Europ. J. Combinatorics (1990) **11**, 459–471

Extended Partial Geometries: Dual 2-Designs

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The study of *EpGs* (extended partial geometries) is continued from [5]; here we consider extended dual 2-designs (EDDs), especially one-point and triangular structures. We concentrate on three interesting cases: semibiplanes, which are extensions of the duals of trivial 2-designs (Section 4); extensions of dual projective geometries (Section 7); extensions of dual affine geometries (Section 8). We find non-existence theorems, examples, uniqueness theorems, and many open questions.

1. INTRODUCTION

In [5] the concept of an *extended partial geometry* (*EpG*) was introduced, and we refer the reader to that paper for basic definitions, terminology, etc. (but we shall recall some of the important definitions and results below). In this paper we continue the study of *EpG*s, especially extensions of dual 2-designs. In particular, we investigate triangular extensions and one-point extensions, and especially for the designs which are duals of the point–line structures of projective and affine geometries. Throughout this paper the dual of the point–block structure $\mathcal{S}$ is indicated by $\mathcal{S}^{\mathrm{T}}$.

In Sections 2 and 3 we establish some basic and simple results for triangular and one-point extensions, respectively. Then in Section 4 we apply this to circle geometries and their duals, which lead to semibiplanes. In Section 5 we study extensions of dual unitals, and in Section 6 a class of dual 2-designs associated (sometimes) with a hyperoval in a projective plane. Finally Sections 7 and 8 are devoted to the duals of projective and affine geometries, respectively. (See [7] for the definition of *biplane* and *semibiplane*.)

DEFINITION 1.1 Let $\mathcal{S}$ be a structure with two types of elements, called *points* and *blocks*:

(a) Two distinct points p and q are *collinear* if there is a block containing them both.

(b) If p is a point of $\mathcal{S}$, then the *residue* $\mathcal{S}_p$ of $\mathcal{S}$ at p is the structure the points of which are collinear with p, and the blocks of which are incident with p.

(c) $\mathcal{S}$ is an *extension* of a family $\mathcal{F}$ of structures if $\mathcal{S}$ is connected and every residue $\mathcal{S}_p$ is in $\mathcal{F}$.

(d) $\mathcal{S}$ is a *one-point extension* of $\mathcal{F}$ if $\mathcal{S}$ is an extension of $\mathcal{F}$ and every two points of $\mathcal{S}$ are collinear.

(e) $\mathcal{S}$ is *triangular* if whenever three points x, y and z are pairwise collinear, then there is a block of $\mathcal{S}$ which contains all three.

(f) If p is a point and y a block of $\mathcal{S}$, then (p, y) is an *antiflag* if p is not on y; by $\varphi(p, y)$ we mean the number of points of y which are collinear with p.

(g) If $\mathcal{S}$ is a structure such that every two points are on at most one common block, every block has $s+1$ points, every point is on $t+1$ blocks (with $s>0$, $t>0$), and $\varphi(p, y)=\alpha>0$ for every antiflag, then $\mathcal{S}$ is a *α-partial geometry of order* (s, t), or a $pG_\alpha(s, t)$, or a pG_α or a pG. The blocks of a pG are usually called *lines*. An extension of the family of pG_α's is an EpG_α, or an EpG.

(h) An EpG is *φ-uniform* (or merely uniform) if every $\varphi(p, y)=0$ or φ; it is *strongly* φ-uniform if every $\varphi(p, y)=\varphi$.

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In [5] it was shown among other things that in an EpG_α $\mathcal{S}$, for any antiflag (p, y) if $\varphi(p, y) \neq 0$, then $\varphi(p, y) \geq \alpha + 1$, and that $\mathcal{S}$ is triangular iff all $\varphi(p, y) = 0$ or $\alpha + 1$; so $\mathcal{S}$ is triangular if it is $(\alpha + 1)$-uniform. Furthermore, if φ_0 is the minimum non-zero value of $\varphi(p, y)$, then the diameter of the point-graph $\Gamma(\mathcal{S})$ of an $EpG_\alpha(s, t)$ is bounded by $\max\{2, s + 5 - 2\varphi_0\}$. Also in [5] will be found:

THEOREM 1.2. *If $\mathcal{S}$ is an $EpG_\alpha(s, t)$, and is strongly φ-uniform with $\varphi \leq s$, then its point-graph Γ is non-trivial strongly regular with parameters*:

$$v = 1 + (s + 1)[1 + st(s + 2)/\varphi\alpha], \qquad k = (s + 1)(1 + st/\alpha),$$
$$\lambda = s + st(\varphi - 1)/\alpha, \qquad \mu = \varphi(1 + st/\alpha).$$

Furthermore, the eigenvalues of Γ are (besides k) $s + 1 - \varphi$ and $-(1 + st/\alpha)$, and α divides st.

Throughout this paper, note that 'pG' means *partial geometry* as above. But since we also consider *projective geometries* in this paper, we point out that $PG(n, q)$ means the projective geometry of (geometric) dimension n over the field of q elements; similarly, $AG(n, q)$ is the affine geometry. We are only interested as a rule in the point–line structure of projective and affine geometries, so these notations usually refer to those 'rank two' structures only. (But in Section 7, in dealing with algebraic varieties, e.g. Grassmanians, we use these notations to refer to the entire projective geometry, with its objects of various dimensions; this should cause no confusion.)

2. TRIANGULARITY

In [5] it is shown that if $\mathcal{S}$ is an EpG_α, then every residue $\mathcal{S}_p$ has the same order (s, t), and so we can refer to $\mathcal{S}$ as an $EpG_\alpha(s, t)$. In fact, it is easy to see that even if $\mathcal{S}$ is an extension of the family of pGs, then the parameters (s, t) are constant for all residues. But an example (due to Pasini) is given in [5] to show that without the insistence that α be constant, it is possible to have residues with different values of α. But another result in that paper, also due to Pasini, shows that if $\mathcal{S}$ is triangular, then α is constant.

DEFINITION 2.1. Let $\mathcal{S}$ be a $pG_\alpha(s, t)$. Then a subset $\mathcal{C}$ of the points of $\mathcal{S}$ is called a *pre-oval* if $|\mathcal{C}| \geq \alpha + 1$ and every point of $\mathcal{C}$ is joined to exactly α other points of $\mathcal{C}$, by α distinct lines (so every line of $\mathcal{S}$ meets $\mathcal{C}$ in ≤ 2 points). A pre-oval $\mathcal{C}$ such that $|\mathcal{C}| = \alpha + 1$ is called a *superoval*. A *triangle* in $\mathcal{S}$ is a set of three points which do not lie on a common line, but such that any two points are collinear.

