Linearized Oscillations for Odd-Order Neutral Delay Differential Equations

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Consider the $n$th order nonlinear neutral delay differential equation

$$\frac{d^n}{dt^n} [x(t) - p(t) g(x(t - \tau))] + q(t) h(x(t - \sigma)) = 0, \quad (1)$$

where $n \geq 1$ is an odd integer. We prove that, under appropriate hypotheses, Eq. (1) oscillates provided that the same is true for an associated linear equation with constant coefficients of the form

$$\frac{d^n}{dt^n} [y(t) - q_0 y(t - \tau)] + q_0 y(t - \sigma) = 0.$$

A partial converse is also presented, where we show that, under appropriate hypotheses, Eq. (1) has a positive solution. © 1990 Academic Press, Inc.

1. INTRODUCTION

Recently a linearized oscillation theory has been developed in [4–7] for nonlinear delay differential equations which in some sense parallels the so called linearized stability theory for differential and difference equations. Roughly speaking, it has been proved that, under appropriate hypotheses, certain nonlinear differential equations have the same oscillatory character as an associated linear equation.

Our aim in this paper is to obtain a linearized oscillation result for the odd-order neutral differential equation

$$\frac{d^n}{dt^n} [x(t) - p(t) g(x(t - \tau))] + q(t) h(x(t - \sigma)) = 0 \quad (1)$$

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under the following hypotheses:

\[ p, q \in C\left([t_0, \infty), \mathbb{R}^+\right), \quad g, h \in C\left[\mathbb{R}, \mathbb{R}\right], \quad \tau \in (0, \infty), \quad \sigma \in [0, \infty), \]

\[ \lim_{t \to \infty} \sup \left\{ p(t) \right\} = P_0 \in (0, 1), \quad \lim_{t \to \infty} \inf \left\{ p(t) \right\} = p_0 \in (0, 1), \]

\[ \lim_{t \to \infty} \quad q(t) = q_0 \in (0, \infty), \]

\[ 0 \leq \frac{g(u)}{u} \leq 1 \quad \text{for} \quad u \neq 0, \quad \lim_{u \to 0} \frac{g(u)}{u} = 1, \]

\[ u h(u) > 0 \quad \text{for} \quad u \neq 0, \quad |h(u)| \geq h_0 > 0 \quad \text{for} \quad |u| \text{ sufficient large} \]

and

\[ \lim_{u \to 0} h(u) = 1. \]

With Eq. (1) we associate the linear equation with constant coefficients

\[
\frac{d^n}{dt^n} \left[ y(t) - P_0 y(t - \tau) \right] + q_0 y(t - \sigma) = 0. \tag{7}
\]

In Section 2 we establish some basic lemmas which we will use in Section 3 and 4 and which are interesting in their own right. In Section 3 we establish sufficient conditions for the oscillation of all solutions of Eq. (1) in terms of the oscillation of all solutions of Eq. (7). Finally, in Section 4 we show that, under appropriate hypotheses, if Eq. (7) has a positive solution, so does Eq. (1).

Let \( m = \max\{\tau, \sigma\} \). By a solution of Eq. (1) we mean a function \( x \in C\left([t_1 - m, \infty), \mathbb{R}\right) \), for some \( t_1 \geq t_0 \), such that \( [x(t) - p(t) g(x(t - \tau))] \) is \( n \) times continuously differentiable on \( [t_1, \infty) \) and such that Eq. (1) is satisfied for \( t \geq t_1 \).

As is customary, a solution of Eq. (1) is said to oscillate if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory. We say that a differential equation oscillates if every solution of the equation oscillates.

2. SOME BASIC LEMMAS

In this section we will present some basic lemmas which are useful in the proofs of the main theorems in Section 3 and 4.

The first two lemmas are extracted from \([1, 2]\), respectively.
Consider the nth order neutral differential equation
\[ \frac{d^n}{dt^n} [y(t) - py(t - \tau)] + qy(t - \sigma) = 0, \]  
(8)

where \( p \) and \( q \) are real numbers and \( \tau \) and \( \sigma \) are nonnegative real numbers. Then every solution of Eq. (8) oscillates if and only if its characteristic equation
\[ \lambda^n - p\lambda^{n-\tau} + qe^{-\lambda\sigma} = 0 \]  
(9)

has no real roots.

