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REGULAR MULTIPLICATION RINGS

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Introduction

Let R be a commutative ring with identity. An R-module M is said to be a multiplication module if every submodule of M is of the form IM, for some ideal I of R. We shall call a ring R a regular multiplication ring if every regular ideal of R is a multiplication R-module. (By a regular ideal I is meant one which contains a regular element of R). A multiplication ring is a ring in which every ideal is a multiplication module. It is well known that multiplication domains are precisely Dedekind domains [5].

The regularity in our definition has allowed us to generalize the known results over Dedekind domains to non-domains. In Section 1, we show that a ring R is a regular multiplication ring if and only if for every regular ideal I of R, R/I is a finite direct sum of special principal ideal rings.

In Section 2, we show that a ring R is a regular multiplication ring if and only if every finitely generated torsion R-module is of finite length and a direct sum of cyclic submodules. We also show that a torsion module over a regular multiplication ring is a direct sum of its primary parts.

We use the following notation: if R is a ring, then MaxSpec R is the set of all maximal ideals of R; if M is an R-module, then $\mathrm{Supp}(M) = \{P \in \mathrm{Spec}\, R \mid M_P \neq 0\}$.

1. Some properties of regular multiplication rings

We begin by noting that a ring R is a regular multiplication ring if and only if every regular ideal of R is invertible. Therefore it follows that a ring R is a regular multiplication ring if and only if ever regular ideal of R is a unique product of powers of finitely many maximal ideals of R [4, Theorem 17].

Theorem 1. *For a ring R the following statements are equivalent:*
(i) *R is a regular multiplication ring.*
(ii) *For each regular ideal I, R/I is a finite direct sum of special principal ideal*

rings. (Recall that a principal ideal ring is called special if it has only one prime ideal and the prime ideal is nilpotent).

(iii) *For each regular nonunit r, R/Rr is a principal ideal ring.*

Proof. (i) ⇒ (ii). Let I be a regular ideal of R. Then $I = P_1^{\upsilon_1} P_2^{\upsilon_2} \dots P_n^{\upsilon_n}$, where $P_1, P_2, \dots, P_n$ are distinct maximal ideals of R and $\upsilon_1, \upsilon_2, \dots, \upsilon_n$ are positive integers. Hence $R/I \simeq \bigoplus_{i=1}^n R/P_i^{\upsilon_i}$. Since for each $i = 1, 2, \dots, n$, $P_i^{\upsilon_i}$ is a regular ideal of R, it follows that every ideal of R containing $P_i^{\upsilon_i}$ is a multiplication R-module. But then clearly each $R/P_i^{\upsilon_i}$ is a multiplication ring. Therefore, each $R/P_i^{\upsilon_i}$ is a special principal ideal ring [1, Theorem 1].

(ii) ⇒ (iii). Follows from the fact that a finite direct sum of principal ideal rings is a principal ideal ring [7, Theorem 33, p. 245].

(iii) ⇒ (i). Let I be a regular ideal of R and r a regular element in I. By hypothesis, $\bar{R} = R/(r^2)$ is a principal ideal ring. Hence $\bar{I}$ is a principal ideal in $\bar{R}$ and $(\bar{r}) \subseteq \bar{I}$ so there is an ideal J of R with $(r) \subseteq J$ such that $\overline{IJ} = (\bar{r})$. But then $r^2 \in IJ$ so $IJ = IJ + (r^2) = (r) + (r^2) = (r)$. Thus I is a factor of a regular principal ideal and hence is invertible. Therefore it follows that R is a regular multiplication ring. □

Corollary. *In a regular multiplication ring R, every regular ideal I can be generated by at most two elements, one of which can be chosen arbitrarily from among the elements of I which are not zero divisors of R.* □

It is clear that every multiplication ring is a regular multiplication ring. But it is not the case that every regular multiplication ring is a multiplication ring. As a counterexample take $R = K[x^2, x^3]/(x^4)$, where K is a field and x is an indeterminate. Clearly, here R is a regular multiplication ring. But R is not a multiplication ring, because R is a local ring and a local multiplication ring is a principal ideal ring [1, p. 761].