(Note that a superoval in a projective plane of order q is thus a set of $q + 2$ points, no three on a line: i.e. a standard hyperoval. A superoval in an affine plane of order q is a set of $q + 1$ points, no three collinear; i.e. an affine oval. But a superoval in a dual affine plane of order q—thought of as a projective plane with a point and all its lines deleted—is, *together with the deleted point*, a set of $q + 2$ points no three collinear; i.e. a hyperoval again. So superovals can exist in the dual of a projective plane or of an affine plane only if the plane has even order; see for example [6]. In Theorem 8.1 there is an important application of this.)

LEMMA 2.2. *Suppose that $\mathcal{S}$ is an $EpG_\alpha(s, t)$ and y is a block not on the point p, but containing points of $\mathcal{S}_p$. Then the set $y \cap S_p$ is a pre-oval in $\mathcal{S}_p$, and is a superoval if $\mathcal{S}$ is triangular. If $\mathcal{S}$ is triangular, then every triangle of $\mathcal{S}_p$ lies in exactly one superoval $y \cap \mathcal{S}_p$.*

PROOF. First suppose that q is a point of $\mathcal{S}_p$, on the set $y \cap \mathcal{S}_p$. Then p is a point in $\mathcal{S}_q$ and y is a line of $\mathcal{S}_q$, with p not on y. So there are α lines of $\mathcal{S}_q$ through p which intersect y, and hence in $\mathcal{S}_p$ q is joined to exactly α points of $y \cap \mathcal{S}_p$, and by α distinct lines. Hence $y \cap \mathcal{S}_p$ is a pre-oval in $\mathcal{S}_p$. If $\mathcal{S}$ is triangular, then $|y \cap S_p| = \alpha + 1$, so $y \cap \mathcal{S}$ is a superoval in $\mathcal{S}_p$. Finally, any triangle in $\mathcal{S}_p$ must, by the triangularity of $\mathcal{S}$, lie in a single block y, and y cannot contain p. (Note that in a generalized quadrangle superovals are simply pairs of collinear points.) □

COROLLARY 2.3. *If $\mathcal{S}$ is a triangular $EpG_\alpha(s, t)$, then $\alpha(\alpha+1)$ divides M, where*

$$M = st(s+1)(t+1)(1+st/\alpha).$$

PROOF. Let N be the number of superovals of the form $y \cap \mathcal{S}_p$ in the pG $\mathcal{S}_p$. We count pairs $(\mathcal{H}, \Delta)$, where $\mathcal{H}$ is such a superoval, and Δ is an ordered triangle in $\mathcal{H}$. For each Δ, there is one $\mathcal{H}$. There are $v = (1+s)(1+st/\alpha)$ points in $\mathcal{S}_p$, each is joined to $s(1+t)$ points, and given two collinear points in $\mathcal{S}_p$, there are $t(\alpha-1)$ points not on their common line but collinear with both. So the number of ordered triangles Δ is $st(s+1)(t+1)(1+st/\alpha)(\alpha-1)$.

On the other hand, each superoval $\mathcal{H}$ contains $(\alpha+1)\alpha(\alpha-1)$ ordered triangles, so $N(\alpha+1)\alpha(\alpha-1) = st(s+1)(t+1)(1+st/\alpha)(\alpha-1)$. If $\alpha > 1$, we have our result; if $\alpha = 1$, the conclusion is trivial. □

This last corollary is rather analogous to the result which says that the number of blocks in a one-point extension must be an integer, and has similarly wide-ranging consequences, as we see later.

Suppose that $\mathcal{S}$ is an extended dual 2-design, which we abbreviate to 'EDD'. By this we shall mean that every residue is a dual 2-design, although often it suffices for our results in this paper to impose the condition that at least one residue has the property; that is usually the case, for instance, when $\mathcal{S}$ is triangular, and when $\mathcal{S}$ is a one-point extension. Let $\mathcal{S}_p$ be the dual of a 2-$(v, k, 1)$, with $r = (v-1)/(k-1)$ lines on a point, and $b = vr/k$ lines in all. Then $\mathcal{S}_p$ is a $pG_k(r-1, k-1)$, with b points and v lines. A superoval in $\mathcal{S}_p$ corresponds to a set of $\mathcal{L}$ of $k+1$ lines in $\mathcal{S}_p^{\mathrm{T}}$ such that any two meet, and any point on one of the lines of $\mathcal{L}$ is on exactly two lines of $\mathcal{L}$. If there *are* 'planes' in $\mathcal{S}_p^{\mathrm{T}}$, then $\mathcal{L}$ and all its points lie in one plane. Note that $\mathcal{S}$ is triangular if it is $(k+1)$-uniform. Substituting $s = r-1$, $t = k-1$, $\alpha = k$ in Corollary 2.3 and simplifying, we have:

COROLLARY 2.4. *If $\mathcal{S}$ is a triangular EDD, and $\mathcal{S}_p^{\mathrm{T}}$ is a 2-$(v, k, 1)$ with b blocks, then $k+1$ divides $b(v-k)$.*

If Corollary 2.4 is applied to the dual $(PG(n, q))^{\mathrm{T}}$ of a projective geometry, it will imply that for each fixed value of n there are only finitely many possible choices for q. But we shall show in Section 7 that a better result can be obtained with a more detailed analysis.

All known triangular EDDs fall into a few classes: they are either semibiplanes (the associated 2-designs of which are trivial; see Section 4), extensions of dual affine geometries (see Section 8), or extensions of dual projective geometries (see Section 7). The existence of dual superovals (indeed of *many* dual superovals) seems to be a powerful restriction on a 2-design, and as pointed out in Theorem (7.1) their existence suggests that the 2-design has 'planes'. There are triangular extended generalized quadrangles (see [2]), and there are triangular extended nets and dual nets (see [5]); these triangular structures are among the 'richest' examples of their classes.

