Ergodic Flows Are Strictly Ergodic

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1. INTRODUCTION

Studying the relationship between topological and measure theoretic properties of dynamical systems the following question is considered to be of interest. Under which conditions is an abstract dynamical system \((\Omega, \mathcal{A}, m, T)\) isomorphic to a strictly ergodic one? Here \((\Omega, \mathcal{A}, m)\) denotes a probability space and \(T\) a measure preserving discrete—or continuous—time transformation group.

In order to get a feeling whether strictly ergodic systems are abundant among all abstract dynamical systems a lot of examples of such systems were constructed in the last ten years. In 1969, R. I. Jewett [6] proved that every weakly mixing invertible (discrete—time) transformation of a Lebesgue measure space is isomorphic to a strictly ergodic system. His conjecture that the result would also hold if the transformation is only ergodic was first proved by W. Krieger [7] in 1970. G. Hansel and J. P. Raoult [4] gave an independent proof. A third proof of this result, in the case of finite entropy, was just finished by one of the authors [2]. In 1970, Jewetts theorem was carried over to continuous-time flows by K. Jacobs [5].

In this paper we show that every ergodic flow on a Lebesgue measure space is isomorphic to a strictly ergodic one. This result is achieved by using an imbedding theorem of one of the authors [3] which assures that the flow can be considered as lying in an appropriate function space. In this space we develop a continuous—time coding technique which we use iteratively.

2. BASIC FACTS ABOUT FLOWS

Let \((\Omega, \mathcal{A}, m)\) denote a measure space and \((S_t)_{t \in \mathbb{R}}\) a one—parameter group of measurable \(m\)—preserving 1—1 mappings of \(\Omega\) onto itself. Then \((\Omega, \mathcal{A}, m, (S_t)_{t \in \mathbb{R}})\) is called a flow.

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This flow is called measurable if the mapping

$$(w, t) \rightarrow S_t w$$

defined on $\Omega \times R$ is measurable with respect to the product-$\sigma$-algebra. We consider only measurable flows on Lebesgue measure spaces, i.e., $(\Omega, \mathcal{A}, m)$ is isomorphic to the unit interval endowed with Lebesgue measure on the Borel sets.

A set $\Omega_0 \subset \Omega$ is $((S_t)_{t \in R})$ invariant if for every $x \in \Omega_0$, $t \in R$ the point $S_t x$ is in $\Omega_0$.

$$\mathcal{O}(x) = \{S_t x \mid t \in R\}$$

is called the orbit of $x$.

A flow is ergodic if there exist no invariant sets except null sets and their complements and it is aperiodic if the set of periodic points is a null set.

Two flows $(\Omega, \mathcal{A}, m, (S_t)_{t \in R})$ and $(\Omega', \mathcal{A}', m', (S'_t)_{t \in R})$ are isomorphic if there is a $1-1$ measure preserving correspondence $\varphi$ between invariant measurable subsets $\Omega_0$, $\Omega_0'$ of measure 1 in $\Omega$, $\Omega'$ respectively such that

$$\varphi \circ S_t(x) = S'_t \circ \varphi(x) \quad (x \in \Omega_0, t \in R).$$

Let $(S_t)_{t \in R}$ denote a one-parameter group of homeomorphisms defined on a compact metric space $X$. We call $(X, (S_t)_{t \in R})$ a continuous flow if the mapping $(x, t) \rightarrow S_t x$ defined on $X \times R$ is continuous. A closed subset $\emptyset \neq X_0 \subset X$ is called minimal invariant if it is invariant and if there is no nonempty closed invariant subset of $X_0$ except $X_0$ itself. Note that if $X_0$ is an invariant set we can always restrict $(S_t)_{t \in R}$ to $X_0$, i.e., consider the flow $(X_0, (S_t)_{t \in R})$.

$(X_0, \mathcal{L}, (S_t)_{t \in R})$ is strictly ergodic if $X_0 \subset X$ is minimal invariant and if there is only one $(S_t)_{t \in R}$-invariant probability measure $m$ on $\mathcal{L}$, the $\sigma$-algebra of Borel sets in $X_0$.

Let $C(X)$ be the normed linear space of continuous real functions on $X$. It is well-known that for a minimal invariant set $\emptyset \neq X_0 \subset X$ the continuous flow $(X_0, \mathcal{L}, (S_t)_{t \in R})$ is strictly ergodic iff for every $f \in C(X)$ and every $x \in X_0$ the time means

$$T^{-1} \int_0^T f(S_{u+t} x) \, dt$$

converge uniformly in $u \in R$ to a constant when $T$ tends to infinity.

This constant is independent of $x \in X_0$ and is $\int f \, dm$ where $m$ is the unique probability measure on $X_0$. If we have shown the uniform
convergence of time means for every \( f \in C(X) \) and every \( x \) in an invariant subset of \( X \) of total \( m \)-measure 1, then the flow \( (X_0, \mathcal{L}, (S_t)_{t \in \mathbb{R}}) \) is strictly ergodic where \( X_0 \) denotes the smallest closed invariant subset with \( m(X_0) = 1 \).

As an introduction to strict ergodicity see for example [8]. A continuous flow \( (X, (S_t)_{t \in \mathbb{R}}) \) is nonwandering iff for every \( n \in \mathbb{N} \) and every open set \( \emptyset \neq U \subseteq X \) there is \( t \geq n \) such that

\[
S_t U \cap U \neq \emptyset.
\]

We note that if \( (X, (S_t)_{t \in \mathbb{R}}) \) is nonwandering then for every open set \( \emptyset \neq U \subseteq X \) there is an unbounded sequence \( t_i (i \in \mathbb{N}) \) with \( 0 < t_i < t_{i+1} \) \( (i \in \mathbb{N}) \) satisfying

\[
S_{t_i} U \cap U \neq \emptyset \quad (i \in \mathbb{N}).
\]

If, in addition, \( (X, (S_t)_{t \in \mathbb{R}}) \) has a dense orbit, then for every open sets \( \emptyset \neq U, V \subseteq X \) there is an unbounded sequence \( s_i \) \( (i \in \mathbb{N}) \) with \( 0 < s_i < s_{i+1} \) \( (i \in \mathbb{N}) \) and

\[
S_{s_i} V \cap U \neq \emptyset \quad (i \in \mathbb{N}).
\]

If \( A > 0 \) then we denote by \( L^A \) the space of \([0, 1]\)-valued functions \( x \) on \( \mathbb{R} \) satisfying the Lipschitz condition

\[
| x(s) - x(t) | \leq A | s - t | \quad (s, t \in \mathbb{R}).
\]

Endowed with the metric

\[
\| x - y \| = \sum_{k=1}^{\infty} 2^{-k} \left( \sup_{t \in [-k, k]} | x(t) - y(t) | \right)
\]

\( L^A \) is a compact metric space.

For every \( t \in \mathbb{R} \)

\[
T_t : L^A \to L^A
\]

defined by

\[
T_t x(s) = x(s + t) \quad (s \in \mathbb{R})
\]

is a group of homeomorphisms such that \( (L^A, (T_t)_{t \in \mathbb{R}}) \) is a continuous flow. It is called the shift flow on \( L^A \). Since \( L^A \subseteq L^{A'} \) for \( A < A' \) we denote the group of homeomorphisms on both spaces with the same symbol \((T_t)_{t \in \mathbb{R}}\).
Now let $f$ be in $C(L^A)$. We say $f$ depends on the interval $I \subset R$ if the following holds:

If $x, y \in L^A$ and $x(t) = y(t)$ \quad ($t \in I$)

then $f(x) = f(y)$.

In part 3 we shall consider a fixed $(T_t)_{t \in R}$-invariant probability Borel measure $\mu$ on $L^A$ and we shall write for every $f \in C(L^A)$, $x \in L^A$ and $s \in [0, \infty[$

$$\Gamma_s(f, x) = \left| s^{-1} \int_0^s f(T_t x) \, dt - \int f \, d\mu \right|.$$ 

3. Improvement of Time Means

Let $(L^A, \mathcal{L}, \mu, (T_t)_{t \in R})$ be as in Chapter 2 for some $A > 0$ where $\mu$ is an ergodic, aperiodic, $(T_t)_{t \in R}$-invariant probability measure on $\mathcal{L}$.

We fix up to 3.9 real functions $f_1, \ldots, f_{N+1} \in C(L^A)$ with the property that $f_k$ depends on $[-k, k]$ ($1 \leq k \leq N + 1$), $\|f_k\| \leq 1$ and we fix also up to 3.9 reals $\epsilon_1, \ldots, \epsilon_N > 0$ as well as reals $M_1, \ldots, M_N > 0$ with $M_1 < M_2 < \cdots < M_N$.

Put $h = M_N + 2N + A^{-1}$ and

$$G = \{x \in L^A \mid \Gamma_{M_k}(f_i, T_t x) < \epsilon_k (1 \leq i \leq k, 1 \leq k \leq N, t \in R)\}.$$ 

Note that $G \in \mathcal{L}$ is $(T_t)_{t \in R}$-invariant.

For $\eta > 0$ and $x \in L^A$ we denote

$$C_\eta(x) = \{y \in L^A \mid \sup_{t \in [-N, M_N + N]} |y(t) - x(t)| < \eta\}$$

and if $X \subset L^A$ we write

$$C_\eta(X) = \bigcup_{x \in X} C_\eta(x).$$

**Lemma 3.1.** If $\emptyset \neq X \subset G$ is closed and $(T_t)_{t \in R}$-invariant then there exists $\eta > 0$ such that for every $y \in C_\eta(X)$

$$\Gamma_{M_k}(f_i, y) \leq \epsilon_k \quad (1 \leq i \leq k, 1 \leq k \leq N).$$
Proof. Let
\[ \epsilon_k(x) = \min_{1 \leq i \leq k} (\epsilon_k - \Gamma_{M_k}(f_i, x)) \quad (1 \leq k \leq N) \]
and
\[ \epsilon_k' = \inf_{x \in X} \epsilon_k(x). \]

Since \( \epsilon_k(.) \) is a continuous function on \( X \) we have \( \epsilon_k' > 0 \) (1 < \( k \) < \( N \)), i.e., \( \epsilon' = \min_{1 \leq k \leq N} \epsilon_k' > 0 \).

