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INTERSECTION THEOREMS FOR SYSTEMS OF FINITE VECTOR SPACES*

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A theorem of Erdös, Ko and Rado states that if S is an n-element set and $\mathcal{F}$ is a family of k-element subsets of S, $k \leq \frac{1}{2}n$, such that no two members of $\mathcal{F}$ are disjoint, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. In this paper we investigate the analogous problem for finite vector spaces.

Let $\mathcal{F}$ be a family of k-dimensional subspaces of an n-dimensional vector space over a field of q elements such that members of $\mathcal{F}$ intersect pairwise non-trivially. Employing a method of Katona, we show that for $n \geq 2k$, $|\mathcal{F}| \leq (k/n)\left[{n \atop k}\right]_q$. By a more detailed analysis, we obtain that for $n \geq 2k+1$, $|\mathcal{F}| \leq \left[{n-1 \atop k-1}\right]_q$, which is a best possible bound. The argument employed is generalized to the problem of finding a bound on the size of $\mathcal{F}$ when its members have pairwise intersection dimension no smaller than r. Again best possible results are obtained for $n \geq 2k+2$ and $n \geq 2k+1$, $q \geq 3$. Application of these methods to the analogous subset problem leads to improvements on the Erdös–Ko–Rado bounds.

1. Introduction

A theorem of Erdös, Ko and Rado [1] states that if S is an n-element set and $\mathcal{F}$ a family of k-element subsets of S, $k \leq \frac{1}{2}n$, such that no two members of $\mathcal{F}$ are disjoint, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. In this paper we consider the analogous problem for finite vector spaces. By $S(r, k, n, q)$ we denote the set of all families $\mathcal{F}$ of k-dimensional subspaces of an n-dimensional vector space V over a finite field F of q elements such that $A, B \in \mathcal{F} \Rightarrow \dim(A \cap B) \geq r$. Suppose that $\mathcal{F} \in S(1, k, n, q)$ with $k \leq \frac{1}{2}n$, what can we say about $|\mathcal{F}|$?

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There are two ways to view the Erdös–Ko–Rado theorem:

(1) that $\binom{n-1}{k-1}$ is an upper bound on $|\mathcal{F}|$, and

(2) that if $\mathcal{S}_k$ is the family of all k-element subsets of S, then $|\mathcal{F}|/|\mathcal{S}_k| \leqslant k/n$. Thus in the finite vector space case, we may expect that either

(1) $|\mathcal{F}| \leqslant \left[{n-1 \atop k-1}\right]_q$, or

(2) $|\mathcal{F}|/|\mathcal{S}_k| \leqslant k/n$, where $\left[{m \atop n}\right]_q$ denotes the Gaussian coefficient, the number of n-dimensional subspaces of an m-dimensional vector space over a finite field of q elements, and $\mathcal{S}_k$ denotes the family of all k-dimensional subspaces of V. The number of k-dimensional subspaces of V containing a specific one-dimensional subspace is $\left[{n-1 \atop k-1}\right]_q$. Thus the inequality in (1) is best possible. Also

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \bigg/ \begin{bmatrix} n \\ k \end{bmatrix}_q = (q^k-1)/(q^n-1) \ll k/n .$$

Hence (1) suggests a much stronger bound on $|\mathcal{F}|$ than (2).

In Section 2 we derive a few basic facts about the Gaussian coefficients that are needed in our study. In Section 3 we prove that $|\mathcal{F}|/|\mathcal{S}_k| \leqslant k/n$ for $n \geqslant 2k$. We also show that if $\mathcal{D}$ is a family of ordered k-tuples with the ith component chosen from $\{1, ..., q_i\}$, $1 \leqslant q_1 \leqslant ... \leqslant q_k$, such that each pair in $\mathcal{D}$ has at least a component in common, then $|\mathcal{D}| \leqslant q_2 ... q_k$. In Section 4 we show that $|\mathcal{F}| \leqslant \left[{n-1 \atop k-1}\right]_q$ for $n \geqslant 2k+1$. We conject that $\left[{n-1 \atop k-1}\right]_q$ should also be a bound on $|\mathcal{F}|$ for $n = 2k$. In Section 5 we apply a generalization of the method developed in Section 4 to $S(r, k, n, q)$. We show that for $n \geqslant 2k+2$, or $n \geqslant 2k+1$, $q \geqslant 3$, any family in $S(r, k, n, q)$ can have size no larger than $\left[{n-r \atop k-r}\right]_q$, a bound that is achieved when the subspaces are chosen to be all those containing some specific r-dimensional subspace. We also show that if $\mathcal{F}$ is a family of k-element subsets of S with pairwise intersection size no smaller than r, then $|\mathcal{F}| \leqslant \binom{n-r}{k-r}$ provided that $n \geqslant r + (r+1)(k-r+1)(k-r)$. The bound on n is a considerable improvement over a previous result of Erdös, Ko and Rado.

2. The Gaussian coefficients

Just as the binomial coefficient $\binom{n}{k}$ counts the number of k-element subsets of an n-element set, the Gaussian coefficient $\left[{n \atop k}\right]_q$ counts the number of k-dimensional subspaces of an n-dimensional vector space V over a finite field of q elements. It is not difficult to derive a formula for $\left[{n \atop k}\right]_q$. Enumerate all ordered bases of k-dimensional subspaces of V as follows: The first vector x_1 can be chosen in $q^n - 1$ ways. There are q vectors dependent upon x_1, so the next vector x_2 can be chosen in $q^n - q$ ways, etc. Thus there are $(q^n - 1)(q^n - q) \dots (q^n - q^{k-1})$ linearly ordered sets of k linearly independent vectors in V. But each k-dimensional subspace has, by the same argument, $(q^k - 1)(q^k - q) \dots (q^k - q^{k-1})$ ordered basis. Thus

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^n - q) \dots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \dots (q^k - q^{k-1})} .$$

Note that as $q \to 1$, $\left[{n \atop k}\right]_q \to \binom{n}{k}$, and thus we can expect that the Gaussian coefficients to share many of the properties of the binomial coefficients. Also note that

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \Big/ \begin{bmatrix} n \\ k \end{bmatrix}_q = (q^k - 1)/(q^n - 1) .$$

Thus for $q > 1$, this proportion is much smaller than k/n. Hence (1) gives a much stronger bound on $|\mathcal{F}|$ than (2).

