L$, $M$, $N$, $P$, with the structure relation $L_\alpha^M \overset{\alpha}{\to} N_\alpha^M \overset{\beta}{\to} P$. Here $M$ and $N$ are given homomorphic images of $L$ with the mappings $\zeta \to \xi$, respectively $\zeta \to \eta$; furthermore $P$ is a homomorphic image both of $M$ and $N$ with the mappings $\xi \to \sigma$, respectively $\eta \to \sigma$. The relation $L \to M \to N \to P$ shows that each element $\xi$ of $\Delta L$ has in $\Delta M$ a uniquely defined image $\xi$, therefore in $\Delta P$ a uniquely defined image $\sigma$. On the other hand it follows from $L \to N \to P$ that each element $\zeta$ of $\Delta L$ has in $\Delta N$ a uniquely defined image $\eta$, therefore in $\Delta P$ a uniquely defined image $\sigma^*$. We assume $\sigma = \sigma^*$ for each element $\xi$ of $\Delta L$.

In other words the structure relation $L_\alpha^M \overset{\alpha}{\to} N_\alpha^M \overset{\beta}{\to} P$ gives two mappings of $\Delta L$ onto $\Delta P$ and these two mappings are identical. The structure relation gives therefore five uniquely defined mappings, namely $\Delta L \to \Delta M$; $\Delta L \to \Delta N$; $\Delta M \to \Delta P$; $\Delta N \to \Delta P$; and finally $\Delta L \to \Delta P$, so that each element $\zeta$ of $\Delta L$ has in $M$, $N$ and $P$ uniquely defined images $\xi(\zeta)$, $\eta(\zeta)$ and $\sigma(\zeta)$.

If $f(\zeta, \xi(\zeta), \eta(\zeta), \sigma(\zeta))$ is defined for each element $\zeta$ of $\Delta L$ and can be written in the form

$$f(\zeta, \xi(\zeta), \eta(\zeta), \sigma(\zeta)) = \nu(\eta(\zeta)) \mu(\xi(\zeta)) \lambda(\xi(\zeta)) \pi(\sigma(\zeta)) \alpha,$$

where $\alpha$ is independent of $\zeta$ and where $\lambda(\xi(\zeta))$, $\mu(\xi(\zeta))$, $\nu(\eta(\zeta))$ and $\pi(\sigma(\zeta))$ are fundamental operators belonging respectively to $L$, $M$, $N$ and $P$, then $\lambda(\xi(\zeta))$, $\mu(\xi(\zeta))$, $\nu(\eta(\zeta))$ and $\pi(\sigma(\zeta))$ are fundamental operators belonging to $L$, so that their product belongs to $L$. Consequently the constant $\alpha$ is uniquely defined and is equal to

$$\alpha = f(L, \xi(L), \eta(L), \sigma(L)).$$

This constant is also denoted by

$$\alpha = f(L, M, N, P).$$
Now we proceed to structures formed by an arbitrary finite or infinite number of neutrices; this number may even be uncountable. These neutrices in combination with certain structure relations form a structure $\mathcal{S}$ if and only if any two neutrices $M$ and $N$ occurring in the structure satisfy the following two structure conditions:

1. If neither of the two neutrices $M$ and $N$ is a homomorphic image (given by the structure) of the other, then the structure contains at least one neutrix $L$ such that $M$ and $N$ are images of $L$ given by the structure.

2. The structure gives at most one mapping of one of the two neutrices $M$ and $N$ onto the other.

It is clear that the four structures $L \rightarrow M$; $L \rightarrow M \rightarrow N$; $L \mathrel{\bowtie} M$ and $L \mathrel{\bowtie} N \rightarrow P$ treated above satisfy these two structure conditions. There is one type of structure with two neutrices, namely $L \rightarrow M$, since the other possible structure $M \rightarrow L$ is obtained by interchanging $L$ and $M$. There are two distinct types of structures with three neutrices, namely $L \rightarrow M \rightarrow N$ and $L \mathrel{\bowtie} M \mathrel{\bowtie} N$; the other structures are obtained by permutation of the three neutrices $L$, $M$ and $N$. There are four distinct types of structures with four neutrices, namely

$$L \rightarrow M \rightarrow N \rightarrow P; \quad L \rightarrow M \mathrel{\bowtie} N; \quad L \mathrel{\bowtie} M \rightarrow P; \quad L \mathrel{\bowtie} N \mathrel{\bowtie} P$$