**Lemma 2 [2].** Let \( F, G, P \in \mathcal{C}([t_0, \infty), \mathbb{R}) \) and \( c \in (0, \infty) \) be such that
\[ F(t) = G(t) - P(t)G(t - c), \quad t \geq t_0 + c. \]
Assume that
\[ F(t) > 0 \quad \text{and} \quad G(t) > 0 \quad \text{for} \quad t \geq t_0, \quad \lim_{t \to \infty} F(t) = 0 \]
and suppose that there exists a \( p \in [0, 1) \) such that
\[ 0 \leq P(t) \leq p < 1. \]
Then
\[ \lim_{t \to \infty} G(t) = 0. \]

The next lemma is obtained by a slight modification in the proof of Lemma 2 in [3]. We will omit the proof.

**Lemma 3.** Assume that \( n \geq 1 \) is a positive integer, \( F, G \in \mathcal{C}([T, \infty), \mathbb{R}^+) \), \( H \in \mathcal{C}([\mathbb{R}^+, \mathbb{R}^+], \mathbb{R}) \), \( \tau \in (0, \infty), \sigma \in [0, \infty) \), \( h(u) \) is nondecreasing in \( u \) and
\[ F(t) > 0 \quad \text{for} \quad t \geq T. \]
(10)
Let \( m = \max\{\tau, \sigma\} \) and suppose that the integral inequality
\[ F(t) z(t - \tau) + \int_t^\infty \int_{s_{n-1}}^\infty \cdots \int_{s_1}^\infty G(s) H(z(s - \sigma)) \, ds \, ds_1 \cdots ds_{n-1} \leq z(t), \quad t \geq T \]
has a continuous positive solution \( z: \left[T - m, \infty\right) \to (0, \infty) \) such that
\[ z(T) < z(t) \quad \text{for} \quad T - m \leq t < T. \]
(11)
Then there exists a continuous positive solution \( x: [T-m, \infty) \to (0, \infty) \) of the corresponding integral equation

\[
F(t) x(t-\tau) + \int_{t-\tau}^{\infty} \cdots \int_{s_{n-1}}^{\infty} G(s) H(x(s-\sigma)) \, ds \, ds_1 \cdots ds_{n-1} = x(t), \quad t \geq T.
\]

By using Lemma 1 and by an argument similar to that in [8, Lemma 2] we can easily establish the following result.

**Lemma 4.** Assume that \( n \geq 1 \) is an odd integer,

\[
p \in (0, 1], \quad q \in (0, \infty), \quad \tau \in (0, \infty), \quad \sigma \in [0, \infty)
\]

and suppose that every solution of Eq. (7) oscillates. Then there exists an \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in [0, \varepsilon_0] \) every solution of the differential equation

\[
\frac{d^n}{dt^n} [y(t) - (p - \varepsilon) y(t-\tau)] + (q - \varepsilon) y(t-\sigma) = 0
\]

also oscillates.

The following corollary is an interesting byproduct of Lemma 4.

**Corollary 1.** Assume that \( n \geq 1 \) is an odd integer and

\[
q \in (0, \infty) \quad \text{and} \quad \tau, \sigma \in [0, \infty).
\]

Then there exists an \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in [0, \varepsilon_0] \) every solution of the differential equation

\[
\frac{d^n}{dt^n} [y(t) - (1 - \varepsilon) y(t-\tau)] + qy(t-\sigma) = 0
\]

oscillates.

**Proof.** By [9, Theorem 2], every solution of the neutral differential equation

\[
\frac{d^n}{dt^n} [y(t) - y(t-\tau)] + qy(t-\sigma) = 0
\]

oscillates. Then by Lemma 4 there exists an \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in [0, \varepsilon_0] \) every solution of the neutral differential equation

\[
\frac{d^n}{dt^n} [y(t) - (1 - \varepsilon) y(t-\tau)] + (q - \varepsilon) y(t-\sigma) = 0
\]
oscillates. Finally by employing the comparison theorem which we established in [3] we see that every solution of (13) oscillates.