We also note that in a regular multiplication ring every regular prime ideal is maximal.

2. Modules over regular multiplication rings

Let R be a ring and M an R-module. An element m of M will be called a 'torsion element' if $rm = 0$ for some non-zero divisor r in R. If we denote by $T(M)$ the set of all torsion elements in M, then $T(M)$ is an R-submodule of M, and will be called the 'torsion submodule' of M. If $T(M) = M$, M will be called a 'torsion' R-module.

Theorem 2. *Let R be a regular multiplication ring and let M be a 'torsion' R-module (in the above sense). For each maximal ideal P of R, write $M^P = \{x \in M \mid P^\upsilon x = 0$, for some positive integer $\upsilon\}$. Then*

$$M = \bigoplus_{P \in \operatorname{Supp}(M)} M^P \simeq \bigoplus_{P \in \operatorname{Supp}(M)} M_P.$$

Proof. Since in this case $\mathrm{Supp}(M) \subseteq \mathrm{MaxSpec}\, R$, we first show that for each maximal ideal P of R, M^P is a submodule of M. But this is straightforward.

We also note that for each P in $\mathrm{Supp}(M)$, every non-zero element of M^P has its annihilator contained only in P but not in any other maximal ideal of R. For if $0 \neq x \in M^P$ and $\mathrm{Ann}(x) \subseteq Q$ for some maximal ideal Q of R, then $P^{\upsilon} \subseteq \mathrm{Ann}(x) \subseteq Q$, which implies that $P = Q$.

Let $\{P_1, P_2, \dots, P_n\}$ be a subset of $\mathrm{Supp}(M)$ and let Q be an element of $\mathrm{Supp}(M)$ such that $Q \notin \{P_1, P_2, \dots, P_n\}$. Since every element of $M^{P_1} + M^{P_2} + \cdots + M^{P_n}$ is annihilated by a product of powers of $P_1, P_2, \dots, P_n$, it follows that $M^Q \cap \{M^{P_1} + M^{P_2} + \cdots + M^{P_n}\} = 0$. Thus the submodules M^P generate their direct sum $\bigoplus_{P \in \mathrm{Supp}(M)} M^P$ in M.

In order to show that $M = \bigoplus_{P \in \mathrm{Supp}(M)} M^P$, we take x to be any non-zero element of M. Since M is a torsion R-module and R is a regular multiplication ring, it follows that $\mathrm{Ann}(x) = P_1^{\upsilon_1} P_2^{\upsilon_2} \dots P_n^{\upsilon_n}$, for some unique set $\{P_1, P_2, \dots, P_n\}$ of invertible maximal ideals of R and for some unique set of positive integers $\upsilon_1, \upsilon_2, \dots, \upsilon_n$. But then we have $Rx \simeq R/\mathrm{Ann}(x) \simeq \bigoplus_{i=1}^n R/P_i^{\upsilon_i}$. That is $Rx = \bigoplus_{i=1}^n Rx_i$, where $Rx_i \simeq R/P_i^{\upsilon_i}$ for $i = 1, 2, \dots, n$. Clearly, for each $i = 1, 2, \dots, n$ every element of Rx_i is annihilated by some power of P_i. Therefore it follows that $Rx_i \subseteq M^{P_i}$ for $i = 1, 2, \dots, n$. Hence $Rx \subseteq \bigoplus_{P \in \mathrm{Supp}(M)} M^P$. But x was taken arbitrarily from among the non-zero elements of M, so it follows that $M = \bigoplus_{P \in \mathrm{Supp}(M)} M^P$.

To see that

$$M = \bigoplus_{P \in \mathrm{Supp}(M)} M^P \simeq \bigoplus_{P \in \mathrm{Supp}(M)} M_P,$$

we consider the canonical R-module homomorphism $f : M \to M \otimes_R R_P$ ($P \in \mathrm{Supp}(M)$). This induces by restriction, an R-module homomorphism $f_P : M^P \to M \otimes_R R_P$. Since M is a torsion R-module and tensor product commutes with direct sum and every element of M whose annihilator not contained in P becomes zero in $M \otimes_R R_P$, we have $M \otimes_R R_P = M^P \otimes_R R_P$. That is, $f_P : M^P \to M^P \otimes_R R_P$. We have $\mathrm{Ker} f_P = \{x \in M^P \mid tx = 0, \text{ for some } t \in R - P\} = 0$. (Because for each $0 \neq x \in M^P$, $\mathrm{Ann}(x) \subseteq P$). Thus $f_P : M^P \to M^P \otimes_R R_P$ is injective.

