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Applied Mathematics Letters 20 (2007) 686–691
Applied Mathematics Letters
www.elsevier.com/locate/aml

Coefficient inequalities and inclusion relations for some families of analytic and multivalent functions

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Received 12 July 2006; accepted 31 July 2006

Abstract

By means of the derivative operator of order m ($m \in \mathbb{N}_0$), we introduce and investigate two new subclasses of p-valently analytic functions of complex order. The various results obtained here for each of these two function classes include coefficient inequalities and inclusion relationships involving the (n, δ)-neighborhood of p-valently analytic functions. Relevant connections with some other recent investigations are also pointed out.

Keywords: Analytic functions; p-Valent functions; Coefficient inequalities; Inclusion relations; Neighborhood properties; (n, δ)-Neighborhood

1. Introduction and definitions

Let $\mathcal{A}_p(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geqq 0;\, n, p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$\mathbb{U} = \{z\colon\ z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Upon differentiating both sides of (1.1) m times with respect to z, we have

$$f^{(m)}(z) = \frac{p!}{(p-m)!} z^{p-m} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m} \quad (n, p \in \mathbb{N};\, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\, p > m). \tag{1.2}$$

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2006.07.009

Now, making use of the function $f^{(m)}(z)$ given by (1.2), we introduce a new subclass $\mathcal{R}_{n,m}^{p}(\lambda, b)$ of the p-valently analytic function class $\mathcal{A}_p(n)$, which consists of functions $f(z)$ satisfying the following inequality:

$$\left|\frac{1}{b}\left(\frac{zf^{(1+m)}(z)+\lambda z^2 f^{(2+m)}(z)}{\lambda z f^{(1+m)}(z)+(1-\lambda) f^{(m)}(z)}-(p-m)\right)\right|<1$$
$$(z \in \mathbb{U};\ p \in \mathbb{N};\ m \in \mathbb{N}_0;\ 0 \leqq \lambda \leqq 1;\ b \in \mathbb{C} \setminus \{0\};\ p > m). \tag{1.3}$$

Next, following the earlier investigations by Goodman [4], Ruscheweyh [9], and others including Altıntaş et al. ([1] and [2]), Murugusundaramoorthy and Srivastava [7], and Raina and Srivastava [8] (see also [5,6,10]), we define the (n, δ)-neighborhood of a function $f(z) \in \mathcal{A}_p(n)$ by (see, for details, [3, p. 1668])

$$N_{n,\delta}(f) := \left\{ g : g \in \mathcal{A}_p(n),\ g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k\,|a_k - b_k| \leqq \delta \right\}. \tag{1.4}$$

It follows from (1.4) that, if

$$h(z) = z^p \quad (p \in \mathbb{N}), \tag{1.5}$$

then

$$N_{n,\delta}(h) := \left\{ g : g \in \mathcal{A}_p(n),\ g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \text{ and } \sum_{k=n+p}^{\infty} k\,|b_k| \leqq \delta \right\}. \tag{1.6}$$

Finally, we denote by $\mathcal{L}_{n,m}^{p}(\lambda, b)$ the subclass of the normalized p-valently analytic function class $\mathcal{A}_p(n)$ consisting of functions $f(z)$ which satisfy the inequality (1.7) below:

$$\left|\frac{1}{b}\left(f^{(1+m)}(z)+\lambda z f^{(2+m)}(z)-(p-m)\right)\right|<p-m$$
$$(z \in \mathbb{U};\ p \in \mathbb{N};\ m \in \mathbb{N}_0;\ 0 \leqq \lambda \leqq 1;\ b \in \mathbb{C} \setminus \{0\};\ p > m). \tag{1.7}$$

The main object of the present work is to investigate the various properties and characteristics of analytic p-valent functions belonging to the subclasses

$$\mathcal{R}_{n,m}^{p}(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^{p}(\lambda, b),$$

which are introduced here by making use of the derivative operator of order m $(m \in \mathbb{N}_0)$ on normalized p-valently analytic functions in $\mathbb{U}$. Apart from deriving a set of coefficient inequalities for each of these two function classes, we establish some inclusion relationships involving the (n, δ)-neighborhoods of analytic p-valent functions belonging to each of these subclasses.

Our definitions of the function classes

$$\mathcal{R}_{n,m}^{p}(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^{p}(\lambda, b)$$

are motivated essentially by several earlier investigations including [2,7,8], in each of which further details and references to other closely related subfamilies of the normalized p-valently analytic function class $\mathcal{A}_p(n)$ can be found.