3. LINEARIZED OSCILLATIONS

The main result in this section is the following theorem.

**Theorem 1.** Assume that (2)-(6) are satisfied and that every solution of Eq. (7) oscillates. Then every solution of Eq. (1) also oscillates.

**Proof.** Assume, for the sake of contradiction, that Eq. (1) has a non-oscillatory solution \( x(t) \). We will assume that \( x(t) \) is eventually positive. The case where \( x(t) \) is eventually negative is similar and will be omitted. Set

\[
z(t) = x(t) - p(t) g(x(t - \tau)).
\]

Then

\[
z^{(n)}(t) = -q(t) h(x(t - \sigma)) \leq 0.
\]

Therefore, \( z^{(n-1)}(t) \) is decreasing and so either

\[
\lim_{t \to \infty} z^{(n-1)}(t) = -\infty
\]

or

\[
\lim_{t \to \infty} z^{(n-1)}(t) = l \in \mathbb{R}.
\]

We claim that (17) holds. Otherwise (16) holds which implies that

\[
\lim_{t \to \infty} z^{(i)}(t) = -\infty \quad \text{for} \quad i = 0, 1, \ldots, n-1.
\]

Then \( x(t) \) is an unbounded function and so there exists a sequence of points \( \{t_k\} \) such that

\[
\lim_{k \to \infty} t_k = \infty, \quad \lim_{k \to \infty} x(t_k) = \infty,
\]

and

\[
x(t_k) = \max_{s \in [t_{k-1}, t_k]} x(s) \quad \text{for} \quad k = 1, 2, \ldots.
\]
Now, by using (3) and (4) we see that
\[ z(t) = x(t) - p(t)g(x(t - \tau)) \]
\[ = x(t) - p(t) \frac{g(x(t - \tau))}{x(t - \tau)} x(t - \tau) \]
\[ \geq x(t) - p(t) \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty \]

which contradicts (18) and so (17) holds. From (15) and (17) it follows that for each \( i = 0, 1, \ldots, n - 1 \) the function \( z^{(i)}(t) \) is monotonic. By integrating both sides of (15) from \( t_i \) to \( \infty \), for \( t_i \) sufficiently large, we obtain
\[ l - z^{(n-1)}(t) = -\int_{t_i}^{\infty} q(s) h(x(s - \sigma)) \, ds. \quad (19) \]

We claim that
\[ \lim \inf_{t \to \infty} x(t) = 0. \quad (20) \]

Otherwise there exists a positive constant \( c \) and a \( t_2 \geq t_1 \) such that \( x(t) \geq c \) for \( t \geq t_2 \). Then from (3) and (5) and for \( t \) sufficiently large, \( q(t) h(x(t - \sigma)) \) is bounded from below by a positive constant. This contradicts (19) and so (20) holds.

Let \( \{ t_k \} \) be a sequence of points such that
\[ \lim_{k \to \infty} t_k = \infty \quad \text{and} \quad \lim_{k \to \infty} x(t_k) = 0. \]

From (14) we see that
\[ z(t_k) \leq x(t_k) \rightarrow 0 \quad \text{as} \quad k \to \infty \]

and also
\[ z(t_k + \tau) \geq -p(t_k + \tau) \frac{g(x(t_k))}{x(t_k)} x(t_k) \rightarrow 0 \quad \text{as} \quad k \to \infty. \]

As \( z(t) \) is monotonic, it follows that
\[ \lim_{t \to \infty} z(t) = 0. \]

From this and (1) it is clear that
\[ \lim_{t \to \infty} z^{(i)}(t) = 0 \quad \text{for} \quad i = 0, 1, \ldots, n - 1 \quad (21) \]
and
\[ z^{(n)}(t) \leq 0, \quad z^{(n-1)}(t) > 0, \ldots, z'(t) < 0, \quad z(t) > 0. \] (22)

We now claim that
\[ \lim_{t \to \infty} x(t) = 0. \] (23)

To this end observe that for \( t \) sufficiently large
\[ z(t) = x(t) - P(t) x(t - \tau), \] (24)

where
\[ P(t) = p(t) \frac{g(x(t - \tau))}{x(t - \tau)}. \]

Let \( p^* \in (P_0, 1) \). Then from (3), (4), and for \( t \) sufficiently large,
\[ 0 < P(t) < p(t) < p^* < 1 \]
and (23) follows by applying Lemma 2 to (24).

