Special Issue of JSC on
"Symbolic Computation in Combinatorics"
—Foreword of the Guest Editors

1. Introductory remarks

The articles in this volume deal with the non-trivial application of symbolic computation algorithms to expressions arising in combinatorial work.

The idea of translating problems from various areas of mathematics into ones that are essentially of "combinatorial manipulative character", i.e. reducible to symbolic computation, has a long tradition. For example, G.W. Leibniz' "De arte combinatoria" (1666) and K.F. Hindenburg's "Sammlung Combinatorisch-Analytischer Abhandlungen", (Vol. 1 and 2, 1796 and 1800).

The systematic application of this idea to problems which are combinatorial by their own nature is much younger. P.A. MacMahon's "Combinatory Analysis" (1916) can be understood as a first, more or less systematic, attempt of modelling discrete structures in symbolic or algebraic terms. The solution of the combinatorial problem is achieved by appropriate formal manipulation of the corresponding expressions.

Computer algebra has developed into an indispensable tool in the applications already addressed by Leibniz and Hindenburg, i.e. for many problems in algebra and analysis. Despite the fact that symbolic computation and the formula-manipulation part of the combinatorialist's work go together and despite some recent major algorithmic breakthroughs, the systematic use of computer algebra is still in its early stages. Thus, as already pointed out in the foreword of the Editor-in-Chief, one important objective of this special issue is to stimulate research in this field. The stimulation will hopefully be in two directions. On the one hand, symbolic computation problems arising from the combinatorialist's work might raise the demand for new computer algebra algorithms. On the other hand, the application of computer algebra algorithms might lead to the study of new and interesting combinatorial structures.

It should be added that the development of computer algebra itself already brought along several new and interesting ideas and concepts which fit exactly under that umbrella. Just to mention two important contributions in the area of indefinite symbolic summation: W. Gosper's algorithm for summing hypergeometric sequences and M. Karr's summation theory formulated in the algebraic frame of difference fields.

In order to complete the picture, one should mention that computer algebra has already been used extensively by several researchers to derive new and important combinatorial results. One prominent example is the work of G. Andrews, connecting the areas of combinatorics, q-series, number theory, and physics, which is cited explicitly in the SIAM report, published 1990, "Future Directions for Research in the Field of Symbolic Computation" (Report of a Workshop on Symbolic and Algebraic Computation, Editors: A. Boyle, B. F. Caviness, Workshop Chairperson: A. C. Hearn).

The editors of this volume were happy to accept B. Buchberger's invitation to serve as guest editors for this special issue. It is our hope that it would help to develop the field of symbolic computation in combinatorics to a well represented topic in the Journal of
Symbolic Computation. Before we briefly discuss the individual contributions, we would like to express our sincerest thanks to all those who helped in the preparation of this volume. This includes the authors as well as the referees, who must remain anonymous.

2. Discussion of the Individual Contributions

The article by Francois and Nantel Bergeron describes how algebraic explorations with a symbolic manipulation package lead to the construction of nice bases for the descent algebras of finite Coxeter groups and how this resulted in a better understanding of the multiplicative structure of many of these descent algebras. Implementational details are discussed. Key procedures are given in form of MAPLE programs, including the complete code for computing the multiplicative structure constants for arbitrary finite dimensional descent algebras.

The article by Frank Garvan and Gaston Gonnet settles the so far open Macdonald-Morris root system constant term conjecture for the affine root systems $S(F_4)$ and $S(F_4)'$. The problem is basically equivalent to showing that the constant term of a certain univariate Laurent polynomial with rational function coefficients, whose construction is related to an affine root system, equals an explicitly given product expression. By applying and refining an algorithmic method, introduced by D. Zeilberger, which reduces the problem to finding and solving a linear system of equations, the authors succeeded in reducing the problem to showing that a certain "seemingly monstrous" rational function in three indeterminates is in fact identically zero. Using their "triangularity result" the authors conclude by discussing those Macdonald-Morris conjectures which have remained open up to now.