Theorem: If $M_1, \ldots, M_m$ denote neutrices occurring in a structure $\mathcal{S}$ on distinct places, then the structure contains at least one neutrix $L$ such that each $M_h (1 \leq h \leq m)$ which does not coincide in the structure with $L$ is an image of $L$ given by the structure.

Proof: In the case $m = 1$ we can choose $L = M_1$. Assume that $m \geq 2$ and that the proof has already been given with $m$ replaced by $m - 1$. Then the structure contains at least one neutrix $N$ such that each $M_h (1 \leq h \leq m - 1)$ which does not coincide with $N$ is an image of $N$ given by the structure.

If $M_m$ is an image of $N$ given by the structure and also if $M_m$ coincides with $N$, then $L = N$ possesses the required property. In the proof we may therefore assume that $M_m$ is not an image of $N$ given by the structure and moreover that $M_m$ does not coincide with $N$. If $N$ is an image of $M_m$ given by the structure then $L = M_m$ possesses the required property. Otherwise the structure contains according to the first structure condition at least one neutrix $L$ such that $M_m$ and $N$ are images of $L$ given by the structure, so that this neutrix $L$ possesses the required property.

3 The neutralized values $f(M_1, \ldots, M_m)$ in the special neutrix calculus

Let $M_1, \ldots, M_m$ be a finite number of neutrices occurring in different places in a given structure $\mathcal{S}$. According to the theorem of the preceding section the structure contains at least one neutrix $L$ such that each $M_h (1 \leq h \leq m)$ which does not coincide with $L$ is a homomorphic image of
L given by the structure. Each element \( \zeta \) of \( \Delta L \) has in \( \Delta M_h \) a uniquely defined image \( \xi_h(\zeta) \). Consider a function \( f(\xi_1, \ldots, \xi_m) \) such that \( f(\xi_1(\zeta), \ldots, \xi_m(\zeta)) \) is defined for each element \( \zeta \) of \( \Delta L \) and can be written in the form

\[
(1) \quad f(\xi_1(\zeta), \ldots, \xi_m(\zeta)) = \mu_1(\xi_1(\zeta)) \ldots \mu_m(\xi_m(\zeta)) \alpha,
\]

where \( \alpha \) denotes an element independent of \( \zeta \) occurring in the fundamental set \( \Gamma \) and where \( \mu_h(\xi_h) \) (\( 1 \leq h \leq m \)) is a fundamental operator belonging to \( M_h \). Then \( \mu_h(\xi_h(\zeta)) \) (\( h = 1, \ldots, m \)) are fundamental operators occurring in \( L \), so that the product of these operators is also a fundamental operator occurring in \( L \). Formula (1) gives therefore

\[
\alpha = f(\xi_1(L), \ldots, \xi_m(L)).
\]

Now the cardinal point in the special neutrix calculus: this constant element \( \alpha \) is independent of the choice of the neutrix \( L \). This problem does not arise in the structures treated above, since each such structure contains only one neutrix with the property imposed on \( L \). This element \( \alpha \) is denoted by \( f(M_1, \ldots, M_m) \).

Let \( K \) be a neutrix occurring in the given structure such that each \( M_h \) (\( 1 \leq h \leq m \)) which does not coincide with \( K \) is a homomorphic image of \( K \) with the mapping \( \sigma \rightarrow \xi_h^*(\sigma) \) and with the property

\[
(2) \quad f(\xi_1^*(\sigma), \ldots, \xi_m^*(\sigma)) = \mu_1^*(\xi_1^*(\sigma)) \ldots \mu_m^*(\xi_m^*(\sigma)) \beta,
\]

where \( \beta \) denotes an element independent of \( \sigma \) occurring in the fundamental set \( \Gamma \) and where \( \mu_h^*(\xi_h^*) \) (\( 1 \leq h \leq m \)) is a fundamental operator belonging to \( M_h \). We must prove that \( \alpha = \beta \).

