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Regularized and inertial algorithms for common fixed points of nonlinear operators

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Abstract

This paper deals with a general fixed point iteration for computing a point in some nonempty closed and convex solution set included in the common fixed point set of a sequence of mappings on a real Hilbert space. The proposed method combines two strategies: viscosity approximations (regularization) and inertial type extrapolation. The first strategy is known to ensure the strong convergence of some successive approximation methods, while the second one is intended to speed up the convergence process. Under classical conditions on the operators and the parameters, we prove that the sequence of iterates generated by our scheme converges strongly to the element of minimal norm in the solution set. This algorithm works, for instance, for approximating common fixed points of infinite families of demicontractive mappings, including the classes of quasi-nonexpansive operators and strictly pseudocontractive ones.

Keywords: Demicontractive; Quasi-nonexpansive; Pseudocontractive; Fixed point; Strongly convergent method; Viscosity method; Inertial type extrapolation; Proximal method

1. Introduction

Throughout, $\mathcal{H}$ is a real Hilbert space endowed with inner product $\langle .,. \rangle$ and induced norm $|.|$. For any mapping $T : \mathcal{H} \to \mathcal{H}$, we denote by $\mathrm{Fix}(T)$ the set of fixed points of T, that is $\mathrm{Fix}(T) := \{x \in \mathcal{H} \mid Tx = x\}$. In this paper, we are interested in solving (common) fixed point problems regarding operators such as quasi-nonexpansive, strictly pseudocontractive, or more general ones. Let us recall that a mapping $T : \mathcal{H} \to \mathcal{H}$ is called:

(i) *quasi-nonexpansive* if $|Tx - q| \leqslant |x - q|$ for all $(x, q) \in \mathcal{H} \times \mathrm{Fix}(T)$;
(ii) *strictly pseudocontractive* if there exists a constant $\rho \in [0, 1)$ such that $|Tx - Ty|^2 \leqslant |x - y|^2 + \rho|x - y - (Tx - Ty)|^2$ for all $(x, y) \in \mathcal{H} \times \mathcal{H}$;
(iii) *demicontractive* (see, e.g., [24,25]), if there exists a constant $k \in [0, 1)$ such that

$$|Tx - q|^2 \leqslant |x - q|^2 + k|x - Tx|^2, \quad \forall (x, q) \in \mathcal{H} \times \mathrm{Fix}(T), \tag{1.1}$$

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which in the light of (2.9) can be equivalently written as

$$\langle x - Tx, x - q\rangle \geqslant \frac{1-k}{2}|x - Tx|^2, \quad \forall (x,q) \in \mathcal{H} \times \mathrm{Fix}(T). \tag{1.2}$$

A mapping satisfying (1.1) or (1.2) will be called k-demicontractive and we denote by $\mathcal{D}_k$ the set of k-demicontractive operators. Fixed point problems plays an important role in nonlinear analysis and optimization. In the setting of Hilbert or Banach spaces, several strongly convergent methods were proposed:

(i) Viscosity methods of Halpern's type [16] for nonexpansive maps [6,9,18,21,26,30,31,33];
(ii) The hybrid steepest descent method for certain quasi-nonexpansive mappings called quasi-shrinking mappings [34];
(iii) Outer approximation methods for certain quasi-nonexpansive mappings called firmly quasi-nonexpansive mappings [7];
(iv) A Mann-type iteration [23] for strictly pseudocontractive maps [19] (see also [10,11]).

Let us emphasize that the class of quasi-nonexpansive mappings is independent of the class of strictly pseudocontractive mappings, but the two of them include the extensively studied class of nonexpansive mappings. Such operators are most difficult for research in the fixed point theory and at the same time most interesting for applications. Other iterative schemes were proposed for approximating fixed points of (special) quasi-nonexpansive or demicontractive maps [14,24,25,29]. Interesting weak convergence results are obtained, but the convergence in norm of the iterates are established under very restrictive conditions regarding for instance the considered operators or the space (demicompactness, continuity, compactness).

It is our purpose to propose a strongly convergent method for approximating (common) fixed points of demicontractive maps, only with classical conditions. It is obviously observed that the class of demicontractive operators contains the classes of quasi-nonexpansive operators and strictly pseudocontractive ones with fixed points. In view of applications, we recall that $\mathcal{D}_0$ contains the class of firmly quasi-nonexpansive maps, including subgradient projection operators which attracts great attention and occurs for instance in signal and image processing [7,12,13,34–36]. In a more general frame, our attention will be focused on a formalism which consists in finding a point in S, a nonempty subset of $\mathcal{H}$, relatively to $(\mathcal{T}_n)$, a sequence of mappings on $\mathcal{H}$, with the following conditions:

(C0) S is a closed and convex subset of $\mathcal{H}$.
(C1) $(\mathcal{T}_n)_{n\geqslant 0} \subset \mathcal{D}_k$, where $k \in [0,1)$.
(C2) $\forall n \geqslant 0$, $S \subset \mathrm{Fix}(\mathcal{T}_n)$.
(C3) For any subsequence $(\mathcal{T}_{n_j})$ of $(\mathcal{T}_n)$, for $(\xi_{n_j}) \subset \mathcal{H}$, for $\xi \in \mathcal{H}$,

$$(\xi_{n_j}) \to \xi \text{ weakly} \quad \text{and} \quad \xi_{n_j} - \mathcal{T}_{n_j}\xi_{n_j} \to 0 \text{ strongly} \quad \Rightarrow \quad \xi \in S.$$

To this end, we examine the following iteration method

$$\left[\begin{array}{l} x_{n+1} := (1-w)v_n + w\mathcal{T}_n v_n, \quad v_n = (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}), \\ \quad x_0, x_1 \in \mathcal{H},\ w \in [0,1),\ (\theta_n) \text{ and } (\alpha_n) \text{ are sequences in } [0,1). \end{array}\right. \tag{1.3}$$

Let us recall that (1.3) is called *inertial*, because of the term $\theta_n(x_n - x_{n-1})$. This algorithm is based upon a discrete version of a second order dissipative dynamical system [4,5] and can be regarded as a procedure of speeding up the convergence properties (see, e.g., [3,28]). It is worth mentioning that the scheme (1.3) was considered in [22], in the special case when $\alpha_n \equiv 0$ and $w = 1$, for solving the above formalism without (C0) and with the condition (C3) replaced by

(C3′) For any $(\xi_n) \subset H$, for $\xi \in H$,

$$\xi \text{ is a weak cluster point of } (\xi_n) \quad \text{and} \quad \xi_n - \mathcal{T}_n\xi_n \to 0 \text{ strongly} \quad \Rightarrow \quad \xi \in S.$$

This latter condition is slightly weaker than (C3), but only weak convergence results were established for a wide class of operators which includes α-averaged quasi-nonexpansive maps. Recall that the condition (C3) and (C3′)

can be regarded as sorts of demi-closedness of the sequence $(\mathcal{T}_n)$ which reduces to the classical demi-closedness property [15] when $\mathcal{T}_n$ is a constant sequence. Let us mention some typical examples covered by our formalism:

