Upper and lower bounds for the solution of the general matrix Riccati differential equation on a time scale

John M. Davis*, Johnny Hendersonb, K. Rajendra Prasad

*Department of Mathematics, Baylor University, Waco, TX 76798, USA
bDepartment of Mathematics, Auburn University, Auburn, AL 36849, USA

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Abstract

We obtain upper and lower bounds for the solution of the general matrix Riccati differential equation on a time scale \( \mathbb{T} \),

\[
R^2(t) = A(t) + B(t)R(t) + R(\sigma(t))B^*(t) - R(\sigma(t))C(t)R(t),
\]

where \( A(t) \) and \( C(t) \) are symmetric \( n \times n \) matrices while \( B(t), V(t), T(t), \) and \( R(t) \) are \( n \times n \) matrices, and \( * \) denotes the transpose of the matrix. We use the quasilinearization technique to obtain these bounds. We also study the monotonicity of the successive approximations. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Differential equations serve as a natural description of continuous time dynamic processes. Many of the known results for continuous time dynamical systems are not readily available in discrete time contexts. Even the results that are known are more analogous to continuous time dynamic processes. The continuous time orbits and the discrete time orbits are topologically different and care should be taken in dealing with such obvious mathematical dichotomies.
The classical Riccati ordinary differential equation
\[ z' + q(t) + \frac{1}{p(t)} z^2 = 0 \quad (1.1) \]
dates back to the late part of the 17th century and the early part of the eighteenth century. This famous equation was studied by (of course) Jacopo Francesco Riccati and also by the Bernoulli brothers. It has remained a subject of current investigation as can be noted by the more than 4000 papers listed on MathSciNet containing the word “Riccati”.

The continuous scalar as well as continuous matrix versions of (1.1) have received considerable recent attention [5,6,18–20,22] in the literature. This is due in large part to the applications of matrix Riccati equations in control theory [13], filter processes [8], and multidimensional transport theory [15,16]. On the other hand, the Riccati differential equation
\[ \Delta z + q(t) + \frac{z^2}{p(t) + z(t)} = 0 \quad (1.2) \]
and its matrix version have also been studied closely. Excellent references for results in this direction are the paper involving continued fraction representations of certain solutions of discrete Riccati equations by Ahlbrandt [1] and Chapter 6 of the text by Ahlbrandt and Peterson [2]. Other authors have studied the continuous and discrete dynamics of Riccati equations separately, but in parallel, as in [10–12].

But just how robust is the dichotomy between (1.1) and (1.2)? Or, more generally, between continuous and discrete dynamics? In 1990, Stefan Hilger [14] gave a unified approach to continuous and discrete calculus, choosing to view the continuous and discrete as but two special cases of time scales or measure chains. This seems to provide an inroad for an affirmative answer to the (now famous) wish of H.L. Turrittin [24]:

On becoming familiar with difference equations and their close relation to differential equations, I was in hopes that the theory of difference equations could be brought completely abreast with that for ordinary differential equations.

Generalizations of ordinary differential equations and finite difference equations to general time scales have become known as dynamic equations on time scales or differential equations on time scales. The forthcoming text by Bohner and Peterson provides a thorough treatment of these equations [7] as does the book by Kaymakcalan et al. [17]. There are numerous other papers which are cited in these aforementioned texts; we recognize [3] and [14] as landmark works.