We now show that f_P is surjective. Let y be any non-zero element of $M^P \otimes_R R_P$. Then y can be written in the form $y = m \otimes 1/t$, for some element m in M^P and t in $R - P$. Since $\mathrm{Ann}(m)$ is contained only in P and $t \in R - P$, it follows that $\mathrm{Ann}(m) + Rt = R$, which implies that $1 = a + bt$, for some $a \in \mathrm{Ann}(m)$ and $b \in R$. Hence $m = am + btm = btm$. Thus

$$y = m \otimes \frac{1}{t} = btm \otimes \frac{1}{t} = bm \otimes \frac{t}{t} = bm \otimes 1.$$

That is, $y = f_p(bm)$. Therefore f_p is surjective. Hence the R-homomorphism

$$f = \sum_{P \in \mathrm{Supp}(M)} f_p : M = \bigoplus_{P \in \mathrm{Supp}(M)} M^P \to \bigoplus_{P \in \mathrm{Supp}(M)} M_P$$

is an isomorphism. □

Before formulating our next statement, we recall a well-known theorem of G. Köthe, I.S. Cohen and I. Kaplansky which states that a ring R is an Artinian principal ideal ring if and only if every R-module is a direct sum of cyclic submodules [6, Theorem 6.7].

Theorem 3. *For a ring R, the following statements are equivalent:*

(i) *R is a regular multiplication ring.*

(ii) *Every R-module whose annihilator contains a regular element is a direct sum of cyclic submodules.*

Proof. (i) ⇒ (ii). Let M be an R-module with $\text{Ann}(M)$ containing a regular element. Then $R/\text{Ann}(M)$ is a finite direct sum of special principal ideal rings and so an Artinian principal ideal ring. Hence M as an $R/\text{Ann}(M)$-module is a direct sum of cyclic submodules. But M as an R-module and as an $R/\text{Ann}(M)$-module is one and the same. Therefore it follows that M as an R-module is a direct sum of cyclic submodules.

(ii) ⇒ (i). Let r be a regular element of R. Then any R/Rr-module is an R-module whose annihilator contains r, and hence is a direct sum of cyclic R-submodules. Therefore it follows that any R/Rr-module is a direct sum of cyclic submodules. Hence by the above remark, R/Rr is an Artinian principal ideal ring. Therefore by Theorem 1, R is a regular multiplication ring. □

Theorem 4. *For a ring R the following statements are equivalent:*

(i) *R is a regular multiplication ring.*

(ii) *Every finitely generated torsion R-module is of finite length and is a direct sum of cyclic submodules.*

Proof. (i) ⇒ (ii). Let M be a finitely generated torsion R-module. Then $\text{Ann}(M)$ contains a regular element and hence by Theorem 3, M is a direct sum of cyclic submodules. Since $R/\text{Ann}(M)$ is an Artinian principal ideal ring and M a finitely generated $R/\text{Ann}(M)$-module, it follows that M is of finite length (both as an $R/\text{Ann}(M)$-module and as an R-module).

(ii) ⇒ (i). Let r be a regular element of R. Since, by hypothesis, as an R-module $\bar{R} = R/(r)$ is of finite length, it follows that $\bar{R}$ is an Artinian ring and so a direct sum of local Artinian rings, say $\bar{R} = \bar{R}_1 \otimes \bar{R}_2 \oplus \cdots \oplus \bar{R}_n$. Since any finitely generated $\bar{R}$-module is a finitely generated torsion R-module, it therefore follows that every finitely generated $\bar{R}$-module is a direct sum of cyclic submodules. But then the same is true for any finitely generated $\bar{R}_i$-module ($1 \le i \le n$) (see [6], pp. 164–165). Therefore each $\bar{R}_i$ is an almost maximal valuation ring [3]. Since each $\bar{R}_i$ is an Artinian almost maximal valuation ring, it is an Artinian principal ideal ring (see [6, p. 185], where a reference is given to A.I. Uzkov). But then it follows that $\bar{R} = R/(r)$ is an Artinian principal ideal ring. Therefore by Theorem 1, R is a regular multiplication ring. □