Next, we rewrite Eq. (1) in the form
\[ \frac{d^n}{dt^n} [x(t) - P(t) x(t - \tau)] + Q(t) x(t - \sigma) = 0, \]
where for \( t \) sufficiently large,
\[ P(t) = p(t) \frac{g(x(t - \tau))}{x(t - \tau)} \quad \text{and} \quad Q(t) = q(t) \frac{h(x(t - \sigma))}{x(t - \sigma)}. \]

Note that because of (23), (3), (4), and (6),
\[ \lim_{t \to \infty} P(t) = p_0 \quad \text{and} \quad \lim_{t \to \infty} Q(t) = q_0. \]

Now one can see by direct substitution that for \( t \) sufficiently large, \( z(t) \) is a solution of the neutral equation
\[ z^{(n)}(t) - P(t - \sigma) \frac{Q(t)}{Q(t - \tau)} z^{(n)}(t - \tau) + Q(t) z(t - \sigma) = 0 \] (25)

and
\[ \liminf_{t \to \infty} \left[ p(t - \sigma) \frac{Q(t)}{Q(t - \tau)} \right] = p_0. \]
Then for any positive number $\varepsilon$ in the interval $0 < \varepsilon < \frac{1}{2} \min\{p_0, q_0\}$, Eq. (25) yields the inequality

$$z^{(n)}(t) - (p_0 - \varepsilon) z^{(n)}(t - \tau) + (q_0 - \varepsilon) z(t - \sigma) \leq 0.$$  

By integrating this inequality from $t$ to $t_1$ and then by letting $t_1 \to \infty$ we find

$$-z^{(n-1)}(t) + (p_0 - \varepsilon) z^{(n-1)}(t - \tau) + (q_0 - \varepsilon) \int_t^{\infty} z(s - \sigma) \, ds \leq 0.$$ 

By repeating the same procedure $n$ times and by noting the fact that $n$ is odd we are led to the inequality

$$(p_0 - \varepsilon) z(t - \tau) + (q_0 - \varepsilon) \int_t^{\infty} \int_{s_{n-1}}^{\infty} \int_{s_{n-2}}^{\infty} \cdots \int_{s_1}^{\infty} z(s - \sigma) \, ds_1 \cdots ds_{n-1} \leq z(t), \quad t \geq T,$$

where $T$ is sufficiently large and $z: [T - m, \infty) \to (0, \infty)$, with $m = \max\{\tau, \sigma\}$, is a continuous and strictly decreasing function. It follows by Lemma 3 that the equation

$$(p_0 - \varepsilon) v(t - \tau) + (q_0 - \varepsilon) \int_t^{\infty} \int_{s_{n-1}}^{\infty} \int_{s_{n-2}}^{\infty} \cdots \int_{s_1}^{\infty} v(s - \sigma) \, ds_1 \cdots ds_{n-1} = v(t)$$

has a continuous positive solution $v: [T - m, \infty) \to (0, \infty)$. Then $v(t)$ is also a positive solution of the neutral equation

$$\frac{d^n}{dt^n} [v(t) - (p_0 - \varepsilon) v(t - \tau)] + (q_0 - \varepsilon) v(t - \sigma) = 0.$$ 

Hence, by Lemma 4 and because of the fact that the $\varepsilon$ is arbitrarily small, it follows that Eq. (7) has a positive solution. This contradicts the hypothesis and completes the proof of the theorem.

4. **Existence of Positive Solutions**

Consider the neutral delay differential equation

$$\frac{d^n}{dt^n} [x(t) - p(t) x(t - \tau)] + q(t) h(x(t - \sigma)) = 0, \quad (26)$$
where \( n \geq 1 \) is an odd integer,
\[
p, q \in C[[t_0, \infty), \mathbb{R}^+], \quad \tau \in (0, \infty), \quad \sigma \in [0, \infty), \quad \text{and} \quad h \in C[\mathbb{R}, \mathbb{R}].
\]
(27)
The next theorem is a partial converse of Theorem 1 and shows that, under appropriate hypotheses, Eq. (26) has a positive solution provided that an associated linear equation has a positive solution.