This brings us to the matrix Riccati differential equation on a general time scale. In an effort to generalize existing theorems related to (1.1) and (1.2) as well as to provide new results which may exist in the continuous case but not the discrete case (or vice versa), we consider such Riccati equations on a general time scale \( \mathbb{T} \). The general matrix Riccati differential equation is associated with a system of two first order differential equations. We are especially interested in its solution when this equation arises from a boundary value problem. We introduce the general matrix Riccati differential equation in a time scales setting and our plan is to obtain upper and lower bounds on the solution using a quasilinearization technique.
To digress for a moment, Bellman and Vasudevan [5] obtained iterative approximations to the (continuous) matrix Riccati differential equation by employing quasilinearization and Laplace transform techniques. They also studied the monotonicity and the nature of convergence of successive approximations. An extensive overview of the method of quasilinearization can be found in [4,25]. Murty et al. [19] established the existence of solutions of the matrix Riccati differential equation using a variation of parameter approach. Later, Murty et al. [20] obtained upper and lower bounds for the solution of the general matrix Riccati differential equation and these same authors went on to apply their quasilinearization techniques to problems in mathematical biology [21]. For discrete-type Riccati equations, Ahlbrandt [1] discussed continued fraction representations of maximal and minimal solutions of these equations. Recently, Prasad [23] established the existence of solutions of scalar and matrix Riccati differential equations on general time scales.

The layout of the paper is as follows. In Section 2, we briefly describe features of time scales, functions defined on time scales, and the needed calculus on time scales. We also formulate the general matrix Riccati differential equation on \( T \). In Section 3, by applying the quasilinearization technique to the general matrix Riccati differential equation on \( T \), we obtain a linear equation. We obtain the solution of the linearized equation in terms of fundamental matrices and thereby obtain an upper bound for the solution of the general matrix Riccati differential equation on \( T \). In Section 4, we obtain a lower bound for the solution of the general matrix Riccati differential equation on \( T \). In Section 5, we study the behavior of the successive approximations.

2. Preliminaries

In this section, we begin by defining certain properties of time scales and operations on time scales.

**Definition 2.1.** Let \( T \) be a closed subset of \( \mathbb{R} \) and let \( T \) have the subspace topology inherited from the Euclidean topology on \( \mathbb{R} \). The set \( T \) is referred to as a measure chain or a time scale. For \( t < \sup T \) and \( t > \inf T \), define the forward jump operator, \( \sigma \), and the backward jump operator, \( \rho \), respectively, by

\[
\sigma(t) = \inf\{\tau \in T \mid \tau > t\} \in T,
\]

\[
\rho(r) = \sup\{\tau \in T \mid \tau < t\} \in T,
\]

for all \( t \in T \). If \( \sigma(t) = t \), \( t \) is said to be right dense, and if \( \rho(r) = r \), \( r \) is said to be left dense.

**Definition 2.2.** For \( x: T \to \mathbb{R} \) and \( t \in T \) (if \( t = \sup T \), assume \( t \) is not left scattered), define the delta derivative of \( x(t) \), denoted by \( x^\Delta(t) \), to be the number (when it exists), with the property that, for any \( \varepsilon > 0 \), there is a neighborhood, \( U \) of \( t \) such that

\[
| [x(\sigma(t)) - x(s)] - x^\Delta(t) [\sigma(t) - s] | \leq \varepsilon |\sigma(t) - s|,
\]

for all \( s \in U \). If \( x \) is delta differentiable for every \( t \in T \), we say that \( x: T \to \mathbb{R} \) is delta differentiable on \( T \).
Definition 2.3. The *graininess* function, $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

Lemma 2.1. If $x$ is continuous at $t$ and $t$ is right scattered, then $x$ is delta differentiable at $t$ with derivative

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$ 

Remark. We make the following remarks concerning the delta derivative and graininess.

1. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta} = f'$ is the (usual) ordinary derivative. The graininess of any $t \in \mathbb{T}$ is $\mu(t) = 0$.
2. If $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta} = \Delta f$ is the usual forward difference operator. The graininess of any $t \in \mathbb{T}$ is $\mu(t) = 1$.