(1) The first one is related to computing zeroes of a maximal monotone set-valued mapping $A : \mathcal{H} \to \mathcal{P}(\mathcal{H})$. The conditions (C0)–(C3) are satisfied with $S = A^{-1}(0)$ and $\mathcal{T}_n = J^A_{\lambda_n}$, where $(\lambda_n) \subset (\lambda, +\infty)$ (for some positive λ) and $J^A_{\lambda_n} := (I + \lambda_n A)^{-1}$ is the resolvent of A of parameter λ_n (see, e.g., [8,20] and the references therein for details on proximal methods). Indeed, (C0) holds because $J^A_{\lambda_n}$ is well known to be nonexpansive, (C1) is satisfied with $k = 0$, (C2) holds since $\mathrm{Fix}(J^A_{\lambda_n}) = A^{-1}(0)$ and (C3) is deduced from the fact that the graph of a maximal monotone mapping is weakly–strongly closed (see, for instance, [8]). Let us mention that the algorithm (1.3) was studied in the special case when $\alpha_n \equiv 0$ and $w = 1$, by Alvarez and Attouch [3] (see also [2,17,27]).

(2) The second one consists in approximating a common fixed point of finitely many maps $(T_i)_{i=0}^N \subset \mathcal{D}_k$ such that each T_i is demi-closed and $\bigcap_{i=0}^N \mathrm{Fix}(T_i) \neq \emptyset$. Letting $U = \sum_{i=0}^N w_i T_i$ where $(w_i)_{i=0}^N \subset (0,1]$ is such that $\sum_{i=0}^N w_i = 1$, we will prove that the conditions (C0)–(C3) hold with $\mathcal{T}_n = U$ and $S = \bigcap_{i=0}^N \mathrm{Fix}(T_i)$ (see Theorem 4.2).

(3) The third one is concerned with the numerical approach to a common fixed point of infinitely many maps $(T_i)_{i\geqslant 0} \subset \mathcal{D}_k$ satisfying $\bigcap_{i\geqslant 0} \mathrm{Fix}(T_i) \neq \emptyset$. Let us consider the special case of our formalism with $\mathcal{T}_n = \sum_{i=0}^n w_{i,n} T_i$ and $S = \bigcap_{i\geqslant 0} \mathrm{Fix}(T_i)$, where each T_i is demi-closed and $(w_{i,n}) \subset [0, +\infty)$ are real numbers such that:

(i) $\forall n \geqslant 0$, $\sum_{i=0}^n w_{i,n} = 1$;
(ii) $\forall i \geqslant 0$, $(w_{i,n})_{n\geqslant 0}$ is bounded away from zero for n large enough (that is: $\forall i \geqslant 0$, $\exists N_i \in \mathbb{N}$ and $\exists w_i > 0$ such that $\forall n \geqslant N_i$, $w_{i,n} \geqslant w_i$).

In this context, we will prove that (C0)–(C3) hold (see Theorem 4.3).

Under the conditions (C0)–(C3) and other suitable conditions on the parameters (w), (α_n) and (θ_n), we prove that the sequence (x_n) generated, with arbitraries x_0 and x_1 in $\mathcal{H}$, by (1.3) converges strongly to $P_S(0)$ where P_S is the metric projection from $\mathcal{H}$ onto S. To this end, for the convenience reader we enumerate the main assumptions used through the rest of the paper:

(H1) (α_n) is a non-increasing sequence in $[0, 1)$.
(H2) $w \in (0, \frac{1-k}{2}]$ (k being the constant occurring in (1.1)).
(H3) $\alpha_n \to 0$.
(H4) (θ_n) is a non-decreasing sequence in $[0, \theta]$, where $\theta \in [0, \frac{1}{3})$.
(SP) $\sum_{n\geqslant 0} \alpha_n = +\infty$ (slow parametrization).

2. Preliminaries

The next lemmas are needed to state our convergence results.

Lemma 2.1. *Let $(\mathcal{T}_n)$ and $S \neq \emptyset$ satisfy the assumptions* (C1)–(C2) *and suppose the conditions* (H1) *and* (H4) *hold. Then the sequence (x_n) given by* (1.3) *satisfies for all $n \geqslant 1$,*

$$P_{n+1} - P_n + (1 - 3\theta_{n+1} - \alpha_n)d_n + \frac{\rho - 1}{2}|x_{n+1} - v_n|^2 \leqslant -\alpha_n \langle x_n, x_n - q\rangle, \tag{2.1}$$

where q is any element in S, $\rho := \frac{1}{w}(1 - k - w)$ and P_n is defined by

$$P_n := \phi_n - \theta_{n-1}\phi_{n-1} + 2\theta_n d_{n-1} + \frac{1}{2}\alpha_n |x_n|^2, \tag{2.2}$$

with $\phi_j := \frac{1}{2}|x_j - q|^2$ and $d_j := \frac{1}{2}|x_{j+1} - x_j|^2$.

Proof. Given any q in S, by (1.3) we have

$$x_{n+1} - q = (v_n - q) - w(v_n - \mathcal{T}_n v_n),$$

hence

$$|x_{n+1}-q|^2 = |v_n-q|^2 + w^2|v_n - \mathcal{T}_n v_n|^2 - 2w\langle v_n - \mathcal{T}_n v_n, v_n - q\rangle,$$

which by the demicontractivity condition (1.2) yields

$$|x_{n+1}-q|^2 \leqslant |v_n-q|^2 - w(1-k-w)|v_n - \mathcal{T}_n v_n|^2. \tag{2.3}$$

By (1.3) we also have $\mathcal{T}_n v_n - v_n = \frac{1}{w}(x_{n+1}-v_n)$. Setting $\rho := \frac{1}{w}(1-k-w)$, by (2.3) we then obtain

$$|x_{n+1}-q|^2 \leqslant |v_n-q|^2 - \rho|x_{n+1}-v_n|^2,$$

or equivalently

$$|x_{n+1}-q|^2 + (\rho-1)|v_n-x_{n+1}|^2 \leqslant |v_n-q|^2 - |v_n-x_{n+1}|^2. \tag{2.4}$$

Let us estimate separately each term in the right-hand side of the previous inequality. Concerning the first term, we have

$$\begin{aligned}|v_n-q|^2 &= \left|(x_n-q)+\theta_n(x_n-x_{n-1})-\alpha_n x_n\right|^2\\ &= |x_n-q|^2 + \left|\theta_n(x_n-x_{n-1})-\alpha_n x_n\right|^2 + 2\left\langle x_n-q, \theta_n(x_n-x_{n-1})-\alpha_n x_n\right\rangle,\end{aligned}$$

that is,

$$|v_n-q|^2 = |x_n-q|^2 + 2\theta_n\langle x_n-q, x_n-x_{n-1}\rangle - 2\alpha_n\langle x_n-q, x_n\rangle + \left|\theta_n(x_n-x_{n-1})-\alpha_n x_n\right|^2. \tag{2.5}$$