To simplify the notation, we shall omit the subscript q and write just $\left[{n \atop k}\right]$ to denote the Gaussian coefficient.

Let $\bar{V}$ be the dual space of linear functions on V. For $A \subset V$, let $A^0 \subset \bar{V}$ be the annihilator of A, i.e., $A^0 = \{f \in \bar{V} : f(A) = 0\}$. If A is a k-dimensional subspace of V, A^0 is an $(n-k)$-dimensional subspace of $\bar{V}$.

Remark 2.1. $\left[{n \atop k}\right] = \left[{n \atop n-k}\right]$.

Proof. $A \subset V \Leftrightarrow A^0 \supset V^0 = (0)$. Thus $\left[{n \atop k}\right]$ = the number of k-dimensional subspaces of V = the number of $(n-k)$-dimensional subspaces of $\bar{V} = \left[{n \atop n-k}\right]$.

Remark 2.2. The number of k-dimensional subspaces of V containing a particular r-dimensional subspace A of V is $\left[{n-r \atop k-r}\right]$.

Proof. $B \supset A \Leftrightarrow B^0 \subset A^0$. Thus the number of k-dimensional subspaces of V containing A = the number of $(n-k)$-dimensional subspaces of $\bar{V}$ contained in $A^0 = \left[{n-r \atop n-k}\right] = \left[{n-r \atop k-r}\right]$.

Thus, if $\left[{n-r \atop k-r}\right]$ is a bound on $|\mathcal{F}|$ for $\mathcal{F} \in S(r, k, n, q)$, then it is a best possible bound.

3. The Katona method

Katona [5] has presented a rather simple proof of the Erdös–Ko–Rado theorem. By employing his technique, we can prove (2).

If $a_1, \ldots, a_t \in V$, we shall use $[a_1, \ldots, a_t]$ to denote the subspace of V spanned by $a_1, \ldots, a_t$. Also we use $\mathcal{S}_k$ to denote the family of all k-dimensional subspaces of V.

Theorem 3.1. *If $k \leqslant \frac{1}{2}n$, $\mathcal{F} \in S(1, k, n, q)$, then $|\mathcal{F}|/|\mathcal{S}_k| \leqslant k/n$.*

Proof. Take a basis $\{x_1, \ldots, x_n\}$ of V. Let $V_i = \{x_{i_1}, \ldots, x_{i_k}\}$, where $i_j \equiv (i-1)k + j \pmod n$ for $i = 1, \ldots, n$. V_i is a k-dimensional subspace of V with basic vectors chosen from $\{x_1, \ldots, x_n\}$. Roughly speaking, the n/k consecutive V_i's intersect only trivially at the origin. Thus every V_i non-trivially intersects at most k other V_j's. Hence if $1 \leqslant i_1 < \ldots < i_d \leqslant n$, and $V_{i_1}, \ldots, V_{i_d}$ intersect pairwise non-trivially, then $d \leqslant k$. (The detail of the argument in Katona's paper can be carried over here in a straightforward fashion.)

Let $\mathcal{H} = \{(\bar{V}_1, \ldots, \bar{V}_n)$, where $\bar{V}_i = [y_{i_1}, \ldots, y_{i_k}]$, $(y_1, \ldots, y_n)$ an ordered basis of V, $i_j \equiv (i-1)k + j \pmod n\}$, i.e. $\mathcal{H} = \{F$: the n-tuple of k-dimensional subspaces obtained from $(V_1, \ldots, V_n)$ by mapping $(x_1, \ldots, x_n)$ onto any ordered basis $(y_1, \ldots, y_n)$ of V with $x_i \to y_i\}$.

From the above, each $F \in \mathcal{H}$ can contain at most k $\bar{V}$'s in $\mathcal{F}$. Each fixed $\bar{V} \in \mathcal{F}$ can be contained in at most

$$n \cdot (q^k - 1) \ldots (q^k - q^{k-1}) \cdot (q^n - q^k)(q^n - q^{k+1}) \ldots (q^n q^{n-1})$$

F's, because there are $(q^k-1)\ldots(q^k-q^{k-1})(q^n-q^k)\ldots(q^n-q^{n-1})$ ways of transforming a fixed V_i onto $\bar{V}$. Thus

$$(q^n-1)\ldots(q^n-q^{n-1}) \cdot k \geqslant |\mathcal{F}| \cdot n \cdot (q^k-1)\ldots(q^k \;\; q^{k-1}) \cdot (q^n-q^k)\ldots(q^n-q^{n-1}),$$

i.e., $|\mathcal{F}| \leqslant (k/n)\left[{n \atop k}\right] = (k/n)|\mathcal{S}_k|$.

Employing the same technique, we can also prove the following theorem for ordered k-tuples.

Theorem 3.2. *Suppose that $\mathcal{G}$ is a family of ordered k-tuples with the ith component chosen from $\{1, ..., q_i\}$, $1 \leqslant q_1 \leqslant ... \leqslant q_k$, such that each pair in $\mathcal{G}$ has at least a component in common, then $|\mathcal{G}| \leqslant q_2 ... q_k$.*

Proof. Let $B_i = (\overbrace{i, ..., i}^{k \text{ copies}})$, $i = 1, ..., q_1$. Note that if $i \neq j$, then B_i, B_j have no common components.

Let $F = (B_1, ..., B_{q_1})$, and $F_{p_1 ... p_k} = (C_1, ..., C_{q_1})$, where p_i is a permutation of $(1, 2, ..., q_i)$ and $C_i = (p_1(i), ..., p_k(i)) = p_1 ... p_k(B_i)$. Thus there can be at most one $C_i \in \mathcal{G}$ in $F_{p_1 ... p_k}$. Counting in two different ways the pairs $(F_{p_1 ... p_k}, A)$, $A \in \mathcal{G}$, we obtain

$$q_1! ... q_k! \geqslant |\mathcal{G}| \cdot q_1 \cdot (q_1 - 1)!(q_2 - 1)! ... (q_k - 1)!,$$

i.e., $|\mathcal{G}| \leqslant q_2 ... q_k$.