Lemma 2.2. If $f$ and $g$ are delta differentiable functions at $t$, then so is $fg$ and

$$(fg)^{\Delta}(t) = f(\sigma(t))g^{\Delta}(t) + f^{\Delta}(t)g(t).$$

Lemma 2.3. If $f$, $g$, and $h$ are delta differentiable functions at $t$, then so is $(fg)h$ and

$$(fgh)^{\Delta}(t) = f(\sigma(t))g(\sigma(t))h^{\Delta}(t) + f(\sigma(t))g^{\Delta}(t)h(t) + f^{\Delta}(t)g(t)h(t).$$

Definition 2.4. If the time scale $\mathbb{T}$ has a maximal element which is also left scattered, then that point is called a *degenerate point*. Any subset of nondegenerate points of $\mathbb{T}$ is denoted by $\mathbb{T}^k$.

Definition 2.5. A function $x : \mathbb{T} \rightarrow \mathbb{R}$ is *right dense continuous* (denoted $x$ is rd-continuous) if it is continuous in every right dense point $t \in \mathbb{T}$, and its left hand limit exists at each left dense point $t \in \mathbb{T}$. Moreover, we say that $f$ is delta differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta} : \mathbb{T}^k \rightarrow \mathbb{R}$ is then called the delta derivative of $f$ on $\mathbb{T}^k$.

Remark. The forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{R}$ is right dense continuous, and more generally if $x : \mathbb{T} \rightarrow \mathbb{R}$ is continuous, then $x(\sigma) : \mathbb{T} \rightarrow \mathbb{R}$ is right dense continuous.

Definition 2.6. A function $F : \mathbb{T}^k \rightarrow \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T}^k \rightarrow \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. We then define the integral of $f$ by

$$\int_a^t f(\tau) \Delta \tau = F(t) - F(a).$$

Lemma 2.4. Let $f : P \rightarrow \mathbb{T}$ be rd-continuous on $P$. Then $f$ possesses an antiderivative on $P$. Moreover, if $f$ is continuous on $P$, then $f(\sigma(t))$ is rd-continuous on $\mathbb{T}$, and hence possesses an antiderivative on $P$, where $P$ denotes a closed subset of $\mathbb{T}$.
In order to obtain uniqueness, we will need the following *regressivity condition* at times

\[ I + \mu \begin{pmatrix} B & A \\ C & -B^* \end{pmatrix} \text{ is invertible.} \]  

(2.1)

The general matrix Riccati differential equation on a time scale can be formulated in the following manner. We state the results without proofs since they parallel those found in [9,14].

**Theorem 2.1.** Assume the regressivity condition (2.1). The matrix Riccati transformation

\[ R(t) = V(t)T^{-1}(t) \]  

transforms the system of two matrix linear equations

\[ V^\Delta(t) = B(t)V(t) + A(t)T(t), \]
\[ T^\Delta(t) = C(t)V(t) - B^*(t)T(t) \]  

(2.3)

into a general matrix Riccati differential equation

\[ R^\Delta(t) = A(t) + B(t)R(t) + R(\sigma(t))B^*(t) - R(\sigma(t))C(t)R(t) \quad \text{(GMRDE)} \]

and conversely, where \( A(t) \) and \( C(t) \) are \( n \times n \) symmetric matrices such that \( A(\sigma(t))A(t) \) and \( C(\sigma(t))C(t) \) are nonnegative definite, and \( B(t), V(t), T(t), \) and \( R(t) \) are \( n \times n \) matrices, and \( * \) denotes the transpose of the matrix.

**Theorem 2.2.** Suppose \( A \) and \( C \) are \( n \times n \) symmetric matrices. The system of two first order linear matrix differential equations (2.3) has a nonsingular solution on the interval \( J \subset \mathbb{T} \) if and only if (GMRDE) has a nonsingular solution defined throughout \( J \).

3. An upper bound on the solution of the GMRDE

**Definition 3.1.** If a real symmetric matrix \( A \) has the property \( (x,Ax) > 0 \), for all nontrivial vectors \( x \), we say that \( A \) is *positive definite* and we write \( A > 0 \). If \( (x,Ax) \geq 0 \), we say that \( A \) is *nonnegative definite* and we write \( A \geq 0 \). The condition of positive definiteness introduces the following partial ordering in real symmetric matrices. We write \( A \geq B \) if \( (A - B) \geq 0 \). Also, we have

1. the matrix \( BAB^{-1} \geq 0 \) if \( A \geq 0 \),
2. if \( A > 0 \), then \( A^{-1} > 0 \).