Concerning the second term, we immediately obtain

$$\begin{aligned}|v_n-x_{n+1}|^2 &= |x_n-x_{n+1}|^2 + 2\theta_n\langle x_n-x_{n+1}, x_n-x_{n-1}\rangle - 2\alpha_n\langle x_n-x_{n+1}, x_n\rangle\\ &\quad + \left|\theta_n(x_n-x_{n-1})-\alpha_n x_n\right|^2.\end{aligned} \tag{2.6}$$

As a consequence, by (2.4)–(2.6) we get

$$\begin{aligned}|x_{n+1}-q|^2 + (\rho-1)|v_n-x_{n+1}|^2 &\leqslant |x_n-q|^2 + 2\theta_n\langle x_n-q, x_n-x_{n-1}\rangle - 2\alpha_n\langle x_n-q, x_n\rangle - |x_n-x_{n+1}|^2\\ &\quad - 2\theta_n\langle x_n-x_{n+1}, x_n-x_{n-1}\rangle + 2\alpha_n\langle x_n-x_{n+1}, x_n\rangle.\end{aligned} \tag{2.7}$$

Using Young's inequality, we have

$$\langle x_n-x_{n+1}, x_n-x_{n-1}\rangle \geqslant -\frac{1}{2}|x_n-x_{n+1}|^2 - \frac{1}{2}|x_n-x_{n-1}|^2,$$

which by (2.7) yields

$$\begin{aligned}&|x_{n+1}-q|^2 - |x_n-q|^2 - \theta_n|x_n-x_{n-1}|^2 + (1-\theta_n)|x_n-x_{n+1}|^2 + (\rho-1)|v_n-x_{n+1}|^2\\ &\quad \leqslant -2\alpha_n\langle x_n-q, x_n\rangle + 2\theta_n\langle x_n-q, x_n-x_{n-1}\rangle + 2\alpha_n\langle x_n-x_{n+1}, x_n\rangle.\end{aligned} \tag{2.8}$$

Furthermore, for any $a, b \in \mathcal{H}$, it is easily checked that

$$\langle a, b\rangle = -\frac{1}{2}|a-b|^2 + \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2, \tag{2.9}$$

so that (2.8) can be equivalently rewritten as

$$\begin{aligned}&|x_{n+1}-q|^2 - |x_n-q|^2 - \theta_n|x_n-x_{n-1}|^2 + (1-\theta_n)|x_n-x_{n+1}|^2 + (\rho-1)|v_n-x_{n+1}|^2\\ &\quad \leqslant -2\alpha_n\langle x_n-q, x_n\rangle + 2\theta_n\left(-\frac{1}{2}|x_{n-1}-q|^2 + \frac{1}{2}|x_n-q|^2 + \frac{1}{2}|x_n-x_{n-1}|^2\right)\\ &\qquad + 2\alpha_n\left(-\frac{1}{2}|x_{n+1}|^2 + \frac{1}{2}|x_{n+1}-x_n|^2 + \frac{1}{2}|x_n|^2\right),\end{aligned} \tag{2.10}$$

that is

$$\begin{aligned}&|x_{n+1}-q|^2-|x_n-q|^2+2\theta_{n+1}|x_{n+1}-x_n|^2-2\theta_n|x_n-x_{n-1}|^2+\theta_n\big(|x_{n-1}-q|^2-|x_n-q|^2\big)\\&\quad+\alpha_n\big(|x_{n+1}|^2-|x_n|^2\big)+(1-\theta_n-2\theta_{n+1}-\alpha_n)|x_n-x_{n+1}|^2+(\rho-1)|v_n-x_{n+1}|^2\\&\quad\leqslant-2\alpha_n\langle x_n-q,x_n\rangle.\end{aligned}\tag{2.11}$$

Assuming (θ_n) is non-decreasing and (α_n) is non-increasing, we deduce

$$\begin{aligned}&|x_{n+1}-q|^2-|x_n-q|^2+2\theta_{n+1}|x_{n+1}-x_n|^2-2\theta_n|x_n-x_{n-1}|^2+\alpha_{n+1}|x_{n+1}|^2-\alpha_n|x_n|^2-\theta_n|x_n-q|^2\\&\quad+\theta_{n-1}|x_{n-1}-q|^2+(1-3\theta_{n+1}-\alpha_n)|x_n-x_{n+1}|^2+(\rho-1)|v_n-x_{n+1}|^2\\&\quad\leqslant-2\alpha_n\langle x_n-q,x_n\rangle,\end{aligned}\tag{2.12}$$

namely

$$\begin{aligned}&\phi_{n+1}-\phi_n+2\theta_{n+1}d_n-2\theta_nd_{n-1}-\theta_n\phi_n+\theta_{n-1}\phi_{n-1}+\frac{1}{2}\alpha_{n+1}|x_{n+1}|^2-\frac{1}{2}\alpha_n|x_n|^2\\&\quad+(1-3\theta_{n+1}-\alpha_n)d_n+\frac{(\rho-1)}{2}|v_n-x_{n+1}|^2\\&\quad\leqslant-\alpha_n\langle x_n-q,x_n\rangle,\end{aligned}\tag{2.13}$$

that is the desired result. □

Lemma 2.2. *Let* $(\mathcal{T}_n)$ *and* $S\neq\emptyset$ *satisfy the assumptions* (C1)–(C2) *and suppose the conditions* (H1)–(H4) *hold. Then there exist some integer* n_0 *and a positive constant* γ *such that for any* q *in* S*, the sequence* (x_n) *given by* (1.3) *satisfies for* $n\geqslant n_0+1$,

$$\Gamma_{n+1}-\Gamma_n+\gamma\mu_{n+1}d_n\leqslant\frac{1}{2}\mu_{n+1}\alpha_n|q|^2,$$

(Γ_n) *being defined by*

$$\Gamma_n:=\mu_n\phi_n+2\mu_n\theta_ne^{\alpha_n}d_{n-1}-\mu_n\theta_{n-1}\phi_{n-1},$$

where $\phi_j:=\frac{1}{2}|x_j-q|^2$, $d_j:=\frac{1}{2}|x_{j+1}-x_j|^2$ *and* $\mu_j:=\exp(\sum_{i=0}^{j}\alpha_i)$.