3. One-point Extensions

Here we consider some one-point extensions. In this case, if $\mathcal{S}$ is a one-point $EpG_\alpha(s, t)$, then $\mathcal{S}$ is a 2-design with $w = 1 + (1+s)(1+st/\alpha)$ points, $m = s+2$ points on a block, and $\lambda = t+1$ on two points. The number of blocks in $\mathcal{S}$ is given by $b^* = \lambda w(w-1)/m(m-1)$; that b^* is an integer is often a very powerful restriction. A one-point EDD is exactly a 'quasi-symmetric 2-design' with intersection numbers 0 and 2, and much additional information will be found about this case in, for example [1, 8, 11] (and see Theorem 3.4).

Theorem 3.1. *For any fixed value of $k > 2$, there are only finitely many values of v for which there is a one-point extension of a dual 2-$(v, k, 1)$.*

Proof. If $\mathcal{S}$ is a one-point extension of a dual 2-$(v, k, 1)$, then $\mathcal{S}$ is a 2-design with $v(v-1)/k(k-1)+$ points, $(v-1)/(k-1)+1$ points on a block, and $\lambda = k$. So the number b^* of blocks in $\mathcal{S}$ is

$$b^* = v[v(v-1) + k(k-1)]/k(v+k-2). \tag{1}$$

Now put $v \equiv -k+2 \pmod{v+k-2}$, in the numerator of (1), to find that

$$2(k-1)^2(k-2) \equiv 0 \pmod{v+k-2}. \tag{2}$$

Since $k > 2$, this implies that $v \leqslant (k-2)(2k^2-4k+1)$. □

For small values of k, there are not many candidates for one-point EDDs; some of the existing ones are interesting, while others are connected to interesting unsettled problems:

Corollary 3.2. *Suppose that $\mathcal{S}$ is a one-point EDD with residue $\mathcal{S}_p$ isomorphic to the dual of a 2-$(v, k, 1)$. We have:*
(i) *if $k = 3$, then $\mathcal{S}$ is the unique* 3-(8, 4, 1);
(ii) *if $k = 4$, then $\mathcal{S}$ is the unique* 2-(21, 6, 4) *isomorphic to the external restriction of the Mathieu design* 3-(22, 6, 1);
(iii) *if $k = 5$, then $\mathcal{S}$ is the unique* 3-(22, 6, 1), *or is a* 2-(100, 12, 5) *with* 375 *blocks*;
(iv) *if $k = 6$, then $v = 46$ or 96 and $\mathcal{S}$ is a* 2-(70, 10, 6) *or a* 2-(305, 20, 6).

Proof. We examine the congruence of (2) in the proof of Theorem 3.1 more carefully:
(i) If $k = 3$, then $v+1$ divides 8, so $v = 7$, hence $\mathcal{S}_p^{\mathrm{T}}$ is the projective plane of order 2, and the conclusion is easy.
(ii) If $k = 4$, then $v+2$ divides 36, and so $v = 7$, 10, 16 or 34. But $v = 7$ implies that $\mathcal{S}_p^{\mathrm{T}}$ is the biplane 2-(7, 4, 2), which has no extension; $v = 10$ and 34 are not possible, since the 2-designs for (10, 4, 1) and (34, 4, 1) do not exist (they would not have an integral number of blocks). But $v = 16$ is possible, and the structure exists as described (see Theorem 8.3 for another discussion of this design).
(iii) If $k = 5$, then $v+3$ divides 96, so $v = 9$, 13, 21, 29, 45 or 93. Since $k(k-1) = 20$ must divide $v(v-1)$, this implies $v = 21$ or $v = 45$. Now $v = 21$ simply gives us for $\mathcal{S}$ the 2-design 2-(22, 6, 5) isomorphic to the Mathieu design 3-(22, 6, 1); $v = 45$ implies that $\mathcal{S}_p^{\mathrm{T}}$ is a 2-(45, 5, 1), with 99 blocks, and $\mathcal{S}$ will be a 2-(100, 12, 5) with 375 blocks (see the comments at the end).
(iv) If $k = 6$, then we proceed as above to find that $v+4$ divides 200, so $v = 16$, 21, 36, 46, 96 or 196. But $v = 16$ or 21 means that $\mathcal{S}_p^{\mathrm{T}}$ is a 2-design violating the Fisher

inequality (see [7]) that $b \geqslant v$; $v = 36$ would imply that $\mathcal{S}_p^{\mathrm{T}}$ is an affine plane of order 6, which does not exist. If $v = 196$, then $\mathcal{S}$ is a 2-(1275, 40, 6), but such a design cannot have an integral number of blocks, so does not exist. This leaves $v = 46$ or 96, and the cases of the Corollary. □

The structures that arise in Corollary 3.2 are interesting. The 3-designs of cases (i) or (iii) have triply transitive groups (respectively $AGL(3, 2)$ and $\overline{M}_{22}$); the 2-design of case (ii) has a doubly transitive group $P\Sigma L(3, 4)$. The 2-(100, 12, 5) of case (ii) would have existed had there been an extended projective plane of order 10, as the set of points off a fixed block and the set of blocks not meeting that block. But its existence still seems to be unsettled. The 2-(46, 6, 1) appears to be unsettled as well, and some doubt has been cast on its existence; hence the 2-(70, 10, 6) might not exist either. And this last design, the quasi-symmetric 2-design for (70, 10, 6), also arises in [1] as a 'smallest' unsettled case. Some of these EDDs can themselves be extended, more than once:

Example 3.3. (i) Let $\mathcal{S}^*$ be the external restriction of the Mathieu 5-design for (24, 8, 1); so $\mathcal{S}^*$ is a 4-(23, 8, 4). Then any residue of $\mathcal{S}^*$ is an extension of the 2-(21, 6, 4) of Corollary 3.2 (ii).
(ii) Let $\mathcal{S}^*$ be the Mathieu 5-design for (24, 8, 1). Then any residue of $\mathcal{S}^*$ is itself an extension of the 3-(22, 6, 1) of Corollary 3.2 (iii).

(These two examples have, respectively, M_{23} and M_{24} as automorphism groups, and the first casts the Mathieu group M_{23} in yet another 'new' role.)

In Theorem 3.1 and Corollary 3.2 there seems to be considerable additional information not really utilized: that the residue $\mathcal{S}_p$ is a dual 2-design. But this simply reflects the fact that $\mathcal{S}$ is a quasi-symmetric 2-design and it appears that this observation, by itself, contributes little new to classifying EDDs. The techniques of Theorem 3.1 can be trivially adapted to prove Theorem 3.4 below, about quasi-symmetric designs; but these are all part of more general results to be found, for example, in [8, 11].