4. The main result

So far we do not have a complete proof for (1). We do obtain the desired bound for $n \geqslant 2k + 1$. Our method, however, does not seem to apply to the case $n = 2k$. We feel that $\left[{n-1 \atop k-1}\right]$ should also be a bound in this case, and probably a different approach has to be considered to handle it.

Let $\mathcal{S}$ be a family of subspaces of V. For $x \in V$, we shall use $\mathcal{S}_x$ to denote the family of subspaces in $\mathcal{S}$ containing x. For $A \subset V$, $\mathcal{S}_A$ is defined in a similar fashion.

Lemma 4.1. *Suppose that $s \geqslant t + k$, then*

$$\begin{bmatrix} s \\ t \end{bmatrix} > (q-1) q^{s-t-k} \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} s-1 \\ t-1 \end{bmatrix},$$

and, in general,

$$\begin{bmatrix} s \\ t \end{bmatrix} > (q-1)^p q^{(s-t-k)p} \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} s-p \\ t-p \end{bmatrix} \qquad \text{for } 1 \leqslant p \leqslant t.$$

Proof.

$$\begin{bmatrix} s \\ t \end{bmatrix} \Big/ \left(\begin{bmatrix} s-1 \\ t-1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix} \right) = \frac{(q^s-1)(q-1)}{(q^t-1)(q^k-1)}$$

$$= (q-1)\left(q^{s-t-k} + \frac{q^{s-t-k}(q^t+q^k-1)-1}{(q^t-1)(q^k-1)}\right)$$

$$> (q-1)q^{s-t-k} .$$

$s-t \geqslant k \Rightarrow (s-p)-(t-p) \geqslant k$, so the general case follows by induction.

Lemma 4.2. *Suppose that* $n \geqslant 2k+1$, *and* $\mathcal{F} \in S(1, k, n, q)$. *If*

$$|\mathcal{F}_x| \leqslant \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}$$

for all $0 \neq x \in V$, *then either*

$$|\mathcal{F}| < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \quad \text{or} \quad |\mathcal{F}_A| \leqslant \begin{bmatrix} k \\ 1 \end{bmatrix}^{p-1} \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix}$$

for all 2-dim $A \subset V$, *where* $1 \leqslant p \leqslant k-1$.

Proof. The assertion is trivial for $p = 1$. Thus assume $p \geqslant 2$. By Lemma 4.1,

$$n \geqslant 2k+1,\ q \geqslant 2 \Rightarrow \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} > q^p \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix} .$$

Thus

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} > \begin{bmatrix} s \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix} \qquad \text{for } 1 \leqslant s \leqslant p.$$

Take a 2-dimensional subspace $[x,y] \subset V$.
If $A \in \mathcal{F} \Rightarrow A \cap [x,y] \neq 0$, then

$$|\mathcal{F}| \leqslant \sum_{\substack{z \subset [x,y] \\ z\ 1\text{-dim}}} |\mathcal{F}_z| \leqslant \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^p \begin{bmatrix} n-1-p \\ k-1-p \end{bmatrix} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} .$$

Thus we can suppose there is some $A_1 \in \mathcal{F}$ such that $A_1 \cap [x,y] = (0)$. Take $0 \neq z_1 \in A_1$. If $A \in \mathcal{F} \Rightarrow A \cap [x,y,z_1] \neq (0)$, then

$$|\mathcal{F}| \leqslant \begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^p\begin{bmatrix}n-1-p\\k-1-p\end{bmatrix} < \begin{bmatrix}n-1\\k-1\end{bmatrix}.$$

Thus we can suppose that there is some $A_2 \in \mathcal{F}$ such that $A_2 \cap [x,y,z_1] = (0)$. Hence $|\mathcal{F}_{x,y,z_1}| \leqslant [{k \atop 1}][{n-4 \atop k-4}]$, and so $|\mathcal{F}_{x,y}| \leqslant [{k \atop 1}]^2[{n-4 \atop k-4}]$.

Suppose that for $1 \leqslant j \leqslant i$, $0 \neq z_j \in A_j$ and $[x,y,z_1,\ldots,z_j] \cap A_{j+1} = (0)$. Take $0 \neq z_{i+1} \in A_{i+1}$. If $A \in \mathcal{F} \Rightarrow A \cap [x,y,z_1,\ldots,z_{i+1}] \neq (0)$, then

$$|\mathcal{F}| \leqslant \begin{bmatrix}i+3\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^p\begin{bmatrix}n-1-p\\k-1-p\end{bmatrix} < \begin{bmatrix}n-1\\k-1\end{bmatrix}.$$

Thus we can suppose that there is some $A_{i+2} \in \mathcal{F}$ such that $A_{i+2} \cap [x,y,z_1,\ldots,z_{i+1}] = (0)$. Hence

$$|\mathcal{F}_{x,y,z_1,\ldots,z_{i+1}}| \leqslant \begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-i-4\\k-i-4\end{bmatrix},$$

and so inductively we obtain

$$|\mathcal{F}_{x,y}| \leqslant \begin{bmatrix}k\\1\end{bmatrix}^{i+2}\begin{bmatrix}n-i-4\\k-i-4\end{bmatrix}.$$

Thus for $1 \leqslant i \leqslant p$, either $|\mathcal{F}| < [{n-1 \atop k-1}]$ or $|\mathcal{F}_{x,y}| \leqslant [{k \atop 1}]^{i-1}[{n-1-i \atop k-1-i}]$. Hence either $|\mathcal{F}| < [{n-1 \atop k-1}]$ or $|\mathcal{F}_{x,y}| \leqslant [{k \atop 1}]^{p-1}[{n-1-p \atop k-1-p}]$.