Proof. By (H2) we have $\rho:=\frac{1}{w}(1-k-w)\geqslant 1$, which by Lemma 2.1 entails

$$\begin{aligned}\phi_{n+1}-\phi_n\leqslant{}&\theta_n\phi_n-\theta_{n-1}\phi_{n-1}-(1-3\theta_{n+1}-\alpha_n)d_n-2\theta_{n+1}d_n+2\theta_nd_{n-1}\\&-\frac{1}{2}\alpha_{n+1}|x_{n+1}|^2+\frac{1}{2}\alpha_n|x_n|^2-\alpha_n\langle x_n,x_n-q\rangle.\end{aligned}$$

In this inequality, it is easily seen that

$$\langle x_n,x_n-q\rangle=-\frac{1}{2}|q|^2+\frac{1}{2}|x_n|^2+\phi_n,$$

hence

$$\phi_{n+1}-\phi_n+\alpha_n\phi_n\leqslant\theta_n\phi_n-\theta_{n-1}\phi_{n-1}-(1-3\theta_{n+1}-\alpha_n)d_n-2\theta_{n+1}d_n+2\theta_nd_{n-1}+\frac{1}{2}\alpha_n|q|^2.\tag{2.14}$$

By a simple calculation we obtain

$$\frac{1}{\mu_{n+1}}(\mu_{n+1}\phi_{n+1}-\mu_n\phi_n)=\phi_{n+1}-\phi_n+\frac{1}{\mu_{n+1}}(\mu_{n+1}-\mu_n)\phi_n\leqslant\phi_{n+1}-\phi_n+\alpha_{n+1}\phi_n,$$

which by (H1) yields

$$\frac{1}{\mu_{n+1}}(\mu_{n+1}\phi_{n+1}-\mu_n\phi_n)\leqslant\phi_{n+1}-\phi_n+\alpha_n\phi_n.\tag{2.15}$$

Thanks to (2.14) and (2.15), we deduce that

$$\frac{1}{\mu_{n+1}}(\mu_{n+1}\phi_{n+1}-\mu_n\phi_n) \leqslant \theta_n\phi_n-\theta_{n-1}\phi_{n-1}-(1-3\theta_{n+1}-\alpha_n)d_n-2\theta_{n+1}d_n + 2\theta_n d_{n-1}+\frac{1}{2}\alpha_n|q|^2. \tag{2.16}$$

As $\mu_n \leqslant \mu_{n+1}$ and $\mu_{n+1}=\mu_n e^{\alpha_{n+1}} \leqslant \mu_n e^{\alpha_n}$ (again with (H1)), we then get

$$\mu_{n+1}\phi_{n+1}-\mu_n\phi_n \leqslant \mu_{n+1}\phi_n\theta_n-\mu_n\phi_{n-1}\theta_{n-1}-\mu_{n+1}(1-3\theta_{n+1}-\alpha_n)d_n-2\mu_{n+1}\theta_{n+1}d_n + 2\mu_n\theta_n e^{\alpha_n}d_{n-1}+\frac{1}{2}\mu_{n+1}\alpha_n|q|^2,$$

namely

$$\mu_{n+1}\phi_{n+1}-\mu_n\phi_n \leqslant \mu_{n+1}\phi_n\theta_n-\mu_n\phi_{n-1}\theta_{n-1}-\mu_{n+1}\big(1-\theta_{n+1}\big(3+2\big(e^{\alpha_{n+1}}-1\big)\big)-\alpha_n\big)d_n - 2\mu_{n+1}\theta_{n+1}e^{\alpha_{n+1}}d_n+2\mu_n\theta_n e^{\alpha_n}d_{n-1}+\frac{1}{2}\mu_{n+1}\alpha_n|q|^2.$$

By (H4), recalling that $\theta_n \in [0,\theta]$ where $\theta \in [0,1/3)$, we have

$$1-\theta_{n+1}\big(3+2\big(e^{\alpha_{n+1}}-1\big)\big)-\alpha_n \geqslant 1-\theta\big(3+2\big(e^{\alpha_{n+1}}-1\big)\big)-\alpha_n.$$

Clearly, for n large enough ($n \geqslant n_0$), it is immediate that there exists a positive constant γ such that

$$1-\theta_{n+1}\big(3+2\big(e^{\alpha_{n+1}}-1\big)\big)-\alpha_n \geqslant \gamma,$$

because $(\alpha_n) \to 0$ by (H3), hence

$$\mu_{n+1}\phi_{n+1}-\mu_n\phi_n \leqslant \mu_{n+1}\phi_n\theta_n-\mu_n\phi_{n-1}\theta_{n-1}-\gamma\mu_{n+1}d_n-2\mu_{n+1}\theta_{n+1}e^{\alpha_{n+1}}d_n+2\mu_n\theta_n e^{\alpha_n}d_{n-1} + \frac{1}{2}\mu_{n+1}\alpha_n|q|^2,$$

which leads to the desired result. □

Lemma 2.3. *Let $(\mathcal{T}_n)$ and $S \neq \emptyset$ satisfy the assumptions* (C1)–(C2) *and suppose the conditions* (H1)–(H4) *hold. Then the sequence (x_n) generated by* (1.3) *is bounded.*

Proof. According to Lemma 2.2, we have for $n \geqslant n_0+1$,

$$\Gamma_{n+1}-\Gamma_{n_0} \leqslant \frac{1}{2}|q|^2\sum_{k=n_0+1}^{n}\mu_{k+1}\alpha_k, \tag{2.17}$$

where $\Gamma_{n+1} := \mu_{n+1}\phi_{n+1}+2\mu_{n+1}\theta_{n+1}e^{\alpha_{n+1}}d_n-\mu_{n+1}\theta_n\phi_n$, hence $\mu_{n+1}(\phi_{n+1}-\theta_n\phi_n) \leqslant \Gamma_{n+1}$, which by (2.17) yields

$$\phi_{n+1}-\theta_n\phi_n \leqslant e^{-t_{n+1}}\Gamma_{n_0}+\frac{1}{2}|q|^2e^{-t_{n+1}}\sum_{k=n_0+1}^{n}\alpha_k e^{t_{k+1}}, \tag{2.18}$$

where $t_n := \sum_{i=0}^{n}\alpha_i$. It is easily checked that $\alpha_k e^{t_{k+1}} \leqslant e^2(e^{t_k}-e^{t_{k-1}})$ (for all $k \geqslant 1$), so that $\sum_{k=n_0+1}^{n}\mu_{k+1}\alpha_k \leqslant e^2e^{t_n}$, which by (2.18) and (H4) leads to

$$\phi_{n+1} \leqslant \theta\phi_n+\left(\Gamma_{n_0}+\frac{1}{2}e^2|q|^2\right).$$

Omitting the details calculation and since $\theta \in [0,1)$, we deduce that

$$\phi_{n+1} \leqslant \theta^{n-n_0}\phi_{n_0+1}+\frac{1}{1-\theta}\left(\Gamma_{n_0}+\frac{1}{2}e^2|q|^2\right),$$

which proves the boundedness of (x_n). □

3. Strong convergence results

This section is devoted to the strong convergence of the sequence generated by (1.3). Under very classical conditions, we prove that (x_n) converges strongly to $P_S(0)$ where P_S is metric projection from $\mathcal{H}$ onto S. The following lemmas are useful to prove our main convergence result.