Theorem 3.4. *For fixed values of ρ and λ, with $\rho \neq \lambda$, $\rho \neq 1$, the family of quasi-symmetric 2-designs for (v, k, λ), with intersection numbers 0 and ρ, is finite.*

4. Circle Geometries, Their Duals, and Semibiplanes

There is a class of trivial pG's the extensions of which have some interest. A 2-design for $(n+2, 2, 1)$ is trivial (as a 2-design), and is a $pG_2(1, n)$; it is sometimes called a *circle geometry*. Its dual is a $pG_2(n, 1)$ (and conversely). These always have extensions:

Example 4.1. (i) Let T be a $pG_2(1, n)$. Then there is an infinite tower of extensions of T, being successively the trivial $(2+j)$-designs for $(n+2+j, 2+j, 1)$; these extensions are unique.
(ii) Let $\mathcal{S}$ be an $EpG_2(n, 1)$. Then $\mathcal{S}$ is a semibiplane with $n+2$ points on a block, and conversely any semibiplane is such an EpG.

Proof. (i) is easy. For (ii) we refer to first principles: in a $pG_2(n, 1)$ every two lines meet once, so in an extension, if two blocks meet, they meet exactly one more time; while every point of a $pG_2(n, 1)$ is on 2 lines, so if two points are collinear, then they are on two blocks. But we could also notice that a $pG_2(1, n)$ is a geometry for the diagram $\circ\overset{\mathrm{c}}{\text{—}}\circ$, and hence a $pG_2(n, 1)$ goes with $\circ\overset{\text{ↄ}}{\text{—}}\circ$. Thus an extension of this last geometry must be a geometry for $\circ\overset{\mathrm{c}}{\text{—}}\circ\overset{\text{ↄ}}{\text{—}}\circ$, which is well known to be the diagram for semibiplanes. □

The case of triangular semibiplanes (i.e. a triangular $EpG_2(1, n)$) is different: the conditions of Corollary 2.3 are satisfied by every semibiplane. But they are not all triangular, since $\varphi(p, y) = 3$ is often violated in semibiplanes (and always in biplanes unless the block size is 3). But, in fact, triangular examples exist of every block size:

EXAMPLE 4.2. (i) Let $\mathscr{B}$ be a triangular semibiplane with incidence matrix B. Then the semibiplane $\mathscr{B}_1$, with incidence matrix

$$\begin{vmatrix} B & I \\ I & B^{\mathrm{T}} \end{vmatrix},$$

is triangular ($\mathscr{B}_1$ is called the *double* of $\mathscr{B}$.)
(ii) There is a triangular semibiplane with $v = 2^{k-1}$ points and k points on a block, for every value of $k \geqslant 3$.

PROOF. That $\mathscr{B}_1$ is a semibiplane is standard; see for instance [7]. To prove (i) we need to see that if p is a point of $\mathscr{B}_1$, then two points in its residue are collinear in $\mathscr{B}_1$ if they are collinear in the residue. Now $\mathscr{B}_1$ is the union of $\mathscr{B}$ and $\mathscr{B}^{\mathrm{T}}$, i.e. if x is a point of $\mathscr{B}$, then x is a point of $\mathscr{B}_1$ and x^{T} is a block of $\mathscr{B}_1$, while if y is a block of $\mathscr{B}$, then y is a block of $\mathscr{B}_1$ and y^{T} is a point of $\mathscr{B}_1$. Incidence is 'natural': that is, x is on x^{T} and is on y if x is on y in $\mathscr{B}$, and so on. We may suppose that p is in $\mathscr{B}$. If c and d are points in the residue of p, then there are three cases: (i) $c, d \in \mathscr{B}$; (ii) $c, d \in \mathscr{B}^{\mathrm{T}}$; (iii) $c \in \mathscr{B}$, $d \in \mathscr{B}^{\mathrm{T}}$. With this much guidance, it is easy to finish the proof, and we omit it.

For (ii) we start with the triangular biplane which is a 2-design for $(4, 3, 2)$, and the construction of (i) produces our infinite family. □

EXAMPLE 4.3. (i) Let $V = V_n(2)$ be the n-dimensional vector space over $GF(2)$, and let D be a spanning set of $k > n$ vectors in V such that no sum of 6 vectors in D is the zero vector. Then if $\mathscr{S}$ is the structure the points of which are the vectors of V and the blocks of which are the point sets $D + \mathbf{v}$ as $\mathbf{v}$ ranges over V, then $\mathscr{S}$ is a triangular semibiplane. The first 'new' example of this sort is for $n = 6$, $k = 8$, with

$$D = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_6, \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_6\},$$

where $\{\mathbf{e}_i\}$ is a basis of V. Then $\mathscr{S}$ has 64 points and block size 8. As n grows, these semibiplanes are smaller than those of Example 4.2.
(ii) The structures of Example 8.2 are triangular semibiplanes, and when $n > 3$, they give examples with a smaller number of points than those obtained from Example 4.2. (e.g. if $n = 4$, this gives $k = 16$, $v = 2^{10}$).

Example 4.3 (i) is based on a suggestion of Peter Wild's; there are other triangular semibiplanes yet, constructed by Wild in [12] for instance. A triangular semibiplane with block-size k is equivalent to a regular graph Γ with the property that for every vertex p, the neighbourhood-graph Γ_p is the complement of a triangular graph $T(k)$. Hence we pose the natural problem (see also the comments following Example 6.3):

PROBLEM 4.4. Find all triangular semibiplanes.

5. Unitals

The original unitals were the 2-designs of absolute points and non-absolute lines of a unitary polarity of a projective plane $PG(2, q^2)$, and the name is now given to any 2-design for $(q^3+1, q+1, 1)$, $q>1$, examples of which exist in non-Desarguesian planes, and even coming from no projective plane at all. Their duals do not appear to have very many extensions. Here Corollary 2.4 works fairly well:

Corollary 5.1. *If $\mathcal{S}$ is a triangular EDD with a residue which is a dual unital, then q is one of* $\{2, 4, 5, 6, 10, 12, 19, 22, 26, 40, 54, 82, 166\}$. *There is no such extension $\mathcal{S}$ which is strongly $(q+2)$-uniform.*

Proof. In this case we have $b = q^2(q^2-q+1)$, so Corollary 2.4 asserts that $q+2$ divides $q^3(q-1)(q^3+1)$. Put $q \equiv -2 \pmod{q+2}$ in this last expression, which implies that $q+2$ divides 168, and we have the first part.