Lemma 4.3. *Suppose that $n \geqslant 2k+1$, and $\mathcal{F} \in S(1,k,n,q)$. If $|\mathcal{F}_x| \leqslant [{k \atop 1}]^{k-1}$ for all $x \in V$, then $|\mathcal{F}| < [{n-1 \atop k-1}]$.*

Proof. For $0 \leqslant i \leqslant k-3$,

$$\frac{q^{n-1}-q^i}{q^{k-1}-q^i} = q^{n-k} + \frac{q^{n-k+i}-q^i}{q^{k-1}-q^i} > q^{n-k} \geqslant q^{k+1},$$

$$\frac{q^{n-1}-q^{k-2}}{q^{k-1}-q^{k-2}} \geqslant \frac{q^{k+2}-1}{q-1} \geqslant q^2\begin{bmatrix}k\\1\end{bmatrix}.$$

Thus

$$\begin{bmatrix}n-1\\k-1\end{bmatrix} = \prod_{i=0}^{k-2}\frac{q^{n-1}-q^i}{q^{k-1}-q^i} \geqslant (q^{k+1})^{k-2}\cdot q^2\begin{bmatrix}k\\1\end{bmatrix} = q^{k^2-k}\begin{bmatrix}k\\1\end{bmatrix} > \begin{bmatrix}k\\1\end{bmatrix}^k.$$

But then $|\mathcal{F}_x| \leqslant [{k \atop 1}]^{k-1}$ for all $x \in V \Rightarrow |\mathcal{F}| \leqslant [{k \atop 1}]^k < [{n-1 \atop k-1}]$.

Theorem 4.4. *If* $n \geqslant 2k+1$, *and* $\mathcal{F} \in S(1, k, n, q)$, *then* $|\mathcal{F}| \leqslant \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$. *In fact, if* $\bigcap \mathcal{F} = (0)$, *then* $|\mathcal{F}| < \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$.

Proof. If $[x] \subset \bigcap \mathcal{F}$ for some $0 \neq x \in V$, then $|\mathcal{F}| \leqslant \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$ by Remark 2.2. Thus we can suppose that $\bigcap \mathcal{F} = (0)$.

Let $x_1 \neq 0$ be such that $|\mathcal{F}_{x_1}| = \max_{x \in V} |\mathcal{F}_x|$.

By our assumption, there is some $A_1 \in \mathcal{F}$ such that $x_1 \notin A_1$. Thus $|\mathcal{F}_{x_1}| \leqslant \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} n-2 \\ k-2 \end{smallmatrix}\right]$. By Lemma 4.3, we can suppose that $k \geqslant 3$.

Suppose that there are two independent vectors $z_1, z_2 \in A_1$ such that $A \in \mathcal{F} \Rightarrow A \cap [x_1, z_i] \neq (0)$ for $i = 1,2$. If $u_i \in [x_1, z_i] \sim [x_1]$, then u_i's are independent. Thus

$$|\mathcal{F}| \leqslant |\mathcal{F}_{x_i}| + \sum_{\substack{u_i \subset [x_1, z_i] \sim [x_1] \\ u_i \text{ 1-dim}}} |\mathcal{F}_{u_1, u_2}|$$

$$\leqslant \begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1\right)^2 \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Thus we can suppose that there is at most one $z \in A_1$ such that $A \in \mathcal{F} \Rightarrow A \cap [x_1, z] \neq (0)$. Suppose that $z \in A_1$ is such. Take $x \in A_1 \sim [z]$, then there is some $A \in \mathcal{F}$ such that $A \cap [x_1, x] = (0)$, and hence $|\mathcal{F}_{x_1, x}| \leqslant \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} n-3 \\ k-3 \end{smallmatrix}\right]$. Thus

$$|\mathcal{F}_{x_1}| \leqslant |\mathcal{F}_{x_1, z}| + \sum_{\substack{x \subset A_1 \sim [z] \\ x \text{ 1-dim}}} |\mathcal{F}_{x_1, x}| \leqslant \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}.$$

But then

$$|\mathcal{F}| \leqslant \sum_{\substack{x \subset [x, z] \\ x \text{ 1-dim}}} |\mathcal{F}_x| \leqslant \begin{bmatrix} 2 \\ 1 \end{bmatrix}\left(\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}\right)$$

$$\leqslant \frac{1}{q}\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix} < \frac{1 + \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]}{q^2}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \leqslant \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Thus we can suppose that for all $x \in A_1$, there is some $A \in \mathcal{F}$ such that $A \cap [x_1, x] = (0)$, and hence $|\mathcal{F}_{x_1, x}| \leqslant \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} n-3 \\ k-3 \end{smallmatrix}\right]$. Thus $|\mathcal{F}_{x_1}| \leqslant \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]^2 \left[\begin{smallmatrix} n-3 \\ k-3 \end{smallmatrix}\right]$ By Lemma 4.3, we can suppose that $k \geqslant 4$.

Take a non-zero vector y_1 in A_1. There is some $A_2 \in \mathcal{F}$ such that $A_2 \cap [x_1, y_1] = (0)$. Suppose that there are three independent vectors z_1, z_2, z_3 in A_2 such that $A \in \mathcal{F} \Rightarrow [x_1, y_1, z_i] \cap A \neq (0)$ for all $A \in \mathcal{F}$ for $i = 1, 2, 3$. If $u_i \in [x_1, y_1, z_i] \sim [x_1, y_1]$, $i = 1, 2, 3$, then the u_i's are independent. Thus

$$|\mathcal{F}| \leqslant \sum_{\substack{x \subset [x_1, y_1] \\ x\ 1\text{-dim}}} |\mathcal{F}_x| + \sum_{\substack{u_i \in [x_1, y_1, z_i] \sim [x_1, y_1] \\ u_i\ 1\text{-dim}}} |\mathcal{F}_{u_1, u_2, u_3}|$$

$$\leqslant \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-3\\k-3\end{bmatrix} + \left(\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right)^3\begin{bmatrix}n-3\\k-3\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-3\\k-3\end{bmatrix} + q^6\begin{bmatrix}n-3\\k-3\end{bmatrix}$$

$$< \left(\begin{bmatrix}2\\1\end{bmatrix} + 1\right)\begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-3\\k-3\end{bmatrix} < \begin{bmatrix}n-1\\k-1\end{bmatrix}.$$