Choose \( \eta > 0 \) such that
\[ |f_i(x) - f_i(y)| < \epsilon' \quad (1 \leq i \leq N) \]
for all \( x, y \in L^A \) with \( \|x - y\| < \eta \). Since the \( f_i \) depend only on \([-N, N]\) we conclude: if
\[ \sup_{t \in [-N,N]} |x(t) - y(t)| < \eta \]
then
\[ |f_i(x) - f_i(y)| < \epsilon' \quad (1 \leq i \leq N). \]

If \( y \in C_\eta(X) \) then there is \( x \in X \) such that for all \( x \in [0, M_N] \)
\[ \sup_{t \in [-N,N]} |T_s y(t) - T_s x(t)| = \sup_{t \in [-N,N]} |y(t) - x(t)| < \eta. \]

Now
\[ \Gamma_{M_k}(f_i, y) \leq \left| M_k^{-1} \int_0^{M_k} f_i(T_s y) \, ds - M_k^{-1} \int_0^{M_k} f_i(T_s x) \, ds \right| \]
\[ + \left| M_k^{-1} \int_0^{M_k} f_i(T_s x) \, ds - \int f_i \, d\mu \right| \]
\[ \leq M_k^{-1} M_k \epsilon' + \Gamma_{M_k}(f_i, x) \leq \epsilon_k(x) + \Gamma_{M_k}(f_i, x) \leq \epsilon_k \quad (1 \leq i \leq k, 1 \leq k \leq N). \]

Corollary 3.2. Let \( X, \eta \) be as in 3.1; then
\[ \{ y \in L^A \mid T_t y \in C_\eta(X)(t \in \mathbb{R}) \} \subseteq G. \]

Lemma 3.3. Let \( X, \eta \) be as in 3.1. If \( x, y \in G \) and \( a \in \mathbb{R} \) with
\[ |x(a) - y(a)| < \eta 2^{-1}, \]
are such that there is
\[ z \in X \cap \{ v \in L^A \mid \sup_{t \in [a-h,a]} | v(t) - x(t) | < \eta 2^{-1} \} \]
\[ \cap \{ v \in L^A \mid \sup_{t \in [a,a+M_N+2N]} | v(t) - y(t) | < \eta \}, \]
then there is \( w \in G \)—uniquely constructed in the proof—such that
\[ w(t) = \begin{cases} x(t) & t \in (-\infty, a - A^{-1}] \\ y(t) & t \in [a, \infty[. \end{cases} \]

Remark 3.4. In general, there are many possibilities to construct such a \( w \in G \). But in the applications of 3.3 we will always use the construction given here. This is important in order to get measurability of the mapping \( \varphi \) in 3.10. The same remark holds for the construction in 3.5 and 3.8.

Proof. Shifting \( x, y \) by \( T_a \), we can assume \( a = 0 \). Furthermore we assume \( x(0) \geq y(0) \) and define
\[ u(t) = \begin{cases} y(t) & t \in [0, \infty[ \\ y(0) - At & t \in (-\infty, 0]. \end{cases} \]
Then there is \( t_1 \in [-A^{-1}, 0] \) and \( u(t_1) = x(t_1) \).
Define
\[ w(t) = \begin{cases} x(t) & t \in (-\infty, t_1] \\ u(t) & t \in [t_1, \infty[. \end{cases} \]
then \( w \in L^A \). We have to show
\[ M_k(f, T_s w) < \epsilon_k \quad (1 \leq i \leq k, 1 \leq k \leq N, s \in \mathbb{R}) \]
\[(1) \quad s \in (-\infty, t_1 - M_N - N]: \]
\[ T_{s+t} w(t) = T_{s+t'} x(t) \quad t' \in [0, M_N], \quad t \in (-N, N] \]
\[ \Rightarrow f_i(T_{s+t} w) = f_i(T_{s+t'} x) \quad t' \in [0, M_N], \quad 1 \leq i \leq N \]
\[ \Rightarrow M_k(f, T_s w) = M_k(f, T_s x) < \epsilon_k \quad (1 \leq i \leq k, 1 \leq k \leq N) \]
\[(2) \quad s \in [N, \infty[ : \text{analogous to (1)} \]
\[(3) \quad s \in [t_1 - M_N - N, N]: \]
\[ |w(t) - x(t)| = |y(t) - z(t)| < \eta \quad t \in [0, M_N + 2N] \]
\[ |w(t) - z(t)| \leq |w(t) - x(t)| + |x(t) - z(t)| < \eta \quad t \in [-h, 0] \]
\[ \Rightarrow T_s w \in C_{\eta}(T_s x) \subset C_{\eta}(X). \]
By 3.1 we have \( M_k(f, T_s w) < \epsilon_k \) \( (1 \leq i \leq k, 1 \leq k \leq N) \).
In the same way we can prove a symmetric form of 3.3. It reads

**Lemma 3.5.** Let $X, \eta$ be as in 3.1. If $x, y \in G$ and $a \in \mathbb{R}$ with

$$|x(a) - y(a)| < \eta 2^{-1},$$

are such that there is

$$z \in X \cap \{v \in L^\mathcal{A} | \sup_{t \in [a - M_N - 2N, a]} |v(t) - x(t)| < \eta\} \cap \{v \in L^\mathcal{A} | \sup_{t \in [a, a + h]} |v(t) - y(t)| < \eta 2^{-1}\},$$

then there is $w \in G$—constructed in a unique manner—such that

$$w(t) = \begin{cases} x(t) & t \in [\alpha, a] \\ y(t) & t \in [a + A^{-1}, \infty] \end{cases}.$$

**Lemma 3.6.** Let $X$ and $\eta$ be as in 3.1. Then there exists $T > 0$ with the following property: Let $x \in G$ and $a \in \mathbb{R}$ such that there is $x' \in X$ with

$$|x(t) - x'(t)| < \eta 2^{-1} \quad t \in [a - M_N - 2N - T, a + M_N + 2N]$$

and let $T'' \in [0, T]$. Then for every $z \in L^{2\mathcal{A}}$ satisfying

$$z(t) = \begin{cases} x(t + T'') & t \in [\alpha, a - T''] \\ x(t) & t \in [a, \infty] \end{cases}$$

we have

$$\Gamma_{M_k}(f_i, T_s z) < \epsilon_k \quad (1 \leq i \leq k, 1 \leq k \leq N, s \in \mathbb{R})$$

**Remark 3.7.** If for $z$ in 3.6

$$z \in L^\mathcal{A} C L^{2\mathcal{A}}$$

then

$$z \in G.$$

**Proof.** Choose $T > 0$ such that $TA < \eta 4^{-1}$, then

$$|z(t) - x(t)| < \eta 2^{-1} \quad t \in \mathbb{R}.$$
(2) $s \in ]-\infty, a - T'' - M_N - N]:$

$$T_{s+t}x(t') = T_{s+t+t''}x(t') \quad t \in [0, M_N], \quad t' \in [-N, N]$$

$$= f_i(T_{s+t}x) = f_i(T_{s+t+t''}x) \quad t \in [0, M_N], \quad 1 \leq i \leq N$$

$$\Rightarrow \Gamma_{M_k}(f_i, T_s x) = \Gamma_{M_k}(f_i, T_{s+t+t''}x) \leq \epsilon_k \quad (1 \leq i \leq k, 1 \leq k \leq N)$$

(3) $s \in [a - T'' - M_N - N, a - N]$: 

$$|z(t) - x'(t)| \leq |z(t) - x(t)| + |x(t) - x'(t)| < \eta$$

$$t \in [a - M_N - 2N - T'', a + M_N + 2N]$$

$$\Rightarrow T_s x \in C_{s''}(T_{s'} x') \subset C_{s}(X)$$

$$\Rightarrow \Gamma_{M_k}(f_i, T_s x) \leq \epsilon_k \quad (1 \leq i \leq k, 1 \leq k \leq N)$$

by 3.1.

**Lemma 3.8.** Let $X, \eta$ be as in 3.1. If $X$ has a dense orbit and $X$ is nonwandering then there is a constant $S > 0$ with the following property: For every $x, y \in X$ there exists $z \in G$ such that

$$z(t) = \begin{cases} x(t + S) & t \in ]-\infty, -S] \\ y(t) & t \in [0, \infty] \end{cases}$$

and the additional property that for every $s \in \mathbb{R}$ there is $z^s \in X$ such that

$$|z(s + t) - z^s(t)| < \eta 4^{-1} \quad t \in [-h, h].$$

**Proof.** (1) For $\omega \in X$ we define

$$F(\omega) = \{v \in G | \sup_{t \in [-h, h]} |v(t) - \omega(t)| < \eta 16^{-1}\}.$$

Since $X$ is compact there are $F(\omega_i) (1 \leq i \leq r)$ such that

$$X \subset \bigcup_{i=1}^{r} F(\omega_i).$$

By the assumptions on $X$ we find for every pair $i, j \in \{1, \ldots, r\}$

$$q_{ij} \in [2h, \infty] \quad \text{and}$$

$$x_{ij} \in F(\omega_i) \cap T_{q_{ij}} F(\omega_j) \cap X.$$

Define $q = \max_{1 \leq i, j \leq r} q_{ij}$ and choose some $T'' \in ]0, T2^{-1}]$ where $T$ is as in 3.6 such that $q(T'')^{-1} \in \mathbb{N}$. We define

$$S = (1 + q(T'')^{-1})q.$$
(2) Let $i, j \in \{1, \ldots, r\}$ be fixed. We choose the unique $k \in \mathbb{N}$ such that

$$S \in [q_{ij} + kg_{jj}, q_{ij} + (k + 1)g_{jj}].$$

Denote $a = S - q_{ij} - kg_{jj}$ then $a \in [0, q_{jj}].$

Since $g_{ij}(T^n)^{-1} \leq q(T^n)^{-1} \leq k$ we conclude

$$q_{ij}k^{-1} \leq T^n$$

and

$$t_0 := ak^{-1} \leq T2^{-1}.$$

Take $x_{ij} \in F(\omega_i) \cap T_{q_{ij}}F(\omega_j) \cap X \quad x_{ij} \in F(\omega_j) \cap T_{q_{ij}}F(\omega_j) \cap X$ as defined in (1).

We use 3.3 $k$-times in order to get $y_{ij} \in G$ such that

$$y_{ij}(t) = x_{ij}(t) \quad t \in [-q_{ij}, \infty[$$

$$y_{ij}(t) = x_{ij}(t + q_{ij} + lq_{jj}) \quad t \in [-q_{ij} - (l + 1)q_{jj}, -q_{ij} - lq_{jj} - A^{-1}]$$

$$0 \leq l \leq k - 1$$

$$y_{ij}(t) = x_{ij}(t + q_{ij} + (k - 1)q_{jj}) \quad t \in ]-\infty, -q_{ij} - (k - 1)q_{jj}].$$

From $q_{ij}, q_{jj} \in [2h, \infty[$ and the definition of the $F(\omega_i)$ $(1 \leq i \leq r)$ we have: for every $s \in \mathbb{R}$ there is $Z_{ij}^s \in X$ such that

$$|y_{ij}(t + t) - Z_{ij}^s(t)| < \eta 8^{-1} \quad t \in [-h, h]. \quad(*)$$

Now we define

$$y_{ij}^l(t) = \begin{cases} y_{ij}(t) & t \in [-q_{ij}, \infty[ \\ y_{ij}(-q_{ij}) & t \in [-q_{ij} - t_0, -q_{ij}] \\ y_{ij}(t + t_0) & t \in ]-\infty, -q_{ij} - t_0] \end{cases}$$

and for $2 \leq l \leq k$

$$y_{ij}^l(t) = \begin{cases} y_{ij}^{l-1}(t) & t \in [-q_{ij} - (l - 1)q_{jj} - (l - 1)t_0, \infty[ \\ y_{ij}^{l-1}(-q_{ij} - (l - 1)q_{jj} - (l - 1)t_0) & t \in [-q_{ij} - (l - 1)q_{jj} - (l - 1)t_0 - q_{ij} - (l - 1)q_{jj} - (l - 1)t_0 - q_{ij} - (l - 1)q_{jj} - (l - 1)t_0] \\ y_{ij}^{l-1}(t + t_0) & t \in ]-\infty, -q_{ij} - (l - 1)q_{jj} - t_0] \end{cases}$$