For the second part apply Theorem 1.2, and compute the multiplicities of the eigenvalues: none of the values of q give integral multiplicities. □

We have little idea whether any of these extensions exist: it is a consequence of Theorem 7.1 below that such an $\mathcal{S}$ does not exist for $q=2$. Since unitals themselves are known to exist for q a prime power, and a few other values such as $q=6$, it might be worthwhile to investigate at least the cases $q=4$, 5, 6 and 19. The discussion preceding Corollary 2.4 implies that the unital contains a dual superoval, i.e. a set $\mathcal{L}$ of $q+2$ lines such that each point on any of these lines is on exactly two of the lines. Such substructures might not be easy to find.

Theorem 5.2. *No dual unital has a one-point extension.*

Proof. If $\mathcal{S}$ is a one-point extension of a dual unital, then it is a

$$2\text{-}(q^2(q^2-q+1)+1,\ q^2+1,\ q+1),$$

and so has $b^* = (q+1)(q^4-q^3+q^2+1)(q^2-q+1)/(q^2+1)$ blocks. In the standard way, put $q^2 \equiv -1 \pmod{q^2+1}$ in the numerator of b^*; this gives $2 \equiv 0 \pmod{q^2+1}$. This is impossible. □

6. A Class Associated with Hyperovals

If $\mathcal{P}$ is a projective plane of even order q, with a hyperoval $\mathcal{Y}$ (a set of $q+2$ points, no three collinear), then the structure of the q^2-1 points not on $\mathcal{Y}$ and of the $q(q-1)/2$ lines which do not meet $\mathcal{Y}$ is itself a dual 2-design. Its dual is a $2\text{-}(q(q-1)/2, q/2, 1)$; however, such designs do not have to come from projective planes. We consider them next.

Corollary 6.1. *Suppose $\mathcal{S}$ is a triangular EDD with a residue which is the dual of a* $2\text{-}(q(q-1)/2, q/2, 1)$, $q>2$. *Then* $q \in \{4, 6, 10, 22\}$. *If, in addition, $\mathcal{S}$ is strongly $(q+2)/2$-uniform, then $q=6$.*

Proof. As in Corollary 5.1: here $b = q^2-1$, so $q+2$ must divide $q(q^2-1)(q-2)$, which implies that $q+2$ divides 24, so q has one of the given values.

If $\mathcal{S}$ is strongly uniform as well, then we refer to Theorem 1.2 and compute the multiplicities m_1, m_2 of the eigenvalues ρ_1 and ρ_2 (which are, respectively, $s+1-\varphi =$

$q/2$ and $-(1+st/\alpha)=-q+1$) from the equations:

$$m_1+m_2=v-1=(q+1)(2q-3),$$
$$(q/2)m_1-(q-1)m_2=-k=-(q+1)(q-1).$$

This gives $(3q-2)m_1=4(q^2-1)(q-2)$; but for $q=10$ or 22, $3q-2$ does not divide $4(q^2-1)(q-2)$.

If $q=4$, then the graph $\Gamma(\mathscr{S})$ would be SR with parameters $(26, 15, 8, 9)$, and the structure would be in fact a semibiplane with block size 6 (see Example 4.1 (ii)). Its 26 blocks must come from 26 6-cliques in this graph, but no SR graph with these parameters has more than 13 such 6-cliques (private communication from Andries Brouwer). □

There is no projective plane for $q=6$, 10 or 22, but the 2-designs may exist, (as indeed they do for $q=6$) and may have extensions (triangular or not). The cases $q=4$ and 6 are interesting:

Example 6.2. For $q=6$, the dual of the 2-$(15, 3, 1)$ isomorphic to the projective geometry $PG(3, q)$ has:
(i) a strongly 4-uniform extension with 64 points, given in Example 7.3 (i) with $n=3$ (or also in 7.3 (iii)(a)); and
(ii) a triangular but not strongly uniform extension given in Example 7.3 (iii)(b), with 72 points.

Example 6.3. For $q=4$, the semibiplane of Example 4.2 (ii) with block size 6 is a triangular extension of a dual 2-$(6, 2, 1)$, with 32 points.

A triangular semibiplane with block size 6 has at least 26 points (and at most 32), and has 26 points if it is strongly uniform; as pointed out in the proof of Corollary 6.1 this structure does not exist. It can be shown that there is no triangular semibiplane of block size 6 with 27 or 28 points, and it seems plausible that the example with 32 points is unique. For $k=3$, 4, 5 the semibiplanes with 4, 8, 16 points, respectively, given by Example 4.2 can be shown to be the only triangular examples (see Problem 4.4).

Now we consider the one-point extensions of our class of dual 2-designs:

Corollary 6.4. *If $\mathscr{S}$ is a one-point EDD, one of the residues $\mathscr{S}_p$ of which is isomorphic to the dual of a 2-$(q(q-1)/2, q/2, 1)$, then $q=4$ and $\mathscr{S}$ is a biplane 2-$(16, 6, 2)$ (and conversely), or $q=10$ and $\mathscr{S}$ is a 2-design for $(100, 12, 5)$.*

Proof. A 2-$(q(q-1)/2, q/2, 1)$ has q^2-1 blocks and $q+1$ blocks on a point. So $\mathscr{S}$ is a 2-$(q^2, q+2, q/2)$, and its number of blocks is given by $q^3(q-1)/2(q+2)$. As usual, we put $q\equiv -2 \pmod{q+2}$ in the numerator to find that $q=4$, 6, 10, or 22. But $q=6$ or 22 does not give an integral number of blocks for $\mathscr{S}$.