Thus we can suppose that there exist at most two such z's, and so

$$|\mathcal{F}_{x_1, y_1}| \leqslant \begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-4\\k-4\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}.$$

But then

$$|\mathcal{F}_{x_1}| \leqslant \begin{bmatrix}k\\1\end{bmatrix}^3\begin{bmatrix}n-4\\k-4\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}.$$

By Lemma 4.2, $|\mathcal{F}_{x,y}| \leqslant [{k \atop 1}][{n-3 \atop k-3}]$ for all 2-dim $[x, y] \subset V$. Suppose that there do exist two such z's, say z_1, z_2. Then

$$|\mathcal{F}| \leqslant \sum_{\substack{x \subset [x_1, y_1] \\ x\ 1\text{-dim}}} |\mathcal{F}_x| + \sum_{\substack{u_i \subset [x_1, y_1, z_i] \sim [x_1, y_1] \\ u_i\ 1\text{-dim}}} |\mathcal{F}_{u_1, u_2}|$$

$$\leqslant \begin{bmatrix}2\\1\end{bmatrix}\left(\begin{bmatrix}k\\1\end{bmatrix}^3\begin{bmatrix}n-4\\k-4\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}\right) + \left(\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right)^2\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^3\begin{bmatrix}n-4\\k-4\end{bmatrix} + \left(\begin{bmatrix}2\\1\end{bmatrix}^2 + q^4\right)\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}$$

$$\leqslant \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^3\begin{bmatrix}n-4\\k-4\end{bmatrix} + q\begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-3\\k-3\end{bmatrix} \leqslant \left(\frac{[{2 \atop 1}]}{q^3} + \frac{1}{q}\right)\begin{bmatrix}n-1\\k-1\end{bmatrix} < \begin{bmatrix}n-1\\k-1\end{bmatrix}.$$

Hence we can suppose that there exist at most one such z's. Suppose that z_1 is such. We have

$$|\mathcal{F}_{x_1, y_1}| \leqslant \begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-4\\k-4\end{bmatrix} + \begin{bmatrix}n-3\\k-3\end{bmatrix},$$

and so

$$|\mathcal{F}_{x_1}| \leqslant \begin{bmatrix}k\\1\end{bmatrix}^3\begin{bmatrix}n-4\\k-4\end{bmatrix} + \begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}.$$

Thus

$$|\mathcal{F}| \leqslant \sum_{\substack{x \subset [x_1, y_1, z_1] \\ x \text{ 1-dim}}} |\mathcal{F}_x| \leqslant \begin{bmatrix}3\\1\end{bmatrix}\left(\begin{bmatrix}k\\1\end{bmatrix}^3 \begin{bmatrix}n-4\\k-4\end{bmatrix} + \begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-3\\k-3\end{bmatrix}\right)$$

$$\leqslant \frac{\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]}{q^3}\begin{bmatrix}n-1\\k-1\end{bmatrix} + \frac{1}{q^2}\begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-3\\k-3\end{bmatrix} < \begin{bmatrix}n-1\\k-1\end{bmatrix}.$$

Thus we can suppose that $x \in A_2 \Rightarrow$ there is some $A \in \mathcal{F}$ such that $A \cap [x_1, y_1, x] = (0)$. Hence $|\mathcal{F}_{x_1}| \leqslant \left[\begin{smallmatrix}k\\1\end{smallmatrix}\right]^3 \left[\begin{smallmatrix}n-4\\k-4\end{smallmatrix}\right]$.

In general, suppose that for $1 \leqslant p \leqslant k-3$, we have non-zero vectors $y_1, \ldots, y_p \in V$ and $A_1, \ldots, A_{p+1} \in \mathcal{F}$ such that $y_i \in A_i$ and $A_{i+1} \cap [x_1, y_1, \ldots, y_i] = (0)$ for $1 \leqslant i \leqslant p$. Thus

$$|\mathcal{F}_{x_1, y_1, \ldots, y_p}| \leqslant \begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-p-2\\k-p-2\end{bmatrix},$$

and so inductively we obtain that

$$|\mathcal{F}_{x_1}| \leqslant \begin{bmatrix}k\\1\end{bmatrix}^{p+1}\begin{bmatrix}n-p-2\\k-p-2\end{bmatrix}.$$

By Lemma 4.3,

$$|\mathcal{F}_{x,y}| \leqslant \begin{bmatrix}k\\1\end{bmatrix}^{p}\begin{bmatrix}n-p-2\\k-p-2\end{bmatrix}$$

for all 2-dimensional $[x, y] \subset V$.

Suppose that there are $p+2$ linearly independent vectors $z_1, \ldots, z_{p+2}$ in A_{p+1} such that $[x_1, y_1, \ldots, y_p, z_i] \cap A \neq (0)$ for $A \in \mathcal{F}$ for $i = 1, \ldots, p+2$. Let $u_i \in [x_1, y_1, \ldots, y_p, z_i] \sim [x_1, y_1, \ldots, y_p]$, $i = 1, \ldots, p+2$, then $u_1, \ldots, u_{p+2}$ are independent. Thus

$$|\mathcal{F}| \leqslant \sum_{\substack{x \subset [x_1, y_1, \ldots, y_p] \\ x \text{ 1-dim}}} |\mathcal{F}_x| + \sum_{\substack{u_i \subset [x_1, y_1, \ldots, y_p, z_i] \sim [x_1, y_1, \ldots, y_p] \\ u_i \text{ 1-dim}}} |\mathcal{F}_{u_1, \ldots, u_{p+2}}|$$

$$\leqslant \begin{bmatrix}p+1\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{p+1}\begin{bmatrix}n-p-2\\k-p-2\end{bmatrix} + \left(\begin{bmatrix}p+2\\1\end{bmatrix} - \begin{bmatrix}p+1\\1\end{bmatrix}\right)^{p+2}\begin{bmatrix}n-p-2\\k-p-2\end{bmatrix}$$

$$\leqslant \begin{bmatrix}p+1\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{p+1}\begin{bmatrix}n-p-2\\k-p-2\end{bmatrix} + q^{(p+1)(k-1)}\begin{bmatrix}n-p-2\\k-p-2\end{bmatrix}$$

$$\leqslant \left(\begin{bmatrix}p+1\\1\end{bmatrix} + 1\right)\begin{bmatrix}k\\1\end{bmatrix}^{p+1}\begin{bmatrix}n-p-2\\k-p-2\end{bmatrix} < \begin{bmatrix}n-1\\k-1\end{bmatrix}.$$