By 3.6 and 3.7 $y_{ij}^l \in G$ $(1 \leq l \leq k)$ and

$$y_{ij}^k(t) = \begin{cases} x_{ij}(t) & t \in [-q_{ij}, \infty[ \\ x_{ij}(t + q_{ij} + kg_{jj} + kt_0) = x_{ij}(t + S) & t \in ]-\infty, -S]. \end{cases}$$
Since \( q_{ij}, q_{ij} \in [2h, \infty[ \) and for all \( x \in L^4 \)

\[
\| T_t x - x \| \leq t_0 A \leq TA^{-1} < \eta_8^{-1}
\]

we conclude from (*) the following: for every \( s \in \mathbb{R} \) there is \( V_{ij}^s \in X \) such that

\[
| y_{ij}^k(s + t) - V_{ij}^s(t) | < \eta_4^{-1} \quad t \in [-h, h]. \tag{**}
\]

From the definition of \( y_{ij}^1 \) \((1 \leq 1 \leq k)\) we see furthermore that for all \( s \in ]-\infty, -S + h[ \cup [-h, \infty[ \) \( \cap \) \( V_{ij}^s \) can be chosen in such a way that

\[
| y_{ij}^k(s + t) - V_{ij}^s(t) | < \eta_8^{-1} \quad t \in [-h, h]. \tag{***}
\]

(3) Denote \( F_i = F(\omega_i) \setminus \bigcap_{k=1}^{r} F(\omega_k) \) \((1 \leq i \leq r)\) and let \( x, y \in X \) be given, then there exists a unique pair \( i, j \in \{1, \ldots, r\} \) such that \( x \in F_i \), \( y \in F_j \). We use 3.3 for \( y_{ij}^k \), \( y \in G \) and \( a = 0 \) and get \( z' \in G \) such that

\[
z'(t) = \begin{cases} y(t) & t \in [0, \infty[ \\ y_{ij}^k(t) & t \in ]-\infty, -A^{-1}[. \end{cases}
\]

Then we use 3.5 for \( z' \), \( T_3 x \in G \) and \( a = -S \) and get \( z \in G \) such that

\[
z(t) = \begin{cases} z'(t) & t \in [-S + A^{-1}, \infty[ \\ x(t + S) & t \in ]-\infty, -S[. \end{cases}
\]

Since \( x, y \in X \) we conclude from (***), (***) for every \( s \in \mathbb{R} \) there is \( Z_s \in X \) such that

\[
| z(s+t) - Z_s(t) | < \eta_8^{-1} \quad t \in [-h, h].
\]

**Lemma 3.9.** Let \( X, \eta \) be as in 3.1. If \( X \) has a dense orbit, \( X \) is a non-wandering set and \( \mu(X) = 1 \) then for every \( \delta > 0 \) there exists \( \omega \in G \) and \( I_0 \in \mathbb{R} \) such that

\[
I_i(f_i, \omega) < \delta \quad (1 \leq i \leq N + 1)
\]

for all \( I \in \{I_0, \infty\} \). \( \omega \) has the property that for every \( s \in \mathbb{R} \) there is \( x^s \in X \) such that

\[
| x^s(t) - \omega(s + t) | < \eta_4^{-1} \quad t \in [-h, h].
\]

**Proof.** Let \( S > 0 \) be as in 3.8. By the individual ergodic theorem we can choose \( y \in X \) and \( I' \in \mathbb{R} \) such that

\[
I' \in ]2h, \infty[, \quad 2S(I' + S)^{-1} < \delta_4^{-1}
\]

\[
I_i(f_i, y) < \delta_4^{-1} \quad (1 \leq i \leq N + 1)
\]
We use 3.8 for $T_t y$, $y \in X$ in order to get $z \in G$ such that

$$z(t) = \begin{cases} T_t y(t + S) & t \in [-\infty, -S] \\ y(t) & t \in [0, \infty] \end{cases}$$

and such that for all $s \in \mathbb{R}$ there is $z^s \in X$ with

$$|z(s + t) - z^s(t)| < \eta 4^{-1} \quad t \in [-h, h]. \quad (*)$$

Now we define for every $k \in \mathbb{Z}$

$$\omega(t + k(I' + S)) = \begin{cases} y(t) & t \in [0, I'] \\ z(t - I' - S) & t \in [I', I' + S]. \end{cases}$$

Then $\omega \in G$ and from (*) we conclude: for every $s \in \mathbb{R}$ there is $x^s \in X$ such that

$$|\omega(s + t) - x^s(t)| < \eta 4^{-1} \quad t \in [-h, h].$$

In the following we use the symbol $[a]$ for the largest integer less than or equal to the real $a$. Choose $I_0 \in \mathbb{R}$ such that $(I' + S)I_0^{-1} < \delta 4^{-1}$ then for $I \in [I_0, \infty]$, $1 \leq i \leq N + 1$

$$T_i(f_i, \omega) = \left| I^{-1}[I(I' + S)^{-1}] \int_0^{I'} f_i(T_t y) \, dt + I'^{-1}[I(I' + S)^{-1}] \int f_i \, d\mu \right. \\
+ \left. I^{-1}[I(I' + S)^{-1}] \int_{I'}^{I'+S} f_i(T_t \omega) \, dt + I^{-1} \int_0^{I-[I(I' + S)^{-1}]I(I'+S)} f_i(T_t \omega) \, dt \\
+ I'^{-1}[I(I' + S)^{-1}] \int f_i \, d\mu - \int f_i \, d\mu \right|$$

$$\leq \delta I^{-1}[I(I' + S)^{-1}] 4^{-1} + I^{-1}[I(I' + S)^{-1}]S \\
+ I^{-1}(I' + S) + |I'^{-1}[I(I' + S)^{-1}] - 1| \\
\leq \delta 4^{-1} + 4^{-1} + I^{-1}(S - I')(I(I' + S)^{-1}) + 1 \\
\leq \delta 2^{-1} + I^{-1}(S + I') + 2S(I' + S)^{-1} < \delta.$$

**Lemma 3.10 (Main Lemma).** Let $(L^A, \mathcal{L}, \mu, (T_t)_{t \in \mathbb{R}})$ be the flow defined in Chapter 2 where $\mu$ is an ergodic, aperiodic, $(T_t)_{t \in \mathbb{R}}$-invariant probability measure on $L$. Let $f_k \in C(L^A)$ ($1 \leq k \leq N + 1$) be functions such that $f_k$ depends only on $[-k, k]$ and $\|f_k\| \leq 1$. Let

$$\epsilon_j > 0 \quad (1 \leq j \leq N)$$

$$M_j > 0 \quad (1 \leq j \leq N) \text{ with } M_j < M_{j+1} \text{ be reals.}$$
If there is \( Z \in \mathcal{L} \), with \( \mu(Z) = 1 \) and \( T_t Z = Z \ (t \in \mathbb{R}) \) such that
\[
\Gamma_{M_k}(f_i, x) < \sum_{i=k}^{N-1} \epsilon_i + \epsilon_N 2^{-1} \quad (x \in Z, \ 1 \leq i \leq k, \ 1 \leq k \leq N)
\]
then for every \( \epsilon_{N+1} > 0 \), \( L \in \mathbb{N} \) and \( A' \in [0, A] \) there exists a real number \( M_{N+1} > M_N \), a set \( Y \in \mathcal{L}, \mu(Y) = 1 \), \( T_t Y = Y \ (t \in \mathbb{R}) \) and a 1-1 Borel mapping
\[
\varphi: Y \to L^{A+A'}
\]
such that \( \varphi \circ T_t = T_t \circ \varphi \) and

(a) \[
\Gamma'_{M_k}(f_i, \varphi x) < \sum_{i=k}^{N} \epsilon_i + \epsilon_{N+1} 2^{-1} \quad (x \in Y, \ 1 \leq i \leq k, \ 1 \leq k \leq N + 1)
\]

where \( \Gamma'_s(f, y) = |s^{-1} \int_0^s f(T_t y) \ dt - \int f d(\varphi \mu)| \)

(b) there is \( B \in \mathcal{L} \) such that
\[
\mu(B) > 1 - \epsilon_{N+1}
\]
and
\[
x(t) = \varphi x(t) \quad (t \in [-L, L], \ x \in B).
\]

(c) there are reals \( p_1, \ r > 0 \) such that for \( x, y \in Y \) and \( t \in \mathbb{R} \) with \( x(t) \neq y(t) \) there exists
\[
s \in [t - p_1 - r, \ t + p_1 + r]
\]
satisfying
\[
\varphi x(s) \neq \varphi y(s).
\]

Proof. Let us be given arbitrary \( \delta > 0 \) of which we will dispose later on. Define \( X \) to be the smallest closed set in \( \mathcal{L} \) such that
\[
T_t X = X \quad (t \in \mathbb{R})
\]
and
\[
\mu(X) = 1.
\]
Define
\[
\Omega = \left\{ x \in L^A \mid \Gamma_{M_k}(f_i, T_t x) < \sum_{i=k}^{N-1} \epsilon_i + 3 \epsilon_N 4^{-1} \ (1 \leq i \leq k, \ 1 \leq k \leq N, \ t \in \mathbb{R}) \right\}
\]
then \( T_t \Omega = \Omega (t \in \mathbb{R}) \) and by the assumption on \( Z \)
\[
\emptyset \neq X \subset \Omega.
\]
We are just in the situation of 3.1 where $G$ is replaced by $Q$ and $\epsilon_k$ by $\sum_{i=k}^{N-1} \epsilon_i + 3\epsilon_N 4^{-1}$. Since by the ergodicity of $\mu$ $X$ has a dense orbit and by the minimality of $X$ $X$ is nonwandering we can use all the results of 3.1–3.9. Let $\eta$, $T$, $S$ be the constants constructed there. As an abbreviation we use again
\[ h = M_N + 2N + A^{-1}. \]

(1) By 3.9 we can find $\omega_0 \in \Omega$ and $I_0 \in \mathbb{R}$ such that
\[ I(f_i, \omega_0) < \delta \quad I \in [I_0, \infty[, \quad 1 \leq i \leq N + 1 \]
and for all $s \in \mathbb{R}$ there is $x^s \in X$ such that
\[ |\omega_0(s + t) - x^s(t)| < \eta 4^{-1} \quad t \in [-h, h]. \]

(2) We choose $x_0 \in X$ such that $x_0(0) < 1$ and $T^s \in ]0, T[$ such that
\[ T^s 2^{-1}(A + A') < 1 - x_0(0). \]
Define
\[
x_1(t) = \begin{cases} 
  x_0(t) & t \in [0, \infty[ \\
  x_0(0) - (A + A')t & t \in [-T^s 2^{-1}, 0] \\
  x_0(0) + (A + A') T^s + (A + A')t & t \in [-T^s, -T^s 2^{-1}] \\
  x_0(t + T^s) & t \in ]-\infty, -T^s[ 
\end{cases}
\]
then by 3.6
\[ I_M(f_i, Tx_1) < \sum_{i=k}^{N-1} \epsilon_i + 3\epsilon_N 4^{-1} \quad (1 \leq i \leq k, 1 \leq k \leq N, t \in \mathbb{R}). \]

(3) Define
\[ F = \{ \psi \in L^A \mid |\psi(t) - x_0(t)| < \eta 4^{-1} \quad t \in [0, h] \} \]
and
\[ F(t) = \sup_{\psi \in F} \psi(t) \quad (t \in \mathbb{R}). \]
Let $\psi \in F$ be given and assume $\psi(0) \leq x_0(0)$, $\psi(h) \geq x_0(h)$. We define
\[
\psi^s(t) = \begin{cases} 
  \psi(0) - At & t \in ]-\infty, 0[ \\
  \psi(t) & t \in [0, h] \\
  \psi(h) - At & t \in [h, \infty[.
\end{cases}
\]
then there are \( t_1' \in [-A^{-1}, 0] \) and \( t_2' \in [h, h + A^{-1}] \) such that
\[
\begin{align*}
v''(t_1') &= x_0(t_1') \\
v''(t_2') &= x_0(t_2').
\end{align*}
\]
Put
\[
v'(t) = \begin{cases} x_0(t) & t \in ]-\infty, t_1'] \cup [t_2', \infty[ \\ v''(t) & t \in [t_1', t_2'] \end{cases}
\]
then
\[
|v'(t) - x_0(t)| < \eta 4^{-1} \quad (t \in \mathbb{R})
\]
hence
\[
T_1v' \in C_0(X) \quad (t \in \mathbb{R})
\]
and by 3.2 \( v' \in \Omega \).