Lemma 3.1. *Let $\mathcal{T}_n : \mathcal{H} \to \mathcal{H}$ and $S \neq \emptyset$ satisfy the assumptions* (C2)–(C3). *Suppose the condition* (H3) *holds and assume the sequence (x_n) given by* (1.3) *is bounded and satisfies $|x_{n+1} - x_n| \to 0$. Then any weak cluster point of (x_n) is in S. If in addition the condition* (C0) *holds, we have*

$$\liminf_{n\to\infty} \langle x_n - x_\infty, x_\infty\rangle \geqslant 0, \tag{3.1}$$

where x_∞ is the element of minimal norm in S (that is $x_\infty := P_S(0)$).

Proof. Let (x_{n_k}) be a subsequence of (x_n) which converges weakly to an element u in $\mathcal{H}$. Assuming $|x_{n+1} - x_n| \to 0$, $\alpha_n \to 0$ and (x_n) is bounded, we easily deduce that (v_{n_k}) converges weakly to u (since $v_n := x_n + \theta_n(x_n - x_{n-1})$) and by (1.3) we have $|\mathcal{T}_n v_n - v_n| = \frac{1}{w}|x_{n+1} - v_n| \to 0$. By (C3), we then obtain $u \in S$, so that the set of weak cluster points of (x_n) is included in S. As (x_n) is assumed to be a bounded sequence, so does the quantity $\langle x_n - q, q\rangle$. It is then a simple matter to check that there exists a subsequence of (x_n) (labeled (x_{m_k})) which converges weakly to some element u_* in $\mathcal{H}$ (hence $u_* \in S$) and such that $\liminf_{n\to\infty}\langle x_n - x_\infty, x_\infty\rangle = \lim_{k\to\infty}\langle x_{m_k} - x_\infty, x_\infty\rangle$, hence $\liminf_{n\to\infty}\langle x_n - x_\infty, x_\infty\rangle = \langle u_* - x_\infty, x_\infty\rangle$. Reminding that $x_\infty := P_S(0)$ and $u_* \in S$, we necessarily have $\langle u_* - x_\infty, x_\infty\rangle \geqslant 0$, which ends the proof. □

Lemma 3.2. *Let $\mathcal{T}_n : \mathcal{H} \to \mathcal{H}$ and $S \neq \emptyset$ satisfy the assumptions* (C0) *and* (C2)–(C3). *Assume* (H3) *holds and suppose the sequence (x_n) generated by* (1.3) *has a subsequence (x_{n_k}) such that*:

(i) $(x_{n_k}) \subset \Omega := \{x \in \mathcal{H};\ \langle x - x_\infty, x\rangle \leqslant 0\}$, *where* $x_\infty := P_S(0)$.
(ii) $|x_{n_k+1} - x_{n_k}| \to 0$ *as* $k \to \infty$.
(iii) $\theta_{n_k}|x_{n_k} - x_{n_k-1}| \to 0$ *as* $k \to \infty$.

Then (x_{n_k}) converges strongly to x_∞.

Proof. It is easily checked that Ω is the closed ball of center $\frac{1}{2}x_\infty$ and radius $\frac{1}{2}|x_\infty|$, that is $\Omega = \{x \in \mathcal{H};\ |x - \frac{1}{2}x_\infty| \leqslant \frac{1}{2}|x_\infty|\}$, hence Ω is a nonempty bounded, closed and convex set. Clearly, by the condition (i), we have $(x_{n_k}) \subset \Omega$. Consequently, by extracting from (x_{n_k}) a subsequence (again labeled (x_{n_k})) which converges weakly to some q in $\mathcal{H}$, we also have $|x_{n_k} - x_{n_k+1}| \to 0$ as $k \to \infty$ and $\theta_{n_k}|x_{n_k} - x_{n_k-1}| \to 0$ (by the condition (i) and (ii)). As Ω is a closed and convex set, it is then weakly closed, so that q belongs to Ω. Moreover, by (1.3) we have $|v_{n_k} - \mathcal{T}_{n_k}v_{n_k}| = \frac{1}{w}|x_{n_k+1} - v_{n_k}| \to 0$, since $v_{n_k} = (1 - \alpha_{n_k})x_{n_k} + \theta_{n_k}(x_{n_k} - x_{n_k-1})$, (x_{n_k}) is bounded and $\alpha_{n_k} \to 0$. Furthermore, it is obvious that (v_{n_k}) converges weakly to q. By (C3) we then obtain $q \in S$, so that $q \in \Omega \cap S = \{x_\infty\}$, hence $q = x_\infty$. Moreover, we have

$$|x_{n_k} - x_\infty|^2 = \langle x_{n_k}, x_{n_k} - x_\infty\rangle - \langle x_\infty, x_{n_k} - x_\infty\rangle,$$

hence $|x_{n_k} - x_\infty|^2 \leqslant -\langle x_\infty, x_{n_k} - x_\infty\rangle$, since $(x_{n_k}) \subset \Omega$. Passing to the limit in this last inequality yields $|x_{n_k} - x_\infty| \to 0$. It is then immediate that (x_{n_k}) converges strongly to x_∞, which ends the proof. □

Lemma 3.3. *Let $\mathcal{T}_n : \mathcal{H} \to \mathcal{H}$ and $S \neq \emptyset$ satisfy the assumptions* (C0)–(C3). *Assume the conditions* (H1)–(H4) *and* (SP) *hold and suppose furthermore the sequence (x_n) given by* (1.3) *satisfies*:

(i) $|x_{n+1} - x_n| \to 0$.
(ii) $\lim_{n\to\infty} |x_{n+1} - x_\infty|^2 - \theta_n|x_n - x_\infty|^2$ *exists* (*where* $x_\infty := P_S(0)$).

Then (x_n) converges strongly to x_∞.

Proof. To begin with, we observe that (x_n) is a bounded sequence (see Lemma 2.3). Let us suppose in addition that the quantity $|x_{n+1} - x_\infty|^2 - \theta_n |x_n - x_\infty|^2$ converges to some $\lambda > 0$. According to Lemma 3.1, we also have $\liminf_{n\to\infty} \langle x_n - x_\infty, x_\infty\rangle \geqslant 0$. As a consequence, noting that

$$\langle x_n - x_\infty, x_n\rangle = |x_n - x_\infty|^2 + \langle x_n - x_\infty, x_\infty\rangle,$$

we obtain $\liminf_{n\to\infty} \langle x_n - x_\infty, x_n\rangle \geqslant \lambda$. It is easily deduced from Lemma 2.1 that there exists $n_4 \geqslant 0$ such that for $n \geqslant n_4$, $P_{n+1} - P_n \leqslant -\alpha_n(\lambda)$ (since $\rho \geqslant 1$ by (H2) and since $1 - 3\theta_{n+1} - \alpha_n \geqslant 0$ by (H3) and (H4) (for n large enough)), which yields $\lambda \sum_{k=n_4}^{n} \alpha_k \leqslant P_{n_4} - P_{n+1}$, $\forall n \geqslant n_4$. Clearly, if $\sum \alpha_n = \infty$ (SP), this last inequality is absurd as $n \to \infty$, because its left-hand side tends to $+\infty$, while the right-hand side is supposed to be bounded (because (x_n) is bounded). We conclude that $\lambda = 0$, which by (H4) and by an easy computation leads to the desired result. □

At once, we claim the main result of this section.