Thus we can suppose that there are at most $p+1$ such z_i's. Hence

$$|\mathcal{F}_{x_1,y_1,\ldots,y_p}| \leqslant \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix},$$

and so

$$|\mathcal{F}_{x_1}| \leqslant \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}.$$

Suppose that we do have independent vectors $z_1, z_2 \in A_{p+2}$ such that $A \in \mathcal{F} \Rightarrow A \cap [x_1, y_1, \ldots, y_p, z_i] \neq (0)$ for $i = 1, 2$. Then

$$\begin{aligned}
|\mathcal{F}| &\leqslant \sum_{\substack{x \subset [x_1, y_1, \ldots, y_p] \\ x \text{ 1-dim}}} |\mathcal{F}_x| + \sum_{\substack{u_i \subset [x_1, y_1, \ldots, y_p, z_i] \sim [x_1, y_1, \ldots, y_p] \\ u_i \text{ 1-dim}}} |\mathcal{F}_{u_1,u_2}| \\
&\leqslant \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix} + \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix} \right) \\
&\qquad + \left(\begin{bmatrix} p+2 \\ 1 \end{bmatrix} - \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \right)^2 \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix} \\
&= \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix} + \left(\begin{bmatrix} p+1 \\ 1 \end{bmatrix}^2 + q^{2(p+1)} \right) \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix} \\
&\leqslant \begin{bmatrix} p+1 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix} + q^p \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix} \\
&\leqslant \left(\frac{\begin{bmatrix} p+1 \\ 1 \end{bmatrix}}{q^{p+2}} + \frac{1}{q} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.
\end{aligned}$$

Thus we can suppose that there is at most one such z. Hence

$$|\mathcal{F}_{x_1}| \leqslant \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix} + \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix}.$$

Suppose that $z_1 \in A_{p+1}$ is such a z, then

$$\begin{aligned}
|\mathcal{F}| &\leqslant \sum_{\substack{x \subset [x_1, y_1, \ldots, y_p, z] \\ x \text{ 1-dim}}} |\mathcal{F}_x| \leqslant \begin{bmatrix} p+2 \\ 1 \end{bmatrix} \left(\begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix} + \begin{bmatrix} k \\ 1 \end{bmatrix}^{p} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix} \right) \\
&< \begin{bmatrix} p+2 \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+2} \begin{bmatrix} n-p-3 \\ k-p-3 \end{bmatrix} + \frac{1}{q} \begin{bmatrix} k \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-p-2 \\ k-p-2 \end{bmatrix} \\
&< \left(\frac{\begin{bmatrix} p+2 \\ 1 \end{bmatrix}}{q^{p+2}} + \frac{1}{q^{p+2}} \right) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} < \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.
\end{aligned}$$

Thus we can suppose that for all $z \in A_{p+1}$, there is some $A \in \mathcal{F}$ such that $A \cap [x_1, y_1, \ldots, y_p, z] = (0)$. Take $y_{p+1} \in A_{p+1}$, and let A_{p+2} be such that $A_{p+2} \cap [x_1, y_1, \ldots, y_{p+1}] = (0)$.

Thus for $1 \leqslant p \leqslant k-1$, we have either $|\mathcal{F}_{x_1}| \leqslant \left[{k \atop 1}\right]^p \left[{n-1-p \atop k-1-p}\right]$ or $|\mathcal{F}| < \left[{n-1 \atop k-1}\right]$. Hence either $|\mathcal{F}_{x_1}| \leqslant \left[{k \atop 1}\right]^{k-1}$ or $|\mathcal{F}| < \left[{n-1 \atop k-1}\right]$. By Lemma 4.2, $|\mathcal{F}_{x_1}| \leqslant \left[{k \atop 1}\right]^{k-1} \Rightarrow |\mathcal{F}| < \left[{n-1 \atop k-1}\right]$. Thus we have $|\mathcal{F}| < \left[{n-1 \atop k-1}\right]$ in all cases.

5. $S(r, k, n, q)$

The method developed in the last section can be modified to obtain a bound on the size of families in $S(r, k, n, q)$. Again, there are non-trivial cases where our method fails to apply, but we do have $\left[{n-r \atop k-r}\right]$ as a best possible bound over a fairly wide range.

Lemma 5.1. *If $q \geqslant 3$, $n \geqslant 2k+1$, or if $n \geqslant 2k+2$, then*

$$\begin{bmatrix} n-r \\ k-r \end{bmatrix} > \begin{bmatrix} p+r \\ r \end{bmatrix} \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix}^p \begin{bmatrix} n-r-p \\ k-r-p \end{bmatrix} \quad \textit{for } 1 \leqslant r \leqslant k.$$

Proof. By Lemma 4.1,

$$\begin{bmatrix} n-r \\ k-r \end{bmatrix} > (q-1)^p q^{(n-2k+r-1)p} \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix}^p \begin{bmatrix} n-r-p \\ k-r-p \end{bmatrix}.$$