Since the other cases \((v(0) > x_0(0) \text{ etc.})\) are settled in the same way we have shown the following:
For every \( v \in F \) there is \( v' \in \Omega \)—constructed in a unique manner—such that
\[
v(t) = v'(t) \quad t \in [0, h].
\]

(4) It is easy to see that we can always find \( y_0 \in F \cap L^\infty \) for some \( A'' \in [0, A] \) such that
\[
|y_0(t) - F(t)| > \alpha \quad (t \in [0, h])
\]
for some \( \alpha > 0 \). Fix \( y_0 \).

(5) Let \( \delta_1 \in ]0, \delta[ \) be given and denote
\[
C = 4S + 3h + T'' + I_0.
\]
For \( p \geq C \) put \( q(p) = p - C \).
By the individual ergodic theorem we can find \( p_1 > 0 \) such that for every \( p \geq p_1 \)
\[
\begin{align*}
& \text{(a) if } X_{q(p)} = \{x \in X \mid \Gamma_{q(p)} (f_i, x) < \delta (1 \leq i \leq N + 1)\} \\
& \mu(X_{q(p)}) > 1 - \delta_1 2^{-1} \\
& \text{(b) } q(p)p^{-1} (1 - \delta_1 2^{-1}) > 1 - \delta + 2Lp^{-1}.
\end{align*}
\]
We denote \( q_1 = q(p_1) \) and fix some \( x_2 \in X_{q_1} \).

(6) If we choose the representation of \((T_1)_{t \in \mathbb{R}}\) on \( X \) as given in
Corollary (1.11.) in [3] we see that there is a measurable set $V \subset X$ such that

(a) $V \cap T_i V = \emptyset$ for $t \in [0, p_1[$
(b) $\mu(\bigcup_{t \in [0, p_1]} T_i V) > 1 - \delta_1 2^{-1}$
(c) $\mu(\bigcup_{t \in [0, p_1+r]} T_i V) = 1$ for some $r > 0$.

We put $J = [0, p_1 + r]$ and

$$X_0 = \bigcap_{q \in \mathbb{Q}} T_q \left( \bigcup_{t \in J} T_i V \right)$$

then $\mu(X_0) = 1$.

If $x \in X_0$ and $t \in \mathbb{R}$ then we have for every $s \in \mathbb{Q} \cap ]t + p_1 + r, \infty[

$$T_s x \in \bigcup_{t \in J} T_i V = T_s x \in T_i V \quad \text{for some } l \in J$$

$$= T_{s-l-t} x = T_{t+(s-l-t)} x \in V \quad \text{where } s-l-t > 0$$

$$= s(t) = \inf \{ t' > 0 \mid T_{t+t'} x \in V \}$$

$$\leq s-l-t \leq s-t$$

Since $s$ is arbitrary: $s(t) \leq p_1 + r$. We have shown

$$X_0 \subset Y := \{ x \in X \mid \forall t \in \mathbb{R} \exists \delta(t) \in J: T_{t+s(t)} x \in V \}. $$

$Y$ is $(T_i)_{t \in \mathbb{R}}$-invariant and $\mu(Y) = 1$.

Furthermore

(d) $Y \cap \bigcup_{t \in J} T_i V = \bigcup_{t \in J} T_i (Y \cap V)$
(e) $Y \subset \bigcup_{t \in J} T_i V$
(f) $\Rightarrow Y = \bigcup_{t \in J} T_i (Y \cap V)$

(7) We choose $\delta_0 > 0$ such that if $\| x - y \| < \delta_0$ then

$$|f_i(x) - f_i(y)| < \delta \quad (1 \leq i \leq N + 1, x, y \in L^4)$$

For later use we assume $\delta_0 \in [0, \gamma 4^{-1}]$. We can find some $n \in \mathbb{N}$ such that

(a) $At_n < \delta_0$ where $t_n = p_1 n^{-1}$.

From (a) we have for $x \in L^4$ and for $t \in [0, t_n]$

$$\| x - T_t x \| < \delta_0.$$
In particular if \( x \in V \) and \( T_{t}x \in T_{-c'}X_{q_{1}} \) for some \( t \in [0, t_{n}] \), where
\[
c' := 3S + 3h + T^{*}
\]
then
\[
\| T_{c'+s}x - T_{c'+s+t}x \| < \delta_{0} \quad (s \in [0, q_{1}])
\]
\[
\Rightarrow | f_{i}(T_{c'+s}x) - f_{i}(T_{c'+s+t}x) | < \delta \quad (s \in [0, q_{1}], 1 \leq i \leq N + 1)
\]
\[
\Rightarrow \Gamma_{q_{1}}(f_{i}, T_{c'}x) \leq \left| q_{i}^{-1} \int_{0}^{q_{1}} f_{i}(T_{c'+s}x) \, ds - q_{i}^{-1} \int_{0}^{q_{1}} f_{i}(T_{c'+s+t}x) \, ds \right|
\]
\[
+ \Gamma_{q_{1}}(f_{i}, T_{c'+t}x) < 2\delta.
\]

(8) Denote \( V_{n} = \bigcup_{i \in [0, t_{n}]} T_{i}V \) then we can assume by (5) (a) and (6) (b)
\[
\mu \left( \bigcup_{k=0}^{n-1} T_{k+t_{n}}(V_{n} \cap T_{-c'}X_{q_{1}}) \right) \geq 1 - \delta_{1}.
\]

(9) Now we can define
\[
\varphi : Y \rightarrow L^{A+A'}
\]

By (6) (f) and \( \varphi \circ T_{i} = T_{i} \circ \varphi \) it is sufficient to define \( \varphi \) on \( Y \cap V \).
Let \( l_{k}(x) (k \in \mathbb{Z}) \) denote the entrance times of \( x \in Y \cap V \) into \( V \), i.e.,
\[
l_{0}(x) = 0
\]
and for \( k \geq 0 \) if \( l_{k}(x) \) is defined
\[
l_{k+1}(x) = \inf \{ t > l_{k}(x) \mid T_{t}x \in V \},
\]
for \( k \leq 0 \) if \( l_{k}(x) \) is defined
\[
l_{k-1}(x) = \sup \{ t < l_{k}(x) \mid T_{t}x \in V \}.
\]

It is sufficient to define \( \varphi(x) \in L^{A+A'} \) on \([0, l_{1}(x)]\). For \( t \notin [0, l_{1}(x)] \) we choose \( k \in \mathbb{Z} \) such that \( t \in [l_{k}(x), l_{k+1}(x)] \) and define
\[
\varphi(x)(t) = \varphi(T_{l_{k}(x)}x)(t - l_{k}(x)).
\]
Now

\[
\varphi(x)(t) = \begin{cases}
\varphi_1(x)(t) & t \in [0, S] \\
\varphi_2(x)(t - S) & t \in [S, S + 2h + T]\n\varphi_3(x)(t - S - 2h - T) & t \in [S + 2h + T, 2S + 2h + T] \\
\varphi_4(x)(t - 2S - 2h - T) & t \in [2S + 2h + T, 2S + 3h + T] \\
\varphi_5(x)(t - 2S - 3h - T) & t \in [2S + 3h + T, c'] \\
\varphi_6(x)(t - c') & t \in [c', c' + q_1] \\
\varphi_7(x)(t - c' - q_1) & t \in [c' + q_1, p_1 - I_0] \\
\varphi_8(x)(t - p_1 + I_0) & t \in [p_1 - I_0, l_1(x)]
\end{cases}
\]

(Note that \(p_1 - I_0 = c' + q_1 + S\))

(10) Definition of \(\varphi_2\): Take \(x_1\) as defined in (2) and define

\[
\varphi_2(x)(t) = T_{-h - T}x_1(t) \quad t \in [0, 2h + T]
\]

(11) Definition of \(\varphi_4\): We denote \(\gamma = \min\{\alpha, h(p_1 + r)^{-1}\}\) and define

\[
\varphi_4(x)(t) = \begin{cases}
\gamma A\eta^{-1}(A - A^* x((p_1 + r) h^{-1}t) + y_0(t) & t \in [0, l_1(x)^{-1} h(p_1 + r)^{-1}]
\gamma A\eta^{-1}(A - A^*) l_1(x) + y_0(t) & t \in [l_1(x)^{-1} h(p_1 + r)^{-1}, h]
\end{cases}
\]

where \(y_0, \alpha, A^*\) are as in (4).
(Note that \(|\varphi_4(x)(t) - \varphi_4(x)(t')| \leq A \cdot |t - t'|\) and \(|\varphi_4(x)(t) - x_0(t)| < \gamma h^{4^{-1}}\) \(t \in [0, h]\)).

(12) Definition of \(\varphi_6\): Recall that we chose \(x_2\) in (5).

\[
\varphi_6(x)(t) = \begin{cases}
x(t + c') & t \in [0, q_1] \\
x_2(t) & t \in [0, q_1] \text{ otherwise.}
\end{cases}
\]

(13) Definition of \(\varphi_8\): Recall that we chose \(\omega_0\) in (1).

\[
\varphi_8(x)(t) = \omega_0(t) \quad t \in [0, l_1(x) - p_1 + I_0].
\]

(14) Definition of \(\varphi_1\): For \(x \in X\) and \(\gamma' > 0\) we define

\[
F_{\gamma'}(x) = \{v \in \Omega \mid |v(t) - x(t)| < \gamma' t \in [-h, 0]\}.
\]

Since \(X\) is compact we can find \(Z_i \in X\) \((1 \leq i \leq \rho)\) such that

\[
X \subset \bigcup_{i=1}^\rho F_{\gamma^\rho_0}(Z_i).
\]
\( \omega_0 \) in (1) has the property that for every \( s \in \mathbb{R} \) there is \( x^s \in X \) such that
\[
| T_s \omega_0(t) - x^s(t) | < \eta 4^{-1} \quad t \in [-h, h]
\]
hence
\[
T_s \omega_0 \in \bigcup_{i=1}^{p} F_{\eta^{-1}}(Z_i) \quad (s \in \mathbb{R})
\]
Denote
\[
F_i = F_{\eta^{-1}}(Z_i) \bigcup_{k=1}^{i-1} F_{\eta^{-1}}(Z_k) \quad (1 \leq i \leq p)
\]
then
\[
F_i \cap F_j = \emptyset \quad (i \neq j)
\]
and
\[
T_s \omega_0 \in \bigcup_{i=1}^{p} F_i \quad (s \in \mathbb{R}).
\]
Recall that \( \varphi_2 \) is defined such that
\[
\varphi_2(x)(t) = x_0(-h + t) \quad t \in [0, h]
\]
where \( x_0 \) is as in (2).
By 3.8 there exists for every pair \( Z_i, x_0 \in X \) \( (1 \leq i \leq p) \) \( \omega_i \in \Omega \) with
\[
\omega_i(t) = \begin{cases} 
Z_i(t) & t \in [\infty, 0] \\
\varphi_2(x)(t-S) & t \in [S, S+h].
\end{cases}
\]
Furthermore there exists \( u_i \in X \) with
\[
| u_i(t) - \omega_i(t) | < \eta 4^{-1} \quad t \in [-h, h].
\]
Since \( F_i \cap F_j = \emptyset \) \( (i \neq j) \) there is a unique \( k \in \{1, \ldots, p\} \) such that
\[
T_{-l_{i-1}}(\omega_{i-1}) \omega_0 \in F_k.
\]
Since
\[
| T_{-l_{i-1}}(\omega_{i-1}) \omega_0(0) - \omega_k(0) | < \eta 2^{-1}
\]
and
\[
u_k \in X \cap \{ v \in L^A | \sup_{t \in [-M_{i-1} - 2N, 0]} | v(t) - T_{-l_{i-1}}(\omega_{i-1}) \omega_0(t) | < \eta \}
\cap \{ v \in L^A | \sup_{t \in [0, h]} | v(t) - \omega_k(t) | < \eta 2^{-1} \}
\]
we can use 3.5 for $T_{i-1} \omega_{p_1+k} \omega$ and get $u \in \Omega$ such that

$$u(t) = \begin{cases} T_{-i-1}(\omega_{p_1+k} \omega_0(t)) & t \in ]-\infty, 0] \\ \omega_k(t) & t \in [A^{-1}, \infty[. \end{cases}$$