In the continuous case, we have \( A^2(t) \geq 0 \) for all real symmetric matrices \( A \).

**Definition 3.2.** Any set of \( n \) linearly independent solutions of \( X^\Delta(t) = A(t)X(t) \) is called a *fundamental solution*. Given a set of scalar functions, \( \{x_i(t)\}_{i=1}^n \), such that

\[ \{[x_1(t), x_1^\Delta(t), \ldots, x_1^{\Delta(n-1)}]^*, \ldots, [x_n(t), x_n^\Delta(t), \ldots, x_n^{\Delta(n-1)}]^* \} \]
forms a fundamental solution, then the matrix whose $i$th column is given by

$$[x_i(t), x_i^\Delta(t), \ldots, x_i^{\Delta^{(n-1)}}]^*$$

is called a fundamental matrix solution of the matrix system.

In this section, we consider the GMRDE on time scales given by (GMRDE) with initial condition matrix $R(0) = E$, where $R(t)$, $A(t)$, $B(t)$, and $E$ are $n \times n$ matrices, the components of $A(t)$, $B(t)$ and $C(t)$ are rd-continuous functions on $\mathbb{T}$, $E$ is a nonsingular constant matrix, $A$ and $C$ are nonnegative definite matrices, and * denotes the transpose of the matrix.

**Remark.** We point out the following.

1. If $\mathbb{T} = \mathbb{R}$, then (GMRDE) becomes

$$R'(t) = A(t) + B(t)R(t) + R(t)B^*(t) - R(t)C(t)R(t),$$

which is a matrix Riccati differential equation in the continuous case.

2. If $\mathbb{T} = \mathbb{Z}$ then (GMRDE) becomes a Riccati difference equation

$$\Delta R(t) = A(t) + B(t)R(t) + R(t + 1)B^*(t) - R(t + 1)C(t)R(t).$$

We now apply the quasilinearization technique to (GMRDE). Consider the identity

$$R(\sigma(t))C(t)R(t) = [T(t) + R(\sigma(t)) - T(t)]C(t)[T(t) + R(t) - T(t)]$$

$$= T(t)C(t)T(t) + T(t)C(t)[R(t) - T(t)] + [R(\sigma(t)) - T(t)]C(t)T(t)$$

$$+ [R(\sigma(t)) - T(t)]C(t)[R(t) - T(t)]$$

$$\geq -T(t)C(t)T(t) + T(t)C(t)R(t) + R(\sigma(t))C(t)T(t),$$

for all symmetric continuous matrices $T(t)$. The equality holds only if $R(t) = T(t)$. Therefore, (GMRDE) becomes

$$R^\Delta(t) \leq [B(t) - T(t)C(t)]R(t) + R(\sigma(t))[B^*(t) - C(t)T(t)]$$

$$+ A(t) + T(t)C(t)T(t)$$

(3.1)

with $R(0) = E$. The associated equation for the above inequality is

$$U^\Delta(t) = [B(t) - T(t)C(t)]U(t) + U(\sigma(t))[B^*(t) - C(t)T(t)]$$

$$+ A(t) + T(t)C(t)T(t)$$

(3.2)

with $U(0) = E$. 
Theorem 3.1. Assume the system
\[ U^\Delta(t) = [B(t) - T(t)C(t)]U(t) + U(\sigma(t))[B^*(t) - C(t)T(t)] \] (3.3)
is regressive. Then any solution of (3.3) is of the form \( \Phi(t)W\Psi^*(t) \), where \( \Phi(t) \) and \( \Psi(t) \) are fundamental matrix solutions of
\[ U^\Delta(t) = [B(t) - T(t)C(t)]U(t) \]
and
\[ U^\Delta(t) = [B^*(t) - C(t)T(t)]^*U(\sigma(t)), \]
respectively. Here, \( W \) is a nonsingular \( n \times n \) constant matrix.