Now

$$\begin{bmatrix} p+r \\ r \end{bmatrix} = \begin{bmatrix} p+r \\ p \end{bmatrix} = \prod_{i=0}^{p-1} \frac{q^{p+r}-q^i}{q^p-q^i},$$

and it can be easily checked that if either $q \geqslant 3$, $n \geqslant 2k+1$, or $n \geqslant 2k+2$, then

$$(q-1)\, q^{n-2k+r-1} \geqslant \frac{q^{p+r}-q^i}{q^p-q^i} \qquad \text{for } i = 0, 1, \ldots, p-1.$$

Thus

$$\begin{bmatrix} n-r \\ k-r \end{bmatrix} > \begin{bmatrix} p+r \\ r \end{bmatrix} \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix}^p \begin{bmatrix} n-r-p \\ k-r-p \end{bmatrix}.$$

Theorem 5.2. *If $n \geqslant 2k+2$ or if $n \geqslant 2k+1$ and $q \geqslant 3$, then $|\mathcal{F}| \leqslant \left[{n-r \atop k-r}\right]$ for $\mathcal{F} \in S(r, k, n, q)$. In fact, if $\dim(\bigcap \mathcal{F}) < r$, then $|\mathcal{F}| < \left[{n-r \atop k-r}\right]$.*

Proof. Take $\mathcal{F} \in S(r, k, n, q)$. If $\dim(\bigcap \mathcal{F}) \geq r$, then $|\mathcal{F}| \leq \left[\begin{smallmatrix} n-r \\ k-r \end{smallmatrix}\right]$ by Remark 2.1. Thus we can suppose $\dim(\bigcap \mathcal{F}) < r$.

Let $[x_1, \ldots, x_r] \subset V$ be such that

$$|\mathcal{F}_{x_1,\ldots,x_r}| = \max_{\substack{A \subset V \\ A\ r\text{-dim}}} |\mathcal{F}_A| .$$

By our assumption, there is some $A_1 \in \mathcal{F}$ such that $[x_1, \ldots, x_r] \not\subset A_1$. Hence there exists a $(k-r+1)$-dimensional subspace B_1 of A_1 such that $B_1 \cap [x_1, \ldots, x_r] = (0)$. For all A in $\mathcal{F}$, $\dim(A \cap A_1) \geq r$, so $A \cap B_1 \neq (0)$. Thus

$$|\mathcal{F}_{x_1,\ldots,x_r}| \leq \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-r-1 \\ k-r-1 \end{bmatrix} .$$

Take $y_1 \in B_1$. If $\dim(A \cap [x_1, \ldots, x_r, y_1]) \geq r$ for all A in $\mathcal{F}$, then

$$|\mathcal{F}| \leq \sum_{\substack{C \subset [x_1,\ldots,x_r,y_1] \\ C\ r\text{-dim}}} |\mathcal{F}_C| \leq \begin{bmatrix} r+1 \\ r \end{bmatrix} \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-r-1 \\ k-r-1 \end{bmatrix} < \begin{bmatrix} n-r \\ k-r \end{bmatrix}$$

by Lemma 5.1.

Hence we can suppose that there is some A_2 in $\mathcal{F}$ such that $\dim(A_2 \cap [x_1, \ldots, x_r, y_1]) \leq r-1$. Thus there is a $(k-r+1)$-dimensional subspace B_2 of A_2 such that $B_2 \cap [x_1, \ldots, x_r, y_1] = (0)$. But $B_2 \cap A \neq (0)$ for all $A \in \mathcal{F}$, so

$$|\mathcal{F}_{x_1,\ldots,x_r,y_1}| \leq \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-r-2 \\ k-r-2 \end{bmatrix} .$$

Hence

$$|\mathcal{F}_{x_1,\ldots,x_r}| \leq \sum_{\substack{y \subset B_1 \\ y\ 1\text{-dim}}} |\mathcal{F}_{x_1,\ldots,x_r,y}| \leq \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-r-2 \\ k-r-2 \end{bmatrix} .$$

In general, for $0 \leq p \leq k-r-1$, either (i) $|\mathcal{F}| < \left[\begin{smallmatrix} n-r \\ k-r \end{smallmatrix}\right]$, or (ii) for every $y_i \in B_i$, there is some $A_{i+1} \in \mathcal{F}$ such that $\dim([x_1, \ldots, x_r, y_1, \ldots, y_i] \cap A_{i+1}) \leq r-1$; thus there is a $(k-r+1)$-dimensional subspace $B_{i+1} \subset A_{i+1}$ such that $[x_1, \ldots, x_r, y_1, \ldots, y_i] \cap B_{i+1} = (0)$ for $i = 0, 1, \ldots, p$.

Suppose that (ii) holds. $A \in \mathcal{F} \Rightarrow \dim(A \cap A_{p+1}) \geq r \Rightarrow A \cap B_{p+1} \neq (0)$. Thus if $y_p \in B_p$, then

$$|\mathcal{F}_{x_1,\dots,x_r,y_1,\dots,y_p}| \leqslant \sum_{\substack{y \subset B_{p+1} \\ y \text{ 1-dim}}} |\mathcal{F}_{x_1,\dots,x_r,y_1,\dots,y_p,y}|$$
$$\leqslant \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix} \begin{bmatrix} n-r-p \\ k-r-p \end{bmatrix}.$$

Hence

$$|\mathcal{F}_{x_1,\dots,x_r,y_1,\dots,y_{p-1}}| \leqslant \sum_{\substack{y_p \subset B_p \\ y_p \text{ 1-dim}}} |\mathcal{F}_{x_1,\dots,x_r,y_1,\dots,y_{p-i}}|$$
$$\leqslant \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-r-p-1 \\ k-r-p-1 \end{bmatrix} \quad \text{for all } y_{p-1} \in B_{p-1}.$$

Inductively, we obtain that for $i = 0, 1, \dots, p$, and $y_{p-i} \in B_{p-i}$,

$$|\mathcal{F}_{x_1,\dots,x_r,y_1,\dots,y_{p-i}}| \leqslant \begin{bmatrix} k-r+1 \\ r \end{bmatrix}^{i+1} \begin{bmatrix} n-r-p-1 \\ k-r-p-1 \end{bmatrix}.$$

Thus

$$|\mathcal{F}_{x_1,\dots,x_r}| \leqslant \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-r-p-1 \\ k-r-p-1 \end{bmatrix}.$$

Take $y_{p+1} \in B_{p+1}$. If $\dim(A \cap [x_1, \dots, x_r, y_1, \dots, y_{p+1}]) \geqslant r$ for all A in $\mathcal{F}$, then

$$|\mathcal{F}| \leqslant \begin{bmatrix} r+p+1 \\ r \end{bmatrix} \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix}^{p+1} \begin{bmatrix} n-r-p-1 \\ k-r-p-1 \end{bmatrix} < \begin{bmatrix} n-r \\ k-r \end{bmatrix}$$

by Lemma 5.1.