We define

$$\varphi_1(x)(t) = u(t) \quad t \in [0, S].$$

(Note that $\varphi_1(x)(S) = \varphi_2(x)(0)$ and

$$\varphi_4(x)(0) = T_{-i-1}(\omega_{p_1+k} \omega_0(0)) = \varphi_6(T_{i-1}(x)(-L_1(x) - p_1 + I_0) = \varphi(T_{i-1}(\omega x)(-L_1(x))).$$

(15) Definition of $\varphi_3$ : Recall that $\varphi_2$ is defined such that

$$\varphi_2(x)(t) = x_0(t - h - T') \quad t \in [h + T', 2h + T']$$

By 3.8 there exists for $T_h x_0, x_0 \in X \omega' \in \Omega$ such that

$$\omega'(t) = \begin{cases} T_h x_0(t) & t \in ]-\infty, 0] \\ x_0(t - S) & t \in [S, \infty[. \end{cases}$$

Since

$$| \varphi_4(x)(t) - x_0(t) | < \eta A^{-1} \quad t \in [0, h]$$

there exists—as we have shown in (3)—$v \in \Omega$ such that

$$v(t) = \varphi_4(x)(t - S) \quad t \in [S, S + h]$$

We use 3.3 for $\omega', v \in \Omega, a = S$ and get $v' \in \Omega$ such that

$$v'(t) = \begin{cases} v(t) & t \in [S, \infty[ \\ \omega'(t) & t \in ]-\infty, S - A^{-1}]. \end{cases}$$

Define

$$\varphi_3(x)(t) = v'(t) \quad t \in [0, S].$$

(Note that $\varphi_3(x)(0) = \varphi_6(x)(2h + T')$ and $\varphi_3(x)(S) = \varphi_4(x)(0)$.

(16) Definition of $\varphi_5$ : For $x \in X$ and $\gamma' > 0$ we define

$$E_{\gamma'}(x) = \{v \in \Omega \mid |v(t) - x(t)| < \gamma' \, t \in [0, q_i]\}$$
Since $X$ is compact we can find $z'_i \in X$ (1 \leq i \leq p') such that
\[
X \subseteq \bigcup_{i=1}^{p'} E_{\eta^{-1}}(z'_i)
\]

Denote
\[
E_i = E_{\eta^{-1}}(z'_i) \bigcup_{k=0}^{i-1} E_{\eta^{-1}}(z'_k) \quad (1 \leq i \leq p').
\]

Let us note that there is a unique $u_x \in X$ such that
\[
u_x(t) = \varphi_0(x)(t) \quad t \in [0, q_1].
\]

This is obvious if $\varphi_0(x)(t) = T_c x(t)$ $t \in [0, q_1]$ since then we take $u_x = T_c x \in X$.

In the second case, i.e., if $\varphi_0(x)(t) = x_2(t)$ $t \in [0, q_1]$, we take $u_x = x_2 \in X$.

The $E_i's$ (1 \leq i \leq p') are disjoint, which means that there is a unique $k \in \{1, ..., p'\}$ such that $u_x \in E_k$.

By 3.8 we find for the pair $z'_k$, $T_h x_0 \in X \omega_k' \in \Omega$ with
\[
\omega_k'(t) = \begin{cases} T_h x_0(t) & t \in ]-\infty, 0] \\ z'_k(t-S) & t \in [S, \infty[ \end{cases}
\]

Now let $T_{s+h} v \in \Omega$ be as in (15), i.e.,
\[
T_{s+h} v(t) = \varphi_0(x)(t+h) \quad t \in [-h, 0].
\]

We use 3.5 for $T_{s+h} v$, $\omega_k' \in \Omega$ and $a = 0$ in order to get $u' \in \Omega$ such that
\[
u'(t) = \begin{cases} T_{s+h} v(t) & t \in ]-\infty, 0] \\ \omega_k'(t) & t \in [A^{-1}, \infty[ \end{cases}
\]

Finally we use 3.3 for $u'$, $T_{-s} u_x \in \Omega$ and $a = S$ in order to get $u'' \in \Omega$ such that
\[
u''(t) = \begin{cases} u'(t) & t \in ]-\infty, S - A^{-1}] \\ T_{-s} u_x(t) & t \in [S, \infty[ \end{cases}
\]
Define
\[ \varphi_0(x)(t) = u^*(t) \quad t \in [0, S]. \]

(It's obvious that \( \varphi_0(x)(h) = \varphi_0(x)(0) \) and \( \varphi_0(x)(S) = \varphi_0(x)(0) \)).

**Remark.** It would also be correct to connect \( u_x \) and \( T_h x_0 \) directly by 3.8 without using the covering \( E_i \) (1 \( \leq i \leq p' \)). The only reason for introducing the covering here is to make obvious that \( \varphi \) is measurable. The same holds for the definition of \( \varphi_7 \).

(17) Definition of \( \varphi_7 \): First we see by the construction of \( \omega_0 \) that there is \( y_2 \in X \) with
\[ \omega_0(t) = y_2(t) \quad t \in [0, h]. \]

Let \( u_x, z_k' \in X \) be as in (16) then we find by 3.8 for the pair \( T_{q_1} z_k' \), \( y_2 \in X \) \( \omega_k' \in \Omega \) such that
\[ \omega_k'(t) = \begin{cases} T_{q_1} z_k'(t) & t \in [-\infty, 0] \\ y_2(t - S) & t \in [S, \infty] \end{cases} \]

We use 3.5 for \( T_{q_1} u_x, \omega_k \in \Omega \) and \( a = 0 \) in order to get \( \omega' \in \Omega \) such that
\[ \omega'(t) = \begin{cases} T_{q_1} u_x(t) & t \in [-\infty, 0] \\ \omega_k'(t) & t \in [A^{-1}, \infty] \end{cases} \]

Define
\[ \varphi_7(x)(t) = \omega'(t) \quad t \in [0, S] \]

(It is obvious that \( \varphi_0(x)(q_1) = \varphi_7(x)(0) \) and \( \varphi_7(x)(S) = \varphi_0(x)(0) \)).

(18) \( \varphi \colon Y \subset L^4 \to L^{4 + A'} \) is a 1-1 Borel mapping with \( \varphi \circ T_i = T_i \circ \varphi \) \( (t \in \mathbb{R}) \): Let \( x, y \in Y \) be given and \( x \neq y \).

**Case 1.** \( \{l_k(x) \mid k \in \mathbb{Z}\} \neq \{l_k(y) \mid k \in \mathbb{Z}\} \). We can assume that there is \( k \in \mathbb{Z} \) such that \( l_k(x) \notin \{l_k(y) \mid k \in \mathbb{Z}\} \). By the definition of \( \varphi_2 \) we have the following
\[ T_{l_k(x) + s + h} \varphi(x)(t) = T_{l_k(x) + s + h} \varphi(x)(0) + (A + A') t \quad t \in [0, T^2 t^{-1}] \]
\[ T_{l_k(x) + s + h} \varphi(x)(t) = T_{l_k(x) + s + h} \varphi(x)(T^2 t^{-1}) - (A + A') t \quad t \in [T^2 t^{-1}, T^2]. \]

On the other hand, there is by the definition of \( \varphi_1 - \varphi_2 \) an interval \( J_0 \subset [l_k(x) + S + h, l_k(x) + S + h + T^2] \) of positive length such that
\[ | \varphi(y)(t) - \varphi(y)(t') | \leq A | t - t' | \quad (t, t' \in J_0). \]
We conclude \( \varphi(x) \neq \varphi(y) \)

**Case 2.** \( \{l_k(x) \mid k \in \mathbb{Z}\} = \{l_k(y) \mid k \in \mathbb{Z}\} \). Since \( x \neq y \) there is \( t_0 \in \mathbb{R} \) such that \( x(t_0) \neq y(t_0) \). Pick \( k \in \mathbb{Z} \) with \( t_0 \in [l_k(x), l_{k+1}(x)] \) then we have

\[
\varphi(x)(l_k(x) + 2S + 2h + T^\sigma + h(p_1 + r)^{-1}(t_0 - l_k(x))) \\
- \varphi(x)(T_{l_k(x)}^\omega(x))(h(p_1 + r)^{-1}(t_0 - l_k(x))) \\
= \gamma A^{-1}(A - A^\sigma) T_{l_k(x)}^\omega(x(t_0 - l_k(x)) + y_0(h(p_1 + r)^{-1}(t_0 - l_k(x))) \\
- \gamma A^{-1}(A - A^\sigma) x(t_0) + y_0(h(p_1 + r)^{-1}(t_0 - l_k(x))) \\
= \cdots = \varphi(y)(l_k(x) + 2S + 2h + T^\sigma + h(p_1 + r)^{-1}(t_0 - l_k(x))). \\
\Rightarrow \varphi(x) \neq \varphi(y).
\]

In order to show the measurability of \( \varphi \) it is sufficient—by theorem 3.2 of [5]—to show that the a.e. defined mapping

\[
\psi: L^4 \to [0, 1] \\
x \mapsto \varphi(x)(0)
\]

is measurable. But this is obvious since the entrance times (in \( V \)) are measurable functions.

That \( \varphi \circ T_t(x) = T_t \circ \varphi(x) \ (t \in \mathbb{R}, x \in Y) \) is clear by construction.