According to the first structure condition the structure contains at least one neutrix \( N \) with \( N = K \rightarrow L \) or \( N = L \rightarrow K \) or with the property that \( K \) and \( L \) are homomorphic images of \( N \). Each element \( \eta \) of \( \Delta N \) possesses therefore in \( \Delta K \) a uniquely defined image \( \sigma(\eta) \) and possesses in \( \Delta L \) a uniquely defined image \( \zeta(\eta) \). If \( N \) coincides with \( K \), then \( \sigma(\eta) = \eta \) and if \( N \) coincides with \( L \), then \( \zeta(\eta) = \eta \).

Since the structure relation \( N \xrightarrow{\xi_h^*} M_h \) gives the same mapping of \( \Delta N \) into \( \Delta M_h \) one has

\[
\xi_h^*(\sigma(\eta)) = \xi_h(\zeta(\eta)) \quad (h = 1, \ldots, m)
\]

for each element \( \eta \) of \( \Delta N \). Applying (1) with \( \zeta = \zeta(\eta) \) and applying (2) with \( \sigma = \sigma(\eta) \) one obtains for each element \( \eta \) of \( \Delta N \)

\[
\mu_1(\xi_1(\zeta(\eta))) \ldots \mu_m(\xi_m(\zeta(\eta))) \alpha = \mu_1^*(\xi_1(\zeta(\eta))) \ldots \mu_m^*(\xi_m(\zeta(\eta))) \beta.
\]

Here \( \mu_h(\xi_h(\zeta(\eta))) \) and \( \mu_h^*(\xi_h(\zeta(\eta))) \) are fundamental operators belonging to \( N \), so that \( \alpha = \beta \). This completes the proof.

4. General neutrix calculus

In the preceding section we have assigned in the special neutrix calculus a value to the symbol \( f(M_1, \ldots, M_m) \) only for particular functions \( f \).
In this section mathematical objects, denoted by \( f(M_1, \ldots, M_m) \), are introduced for each choice of the function \( f \). Any two such objects \( f(M_1, \ldots, M_m) \) and \( g(N_1, \ldots, N_n) \) are said to be equal in the analysis with structure \( \mathcal{C} \) if and only if the structure contains at least one neutrix \( L \) with the following properties:

1. Each \( M_h \) \((1 \leq h \leq m)\) which does not coincide with \( L \) is a homomorphic image of \( L \) given by the structure with the mapping \( \sigma \to \xi_h(\sigma) \).
2. Each \( N_k \) \((1 \leq k \leq n)\) which does not coincide with \( L \) is a homomorphic image of \( L \) given by the structure with the mapping \( \sigma \to \eta_k(\sigma) \).
3. The functions \( f(\xi_1(\sigma), \ldots, \xi_m(\sigma)) \) and \( g(\eta_1(\sigma), \ldots, \eta_n(\sigma)) \) are defined for each element \( \sigma \) of \( \Delta L \) and satisfy for each element \( \sigma \) of \( \Delta L \) the condition

\[
\lambda(\sigma) f(\xi_1(\sigma), \ldots, \xi_m(\sigma)) = \lambda^*(\sigma) g(\eta_1(\sigma), \ldots, \eta_n(\sigma)),
\]

where \( \lambda(\sigma) \) and \( \lambda^*(\sigma) \) denote suitably chosen fundamental operators belonging to \( L \).

This notion of equality is reflexive, since in the case

\[
m = n; \quad M_h = N_h, \quad \xi_h = \eta_h, \quad f(\xi_1, \ldots, \xi_m) = g(\xi_1, \ldots, \xi_m)
\]

formula (1) holds with \( \lambda(\sigma) = 1; \lambda^*(\sigma) = 1 \).