2. A set of coefficient inequalities

In this section, we prove the following results which yield the coefficient inequalities for functions in the subclasses

$$\mathcal{R}_{n,m}^{p}(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^{p}(\lambda, b).$$

Theorem 1. *Let $f(z) \in \mathcal{A}_p(n)$ be given by (1.1). Then $f(z) \in \mathcal{R}_{n,m}^{p}(\lambda, b)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k+|b|-p)k![\lambda(k-m-1)+1]}{(k-m)!}\, a_k \leqq \frac{|b|\, p![\lambda(p-m-1)+1]}{(p-m)!}. \tag{2.1}$$

Proof. Let a function $f(z)$ of the form (1.1) belong to the class $\mathcal{R}^p_{n,m}(\lambda, b)$. Then, in view of (1.2) and (1.3), we obtain the following inequality:

$$\Re\left(\frac{\sum_{k=n+p}^{\infty}\frac{(p-k)k![\lambda(k-m-1)+1]}{(k-m)!}a_k z^{k-m}}{\frac{p![\lambda(p-m-1)+1]}{(p-m)!}z^{p-m}-\sum_{k=n+p}^{\infty}\frac{(p-k)k![\lambda(k-m-1)+1]}{(k-m)!}a_k z^{k-m}}\right) > -|b| \quad (z \in \mathbb{U}). \tag{2.2}$$

Setting $z = r$ $(0 \leqq r < 1)$ in (2.2), we observe that the expression in the denominator on the left-hand side of (2.2) is positive for $r = 0$ and also for all r $(0 < r < 1)$. Thus, by letting $r \to 1-$ through *real* values, (2.2) leads us to the desired assertion (2.1) of Theorem 1.

Conversely, by applying (2.1) and setting $|z| = 1$, we find from (1.2) that

$$\left|\frac{zf^{(1+m)}(z)+\lambda z^2 f^{(2+m)}(z)}{\lambda z f^{(1+m)}(z)+(1-\lambda)f^{(m)}(z)}-(p-m)\right|$$
$$= \left|\frac{\sum_{k=n+p}^{\infty}\frac{(p-k)k![\lambda(k-m-1)+1]}{(k-m)!}a_k z^{k-m}}{\frac{p![\lambda(p-m-1)+1]}{(p-m)!}z^{p-m}-\sum_{k=n+p}^{\infty}\frac{(p-k)k![\lambda(k-m-1)+1]}{(k-m)!}a_k z^{k-m}}\right|$$
$$\leqq \frac{|b|\left\{\frac{p![\lambda(k-m-1)+1]}{(p-m)!}-\sum_{k=n+p}^{\infty}\frac{(p-k)k![\lambda(k-m-1)+1]}{(k-m)!}a_k\right\}}{\frac{p![\lambda(k-m-1)+1]}{(p-m)!}-\sum_{k=n+p}^{\infty}\frac{(p-k)k![\lambda(k-m-1)+1]}{(k-m)!}a_k} = |b|.$$

Hence, by the *maximum modulus principle*, we infer that

$$f(z) \in \mathcal{R}^p_{n,m}(\lambda, b),$$

which evidently completes the proof of Theorem 1. □

Remark 1. In its special case when

$$m = 0, \quad p = 1 \quad \text{and} \quad b = \beta\gamma \quad (0 < \beta \leqq 1; \gamma \in \mathbb{C} \setminus \{0\}), \tag{2.3}$$

Theorem 1 yields a result given earlier by Altıntaş et al. [2, p. 64, Lemma 1].

Similarly, we can prove the following theorem.

Theorem 2. *Let $f(z) \in \mathcal{A}_p(n)$ be given by (1.1). Then $f(z) \in \mathcal{L}^p_{n,m}(\lambda, b)$ if and only if*

$$\sum_{k=n+p}^{\infty}\binom{k}{m}(k-m)[\lambda(k-m-1)+1]a_k \leqq (p-m)\left[\frac{|b|-1}{m!}+\binom{p}{m}[\lambda(p-m-1)+1]\right]. \tag{2.4}$$

Remark 2. Making use of the same parametric substitutions as were mentioned above in (2.3), Theorem 2 yields another known result due to Altıntaş et al. [2, p. 65, Lemma 2].

3. Inclusion relations involving the (n, δ)-neighborhoods

In this section, we establish several inclusion relations for the normalized p-valently analytic function classes

$$\mathcal{R}^p_{n,m}(\lambda, b) \quad \text{and} \quad \mathcal{L}^p_{n,m}(\lambda, b)$$

involving the (n, δ)-neighborhood defined by (1.6).

Theorem 3. *If*

$$\delta := \frac{|b|\, p!(n+p-m)!\,[\lambda(p-m-1)+1]}{(n+|b|)(p-m)!(n+p-1)!\,[\lambda(n+p-m-1)+1]} \quad (p > |b|), \tag{3.1}$$

then

$$\mathcal{R}^{p}_{n,m}(\lambda, b) \subset N_{n,\delta}(h). \tag{3.2}$$

Proof. Let $f(z) \in \mathcal{R}^{p}_{n,m}(\lambda, b)$. Then, in view of the assertion (2.1) of Theorem 1, we have

$$\frac{(n+|b|)(n+p)![\lambda(n+p-m-1)+1]}{(n+p-m)!} \sum_{k=n+p}^{\infty} a_k \leqq \frac{|b|\, p![\lambda(p-m-1)+1]}{(p-m)!}, \tag{3.3}$$

which readily yields

$$\sum_{k=n+p}^{\infty} a_k \leqq \frac{|b|\, p!(n+p-m)![\lambda(p-m-1)+1]}{(n+|b|)(n+p)!(p-m)![\lambda(n+p-m-1)+1]}. \tag{3.4}$$