(19) Calculation of time means: Let \( x \in Y \) be given. We consider

\[
\varphi(x)(s) \quad s \in [-N, M_N + N].
\]

If \( x \) is such that only \( \varphi_1 \) or \( \varphi_2 - \varphi_3 \) intervene in the definition of \( \varphi(x)(s) \ s \in [-N, M_N + N] \) then there is \( v \in \Omega \) with

\[
v(s) = \varphi(x)(s) \quad s \in [-N, M_N + N]
\]

hence

\[
\Gamma_{M_k}(f_i, \varphi x) = \Gamma_{M_k}(f_i, v) < \sum_{l=k}^{N-1} \epsilon_l + 3\epsilon_N A^{-1} \quad (1 \leq i \leq k, 1 \leq k \leq N).
\]

If \( \varphi_2 \) intervenes in the definition of \( \varphi(x)(s) \ s \in [-N, M_N + N] \) then we have by (2)

\[
\Gamma_{M_k}(f_i, \varphi x) < \sum_{l=k}^{N-1} \epsilon_l + 3\epsilon_N A^{-1} \quad (1 \leq i \leq k, 1 \leq k \leq N).
\]
Before defining $M_{N+1}$, we calculate for $x \in Y \cap V$ and $1 \leq i \leq N + 1$

$$\Gamma_{M_1}(f_i, \varphi x)$$

$$= l_1(x)^{-1} \int_0^{c_1} f_i(T_0 \varphi x) \, dt + l_1(x)^{-1} \int_{c_1}^{c_1 + q_1} f_i(T_0 \varphi x) \, dt$$

$$+ l_1(x)^{-1} \int_{c_1 + q_1}^{c_1 + q_1 + S} f_i(T_0 \varphi x) \, dt + l_1(x)^{-1} \int_{c_1 + q_1 + S}^{l_1(x)} f_i(T_0 \varphi x) \, dt - \int f_i \, d\mu$$

$$\lesssim l_1(x)^{-1}(c' + S) + q_1 l_1(x)^{-1} \left| \int_0^{q_1} f_i(T_i T_0 \varphi x) \, dt - \int f_i \, d\mu \right|$$

$$+ (l_1(x) - p_1 + I_0) l_1(x)^{-1} \left| \int_0^{l_1(x) - p_1 + I_0} f_i(T_0 \varphi x) \, dt - \int f_i \, d\mu \right|$$

$$+ \left| \int f_i \, d\mu \right| \left| q_1 l_1(x)^{-1} + (l_1(x) - p_1 + I_0) l_1(x)^{-1} - 1 \right|$$

$$\leq p_1^{-1}(c' + S) + q_1 l_1(x)^{-1} 28 + (l_1(x) - p_1 + I_0) l_1(x)^{-1} \delta$$

$$+ l_1(x)^{-1} | q_1 - p_1 + I_0 |$$

$$\leq p_1^{-1}(c' + S) + 28 + \delta + p_1^{-1}(c' + S)$$

$$\leq 2(p_1 - q_1) p_1^{-1} + 3\delta < 5\delta.$$ 

Note that $(p_1 - q_1)p_1^{-1} < \delta$ is true by (5)(b). Choose $M_{N+1} \in \mathbb{Z}, \infty$ such that

$$2(p_1 + r) M_{N+1}^3 < \delta$$

then we have since $| l_{k+1}(x) - l_k(x) | \leq p_1 + r$ ($k \in \mathbb{Z}, x \in Y \cap V$)

$$\Gamma_{M_{N+1}}(f_i, \varphi x) < 6\delta \quad (x \in Y, 1 \leq i \leq N + 1).$$

Pick $y_1 \in Y$ such that

$$\lim_{M \to \infty} M^{-1} \int_0^M f_i(T_0 \varphi y_1) \, dt = \int f_i \, d\mu \quad (1 \leq i \leq N + 1)$$

then since

$$\Gamma_M(f_i, \varphi x) < 6\delta \quad (x \in Y, 1 \leq i \leq N + 1, M \geq M_{N+1})$$
and
\[
\left| \int f_i \, d\mu - \int f_i d(\varphi \mu) \right| = \left| M^{-1} \int_0^M f_i(T_i \varphi y_1) \, dt - \int f_i \, d\mu - \left( M^{-1} \int_0^M f_i(T_i \varphi y_1) \, dt - \int d(\varphi d \mu) \right) \right|
\]
we have
\[
\left| \int f_i \, d\mu - \int f_i d(\varphi \mu) \right| < 6\delta.
\]
Therefore
\[
\Gamma'_{M_k}(f_i, \varphi x) \leq M_k^{-1} \int_0^M f_i(T_i \varphi x) \, dt - \int f_i \, d\mu \left| + \left| \int f_i \, d\mu - \int f_i d(\varphi \mu) \right| < \sum_{i=k}^{N-1} \epsilon_i + 3\epsilon_N 4^{i-1} + 6\delta \quad (1 \leq i \leq k, 1 \leq k \leq N, x \in Y)
\]
and
\[
\Gamma'_{M_{N+1}}(f_i, \varphi x) = \Gamma_{M_{N+1}}(f_i, \varphi x) + \left| \int f_i \, d\mu - \int f_i d(\varphi \mu) \right| < 12\delta.
\]
Recall that \( \delta > 0 \) was arbitrary.
For \( \delta \in ]0, \epsilon_{N+1} 24^{-1} \) [we have
\[
\Gamma'_{M_k}(f_i, \varphi x) < \sum_{i=k}^{N} \epsilon_i + \epsilon_{N+1} 2^{-i} \quad (x \in Y, 1 \leq i \leq k, 1 \leq k \leq N + 1).
\]
(20) We show (b) of 3.10: Define
\[
B = \left( \bigcup_{t \in [c'+L, c'+q_1-L]} T_i V \right) \cap \left( \bigcup_{k=0}^{n-1} T_{kt_h} (V_n \cap T_{-c'} X_{q_1}) \right).
\]
From
\[
\mu \left( \bigcup_{t \in [c'+L, c'+q_1-L]} T_i V \right) = (q_1 - 2L) \beta_1^{-1} \mu \left( \bigcup_{t \in [0, p_1]} T_i V \right) > (q_1 - 2L) \beta_1^{-1} (1 - \delta_1 2^{-i})
\]
and
\[
\mu \left( \bigcup_{k=0}^{n-1} T_{kt_h} (V_n \cap T_{-c'} X_{q_1}) \right) > 1 - \delta_1
\]
we conclude by (5)(b)

\[ \mu(B) > 1 - \delta_1 - (1 - (q_1 - 2L) p_1^{-1}(1 - \delta_1 2^{-3})) \]
\[ > 1 - \delta_1 - 1 + 1 - \delta > 1 - 2\delta > 1 - \epsilon_{N+1} \]

Let \( y \in B \) be given then there exists \( t' \in [c' + L, c' + q_1 - L] \) such that \( T_{-t'} y \in V \). As well we find \( k \in \{0, \ldots, n - 1\} \) such that

\[ T_{-kt_n} y \in V_n \cap T_{-c'} X_{q_1}. \]

By the definition of \( V_n \) there is \( s \in [0, t_n] \) such that

\[ T_{-kt_n-s} y \in V. \]

Since \( |t' - kt_n - s| < p_1 \) we have

\[ t' = kt_n + s \]

hence

\[ T_{-t' + s} y \in T_{-c'} X_{q_1}. \]

By the definition of \( \varphi_6 \)

\[ \varphi(y)(t) = y(t) \quad t \in [-t' + c', -t' + c' + q_1]. \]

Now the statement follows from

\[ -t' + c' \leq -c' - L \quad c' = L \]
\[ -t' + c' + q_1 \leq -c' - q_1 + L - c' + q_1 = L. \]

(21) Since (c) of 3.10 is obvious by the definition of \( \varphi(x) \) on intervals \([l_k(x), l_{k+1}(x)] \) of lengths smaller than \( p_1 + r \) the proof of 3.10 is complete.

4. Strictly Ergodic Imbeddings

In this section we shall prove the isomorphism of an ergodic flow to some strictly ergodic one. For preparation, aside of part 3., we need the following lemma 4.1 and theorem 4.2.
Lemma 4.1. There is a sequence $F = \{f_i \mid i \in \mathbb{N}\}$ of continuous real functions on $L^A$ such that

(a) the linear hull of $F$ is dense in $C(L^A)$
(b) $f_i$ depends on $[-i, i]$ ($i \in \mathbb{N}$)
(c) $\|f_i\| = 1$ ($i \in \mathbb{N}$)

Proof. Consider the evaluation functions

$$\pi_t : L^A \rightarrow \mathbb{R}$$

defined by $\pi_t x = x(t)$ ($t \in \mathbb{R}$). Restricting to $t \in \mathbb{Q}$ we know that

$$\{\pi_t \mid t \in \mathbb{Q}\}$$

is a countable set of continuous, point-separating functions on $L^A$, such that every $\pi_t$ depends on some interval and $\|\pi_t\| = \sup_{x \in L^A} |\pi_t x| = 1$.

The same properties hold for finite products of powers of the $\pi_t (t \in \mathbb{Q})$ i.e., for

$$\prod_{i=1}^l (n_i t_i)^{k_i} \quad (k_i \in \mathbb{N}, t_i \in \mathbb{Q} \ (1 \leq i \leq l)).$$

By the Stone–Weierstraß theorem the algebra, generated by $\{\pi_t \mid t \in \mathbb{Q}\}$ is dense in $C(L^A)$.

Theorem 4.2. [3] Every aperiodic flow can be imbedded into $(L^A, \mathcal{L}, (T_t)_{t \in \mathbb{R}})$ with arbitrary Lipschitz constant $A > 0$.

Proof. See (3).

The result of our paper is contained in

Theorem 4.3. (1) Let $A > 0$. Then every aperiodic and ergodic flow $(\Omega, \mathcal{A}, \mu, (S_t)_{t \in \mathbb{R}})$ is isomorphic to a strictly ergodic flow in $(L^A, \mathcal{L}, (T_t)_{t \in \mathbb{R}})$.

(2) Every ergodic flow is isomorphic to some strictly ergodic one.

Proof. Since a periodic and ergodic flow is strictly ergodic and an ergodic flow is either aperiodic or periodic, (2) follows from (1) immediately. Thus we have to show (1), and by 4.2 it suffices to consider the case

$$\Omega \subseteq L^{A^{-1}}.$$

I. The first section of the proof contains an iterated application of 3.10 of part 3.
Choose $\epsilon_n > 0$ ($n \in \mathbb{N}$) such that $\sum_{n=1}^{\infty} \epsilon_n = A2^{-1}$, and denote by $(f_i)_{i \in \mathbb{N}}$ the sequence of continuous functions defined in 4.1.

Put $\mu_1 = \mu$ and $M_1 = 1$. Since for $f \in C(L^A)$ and $x \in L^A$

$$\left| \int_0^1 fT_n x \, dt - \int f \, d\mu_1 \right| < 2,$$

we can apply 3.10 to $(\Omega, \mathcal{F}, \mu, (S_i)_{i \in \mathbb{N}})$, $\epsilon_1 > 0$, $A_1 = A2^{-1} + \epsilon_1$ and $L_1 = 2$ to find $M_2 > 0$, $B_1$, $Y_1 \in \mathcal{L}$, $p_1$, $r_1 \in \mathbb{R}_+$ and an isomorphism

$$\varphi_1: Y_1 \to L^A$$

with the following properties:

$$\mu_1(Y_1) = 1, \quad \mu_1(B_1) > 1 - \epsilon_1,$$

$$\left| \frac{M_2}{M_1} \int_0^{M_1} fT_n x \, dt - \int f \, d(\varphi_1 \mu_1) \right| \leq \epsilon_2^{-1},$$

for $i = 1, 2$ and $x \in \varphi_1 Y_1$,

$$x(t) = \varphi_1 x(t) \quad \text{for} \quad t \in [-2, 2] \text{ and } x \in B_1,$$

for $x, y \in Y_1$ and $t \in \mathbb{R}$ with $x(t) \neq y(t)$ there exists $s \in [t - p_1 - r_1, t + r_1 + p_1]$ with

$$\varphi_1 x(s) \neq \varphi_1 y(s).$$

We proceed by induction. Suppose we have, applying 3.10 to $\epsilon_j > 0$, $L_j > j + \sum_{i=1}^{j-1} (p_i + r_i)$,

$$A_j := A2^{-1} + \sum_{i=1}^{j} \epsilon_i \quad \text{and} \quad (L^A, \mathcal{L}, \mu_j, (T_i)_{i \in \mathbb{N}})$$

for every $1 \leq j \leq n - 1$, $M_j > M_{j-1}(2 \leq j \leq n)$,

$$B_j, Y_j \in \mathcal{L} \quad (1 \leq j \leq n - 1), \quad p_j, r_j \in \mathbb{R}_+ \quad (1 \leq j \leq n - 1)$$

and isomorphisms

$$\varphi_j: Y_j \to L^A \quad (1 \leq j \leq n - 1)$$

such that the following holds:

(i) For $\mu_j := \varphi_{j-1} \circ \cdots \circ \varphi_1 \mu_1$ ($1 \leq j \leq n$)

$$\mu_j(Y_j) = 1 \quad (1 \leq j \leq n - 1)$$

$$\mu_j(B_j) > 1 - \epsilon_j \quad (1 \leq j \leq n - 1)$$
(ii) \[ M_k^{-1} \int_0^M f_i T x \, dt - \int f_i \, d\mu_j \leq \sum_{i=k}^{j-2} \epsilon_i + \epsilon_{j-2} 2^{-1} \] for every \( 2 \leq j \leq n \), every \( 2 \leq k \leq j \), every \( 1 \leq i \leq k \) and every \( x \in \varphi_{j-1} Y_{j-1} \). 