Theorem 3.2. Any solution of (3.2) is of the form
\[ U(t) = \Phi(t)C\Psi^*(t) + \overline{U(t)}, \]
where \( \overline{U(t)} \) is a particular solution of (3.2).

Theorem 3.3. Let \( \Phi(t) \) and \( \Psi(t) \) be as in Theorem 3.1. A particular solution of (3.2) is of the form
\[ \overline{U(t)} = \Phi(t) \left[ \int_0^t \Phi^{-1}(\sigma(s))F(s)\Psi^*\Psi^{-1}(s) \Delta s \right] \Psi^*(t), \]
where \( F(t) = A(t) + T(t)C(t)T(t) \).

Theorem 3.4. Any solution of (3.2) satisfying \( U(0) = E \) is of the form
\[ U(t) = \Phi(t, 0)E\Psi^*(t, 0) + \int_0^t \Phi(t, \sigma(s))[A(s) + T(s)C(s)T(s)]\Psi^*(s, s) \Delta s. \]

Theorem 3.5. An upper bound for the solution of \( \text{GMRDE} \) with \( R(0) = E \) is \( U(T, t) \), where \( U(T, t) \) is the solution of (3.2) with \( U(0) = E \) and is denoted by \( U_{\text{upper}}(T, t) \).

Proof. From (3.1) and (3.2) we have,
\[ [U(t) - R(t)]^\Delta \geq [B(t) - T(t)C(t)][U(t) - R(t)] + [U(\sigma(t)) - R(\sigma(t))][B^*(t) - C(t)T(t)] \]
with \( U(0) = R(0) = 0 \). This inequality can be written as
\[ [U(t) - R(t)]^\Delta = [B(t) - T(t)C(t)][U(t) - R(t)] + [U(\sigma(t)) - R(\sigma(t))][B^*(t) - C(t)T(t)] + Q(t) \] (3.4)
with $U(0) - R(0) = 0$, where $Q(t)$ is a nonnegative definite $n \times n$ matrix. Let $F(t) = U(t) - R(t)$. Then (3.4) becomes

$$F^\Delta(t) = [B(t) - T(t)C(t)]F(t) + F(\sigma(t))[B^*(t) - C(t)T(t)] + Q(t)$$

with $F(0) = 0$. Now using Theorems 3.1–3.4 we see that if $\Phi(t)$ and $\Psi(t)$ are fundamental matrix solutions of

$$F^\Delta(t) = [B(t) - T(t)C(t)]F(t)$$

and

$$F^\Delta(t) = [B^*(t) - C(t)T(t)]^*F(\sigma(t)),$$

respectively, then the solution of (3.5) is

$$F(t) = \int_0^t \Phi(t, \sigma(s))Q(s)\Psi^*(t,s) \Delta s.$$

$F(t) \geq 0$ follows from the fact that $\Psi^*$ is the transpose of $\Phi$; this in turn is the result of [20, Lemma 3.1]. Therefore, $T(t) \geq R(t)$ which says $R(t) \leq U(W,t)$ or $R(t) \leq U_{\text{upper}}(W,t)$. □

4. A lower bound on the solution of the GMRDE

**Theorem 4.1.** The inverse of a nonsingular solution of (GMRDE) satisfies a Riccati equation for

$$R(t) = X^{-1}(t).$$

**Proof.** Let $R(t) = X^{-1}(t)$. Then $R^\Delta(t) = -X^{-1}(\sigma(t))X^\Delta(t)X^{-1}(t)$. Now (GMRDE) becomes

$$X^\Delta(t) = -B^*(t)X(t) - X(\sigma(t))B(t) + C(t) + X(\sigma(t))A(t)X(t)$$

with $X(0) = E^{-1} = E_1$. □

Now applying the quasilinearization technique to the nonlinear term which we stated in the previous section, we get