The article by Ira Gessel presents a study of new ballot number analogs, "super ballot numbers", of generalized Catalan numbers. It provides an excellent example how computer algebra can be used in the process of combinatorial exploration. Interesting properties of super ballot numbers are discussed. One most surprising fact is that they are closely related to the power series coefficients of $1/(1 - x - y - z + 4xyz)$, which have been studied by several authors. A new proof for the positivity of these coefficients is given and a conjecture, including this result as a special case, is stated. In addition, the author relates super ballot numbers to "super Catalan numbers", introduced by E. Catalan in 1874.

The article by Adalbert Kerber, Axel Kohnert and Alain Lascoux presents a review of the object oriented computer algebra system SYMMETRICA, formerly called SYMCHAR, which is devoted to representation theory, invariant theory and combinatorics of finite symmetric groups. Moreover, the system can be used for classical multivariate polynomials via the different actions of the symmetric group on the algebra of polynomials. The review contains a brief introduction to the basic methods used. Schubert polynomials are introduced, examples are given, and some applications are described. In particular, they provide a new algorithm for the evaluation of Littlewood–Richardson coefficients, which play a central role in the representation theories of the symmetric group and of the general linear group, via symbolic computations using integer sequences instead of partitions, tableaux or lattice permutations.

The article by Axel Kohnert deals with an algorithmic application of Schubert polynomials. Schubert polynomials, which play a fundamental role in the computer algebra system SYMMETRICA (see the review article in this volume), were introduced by A. Lascoux and M. P. Schützenberger in order to provide a new combinatorially motivated tool for solving symbolic computation problems involving polynomials in several variables.
Using a procedure to decompose the symmetric part of a Schubert polynomial into a sum of Schur polynomials they gave an algorithm for computing the decomposition of the product of Schur functions as a sum of Schur functions, i.e. the computation of the Littlewood-Richardson coefficients. Kohnert shows that this algorithm also can be used to compute the decomposition of a skew Schur function into a sum of Schur functions. This method, used in SYMMETRICA, is totally different from the well-known Littlewood-Richardson rule, which provides the same decomposition.

The article by Gilbert Labelle treats the enumeration problem for asymmetric structures, i.e. structures with trivial automorphism group. Formulated in the language of Joyal’s theory of species an alternative approach to certain instances of that problem is given. Also included is a brief introduction to the species method, a method for translating combinatorial relations between structures into relations in some power series algebra. As a basic tool a newly introduced “asymmetric index series” is used. It is interesting to point out that for the solution of the corresponding symbolic computation problem, i.e. for the computation of power series coefficients, the multivariate Lagrange–Good inversion formula is used, as well as methods based on a combinatorial approach to Newton-Raphson iteration. The general theory is illustrated by a collection of interesting combinatorial applications, including ordinary, planar, topological, oriented, 1-2-3, cyclic, and permutation (rooted) trees, and several tables computed with help of MAPLE and DARWIN.

The article by Marko Petkovšek describes an algorithm “Hyper” for finding all hypergeometric solutions of linear recurrences with polynomial coefficients. The first-order case $a(n + 1) - a(n) = h(n)$, being equivalent to the hypergeometric summation problem, was solved by W. Gosper. A main ingredient of Gosper’s algorithm is a certain representation of rational functions. A new canonical form is introduced, and an algorithm for its computation is explicitly given, that turns out to be crucial for Hyper. Several applications that illustrate the method are given, including ones that in combination with Zeilberger’s algorithm decide whether a given definite hypergeometric sum can be expressed as a linear combination of hypergeometric terms. Finally, the author discusses the necessity of computing in algebraic extensions of the coefficient field of the equation in order to find all its hypergeometric solutions.

The article by Nobuki Takayama considers D. Zeilberger’s “holonomic system” point-of-view to the zero recognition problem of expressions representing mathematical functions. Based on a Weyl-algebra analog of Buchberger’s algorithm the author presents a quasi-algorithm that recognizes whether a holonomic function is zero or not. The algorithm consists of procedures for computing differential operators that annihilate sums, products and definite integrals with respect to parameters of holonomic functions. For certain cases accelerations and methods to save memory are discussed. One of the goals of this approach is to improve Zeilberger’s original one that used an analog of Sylvester’s dialytic elimination. Several conjectures and open problems are stated.

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