**THEOREM 2.** Assume that (27) holds and that there exist positive constants \( p_0, q_0, \) and \( \delta \) such that
\[
0 < p(t) \leq p_0 < 1, \quad 0 \leq q(t) \leq q_0
\]
(28)
and
\[
\text{either} \quad 0 \leq h(u) \leq u \quad \text{for} \quad 0 \leq u \leq \delta \\
\text{or} \quad 0 \geq h(u) \geq u \quad \text{for} \quad -\delta \leq u \leq 0.
\]
(29, 30)
Suppose also that \( h(u) \) is nondecreasing in \( u \) and that the characteristic equation of Eq. (7)
\[
\lambda^n - p_0 \lambda^n e^{-\lambda t} + q_0 e^{-\lambda \sigma} = 0
\]
(31)
has a real root. Then Eq. (26) has a nonoscillatory solution.

**Proof.** Assume that \( 0 \leq h(u) \leq u \) for \( 0 \leq u \leq \delta \). The case where \( 0 \geq h(u) \geq u \) for \( -\delta \leq u \leq 0 \) is similar and will be omitted. Let \( \lambda_0 \) be a root of Eq. (30). As \( p_0 \in (0, 1) \), it follows that \( \lambda_0 < 0 \). Set \( y(t) = \exp(\lambda_0 t) \). Then there exists a \( T > t_0 \) such that
\[
0 < y(t) \leq \delta \quad \text{for} \quad t \geq T - \delta.
\]
Clearly,
\[
y(t) > 0, \quad y'(t) < 0, \quad y''(t) > 0, \ldots, \quad y^{(n-1)}(t) > 0, \quad y^{(n)}(t) < 0,
\]
\[
\lim_{t \to \infty} y^{(i)}(t) = 0 \quad \text{for} \quad i = 0, 1, 2, \ldots, n
\]
and \( y(t) \) satisfies the neutral equation
\[
\frac{d^n}{dt^n} \left[ y(t) - p_0 y(t - \tau) \right] + q_0 y(t - \sigma) = 0.
\]
(32)
By integrating \( n \) times both sides of (32) from \( t \geq T \) to \( \infty \) we obtain
\[
p_0 y(t - \tau) + \int_t^\infty \int_{\sigma}^\infty \ldots \int_{\sigma}^\infty q_0 y(s - \sigma) \, ds \\ d\sigma = y(t), \quad t \geq T.
\]
In view of (28) and (29), this equation implies that
\[ p(t) y(t-\tau) + \int_{t}^{\infty} \cdots \int_{s_{n-1}}^{\infty} q(s) h(y(s-\sigma)) \, ds \, ds_{1} \cdots ds_{n-1} \]
\[ \leq y(t), \quad t \geq T. \]

By Lemma 3 it follows that the corresponding equation
\[ p(t) x(t-\tau) + \int_{t}^{\infty} \cdots \int_{s_{n-1}}^{\infty} q(s) h(x(s-\sigma)) \, ds \, ds_{1} \cdots ds_{n-1} \]
\[ = x(t), \quad t \geq T \]
has a continuous positive solution. Hence, Eq. (26) also has a positive solution. The proof of the theorem is complete.

By combining Theorems 1 and 2 we obtain the following necessary and sufficient condition for the oscillation of every solution of Eq. (26).

**Corollary 2.** Assume that there exist positive constants \( h_{0}, p_{0}, q_{0}, \) and \( \delta \) such that (27), (5), (6) and either (29) or (30) are satisfied,
\[ 0 < p(t) < p_{0} = \lim_{t \to \infty} p(t) < 1, \quad 0 \leq q(t) < q_{0} = \lim_{t \to \infty} q(t) \]
and \( h(u) \) is nondecreasing in \( u \). Then every solution of Eq. (26) oscillates if and only if every solution of the linear equation (7) oscillates if and only if Eq. (31) has no negative real roots.

**References**