Theorem 3.4. *Let $\mathcal{T}_n : \mathcal{H} \to \mathcal{H}$ and $S \neq \emptyset$ satisfy the assumptions* (C0)–(C3). *Assume the following conditions hold*: (H1), (H3)–(H4), (SP) *and*

(H2)′ $0 < w < \frac{1-k}{2}$.

Then the sequence (x_n) given by (1.3) *converges strongly to $x_\infty := P_S(0)$, where P_S is the metric projection from $\mathcal{H}$ onto S.*

Proof. Clearly, (x_n) is bounded (by Lemma 2.3), so that there exists a positive constant C such that $|\langle x_n, x_n - x_\infty\rangle| \leqslant C$ for all $n \geqslant 0$. Moreover, by (H3) and (H4) there exists a positive constant γ such that $1 - 3\theta_{n+1} - \alpha \geqslant \gamma$ for all $n \geqslant m_0$, where m_0 is some large enough integer. Consequently, by Lemma 2.1 we get for $m \geqslant m_0$,

$$Q_{n+1} - Q_n + \gamma |x_{n+1} - x_n|^2 + \frac{\rho - 1}{2} |x_{n+1} - v_n|^2 \leqslant C\alpha_n \tag{3.2}$$

where $Q_n := \phi_n - \theta_{n-1}\phi_{n-1} + 2\theta_n d_{n-1} + \frac{1}{2}\alpha_n |x_n|^2$, with $\phi_j := \frac{1}{2}|x_j - x_\infty|^2$ and $d_j := \frac{1}{2}|x_{j+1} - x_j|^2$ and $\rho := \frac{1}{w}(1 - k - w)$, hence $\rho > 1$ (by the condition (H2)′). The rest of the proof is divided into two parts:

(1) Assume (Q_n) is a monotonous sequence (that is, for some n_0 large enough, $(Q_n)_{n\geqslant n_0}$ is either non-decreasing or non-increasing). It is then immediate that (Q_n) is convergent, hence $Q_{n+1} - Q_n \to 0$ which by (3.2) yields $|x_{n+1} - x_n| \to 0$. Moreover, it is easily observed that

$$\lim_{n\to+\infty} Q_n = \lim_{n\to+\infty} (\phi_n - \theta_{n-1}\phi_{n-1}),$$

so that $\lim_{n\to+\infty} |x_{n+1} - x_\infty|^2 - \theta_n |x_n - x_\infty|^2$ exists. As a consequence, applying Lemma 3.3 we deduce $\lim_{n\to\infty} |x_n - x_\infty| = 0$.

(2) Assume (Q_n) is not a monotonous sequence and let $\tau : \mathbb{N} \to \mathbb{N}$ be the map defined for all $n \geqslant n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N};\ k \leqslant n,\ Q_k \leqslant Q_{k+1}\}. \tag{3.3}$$

Clearly, $\tau(n)$ is a non-decreasing sequence such that $\tau(n) \to +\infty$ (as $n \to +\infty$) and $Q_{\tau(n)} \leqslant Q_{\tau(n)+1}$ (for $n \geqslant n_0$), which by (3.2) entails

$$\gamma |x_{\tau(n)+1} - x_{\tau(n)}|^2 + (\rho - 1)|x_{\tau(n)+1} - v_{\tau(n)}|^2 \leqslant C\alpha_{\tau(n)} \to 0.$$

It is then easily deduced that $|x_{\tau(n)+1} - x_{\tau(n)}| \to 0$ and $\theta_{\tau(n)}|x_{\tau(n)} - x_{\tau(n)-1}| \to 0$ (since $v_n := x_n + \theta_n(x_n - x_{n-1})$). Note also that for any $j \geqslant 0$ (by Lemma 2.1), we have $Q_{j+1} < Q_j$ when $x_j \notin \Omega := \{x \in \mathcal{H};\ \langle x - x_\infty, x\rangle \leqslant 0\}$, hence $x_{\tau(n)} \in \Omega$ for all $n \geqslant n_0$ (since $Q_{\tau(n)} \leqslant Q_{\tau(n)+1}$). Consequently, by Lemma 3.2 we deduce $|x_{\tau(n)} - x_\infty| \to 0$ and it is immediate that $\lim_{n\to\infty} Q_{\tau(n)} = \lim_{n\to\infty} Q_{\tau(n)+1} = 0$. Furthermore, for $n \geqslant n_0$, it is easily observed that $Q_n \leqslant Q_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, if $\tau(n) < n$), because we necessarily have $Q_j > Q_{j+1}$ for $\tau(n) + 1 \leqslant j \leqslant n - 1$. It follows that for all $n \geqslant n_0$, $Q_n \leqslant \max\{Q_{\tau(n)},\ Q_{\tau(n)+1}\} = Q_{\tau(n)+1} \to 0$, hence $\limsup_{n\to\infty} Q_n \leqslant 0$, that is $\limsup_{n\to\infty}(\phi_{n+1} - \theta_n\phi_n) \leqslant 0$, which by (H4) and by a simple calculation leads to the desired result. □

4. Application to common fixed point problems

In this section, we show how (1.3) works for solving common fixed point problems. To begin with, we make the following remark.

Remark 4.1. Let T be a k-demicontractive self-mapping on $\mathcal{H}$ with $\mathrm{Fix}(T) \neq \emptyset$ and set $T_w := (1-w)I + wT$ for $w \in (0,1]$. It is obviously checked that $\mathrm{Fix}(T) = \mathrm{Fix}(T_w)$ if $w \neq 0$. For an arbitrary $(x,q) \in \mathcal{H} \times \mathrm{Fix}(T)$ and using (1.2), we have

$$\begin{aligned}|T_w x - q|^2 &= \left|(x-q) + w(Tx - x)\right|^2 \\ &= |x-q|^2 - 2w\langle x-q, x-Tx\rangle + w^2|Tx-x|^2 \\ &\leqslant |x-q|^2 - w(1-k-w)|Tx-x|^2.\end{aligned}$$

It is immediate that T_w is quasi-nonexpansive with $\mathrm{Fix}(T) = \mathrm{Fix}(T_w)$, provided that $w \in (0, 1-k]$. As a consequence, $\mathrm{Fix}(T)$ is a closed convex subset of $\mathcal{H}$, as the fixed point set of a quasi-nonexpansive mapping (see [34, Proposition 1]).