If $q=4$, then $\mathscr{S}$ is a 2-$(16, 6, 2)$, i.e. one of the three biplanes with 16 points. Any one of these biplanes is an extension of a dual 2-$(6, 2, 1)$ (see also Example 4.1). Similarly, if $q=10$, then $\mathscr{S}$ is a 2-$(100, 12, 5)$; we have met this design before, in Corollary 3.2(iii), and as expressed in the proof there, the design might well not exist. □

7. Projective Geometries

In this section we consider one of the most interesting kinds of EDDs: extensions of dual projective geometries. If any EDD is triangular, then $\mathscr{S}_p^{\mathrm{T}}$, the dual of a residue, contains a dual superoval (see the discussion following Corollary 2.3); that is, a set of $k+1$ lines such that any two meet, and any point on one of the lines is on exactly two. This is suggestively close to Pasch's axiom; if the 2-design whose dual is extended has block size 3, it is exactly Pasch's axiom:

THEOREM 7.1. *If $\mathcal{S}$ is a triangular EDD and $\mathcal{S}_p^{\mathrm{T}}$ is a 2-$(v, 3, 1)$, then $\mathcal{S}_p^{\mathrm{T}}$ is the point–line structure of $PG(n, 2)$ for some n.*

PROOF. $\mathcal{S}_p^{\mathrm{T}}$ is a structure of points and lines in which any two points are collinear and in which Pasch's axiom holds: for if a, b and c are a triangle of points in $\mathcal{S}_p^{\mathrm{T}}$, then on any of the three lines x, y and z joining them two at a time there is a unique third point. Since the three lines x, y and z correspond to a triangle in $\mathcal{S}_p$, there is a fourth point in $\mathcal{S}_p$ which forms a superoval with x, y and z. But this fourth point in $\mathcal{S}_p$ corresponds to a fourth line in $\mathcal{S}_p^{\mathrm{T}}$ which contains the third point on each of the three lines x, y and z. This is Pasch's axiom and hence (e.g. see [3, 10]) $\mathcal{S}_p^{\mathrm{T}}$ is a projective geometry with three points on every line, and so is a $PG(n, 2)$. □

THEOREM 7.2. *If $\mathcal{S}$ is a triangular EDD with $\mathcal{S}_p$ isomorphic to $(PG(n, q))^{\mathrm{T}}$, then $q = 2$ or 4.*

PROOF. Every triangle of points in $\mathcal{S}_p$ is in a superoval $y \cap \mathcal{S}_p$ of $\mathcal{S}_p$: this implies that every triangle of $PG(n, q)$ is in a dual superoval of $PG(n, q)$. Superovals in $\mathcal{S}_p$ contain $k+1 = q+2$ lines. Any dual superoval lies completely in a plane $PG(2 q)$ of $PG(n, q)$ (and so it is a dual hyperoval of that plane). We shall count pairs $(\mathcal{H}, \Delta)$, where $\mathcal{H}^{\mathrm{T}} = y \cap \mathcal{S}_p$ as above, and Δ is a triangle in $\mathcal{H}$, all lying in one fixed plane of $PG(n, q)$. Suppose that there are T such dual superovals in the plane; each dual superoval contains $(q+2)(q+1)q$ triangles, and each triangle lies in one dual superoval. The number of triangles in a plane $PG(2, q)$ is $(q^2+q+1)(q^2+q)q^2$. So:

$$T(q+2)(q+1)q = (q^2+q+1)(q^2+q)q^2$$

and thus $q+2$ divides $q^2(q^2+q+1)$; this immediately implies that $q+2$ divides 12, so $q = 2$, 4 or 10. There is no projective plane of order 10, so $q = 2$ or 4. □

By an 'affine Grassmanian' construction, we can find triangular EDDs the residues of which are dual to a $PG(n, 2)$, for every $n > 2$. (See [4, 9] for more about Grassmanian varieties.)

EXAMPLE 7.3. Let $\mathcal{V} = \mathcal{V}_N(2)$ be the Grassmanian variety over $GF(2)$ of embedding dimension $N = n(n+1)/2 - 1$ (i.e. $\mathcal{V}$ lies naturally in $PG(N, 2)$), in which the point–line structure of $PG(n, 2)$ is 'dually' embedded, i.e. with points of $PG(n, 2)$ corresponding to the 'special' subspaces $PG(n-1, 2)$ of $\mathcal{V}$ and lines of $PG(n, 2)$ corresponding to points of $\mathcal{V}$ (see [4, 9]).

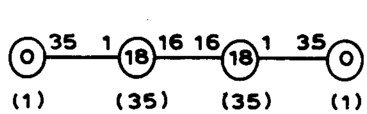
(i) Let $\mathcal{H}$ be a hyperplane in $PG(N+1, 2)$, with $\mathcal{V}$ embedded in $\mathcal{H}$. Let $\mathcal{S}$ be the structure the points of which are the points of $AG(N+1, 2) = PG(N+1, 2) \backslash \mathcal{H}$, and the blocks of which are the subspaces $PG(n, 2)$ which meet $\mathcal{H}$ in one of the special subspaces $PG(n-1, 2)$ of $\mathcal{V}$. Then $\mathcal{S}$ is a triangular EDD, all of whose residues are dual to $PG(n, 2)$.

(ii) In $PG(N+2, 2)$, let $\mathcal{W}$ be an algebraic variety such that some hyperplane $\mathcal{H}$ meets $\mathcal{W}$ in an algebraic variety $\mathcal{W} \cap \mathcal{H}$ (of $\mathcal{H}$) with the property: the set of points of $\mathcal{W} \cap \mathcal{H}$ which are collinear with a fixed point of $\mathcal{W} \backslash \mathcal{H}$, by lines that lie in $\mathcal{W}$ and meet $\mathcal{H}$ in one point only, is the point-set of a $\mathcal{V}$. Let $\mathcal{S}$ be the structure the points of which are the points of $\mathcal{W} \backslash \mathcal{H}$ and the blocks of which are the subspaces $PG(n, 2)$ in $\mathcal{W}$ which meet $\mathscr{H}$ in the special subspaces of some $\mathscr{V}_N(2)$. Then $\mathscr{S}$ is a triangular EDD all of whose residues are dual to $PG(n, 2)$.

(iii) If $n = 3$, then $N = 5$, and $\mathscr{V}_5$ is the quadric $Q_5^+(2)$ (minus some of its isotropic planes, since we only want the 'special' subplanes of one family in our construction, but this need not concern us here). Here we can carry out the construction of (ii) above in (at least) two ways. In each we let $\mathscr{W}$ be the quadric $Q_7^+(2)$, embedded in $PG(7, 2)$, and $\mathscr{H}$ is a hyperplane of $PG(7, 2)$:

(a) Let $\mathscr{H}$ be a tangent hyperplane to $\mathscr{W}$; so $\mathscr{W} \cap \mathscr{H}$ is a cone over a $Q_5^+(2)$, and the structure $\mathscr{S}$ will be an EDD with $\mathscr{S}_p^{\mathrm{T}}$ isomorphic to $PG(3, 2)$. Now $Q_7^+(2)$ has 135 points, and the cone over $Q_5^+(2)$ contains 71 points, so $\mathscr{S}$ has 64 points. This structure is isomorphic to the one constructed in (i) with $n = 3$. Its graph is SR with parameters $(64, 35, 18, 20)$.