The notion of equality is symmetric, since (1) implies

\[
\lambda^*(\sigma) g(\eta_1(\sigma), \ldots, \eta_n(\sigma)) = \lambda(\sigma) f(\xi_1(\sigma), \ldots, \xi_m(\sigma)),
\]

hence

\[
g(N_1, \ldots, N_n) = f(M_1, \ldots, M_m).
\]

That the notion of equality is transitive is the cardinal point of the general neutrix calculus. Assume

\[
f(M_1, \ldots, M_m) = g(N_1, \ldots, N_n) \quad \text{and} \quad g(N_1, \ldots, N_n) = j(P_1, \ldots, P_p).
\]

Then (1) holds. Moreover the structure contains at least one neutrix \( K \) with the following properties:

1. Each \( N_k \) \((1 \leq k \leq n)\) which does not coincide with \( K \) is a homomorphic image of \( K \) given by the structure with the mapping \( \tau \to \eta^*_k(\tau) \).
2. Each \( P_l \) \((1 \leq l \leq p)\) which does not coincide with \( K \) is a homomorphic image of \( K \) given by the structure with the mapping \( \tau \to \zeta_l(\tau) \).
3. The functions \( g(\eta_1(\tau), \ldots, \eta_n(\tau)) \) and \( j(\zeta_1(\tau), \ldots, \zeta_p(\tau)) \) are defined for each element \( \tau \) of \( \Delta K \) and satisfy for each element \( \tau \) of \( \Delta K \) the condition

\[
k(\tau) g(\eta_1(\tau), \ldots, \eta_n(\tau)) = k^*(\tau) j(\zeta_1(\tau), \ldots, \zeta_p(\tau)),
\]

where \( k(\tau) \) and \( k^*(\tau) \) denote suitably chosen fundamental operators belonging to \( K \).

According to the first structure condition the structure contains at least one neutrix \( R \) which either coincides with at least one of the neutrices
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\[ K \text{ and } L \text{ whereas the other of these two neutrices is a homomorphic image of } R \text{ or both } L \text{ and } K \text{ are homomorphic images of } R \text{ given by the structure. In this way we obtain the following structure relation} \]

\[
\begin{array}{c}
M_h/\xi_h \\
L/\sigma \\
R/\nu \\
N_k/\eta_k \\
K/\tau \\
P_l/\eta_l
\end{array}
\]

It is possible that \( L \) coincides with one of the neutrices \( M_h \) or \( N_k \), that \( K \) coincides with one of the neutrices \( N_k \) or \( P_l \), that \( R \) coincides with \( L \) or \( K \).

The notations \( M_h/\xi_h, R/\nu \) and so on indicate that \( \xi_h \) denotes an arbitrary element of \( \Delta M_h \), that \( \nu \) denotes an arbitrary element of \( \Delta R \), and so on.

Each element \( \nu \) of \( \Delta R \) has according to the structure relation \( R \rightarrow L \rightarrow N_k \) in \( \Delta N_k \) an image \( \eta_k(\sigma(v)) \) and according to the structure relation \( R \rightarrow K \rightarrow N_k \) in \( \Delta N_k \) the same image \( \eta_k^*(\tau(v)) \), so that for each element \( \nu \) of \( \Delta R \)

\[ \eta_k(\sigma(v)) = \eta_k^*(\tau(v)) \quad (k = 1, 2, \ldots, n). \]

Applying (1) with \( \sigma = \sigma(v) \) and applying (2) with \( \tau = \tau(v) \) we obtain therefore for each element \( \nu \) of \( R \)

\[
\begin{align*}
&k(\tau(v)) \lambda(\sigma(v)) f(\xi_1(\sigma(v)), \ldots, \xi_m(\sigma(v))) \\
&= k^*(\tau(v)) \lambda^*(\sigma(v)) j(\xi_1(\tau(v)), \ldots, \xi_p(\tau(v))).
\end{align*}
\]

The four fundamental operators occurring in this formula belong to \( R \), so that also the products \( k(\tau(v)) \lambda(\sigma(v)) \) and \( k^*(\tau(v)) \lambda^*(\sigma(v)) \) denote fundamental operators belonging to \( R \). According to the definition of equality formula (3) implies

\[ f(M_1, \ldots, M_m) = j(P_1, \ldots, P_p), \]

so that the notion of equality is transitive.