Lemma 4.1. *Let $(T_i)_{i=0}^N \subset \mathcal{D}_k$ (with $N \in \mathbb{N}$ and $k \in [0,1)$) be such that $\bigcap_{i=0}^N \mathrm{Fix}(T_i) \neq \emptyset$ and set $\mathcal{T} := \sum_{i=0}^N w_i T_i$, where $(w_i)_{i \geqslant 0} \subset [0,+\infty)$ are such that $\sum_{i=0}^N w_i = 1$. Then the following results hold*:

(i1) $\mathrm{Fix}(\mathcal{T}) = \bigcap_{i \in I} \mathrm{Fix}(T_i)$, *where* $I := \{i \in \mathbb{N} \mid i \leqslant N,\ w_i \neq 0\}$.
(i2) $\mathcal{T}$ *belongs to* $\mathcal{D}_k$.
(i3) $\langle x - \mathcal{T}x, x-q\rangle \geqslant \frac{k-1}{2}\sum_{i=0}^N w_i|x - T_i x|^2$, *for all* $(x,q) \in \mathcal{H} \times \mathrm{Fix}(\mathcal{T})$.

Proof. Let us prove (i1). Setting $S := \bigcap_{i=0}^N \mathrm{Fix}(T_i) \neq \emptyset$, we clearly have $S \subset \mathrm{Fix}(\mathcal{T})$, so that $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. Let $q \in \mathrm{Fix}(\mathcal{T})$ and let $p \in S$. It is easily seen that $\sum_{i=0}^N w_i(q - T_i q) = 0$, because $\sum_{i=0}^N w_i = 1$. Consequently, since $(T_i)_{i=0}^N \subset \mathcal{D}_k$ and since p belongs to each $\mathrm{Fix}(T_i)$, we have

$$0 = \sum_{i=0}^N w_i\langle q - T_i q, q-p\rangle \geqslant \frac{k-1}{2}\sum_{i=0}^N w_i|q - T_i q|^2.$$

We then obtain $q - T_i q = 0$ for each $i \in I$, which leads to $\mathrm{Fix}(\mathcal{T}) \subset \bigcap_{i \in I}\mathrm{Fix}(T_i)$, while the converse is obvious. Hence $\mathrm{Fix}(\mathcal{T}) = \bigcap_{i \in I}\mathrm{Fix}(T_i)$, which proves (i1). Let us prove (i2) and (i3). For any $(x,q) \in \mathcal{H} \times \mathrm{Fix}(\mathcal{T})$, we easily observe that

$$\langle x - \mathcal{T}x, x-q\rangle = \left\langle x - \sum_{i=0}^N w_i T_i x, x-q\right\rangle = \sum_{i=0}^N w_i\langle x - T_i x, x-q\rangle;$$

hence, as $(T_i)_{i=0}^N \subset \mathcal{D}_k$, we obtain (iii). Moreover, we obviously have

$$|x - \mathcal{T}x| = \left|\sum_{i=0}^N w_i(x - T_i x)\right| \leqslant \sum_{i=0}^N w_i|x - T_i x|,$$

which by the fact that $\sum_{i=0}^N w_i = 1$ and thanks to Young's inequality leads to

$$|x - \mathcal{T}x|^2 \leqslant \left(\sum_{i=0}^N w_i\right)\left(\sum_{i=0}^N w_i|x - T_i x|^2\right) = \sum_{i=0}^N w_i|x - T_i x|^2.$$

By joining this last inequality to (i3), we get $\langle x - \mathcal{T}x, x-q\rangle \geqslant \frac{k-1}{2}|x - \mathcal{T}x|^2$, so that $\mathcal{T} \in \mathcal{D}_k$, which completes the proof. □

The following theorem is concerned with the computation of a common element of a finite family of mappings.

Theorem 4.2. *Let $(T_i)_{i=0}^N \subset \mathcal{D}_k$, where $k \in [0,1)$, be such that $\bigcap_{i=0}^N \mathrm{Fix}(T_i) \neq \emptyset$ and suppose each T_i is demi-closed. Let $(x_n) \subset \mathcal{H}$ be a sequence such that*

$$\left[\begin{array}{l} x_{n+1} := (1-w)v_n + w\sum_{i=0}^N w_i T_i v_n, \\ \quad \textit{with } v_n := (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}), \quad \forall n \geqslant 1, \end{array}\right. \tag{4.1}$$

where $w \in (0,1]$, (θ_n), $(\alpha_n) \subset [0,1]$ and $(w_i)_{i=0}^N \subset (0,1]$ are real numbers such that $\sum_{i=0}^N w_i = 1$. Assume in addition the following conditions are satisfied:

(W1) *(α_n) is a non-increasing sequence in $(0,1)$, $\alpha_n \to 0$ and $\sum_{n\geqslant 0} \alpha_n = +\infty$.*
(W2) *$w \in (0, \frac{1-k}{2})$.*
(W3) *(θ_n) is a non-decreasing sequence in $[0,\theta]$, where $\theta \in [0, \frac{1}{3})$.*

Then $x_n \to P_S(0)$ strongly in $\mathcal{H}$ as $n \to \infty$, $P_S(0)$ being the element of minimal norm in $S := \bigcap_{i=0}^N \mathrm{Fix}(T_i)$.

Proof. This result is easily deduced from Theorem 3.4. Setting $U = \sum_{i=0}^N w_i T_i$, we just need to prove that the conditions (C0)–(C3) hold with $\mathcal{T}_n = U$ and $S = \bigcap_{i=0}^N \mathrm{Fix}(T_i)$. According to Remark 4.1, each fixed point set $\mathrm{Fix}(T_i)$ is closed and convex, which leads to (C0). By Lemma 4.1, we observe that $(\mathcal{T}_n) \subset \mathcal{D}_k$, hence (C1) holds. Using again Lemma 4.1, we obtain $\mathrm{Fix}(\mathcal{T}_n) = S$, which amounts to (C2). It just remains to prove that (C3) is true. Let (ξ_{n_j}) be a subsequence of (ξ_n) such that $\lim_{j\to 0} |\xi_{n_j} - U\xi_{n_j}| = 0$ and assume that ξ_{n_j} converges weakly to some ξ in $\mathcal{H}$. Clearly, for $q \in S = \mathrm{Fix}(U)$ and using again Lemma 4.1, we easily have

$$\langle \xi_{n_j} - U\xi_{n_j}, \xi_{n_j} - q\rangle \geqslant \frac{1-k}{2}\sum_{i=0}^N w_i |\xi_{n_j} - T_i\xi_{n_j}|^2.$$

Consequently, by the boundedness of (ξ_{n_j}) (thanks to its weak convergence), we easily deduce that

$$\lim_{j\to+\infty} \sum_{i=0}^N w_i |\xi_{n_j} - T_i\xi_{n_j}|^2 = 0;$$

hence, for $i = 0,\dots,N$, we obtain $\lim_{j\to+\infty} |\xi_{n_j} - T_i\xi_{n_j}|^2 = 0$ (since each w_i is positive). As each T_i is assumed to be demi-closed and by the weak convergence of (ξ_{n_j}) to ξ, we conclude that $\xi = T_i\xi$ (for $i = 0,\dots,N$), so that $\xi \in S$, which yields (C3) and completes the proof. □

The following theorem is concerned with the computation of a common element of an infinite family of mappings.