Making use of (2.1) again, in conjunction with (3.4), we get

$$\begin{aligned}
&\frac{(n+p)![\lambda(n+p-m-1)+1]}{(n+p-m)!} \sum_{k=n+p}^{\infty} k a_k \\
&\leqq \frac{|b|\, p![\lambda(p-m-1)+1]}{(p-m)!} + \frac{(p-|b|)(n+p)![\lambda(n+p-m-1)+1]}{(n+p-m)!} \sum_{k=n+p}^{\infty} a_k \\
&\leqq \frac{|b|\, p![\lambda(p-m-1)+1]}{(p-m)!} + \frac{|b|\, p!(p-|b|)[\lambda(p-m-1)+1]}{(p-m)!(n+|b|)} \\
&= \frac{|b|\, p!(n+p)[\lambda(p-m-1)+1]}{(p-m)!(n+|b|)}.
\end{aligned}$$

Hence

$$\sum_{k=n+p}^{\infty} k a_k \leqq \frac{|b|\, p!(n+p-m)!\,[\lambda(p-m-1)+1]}{(n+|b|)(p-m)!(n+p-1)!\,[\lambda(n+p-m-1)+1]} =: \delta, \quad (p > |b|) \tag{3.5}$$

which, by means of the definition (1.6), establishes the inclusion relation (3.2) asserted by Theorem 3. □

In a similar manner, by applying the assertion (2.4) of Theorem 2 instead of the assertion (2.1) of Theorem 1 to functions in the class $\mathcal{L}^{p}_{n,m}(\lambda, b)$, we can prove the following inclusion relationship.

Theorem 4. *If*

$$\delta := \frac{(p-m)\left[\dfrac{|b|-1}{m!} + \dbinom{p}{m}[1+\lambda(p-m-1)]\,(n+p)\right]}{\dbinom{n+p}{m}(n+p-m)\,[1+\lambda(p-m-1)]}, \tag{3.6}$$

then

$$\mathcal{L}^{p}_{n,m}(\lambda, b) \subset N_{n,\delta}(h).$$

Remark 3. Applying the parametric substitutions listed in (2.3), Theorems 3 and 4 would yield a set of known results due to Altıntaş et al. [2, p. 65, Theorem 1; p. 66, Theorem 2].

4. Further neighborhood properties

In this last section, we determine the neighborhood properties for each the following (*slightly modified*) function classes:

$$\mathcal{R}_{n,m}^{p,\alpha}(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^{p,\alpha}(\lambda, b).$$

Here the class $\mathcal{R}_{n,m}^{p,\alpha}(\lambda, b)$ consists of functions $f(z) \in \mathcal{A}_p(n)$ for which there exists another function $g(z) \in \mathcal{R}_{n,m}^{p}(\lambda, b)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \alpha \quad (z \in \mathbb{U}; 0 \leqq \alpha < p). \tag{4.1}$$

Analogously, the class $\mathcal{L}_{n,m}^{p,\alpha}(\lambda, b)$ consists of functions $f(z) \in \mathcal{A}_p(n)$ for which there exists another function $g(z) \in \mathcal{L}_{n,m}^{p}(\lambda, b)$ satisfying the inequality (4.1).

The proofs of the following results involving the neighborhood properties for the classes

$$\mathcal{R}_{n,m}^{p,\alpha}(\lambda, b) \quad \text{and} \quad \mathcal{L}_{n,m}^{p,\alpha}(\lambda, b)$$

are similar to those given already in [2,7] and [10]. Therefore, we skip their proofs here.

Theorem 5. *Let $g(z) \in \mathcal{R}_{n,m}^{p}(\lambda, b)$. Suppose also that*

$$\alpha := p - \frac{\delta(n + |b|)(n + p - 1)!(p - m)![\lambda(n + p - m - 1) + 1]}{(n + |b|)(n + p)!(p - m)![\lambda(n + p - m - 1) + 1] - |b|\, p!(n + p - m)![\lambda(n + p - m) + 1]}. \tag{4.2}$$

Then

$$N_{n,\delta}(g) \subset \mathcal{R}_{n,m}^{p,\alpha}(\lambda, b).$$

Theorem 6. *Let $g(z) \in \mathcal{L}_{n,m}^{p}(\lambda, p)$. Suppose also that*

$$\alpha := p - \frac{\delta\binom{n+p}{m}(n + p - m)[\lambda(n + p - m - 1) + 1]}{(n + p)\left\{\binom{n+p}{m}(n + p - m)[\lambda(n + p - m - 1) + 1] - (p - m)\left[\frac{|b| - 1}{m!} + \binom{p}{m}[\lambda(p - m - 1) + 1]\right]\right\}}. \tag{4.3}$$

Then

$$N_{n,\delta}(g) \subset \mathcal{L}_{n,m}^{p,\alpha}(\lambda, b).$$

Acknowledgements

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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