(b) Let $\mathscr{H}$ be a secant hyperplane to $\mathscr{W}$; so $\mathscr{W} \cap \mathscr{H}$ is a $Q_6(2)$, and the structure $\mathscr{S}$ will be an EDD with $\mathscr{S}_p^{\mathrm{T}}$ isomorphic to $PG(3, 2)$. Since $Q_6(2)$ has 63 points, $\mathscr{S}$ will have 72 points. It is triangular, not strongly uniform, but distance regular. Its point-graph $\Gamma(\mathscr{S})$ has 'picture':



It can be shown that if $\mathscr{S}$ is a triangular extension of a dual $PG(n, 2)$, and also *strongly* uniform, then $n = 3$ or 5 (the proof is like that of Theorem 7.4, but we omit it). The example above with $n = 3$ and 64 points is strongly uniform, but whether there is a strongly 4-uniform extension of a dual $PG(5, 2)$ is not known: it would have 4992 points, and hence is not constructed as in Example 7.3(i), for the example there with $n = 5$ has 2^{15} points.

We know of no example of a triangular EDD which extends a dual $PG(n, 4)$, except for the case $n = 2$, where $\mathscr{S}$ is the Mathieu 3-design for $(22, 6, 1)$. Such an extension would be 6-uniform; it cannot be strongly uniform as well:

THEOREM 7.4. *If $n > 2$, then $(PG(n, 4))^{\mathrm{T}}$ never has a strongly 6-uniform extension.*

PROOF. If such an extension $\mathscr{S}$ exists, then its point–graph Γ is strongly regular. Using Theorem 1.2, we first see that the condition 'α divides st' implies that n is odd. But the more important results come from computing the multiplicities of the eigenvalues. The eigenvalues are $\rho_1 = 4^2(4^{n-3} + 4^{n-4} + \cdots + 4 + 1) - 1$, and $\rho_2 = -[1 + 4^2(4^{n-2} + 4^{n-3} + \cdots + 4 + 1)]$, and the equations for the multiplicities m_1 and m_2 are:

$$m_1 + m_2 = v - 1, \qquad \rho_1 m_1 + \rho_2 m_2 = -k.$$

Multiplying the first of these by $-\rho_2$ and adding to the second, we find an equation of the form $Am_1 = B$, and if $n \geq 3$, then $A \equiv 0 \pmod{32}$ while $B \equiv 16 \pmod{32}$, which is a contradiction (we leave the straightforward but messy details to the reader). □

PROBLEM 7.5. There remain some problems, and possible conjectures:

(1) Is it possible to find algebraic varieties $\mathscr{W}$ of embedding dimension $N + 2$ with the structure of Example 7.3(ii), as in the case $N = 5$? (By analogy with that situation, and with quadrics in general, it seems reasonable to expect a couple of ways to do this, for each N; it also seems reasonable to expect one of them to be isomorphic with the example constructed in 7.3(i).) If $\mathscr{W}$ can be found, what are the graphs $\Gamma(\mathscr{S})$ which result?

(2) Is it reasonable to conjecture that all triangular extensions of $(PG(n,2))^{\mathrm{T}}$, $n>2$, are constructed as in Example 7.3?
(3) Is it reasonable to conjecture that $(PG(n,4))^{\mathrm{T}}$ never has a triangular extension if $n>2$?

Now we consider one-point extensions of $(PG(n,2))^{\mathrm{T}}$. This situation is not as interesting as the triangular case:

THEOREM 7.6. *If $n>2$, then no EDD is a one-point extension of a $(PG(n,q))^{\mathrm{T}}$.*

PROOF. Suppose $\mathcal{S}$ is an extension of a $(PG(n,q))^{\mathrm{T}}$. Each residue of $\mathcal{S}$ has b points, and has r points on a block, where

$$b=(q^{n+1}-1)(q^n-1)/(q+1)(q-1)^2, \qquad r=(q^n-1)/(q-1).$$

Hence $\mathcal{S}$ is a 2-$(b+1, r+1, q+1)$, and its number of blocks is

$$b^*=(b+1)b(q+1)/(r+1)r. \tag{1}$$

This unpleasant expression can be simplified to:

$$b^*=(q^{n+1}-1)[(q^{n+1}-1)(q^n-1)+(q+1)(q-1)^2]/(q+1)(q-1)^2(q^n+q-2), \tag{2}$$

which is also not very attractive.

If we put $q^n\equiv -q+2 \pmod{q^n+q-2}$ in the numerator of (2) then we find that

$$2(q+1)^2(q-1)^3\equiv 0 \pmod{q^n+q-2}, \tag{3}$$

and since $q^n+q-2=(q-1)(q^{n-1}+q^{n-2}+\cdots+q+2)$, this becomes

$$2(q^2-1)^2\equiv 0 \pmod{q^{n-1}+q^{n-2}+\cdots+q+2}.$$

Then $q^{n-1}<q^{n-1}+q^{n-2}+\cdots+q+2\leqslant 2(q^2-1)^2<2q^4$, and so $n\leqslant 5$. For $n=3$, (4) implies that q^2+q+2 divides $2(3q+2)$, so $q=2$; but $q=2$ and $n=3$ do not give integral b^* in (2). If $n=4$, (4) becomes $2q\equiv 0 \pmod{q^3+q^2+q+2}$, which is not possible. If $n=5$, then (4) implies that $q^4+q^3+q^2+q+2$ divides $2(q^3+3q^2+q+1)$, and so is bounded by it; hence $q=2$, which does not satisfy the congruence (4). □

Now we return to (7.5):

PROBLEM 7.5 (continued):
(4) What other extensions of $(PG(n,q))^{\mathrm{T}}$ are there (i.e. not triangular or one-point)?
(5) Is it possible that every extension of a $(PG(n,q))^{\mathrm{T}}$ has $q=2$ or 4?