Theorem 4.3. *Let $(T_i)_{i\geqslant 0} \subset \mathcal{D}_k$, where $k \in [0,1)$, be such that $\bigcap_{i\geqslant 0} \mathrm{Fix}(T_i) \neq \emptyset$ and suppose each T_i is demi-closed. Let $(x_n) \subset \mathcal{H}$ be a sequence such that*

$$\left[\begin{array}{l} x_{n+1} := (1-w)v_n + w\sum_{i=0}^n w_{i,n} T_i v_n, \\ \quad \textit{with } v_n := (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}), \quad \forall n \geqslant 1, \end{array}\right. \tag{4.2}$$

where $w \in (0,1]$, (θ_n), $(\alpha_n) \subset [0,1]$ and $(w_{i,n}) \subset [0,+\infty)$ are real numbers such that:

(i) $\forall n \geqslant 0$, $\sum_{i=0}^n w_{i,n} = 1$;
(ii) *for all $i \geqslant 0$, $(w_{i,n})_{n\geqslant 0}$ is bounded away from zero for n large enough* (*that is*: $\forall i \geqslant 0$, $\exists N_i \in \mathbb{N}$ *and* $\exists w_i > 0$ *such that* $\forall n \geqslant N_i$, $w_{i,n} \geqslant w_i$).

Assume in addition the conditions (W1)–(W3) *in Theorem* 4.3 *hold. Then $x_n \to P_S(0)$ strongly in $\mathcal{H}$ as $n \to \infty$, $P_S(0)$ being the element of minimal norm in $S := \bigcap_{i\geqslant 0} \mathrm{Fix}(T_i)$.*

Proof. The proof follows the same lines as Theorem 4.2 and it is given for the sake of completeness. Let us prove that the conditions (C0)–(C3) in Theorem 3.4 are satisfied with $\mathcal{T}_n = \sum_{i=0}^{n} w_{i,n} T_i$ and $S = \bigcap_{i\geqslant 0} \mathrm{Fix}(T_i)$. From Remark 4.1 and Lemma 4.1, we deduce that (C0) and (C1) hold. Using again Lemma 4.1, we obtain $\mathrm{Fix}(\mathcal{T}_n) = \bigcap_{i\in I_n} \mathrm{Fix}(T_i)$ for all $n \geqslant 0$, where $I_n := \{i \in \mathbb{N} \mid i \leqslant n,\ w_{i,n} \neq 0\}$. Noting that $S \subset \bigcap_{i\in I_n} \mathrm{Fix}(T_i)$, we deduce that $S \subset \mathrm{Fix}(\mathcal{T}_n)$, that is (C2). It just remains to prove that (C3) is true. Let $(\mathcal{T}_{n_j})$ be a subsequence of $(\mathcal{T}_n)$ and let $(\xi_{n_j}) \subset H$ be such that $\lim_{j\to 0} |\xi_{n_j} - \mathcal{T}_{n_j}\xi_{n_j}| = 0$ and assume that ξ_{n_j} converges weakly to some ξ in $\mathcal{H}$. Clearly, for $q \in \mathrm{Fix}(\mathcal{T}_n)$ and by Lemma 4.1, we easily obtain $\langle \xi_{n_j} - \mathcal{T}_{n_j}\xi_{n_j}, \xi_{n_j} - q\rangle \geqslant (1/2)(1-k)\sum_{i=0}^{n_j} w_{i,n_j}|\xi_{n_j} - T_i\xi_{n_j}|^2$. Consequently, by the boundedness of (ξ_{n_j}) (thanks to its weak convergence), we easily deduce that $\lim_{j\to+\infty}\sum_{i=0}^{n_j} w_{n_j,i}|\xi_{n_j} - T_i\xi_{n_j}|^2 = 0$; hence, for all $i \geqslant 0$, we obtain $\lim_{j\to+\infty} w_{n_j,i}|\xi_{n_j} - T_i\xi_{n_j}|^2 = 0$, which by (ii) leads to $\lim_{j\to+\infty}|\xi_{n_j} - T_i\xi_{n_j}| = 0$. Assuming that each T_i is demi-closed and by the weak convergence of (ξ_{n_j}) to ξ, we conclude that $\xi = T_i\xi$ (for all $i \geqslant 0$), so that $\xi \in S$, which yields (C3) and completes the proof. □

Remark 4.2. Let us observe that conditions (i) and (ii) in Theorem 4.3 are satisfied by $w_{i,n} = \frac{\gamma_i}{\sum_{k=0}^{n}\gamma_k}$ for $0 \leqslant i \leqslant n$, where (γ_k) is any sequence in (0, 1) such that $\sum_{k\geqslant 0}\gamma_k < \infty$. Indeed, (i) is immediate, while (ii) is deduced from the fact that $w_{i,n} \geqslant \frac{\gamma_i}{\sum_{k\geqslant 0}\gamma_k} > 0$, for all $i \geqslant 0$.

Remark 4.3. The work of this paper can be extend to more general convergence results. Indeed, when v_n in (1.3) is replaced by $v_n := (1-\alpha_n)x_n + \alpha_n a + \theta_n(x_n - x_{n-1})$, where a is an arbitrary but fixed element in $\mathcal{H}$, one may expect to get the strong convergence of the sequence (x_n) to $P_S(a)$ (in Theorem 3.4), P_S being the metric projection from $\mathcal{H}$ onto S.

Remark 4.4. Another class of mappings which is also extensively studied and more general than nonexpansive ones is the so-called class of asymptotically nonexpansive mappings (see, for instance, [1,32]). Interesting strong convergence results are proved, for this latter class of operators, regarding some fixed point methods. In particular, an algorithm which combines viscosity and outer approximations was proposed in [32]. It is worth noting that this latter algorithm can be adapted so that it converges to the element of minimal norm in the fixed point set of a given asymptotically nonexpansive mapping T. This limit is the same attained by the method (4.1) in the case of a single demicontractive mapping T (i.e., $N = 0$ and $T_0 = T$). However the products T^n of a quasi-nonexpansive mapping T are obviously quasi-nonexpansive, hence not necessarily continuous (for n large enough), on the contrary to the case when T is asymptotically nonexpansive mapping. In this latter frame, the operator T^n is Lipschitz continuous for n large enough, which is an important property needed in the convergence analysis of the related algorithms for computing fixed points. This can explain why the technique for analysis used in this paper for demicontractive maps is completely different from the existing ones for asymptotically nonexpansive maps.

Remark 4.5. To the best of our knowledge, there is no known existence result for a common fixed point of an infinite family of strictly pseudocontractive mappings. Thus the assumption $\bigcap_{n\geqslant 0}\mathrm{Fix}(\mathcal{T}_n) \neq \emptyset$ in Theorem 3.4 as well as the assumption $\bigcap_{i\geqslant 0}\mathrm{Fix}(T_i) \neq \emptyset$ in Theorem 4.3 are not warranted. Then it would be interesting to study the possible existence of a common fixed point of infinitely many demicontractive operators. However this is out of scope of the present paper.

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