Let M be an R-module and x an element of M. Then x is said to be regular if $\mathrm{Ann}_R(x)=0$. If the module M has a regular element, then we call M a regular R-module.

Let M and N be two R-modules with M a submodule of N. We say that $M \subseteq N$ is distributive if $M \cap (X+Y)=(M \cap X)+(M \cap Y)$ for all submodules X, Y of N.

Theorem 5. *Let R be a ring and M a regular R-module. Suppose that every regular submodule of M is a multiplication R-module. Then*

(i) *R is a regular multiplication ring.*

(ii) *Every regular submodule of M is a distributive submodule of M.*

(iii) *Every finitely generated regular submodule of M is projective of rank one.*

Proof. (i) Let $x \in M$ be regular, so $Rx \simeq R$ and every regular submodule of Rx is a multiplication R-module. Hence, since $Rx \simeq R$, every regular ideal of R is a multiplication R-module. Therefore R is a regular multiplication ring.

(ii) Let X be any regular submodule of M and let P be any maximal ideal of R such that $X_P \subsetneqq M_P$. Then there exists an element m in M such that $m/1 \in M_P$ and $m/1 \notin X_P$. Hence a fortiori such an m is not in X. So we have $X \subsetneqq X+Rm$. Since X is a regular submodule of M, it follows that $X+Rm$ is a regular submodule of M. Therefore $X+Rm$ is a multiplication R-module. Hence $X=I(X+Rm)$, for some ideal I of R. Now by localizing $X=I(X+Rm)$ at P, we get $X_P=I_PX_P+I_P(m/1)$. Clearly $I_P \neq R_P$ (because $m/1 \notin X_P$). That is, I_P is contained in the maximal ideal P_P of R_P. Since X is a regular submodule of M, X is a multiplication R-module. Hence X_P is a multiplication R_P-module [1, pp. 760–761]. Therefore, X_P is a cyclic R_P-module [1, Theorem 1]. Hence by Nakayama's Lemma, it follows that $X_P= I_P(m/1)$ and so $X_P=I_P(m/1) \subseteq R_P(m/1)$. Hence by [2, Lemma 2.7], it follows that $X_P \subseteq M_P$ is R_P-distributive, for all $P \in \mathrm{MaxSpec}\, R$. Therefore $X \subseteq M$ is R-distributive [2, Lemma 2.6].

(iii) Let X be a finitely generated regular submodule of M. Then as in the proof of (ii) above, X_P is a cyclic R_P-module, for all $P \in \mathrm{MaxSpec}\, R$. Since X_P is a cyclic regular multiplication R_P-module, it follows that $X_P \simeq R_P$, for all $P \in \mathrm{MaxSpec}\, R$. Hence X is a projective R-module of rank one. □

References

[1] D.D. Anderson, Multiplication ideals, multiplication rings, and the ring $R(X)$, Canad. J. Math. 28 (1976) 760–768.

[2] T.M.K. Davison, Distributive homomorphisms of rings and modules, J. Reine Angew. Math. 271 (1974) 28–34.

[3] D.T. Gill, Almost maximal valuation rings, J. London Math. Soc. 4 (2) (1971) 140–146.

[4] M. Griffin, Prüfer rings with zero divisors, J. Reine Angew. Math. 239/240 (1969) 55–67.

[5] M.D. Larsen and P.J. McCarthy, Multiplicative Theory of Ideals (Academic Press, New York, 1971).

[6] D.W. Sharpe and P. Vámos, Injective modules (Cambridge University Press, Cambridge, 1972).

[7] O. Zariski and P. Samuel, Commutative Algebra, Vol. 1 (Springer, Berlin, 1958).