(iii) If \( 1 \leq j \leq n - 1 \) and \( x \in B_j \), then 
\[ x(t) = \varphi_j x(t) \quad \text{for} \quad t \in [-L_j, L_j] \]

(iv) If \( 1 \leq j \leq n - 1 \), \( x, y \in Y_j \) and \( t \in \mathbb{R} \) with \( x(t) \neq y(t) \), then there is \( s \in [t - p_j - r_j, t + p_j + r_j] \) satisfying 
\[ \varphi_j x(s) \neq \varphi_j y(s). \]

Note that (i) and (ii) assures the conditions of 3.10 again for \((L^{n-1}, \varphi, \mu_n, (T_i)_{i \in \mathbb{N}})\). Thus using 3.10 applied to \( \epsilon_n \), \( A_n := A_{n-1} + \epsilon_n \) and \( L_n > n + \sum_{i=1}^{n-1} (p_i + r_i) \) we get \( M_{n+1} > M_n \), \( p_n \), \( r_n \in \mathbb{R}_+ \), \( Y_n \), \( B_n \in \mathcal{L} \) and an isomorphism 
\[ \varphi_n: Y_n \to L^n \]

such that the following holds:

(i') \( \mu_n(Y_n) = 1; \mu_n(B_n) > 1 - \epsilon_n \)

(ii') Put \( \mu_{n+1} = \varphi_n \mu_n \). Then 
\[ \left| M_k^{-1} \int_0^M f_i T x \, dt - \int f_i \, d\mu_{n+1} \right| \leq \sum_{i=k}^{n-1} \epsilon_i + \epsilon_n 2^{-1} \]

for every \( 2 \leq k \leq n + 1 \), every \( 1 \leq i \leq k \) and every \( x \in \varphi_n Y_n \)

(iii') \( x(t) = \varphi_n x(t) \) for \( t \in [-L_n, L_n] \) and \( x \in B_n \)

(iv') For \( x, y \in Y_n \) and \( t \in \mathbb{R} \) with \( x(t) \neq y(t) \) there exists \( s \in [t - p_n - r_n, t + p_n + r_n] \) such that 
\[ \varphi_n x(s) \neq \varphi_n y(s). \]

Therefore (i)–(iv) holds with \( n + 1 \) instead of \( n \) and the induction is finished.

To state it explicitly once more, what we have got so far: The iterated application of 3.10 shows the existence of \( M_n, L_n, r_n, p_n \in \mathbb{R}_+ \), \( Y_n \), \( B_n \in \mathcal{L} \) and isomorphisms
\[ \varphi_n: Y_n \to Y_{n+1} \quad (n \in \mathbb{N}) \]
(Note, that we can assume \( \varphi_n Y_n = Y_{n+1} \)) such that for every \( n \in \mathbb{N} \) the following holds:

1. \( \mu_n = \varphi_{n-1} \circ \cdots \circ \varphi_1 \mu_1 \)
   \[ \mu_n(Y_n) = 1; \quad \mu_n(B_n) > 1 - \epsilon_n \]

2. \[ |M_k^{-1} \int_0^M \int_0^T \int_0^\infty f_i T x dt - \int f_i d\mu_n| \leq \sum_{l-k} \epsilon_l + \epsilon_{n-1} 2^{-1} \]
   for every \( 2 \leq k \leq n \), every \( 1 \leq i \leq k \) and every \( x \in Y_n \)

3. \( x(t) = \varphi_n x(t) \) for \( t \in [-L_n, L_n] \) and \( x \in B_n \)

4. For \( x, y \in Y_n \) and \( t \in \mathbb{R} \) with \( x(t) \neq y(t) \), there is \( s \in [t - r_n, t + r_n] \) such that
   \[ \varphi_n x(s) \neq \varphi_n y(s) \]

II. This second part is devoted to the definition of a map \( \varphi \), which turns out in part III. to be the desired isomorphism. Define inductively

\[ \psi_0 = \text{id} \]
\[ \psi_n := \varphi_n \circ \psi_{n-1} \quad (n \in \mathbb{N}) \]

and put \( C_1 : = B_1 \), \( C_n := \psi_{n-1} B_n (n \geq 2) \).

Note that \( \psi_n : Y_1 \to Y_{n+1} (n \in \mathbb{N}) \) is an isomorphism with \( \psi_n \mu_{n+1} = \mu_{n+1} \).

Therefore we have

\[ \mu_1(C_n) > 1 - \epsilon_n \quad (n \in \mathbb{N}) \]

Put

\[ W_0 := \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} C_m \]

and

\[ W := \bigcup_{t \in \mathbb{R}} T_t W_0 \]

Since \( \sum_{n=1}^{\infty} \epsilon_n < \infty \), we know that \( \mu_1(W) = 1 \). Because \( W \) is \( (T_t)_{t \in \mathbb{R}} \)-invariant, it follows for

\[ Y := Y_1 \cap W \]

immediately that \( \mu_1(Y) = 1 \) and \( T_t Y = Y(t \in \mathbb{R}) \). We shall show now, that for \( y \in Y \)

\[ \lim_{n \to \infty} \psi_n y \]

exists pointwise, more precisely we shall show a little bit more:
(5) For every $K \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for $m, n \geq n_0$
\[ \psi_m y(t) = \psi_n y(t) \quad t \in [-K, K]. \]

To show this, let $y \in Y$. Pick $t_y \in \mathbb{R}$ and $n_y \in \mathbb{N}$ such that
\[ T_{t_y} y \in \bigcap_{m=n_y}^{\infty} C_m \cap Y. \]

Let $K \in \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ such that $n_0 \geq n_y$ and $L_{n_0} \geq |t_y| + K$.

Let $n \geq n_0$ and $m > n$. By definition we have
\[ \psi_m = \varphi_m \circ \cdots \circ \varphi_{n+1} \circ \psi_n \]
and, since $T_{t_y} y \in C_k$ for $k \geq n_0$
\[ \psi_{k-1} T_{t_y} y \in B_k \quad \text{for } k \geq n_0. \]

Therefore
\[ \psi_n T_{t_y} y(t) = \varphi_m \circ \cdots \circ \varphi_{n+1} \circ \psi_n T_{t_y} y(t) = \psi_n T_{t_y} y(t) \]
for $t \in [-L_{n+1}, L_{n+1}]$.

Because by the choice of $n_0$ we have
\[ t - t_y \in [-L_{n_0}, L_{n_0}] \subset [-L_{n+1}, L_{n+1}] \]
for every $t \in [-K, K]$, hence
\[ \psi_m y(t) = \psi_n T_{t_y} y(t - t_y) = \psi_n T_{t_y} y(t - t_y) = \psi_n y(t) \]
for every $t \in [-K, K]$.

Thus, we have shown (5) and the map
\[ \varphi: Y \to L^A, \quad \text{defined by} \]
\[ \varphi y = \lim_{n \to \infty} \psi_n y \quad (\text{pointwise}), \]
is well defined.

To see that $\varphi Y \subset L^A$, note that for every $n \in \mathbb{N}$ $\psi_n Y \subset L^A \subset L^A$.

III. $\varphi$ is an isomorphism onto a strictly ergodic flow.

(1) $\varphi$ is injective. Suppose $x, y \in Y$ and $x \neq y$, i.e., $x(t_0) \neq y(t_0)$ for some $t_0 \in \mathbb{R}$. By iterated application of I. (4) for every $n \in \mathbb{N}$ we find
\[ s_n \in \left[ t_0 - \sum_{i=1}^{n-1} (p_i + r_i), t_0 + \sum_{i=1}^{n-1} (p_i + r_i) \right] \]
such that

$$\psi_n(x(s_n)) \neq \psi_n(y(s_n)).$$

Pick $n_x, n_y \in \mathbb{N}$ and $t_x, t_y \in \mathbb{R}$ such that

$$T_{t_x}x \in \bigcap_{m=n_x}^{\infty} C_m$$

and

$$T_{t_y}y \in \bigcap_{m=n_y}^{\infty} C_m.$$

Choose $n_0 = |t_0| + |t_x| + |t_y| + |n_x| + |n_y|$ and let $n \geq n_0$.

Using I. (3) we get

$$\psi_n T_{t_x}x(t) = \psi_{n_0-1} T_{t_x}x(t)$$

and

$$\psi_n T_{t_y}y(t) = \psi_{n_0-1} T_{t_y}y(t)$$

for every $t \in [-L_{n_0}, L_{n_0}]$, i.e.,

$$\psi_n x(t) = \psi_{n_0-1} x(t) \quad \text{for} \quad t \in [-L_{n_0}, L_{n_0}].$$

and

$$\psi_n y(t) = \psi_{n_0-1} y(t) \quad \text{for} \quad t \in [-L_{n_0}, L_{n_0}].$$

Since by the choice of $L_{n_0}$ for $z \in \{x, y\}$

$$-L_{n_0} + |t_z| \leq -|t_0| - \sum_{i=1}^{n_0-1} (p_i + r_i) \leq t_0 - \sum_{i=1}^{n_0-1} (p_i + r_i) \leq s_{n_0-1} \leq t_0 + \sum_{i=1}^{n_0-1} (p_i + r_i) \leq |t_0| + \sum_{i=1}^{n_0-1} (p_i + r_i) \leq L_{n_0} - |t_z|,$$

we get

$$\psi_n x(s_{n_0-1}) = \psi_{n_0-1} x(s_{n_0-1})$$

and

$$\psi_{n_0-1} y(s_{n_0-1}) = \psi_n y(s_{n_0-1}).$$

It follows, since $n \geq n_0$ was chosen arbitrarily,

$$\varphi x(s_{n_0-1}) \neq \varphi y(s_{n_0-1}).$$

Hence $\varphi$ is injective.
(2) Measurability of \( q^{-1} \) and \( q^{-1} \). Let \( F \in \mathcal{L} \) depending on an interval \([-K, K]\), i.e., if \( x, y \in L^4 \) and \( x(t) = y(t) \) \((t \in [-K, K])\) then \( x \in F \) iff \( y \in F \). Note that these sets depending on intervals generate \( \mathcal{L} \) (see 4.1).