In the particular case that (1) holds, where \( g(\eta_1(\sigma), \ldots, \eta_n(\sigma)) \) is for each element \( \sigma \) of \( \Delta L \) equal to an element \( \alpha \) independent of \( \sigma \) belonging to the fundamental set \( \Gamma \), then \( f(M_1, \ldots, M_m) \) is identified with \( \alpha \). This has the consequence that each element \( \alpha \) of the fundamental set is equal to a neutralized value \( f(M_1, \ldots, M_m) \), so that the special neutrix calculus is a particular case of the general neutrix calculus.
5. Operators and pseudo operators

Let \( p \) be a positive integer. A system \( \{x_1, \ldots, x_p\} \) will be denoted by \( \{x_k\} \), where \( k \) assumes the values 1, 2, \ldots, \( p \). Let \( u \) denote an operator admitted to the fundamental analysis which can be applied on some systems \( \{x_k\} \) formed by \( p \) elements of the fundamental set \( \Gamma \). This means that the result \( u\{x_k\} \) obtained by application of \( u \) on a system \( \{x_k\} \) is an element of \( \Gamma \) whenever this application is possible.

The purpose of this section is to introduce in the analysis with a structure \( \mathfrak{S} \) symbols of the form \( u\{f_k(M_1, \ldots, M_m)\} \), where \( f_k(M_1, \ldots, M_m) \) \( (k=1, \ldots, p) \) denote neutralized values; if \( u\{f_k(M_1, \ldots, M_m)\} \) exists, then it represents again a neutralized value belonging to the analysis with structure \( \mathfrak{S} \).

Here we have to distinguish two cases.

1. In the analysis with structure \( \mathfrak{S} \) \( u \) denotes an operator applied on systems \( \{f_k(M_1, \ldots, M_m)\} \) formed by \( p \) neutralized values. This means: If

\[
f_k(M_1, \ldots, M_m) = g_k(N_1, \ldots, N_n) \quad (k=1, \ldots, p)
\]

and \( u\{g_k(N_1, \ldots, N_n)\} \) exists, then \( u\{f_k(M_1, \ldots, M_m)\} \) exists and represents the same neutralized value.

In particular: if

\[
f_k(M_1, \ldots, M_m) = x_k \quad (k=1, \ldots, p),
\]

where \( x_k \) \( (k=1, \ldots, p) \) is an element of the fundamental set \( \Gamma \) and if the operator \( u \) admitted to the fundamental analysis has the property that it can be applied on the system \( \{x_k\} \), then the operator \( u \) can be applied on the system \( \{f_k(M_1, \ldots, M_m)\} \) and one has

\[
u\{f_k(M_1, \ldots, M_m)\} = u\{x_k\}.
\]

This phenomenon implies the law of permanence according to which a formula valid in the fundamental analysis remains true in the analysis with structure \( \mathfrak{S} \) when in this formula each element \( \alpha \) of the fundamental set \( \Gamma \) is replaced by the neutralized value which is equal to \( \alpha \) and when simultaneously the operator \( u \) admitted to the fundamental analysis is replaced by the operator \( u \) introduced into the analysis with structure \( \mathfrak{S} \).

As an example how sometimes \( u\{f_k(M_1, \ldots, M_m)\} \) can be defined we assume that it is possible to find \( p \) neutralized values \( g_k(N_1, \ldots, N_n) \) with properly (1) such that the operator \( u \) admitted to the fundamental analysis can be applied on the system \( \{g_k(\eta_1, \ldots, \eta_n)\} \) for each element \( \eta_h \) of \( \Delta N_h \) \( (h=1, \ldots, n) \), say

\[
(2) \quad u\{g_k(\eta_1, \ldots, \eta_n)\} = j(\eta_1, \ldots, \eta_n).
\]

Then we put

\[
u\{f_1(M_1, \ldots, M_m)\} = j(N_1, \ldots, N_n),
\]

provided that the neutralized value occurring on the right-hand side is
independent of the choice of the neutrices \( N_1, \ldots, N_m \) and independent of the choice of the functions \( g_k(\eta, \ldots, \eta_n) \) \((k=1, \ldots, p)\).