Thus we can suppose that there is some $A_{p+2} \in \mathcal{F}$ such that $\dim(A_{p+2} \cap [x_1, \dots, x_r, y_1, \dots, y_{p+1}]) \leqslant r-1$. Hence there is a $(k-r+1)$-dimensional subspace B_{p+2} of A_{p+2} such that $B_{p+2} \cap [x_1, \dots, x_r, y_1, \dots, y_{p+1}] = (0)$.

We conclude that for all p such that $0 \leqslant p \leqslant k-r-1$, either (i) $|\mathcal{F}| < \left[\begin{smallmatrix} n-r \\ k-r \end{smallmatrix}\right]$ or (ii) $|\mathcal{F}_{x_1,\dots,x_r}| \leqslant \left[\begin{smallmatrix} k-r+1 \\ 1 \end{smallmatrix}\right]^{p+1} \left[\begin{smallmatrix} n-r-p-1 \\ k-r-p-1 \end{smallmatrix}\right]$. Thus either (i) $|\mathcal{F}| < \left[\begin{smallmatrix} n-r \\ k-r \end{smallmatrix}\right]$ or (ii) $|\mathcal{F}_{x_1,\dots,x_r}| \leqslant \left[\begin{smallmatrix} k-r+1 \\ 1 \end{smallmatrix}\right]^{k-r}$. Suppose that (ii) holds. Take any $A \in \mathcal{F}$, we have

$$|\mathcal{F}| \leqslant \sum_{\substack{B \subset A \\ B \text{ } r\text{-dim}}} |\mathcal{F}_B| \leqslant \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix}^{k-r} = \begin{bmatrix} (k-r)+r \\ r \end{bmatrix} \begin{bmatrix} k-r+1 \\ 1 \end{bmatrix}^{k-r}$$
$$< \begin{bmatrix} n-r \\ k-r \end{bmatrix} \quad \text{by Lemma 5.1.}$$

Thus $|\mathcal{F}| < \left[{n-r \atop k-r}\right]$ in all cases.

Remark 5.3. If $n \geqslant 2k$, and $k = r+1$, then $|\mathcal{F}| \leqslant \left[{n-r \atop k-r}\right]$ for all $\mathcal{F} \in S(r,k,n,q)$. In fact, if $A, B \in \mathcal{F}$ such that $\dim(A \cap B) = r = k-1$, then $D \in \mathcal{F} \Rightarrow D \subset \operatorname{span}(A \cup B)$, and so

$$|\mathcal{F}| \leqslant \begin{bmatrix} 2k-r \\ k \end{bmatrix} = \begin{bmatrix} 2k-r \\ k-r \end{bmatrix} \leqslant \begin{bmatrix} n-r \\ k-r \end{bmatrix} .$$

The method in Theorem 5.2 can be applied to the analogous problem for subsets. By $S(r,k,n)$ we denote the set of all families $\mathcal{F}$ of k-element subsets of an n-element set S such that $A, B \in \mathcal{F} \Rightarrow |A \cap B| \geqslant r$. We have the following result:

Theorem 5.4. *If $n \geqslant r + (r+1)(k-r+1)(k-r)$, and $\mathcal{F} \in S(r,k,n)$, then $|\mathcal{F}| \leqslant \binom{n-r}{k-r}$.*

Proof. Checking over the proof for Theorem 5.2, we note that if

$$(*) \qquad \binom{r+p}{r}\binom{k-r+1}{1}^p\binom{n-r-p}{k-r-p} \leqslant \binom{n-r}{k-r} \qquad \text{for } 0 \leqslant p \leqslant k-r ,$$

then $|\mathcal{F}| \leqslant \binom{n-r}{k-r}$.

Now $(n-r-i)/(k-r-i) \geqslant (n-r)/(k-r)$ for $0 \leqslant i \leqslant p-1$. Thus

$$\binom{n-r}{k-r}\Big/\binom{n-r-p}{k-r-p} = \prod_{i=0}^{p-1} \frac{n-r-i}{k-r-i} \geqslant \left(\frac{n-r}{k-r}\right)^p .$$

Also, $\binom{r+p}{r} = \binom{r+p}{p} \leqslant (r+1)^p$. Hence if $n \geqslant r+(r+1)(k-r+1)(k-r)$, then $(n-r)/(k-r) \geqslant (r+1)(k-r+1)$, and so

$$\begin{aligned}\binom{n-r}{k-r} &\geqslant \left(\frac{n-r}{k-r}\right)^p\binom{n-r-p}{k-r-p} \geqslant (r+1)^p\binom{k-r+1}{1}^p\binom{n-r-p}{k-r-p} \\ &\geqslant \binom{r+p}{r}\binom{k-r+1}{1}^p\binom{n-r-p}{k-r-p}\end{aligned}$$

for $0 \leqslant p \leqslant k-r$, which is $(*)$.

Remark 5.5. [1, Theorem 2] states that if $n \geqslant r+(k-r)\binom{k}{r}^3$ and $\mathcal{F} \in S(r,k,n)$, then $|\mathcal{F}| \leqslant \binom{n-r}{k-r}$. Now $\binom{k}{r}^3 \geqslant \binom{k}{1}^3 = k^3 \geqslant (r+1)(k-r+1)$. Thus our result is a considerable improvement over that of Erdös–Ko–Rado's. It seems that the bound on n could be improved further.

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