The equation

\[
q^{-1}F = \{ y \in Y \mid qy \in F \} = \{ y \in Y \mid \lim_{n \to \infty} \psi_n y \in F \}
\]

\[
= \bigcup_{k=K}^{\infty} \left\{ y \in \bigcap_{m=k}^{\infty} C_m \cap Y \mid \lim_{n \to \infty} \psi_n y \in F \right\}
\]

\[
= \bigcup_{k=K}^{\infty} \left\{ y \in \bigcap_{m=k}^{\infty} C_m \cap Y \mid \psi_{k-1} y \in F \right\}
\]

\[
= \bigcup_{k=K}^{\infty} \left( \bigcap_{m=k}^{\infty} C_m \cap Y \cap \psi_{k-1}^{-1}F \right)
\]

clearly implies \( q^{-1}F \in \mathcal{L} \), i.e., measurability of \( q \).

We prove measurability of \( q^{-1} \) by showing that for every \( E \in \mathcal{L} \) there is \( E' \in \mathcal{L} \) with \( q^{-1}E' = E \). It suffices to prove it for \( F \in \mathcal{L} \) depending on an interval \([-K, K]\).

Using I. (4) one sees that for every \( n \in \mathbb{N} \) \( \psi_{n-1}F \) as a subset of \( Y_n \) depends on

\[
\left[-K - \sum_{i=1}^{n-1} (p_i - r_i), K + \sum_{i=1}^{n-1} (p_i + r_i)\right],
\]

i.e., if \( x, y \in Y_n \) and

\[
x(t) = y(t) \quad \text{ for } t \in \left[-K - \sum_{i=1}^{n-1} (p_i + r_i), K + \sum_{i=1}^{n-1} (p_i + r_i)\right]
\]

then \( x \in \psi_{n-1}F \) iff \( y \in \psi_{n-1}F \).

Therefore we can write

\[
\psi_{n-1}F = F_n \cap Y_n \quad (n \in \mathbb{N}),
\]

where \( F_n \) depends on \([-K - \sum_{i=1}^{n-1} (p_i + r_i), K + \sum_{i=1}^{n-1} (p_i + r_i)]\). Let \( n \geq K, n \in \mathbb{N} \) be fixed, \( K \geq n \) and \( x \in \bigcap_{j=n}^{\infty} C_j \). Then by I. (3)

\[
\psi_{k}x(t) = \psi_{n-1}x(t) \quad \text{ for } t \in [-L_n, L_n]
\]

and since \( L_n \geq K + \sum_{i=1}^{n-1} (p_i + r_i) \)

\[
\psi_{k}x(t) = \psi_{n-1}x(t) \quad \text{ for } t \in \left[-K - \sum_{i=1}^{n-1} (p_i + r_i), K + \sum_{i=1}^{n-1} (p_i + r_i)\right].
\]
It follows
\[ \psi_k x \in F_n \quad \text{iff} \quad \psi_{n-1} x \in F_n , \]
i.e.,
\[ \psi_k^{-1} F_n \cap \bigcap_{m=n}^{\infty} C_m = \psi_{n-1}^{-1} F_n \cap \bigcap_{m=n}^{\infty} C_m . \]

Computing \( \varphi^{-1} F_n \) we get
\[
\varphi^{-1} F_n = \{ y \in Y \mid \lim_{j \to \infty} \psi_j y \in F_n \}
= \bigcup_{k=n}^{\infty} \{ y \in \bigcap_{j=k}^{\infty} C_j \mid \psi_{k-1} y \in F_n \}
= \bigcup_{k=n}^{\infty} \left( \bigcap_{j=k}^{\infty} C_j \cap \psi_{k-1}^{-1} F_n \right)
= \bigcup_{k=n}^{\infty} \left( \left( \bigcap_{j=k}^{\infty} C_j \cap \bigcap_{m=n}^{\infty} C_m \right) \cap \psi_{k-1}^{-1} F_n \right) \cup \bigcup_{k=n}^{\infty} \left( \bigcap_{j=n}^{\infty} C_j \cap \psi_{k-1}^{-1} F_n \right)
= \left( \bigcup_{m=n}^{\infty} C_m \right) \cup \left( F \cap \bigcap_{m=n}^{\infty} C_m \right).
\]

Let \( \epsilon > 0 \). Pick \( n \in \mathbb{N} \) such that \( n \geq K \) and
\[
\mu_1 \left( \frac{Y}{\bigcap_{m=n}^{\infty} C_m} \right) < \frac{\epsilon}{2}.
\]
Then
\[
\mu_1(F \Delta \varphi^{-1} F_n) = \mu_1(F \setminus \varphi^{-1} F_n) + \mu_1(\varphi^{-1} F_n \setminus F)
\leq \mu_1 \left( \frac{Y}{\bigcap_{m=n}^{\infty} C_m} \right) + \mu_1 \left( \frac{Y}{\bigcap_{m=n}^{\infty} C_m} \right) < \epsilon.
\]
Hence we have
\[ F = \bigcap_{n=K}^{\infty} \bigcup_{m=n}^{\infty} \varphi^{-1} F_n . \]

(3) \( (T_t)_{t \in \mathbb{R}} \)–invariance. Let \( y \in Y \) and \( s_0 \), \( t_0 \in \mathbb{R} \). Choose \( t_y \in \mathbb{R} \) and \( n_y \in \mathbb{N} \) such that
\[ T_{t_y} y \in \bigcap_{m=n_y}^{\infty} C_m . \]
Pick \( n_0 \geq n_y \) such that
\[
L_{n_0} \geq |t_y| + |s_0| + |t_0|,
\]
hence
\[
\psi_n T_{t_0} y(s) = \psi_{n_0-1} T_{t_0} y(s)
\]
for every \( s \in [-L_{n_0}, L_{n_0}] \) and every \( n \geq n_0 \), i.e.,
\[
\psi_n y(s) = \psi_{n_0-1} y(s)
\]
for every \( s \in [-L_{n_0} + |t_y|, L_{n_0} - |t_y|] \).
Since
\[
L_{n_0} + |t_y| = -|s_0| - |t_0| \leq -|s_0| \leq |s_0| \leq |s_0| + |t_0| \leq L_{n_0} - |t_y|
\]
we get
\[
\psi_n T_{t_0} y(s_0) = \psi_n y(s_0 + t_0) = \psi_{n_0-1} T_{t_0} y(s_0)
\]
and
\[
\varphi T_{t_0} y(s_0) = \psi_{n_0-1} T_{t_0} y(s_0) = T_{t_0} \varphi y(s_0).
\]
Hence because \( s_0, t_0 \in \mathbb{R}, y \in Y \) was arbitrary
\[
\varphi \circ T_t = T_t \circ \varphi \quad (t \in \mathbb{R}).
\]

(4) So far we have proved that \( \varphi \) is an isomorphism. Put \( m = \varphi \mu \).
It is left to show that \((L^A, \mathcal{L}, m, (T_t)_{t \in \mathbb{R}})\) is strictly ergodic. It suffices to prove for every \( f_i (i \in \mathbb{N}) \) (see 4.1) and every \( \varepsilon > 0 \) the existence of \( M \in \mathbb{R}_+ \) such that for every \( y \in \varphi Y \)
\[
|M^{-1} \int_0^M f_i T_t y \, dt - \int f_i \, dm| < \varepsilon.
\]
Fix \( f_i (i \in \mathbb{N}) \) and \( \varepsilon > 0 \). In a first step we prove: For every \( \eta > 0 \) there is \( n_n \in \mathbb{N} \) such that
\[
\left| \int f_i \, dm - \int f_i \, d\mu_n \right| < \eta \quad \text{for } n \geq n_n.
\]
Pick \( y \in \varphi Y \) such that \( \varphi^{-1} y \in W_0 \) and

\[
\lim_{N \to \infty} N^{-1} \int_0^N f_t y \, dt = \int f_t \, dm.
\]

Let \( \eta > 0 \). Choose \( i \leq N \in \mathbb{N} \) such that

\[
\left| \frac{1}{MN} \int_0^{MN} f_t y \, dt - \int f_t \, dm \right| < \eta 2^{-1},
\]

\( \varphi^{-1} y \in \bigcap_{m=-N}^{\infty} C_m \) and \( \sum_{l=N}^{\infty} \epsilon_l < \eta 2^{-1} \).

Define \( n_0 \geq N \) by \( L_{n_\eta} \geq MN + i \).

Then for \( n \geq n_0 \)

\[
\psi_{n-1} \varphi^{-1} y(t) = \psi_{n-2} \varphi^{-1} y(t) \quad t \in [-L_n, L_n], \quad h \geq n.
\]

Therefore we get

\[
y(t) = \psi_{n-1} \varphi^{-1} y(t) \quad \text{for} \quad t \in [-i, MN + i]
\]

and \( T_s y(t) = \psi_{n-1} T_s \varphi^{-1} y(t) \) for \( t \in [-i, i], \ s \in [0, MN] \). Thus, \( f_t T_s y = f_t T_s \varphi^{-1} y \) for every \( s \in [0, MN] \).

Now it is easy to conclude (use I.(2)):

\[
\left| \int f_t \, dm - \int f_t \, d\mu_n \right| \leq \left| \int f_t \, dm - MN^{-1} \int_0^{MN} f_t T_s y \, ds \right| + \left| MN^{-1} \int_0^{MN} f_t T_s \varphi^{-1} y \, ds - \int f \, d\mu_n \right| < \eta \cdot 2^{-1} + \sum_{l=N}^{\infty} \epsilon_l \leq \eta.
\]

We shall show strict ergodicity now.

Pick \( m \in \mathbb{N} \) such that \( \sum_{l=m}^{\infty} \epsilon_l < \epsilon \cdot 2^{-1} \) and \( m \geq i \). Let \( y \in Y \). Choose \( t_y \in \mathbb{R} \) and \( n_y \in \mathbb{N} \) with

\[
T_{t_y} y \in \bigcap_{m=n_y}^{\infty} C_m,
\]

and let \( n_0 \geq \max(n_y, n_{e,2^{-1}}, m) \) with

\[
L_{n_0} \geq |t_y| + m + M_m.
\]
By the same reason as in step 1
\[ \psi_k T_y y(t) = \psi_{n_0-1} T_y y(t) \quad \text{for} \quad t \in [-L_{n_0}, L_{n_0}] \]
and for every \( k \geq n_0 \).
Hence we get
\[ qy(t) = \psi_{n_0-1} y(t) \]
for every \( t \in [-m, m + M_m] \), because
\[ -L_{n_0} + |t_y| \leq -m \leq M_m + m \leq L_{n_0} - |t_y|. \]
It follows
\[ f_i T_s qy = f_i T_s \psi_{n_0-1} y \quad \text{for} \quad s \in [0, M_m]. \]
Using I. (2) again we have
\[
\left| M_m^{-1} \int_0^{M_m} f_i T_s qy \, ds - \int f_i \, dm \right|
\leq \left| M_m^{-1} \int_0^{M_m} f_i T_s \psi_{n_0-1} y \, ds - \int f_i \, d\mu_{n_0} \right| + \left| \int f_i \, d\mu_{n_0} - \int f_i \, dm \right|
\leq \sum_{i=m}^{\infty} \epsilon_i + \epsilon \cdot 2^{-1} \leq \epsilon.
\]
Note that \( m \) did not depend on \( y \). Thus we have shown strict ergodicity.

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