It is clear that \( u \) represents here an operator, for if

\[
\hat{f}_k(M_1, \ldots, M_m) = \hat{f}_k(M_1^*, \ldots, M_q^*) \quad (k=1, \ldots, p),
\]

then we can use in the definition of \( u\{\hat{f}_k(M_1^*, \ldots, M_q^*)\} \) the same neutralized values \( g_k(N_1, \ldots, N_n) \) \((k=1, \ldots, p)\) as above so that we obtain for \( u\{\hat{f}_k(M_1^*, \ldots, M_q^*)\} \) the same result as for \( u\{\hat{f}_k(M_1, \ldots, M_m)\} \).

2. Assume that the operator \( u \) occurring in the fundamental analysis can be applied for each element \( \xi_h \) of \( \Lambda M_h \) \((h=1, \ldots, m)\) on the system \( \{f_k(\xi_1, \ldots, \xi_m)\} \), say

\[
(3) \quad u\{f_k(\xi_1, \ldots, \xi_m)\} = f(\xi_1, \ldots, \xi_m).
\]

If the neutrices \( M_1, \ldots, M_m \), the functions \( f_k(\xi_1, \ldots, \xi_m) \) \((k=1, \ldots, p)\) and the operator \( u \) admitted to the fundamental analysis are given, then the neutralized value \( f(M_1, \ldots, M_m) \) is uniquely defined. We denote this neutralized value by

\[
(4) \quad u\{f_k(M_1, \ldots, M_m)\} = f(M_1, \ldots, M_m).
\]

In this case \( u \) is not necessarily an operator applied on the system \( \{f_k(M_1, \ldots, M_m)\} \). In fact, if (1) holds, we have no guarantee that \( u\{g_k(N_1, \ldots, N_n)\} \) exists and even if it exists we have no guarantee that it has the same meaning as \( u\{f_k(M_1, \ldots, M_m)\} \). In this case \( u \) is therefore called a pseudo operator. The left side of (4) is to be considered as one symbol, under certain circumstances uniquely defined if the operators \( M_1, \ldots, M_m \), the functions \( f_1, \ldots, f_p \) and the operator \( u \) admitted to the fundamental analysis are given.

Also the introduction of the pseudo operators maintains the law of permanence, for if (3) is valid in the fundamental analysis, then the corresponding formula (4) holds in the analysis with structure \( \mathcal{G} \).

The operator \( u \) and the pseudo operator \( u \) may be used simultaneously if both have a meaning. In other words: if \( u\{f_k(M_1, \ldots, M_m)\} \) has a meaning for the operator \( u \) and also for the pseudo operator \( u \), then it represents in both cases the same neutralized value. Indeed, in that case one can choose in the definition of the operator \( u \)

\[
g_k(N_1, \ldots, N_n) = f_k(M_1, \ldots, M_m) \quad (k=1, \ldots, p); \quad n = m; \quad \xi_h = \eta_h; \quad M_h = N_h
\]

\((h=1, \ldots, m), \)

so that the formulas (2) and (3) yield the same function \( f(\xi_1, \ldots, \xi_m) \), therefore the same neutralized value \( f(M_1, \ldots, M_m) \).

The analysis with structure \( \mathcal{G} \) is formed by the relations which are valid between the neutralized values and which involve operators and pseudo operators. The high degree of freedom in the choice of the fundamental operators and the structure guarantees the generality of the new analysis.
If in the fundamental set $\Gamma$ a certain order ($x < \beta$), a certain distance between any two elements, a certain norm or a certain topology are given, then under general conditions it is possible to introduce into the set formed by the neutralized values also an order, distance, norm or topology. However the only purpose of this introduction is to give an abstract definition of the analysis with an arbitrary structure. The treatment of some calculi with given structure is left to the future.

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