8. AFFINE GEOMETRIES

Here we consider $(AG(n,q))^{\mathrm{T}}$, again for the triangular and one-point cases. (See [5] for more about the case $n=2$.) The results are certainly not complete in the triangular case:

THEOREM 8.1. *Suppose that $\mathcal{S}$ is a triangular extension of $(AG(n,q))^{\mathrm{T}}$. Then q is even.*

PROOF. As in Theorem 7.2, we consider the superovals $y\cap\mathcal{S}_p$ in $\mathcal{S}_p$: these become sets of $q+1$ lines in $\mathcal{S}_p^{\mathrm{T}}=AG(n,q)$ such that every point is on 0 or 2 of the lines. Since there are planes in $AG(n,q)$, it will follow that these $q+1$ lines all lie in a plane

$AG(2, q)$, and hence they form a dual oval, in the projective plane sense of the term. The line at infinity of this $AG(2, q)$ contains no one of the $(q+1)q/2$ of the points which lie on two lines of the dual oval, so adjoining it, we have a set of $q+2$ lines, such that every point of the *projective* plane lies on 0 or 2 of the lines. This is a dual hyperoval, again in the projective plane sense of the term, and so q is even (see [6]). Also see the comments following Definition 2.1 about this dual superoval/hyperoval. □

EXAMPLE 8.2. In the embedding of Example 7.3 of the points and lines of $PG(n, 2)$ as the special subspaces $PG(n-1, 2)$ and points of the Grassman variety $\mathscr{V}_N(2)$, we can 'restrict' the embedding to the points and lines of an affine geometry $AG(n, 2)$ in $PG(n, 2)$, and obtain an embedding of the points and lines of $AG(n, 2)$ as certain special subspaces and certain points, in $\mathscr{V}_N(2)$. Call this substructure $\mathscr{B}_N$. Then embedding $\mathscr{B}_N$ in a hyperplane $\mathscr{H}$ of $PG(N+1, 2)$, we can exactly copy the construction of 7.3(i), and obtain a triangular extension of the dual of $AG(n, 2)$, for each $n > 2$. This EDD is also a triangular semibiplane, and has block size 2^n, with 2^{N+1} points.

(We thank Antonio Pasini for putting the finishing touches to the above construction for us.) We do not know of any example of such an extension for $n > 2$ with $q > 2$. However, for $n = 2$, Example 4.6 of [5] gives triangular examples for every $q = 2^m$, so perhaps we should expect to find examples for even q with $n > 2$ as well.

The one-point case is both curious and interesting:

THEOREM 8.3. *Suppose that $\mathscr{S}$ is a one-point extension of an $(AG(n, q))^{\mathrm{T}}$, with $n > 1$. Then n is even, $q = 2$, and $\mathscr{S}$ is a biplane with block size 2^n, or $\mathscr{S}$ is a* 2-(21, 6, 4) *isomorphic to the external restriction at a point of the Mathieu design* 3-(22, 6, 1). *Conversely, every biplane with block size 2^n is an example of such an EDD.*

PROOF. An $AG(n, q)$ has $b = q^{n-1}(q^n - 1)/(q-1)$ blocks, and $r = (q^n - 1)/(q-1)$ blocks on a point, so a one-point extensions of its dual will be a 2-design for $(b+1, r+1, q)$, with $b^* = qb(b+1)/(r+1)r$ blocks. Substituting, we find:

$$b^* = q^n[q^{n-1}(q^n - 1) + q - 1]/(q^n + q - 2). \tag{1}$$

Then in the standard way, we put $q^n \equiv -q + 2 \pmod{q^n + q - 2}$ in the numerator of (1) and successively simplify to obtain:

$$2(q^{n-1} - q^2 + 4q - 4) \equiv 0 \pmod{q^n + q - 2}. \tag{2}$$

Thus, in particular:

$$q^n + q - 2 \leqslant 2(q^{n-1} - q^2 + 4q - 4) \quad \text{or} \quad q^{n-1} - q^2 + 4q - 4 = 0. \tag{3}$$

The second possibility of (3) means that $q^{n-1} = (q-2)^2$, so $n = 2$ and $q = 4$; then it is straightforward that $\mathscr{S}$ is a 2-(21, 6, 4); this design exists and is the external restriction of the unique 3-(22, 6, 1), and this can be proved directly, or be found in [5], Theorem 4.1. (See also Example 3.3: it can be extended twice.)

If the first possibility of (3) happens, then since $2(q^{n-1} - q^2 + 4q - 4) \leqslant 2q^{n-1}$, and $q^n + q - 2 \geqslant q^n$, we have $q \leqslant 2$; hence $q = 2$. Then $b^* = 2^{n-1}(2^n - 1) + 1 = b + 1$, and $\mathscr{S}$ is a biplane; the number of points on a block is $r + 1 = 2^n$. Using the Bruck–Ryser–Chowla Theorem (e.g. see [7]), it follows that a biplane with block size 2^n must have $n \equiv 0 \pmod 2$.

Now the point–line structure of $AG(n, 2)$ is trivial: every pair of points are on a line, and it is a 2-(2^n, 2, 1). So it is straightforward that *any* biplane with blocksize 2^n can be identified as an extension of an $(AG(n, 2))^T$, as already explained in Example 4.1. □

While the point–line structure $AG(n, 2)$ is trivial, there is a lot of structure in the geometry and it might be possible to exploit this to find more biplanes. Notice that the first 'unknown' biplane covered by this situation is also the first case where existence of a biplane is unsettled: $k = 16$, that is, a 2-(121, 16, 2).

9. Comments

There are obviously many more 2-designs the duals of which might be extended; there are also many possibilities of extensions not studied here, i.e. non-triangular and not one-point. It is not easy to say which of these will prove interesting. The problems raised in Example 7.3(ii) concerning algebraic varieties should certainly be resolved.

Acknowledgements

A number of colleagues have contributed useful comments at various stages in writing this paper, and while they are often referred to at the appropriate point above, the author is pleased to thank them all here: Andries Brouwer, Frank De Clerck, Alberto Del Fra, Dina Ghinelli, James Hirschfeld, Antonio Pasini, Mohan Shrikhande, Giuseppe Tallini, Joseph Thas and Peter Wild.

Much of this work was carried out while the author was a Visiting Professor at the Università di Roma 'La Sapienza', supported by the Consiglio Nazionale delle Ricerche.

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Received 26 December 1989 and accepted in revised form 13 April 1990

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