$$X(\sigma(t))A(t)X(t) \geq T(t)A(t)X(t) + X(\sigma(t))A(t)T(t) - T(t)A(t)T(t),$$

for all continuous symmetric matrices $T(t)$, and equality holds only if $T(t) = X(t)$. 
Now (4.2) becomes
\[
X^\Delta(t) \leq - [B^*(t) + T(t)A(t)]X(t) - X(\sigma(t))[B(t) + A(t)T(t)] + C(t) + T(t)A(t)T(t)
\]  
(4.3)
with \(X(0) = E_1\). Now the associated equation for (4.3) becomes
\[
U^\Delta(t) = - [B^*(t) + T(t)A(t)]U(t) - U(\sigma(t))[B(t) + A(t)T(t)] + C(t) + T(t)A(t)T(t)
\]  
(4.4)
with \(U(0) = E_1\).

**Theorem 4.2.** The solution of (4.4) with \(U(0) = E_1\) is of the form
\[
U(t) = \Phi_1(t,0)E_1 \Psi_1^*(t,0) + \int_0^t \Phi_1(t, \sigma(s))[C(s) + T(s)A(s)T(s)]\Psi_1^*(t,s) \Delta s,
\]
where \(\Phi_1(t,s) = \Phi_1(t)\Phi_1^{-1}(s)\) and \(\Psi_1(t,s) = \Psi_1(t)\Psi_1^{-1}(s)\), \(\Phi_1(t)\) and \(\Psi_1(t)\) are fundamental matrix solutions of
\[
U^\Delta(t) = - [B^*(t) + T(t)A(t)]U(t)
\]
and
\[
U^\Delta(t) = - [B(t) + A(t)T(t)]^*U(\sigma(t)),
\]
respectively.

**Proof.** Using Theorems 3.1–3.3, we obtain the solution of (4.4) as
\[
U(t) = \Phi_1(t)C\Psi_1^*(t) + \int_0^t \Phi_1(t, \sigma(s))[C(s) + T(s)A(s)T(s)]\Psi_1^*(t,s) \Delta s.
\]
Applying the initial condition \(U(0) = E_1\), we obtain
\[
C = \Phi_1^{-1}(0)E_1 \Psi_1^{*-1}(0).
\]
Thus,
\[
U(t) = \Phi_1(t,0)E_1 \Psi_1^*(t,0) + \int_0^t \Phi_1(t, \sigma(s))[C(s) + T(s)A(s)T(s)]\Psi_1^*(t,s) \Delta s. \quad \square
\]

**Theorem 4.3.** A lower bound for the solution of (GMRDE) is \(U^{-1}(T,t)\), where \(U(T,t)\) is the solution of (4.4). We denote \(U(T,t)\) by \(U_{\text{lower}}(T,t)\).

**Proof.** From (4.3) and (4.4), we have,
\[
[U(t) - X(t)]^\Delta \geq -[B^*(t) + T(t)C(t)][U(t) - X(t)]
- [U(\sigma(t)) - X(\sigma(t))][B(t) + C(t)T(t)]
\]
with \( U(0) - X(0) = 0 \). Similar to the methods used in Theorem 3.5 (see [9]), we obtain \( U(t) - X(t) \geq 0 \) so that \( X^{-1}(t) \geq U^{-1}(T, t) \), or \( R(t) \geq U_{\text{lower}}(T, t) \). □

Combining the arguments for the upper and lower bounds we have the following theorem.

**Theorem 4.4.** For any two continuous symmetric matrices \( T_1 \) and \( T_2 \), we have

\[
U_{\text{lower}}(T_1, t) \leq R(t) \leq U_{\text{upper}}(T_2, t).
\]

5. Monotonicity of the successive approximations

Consider the associated Eq. (3.2) with \( U(0) = E \). Let us denote \( U(t) \) the solution of (3.2) as \( U_1(t) \) the estimate at the first stage of approximation. Next, let us replace \( T(t) \) in (3.2) by \( U_1(t) \) and write the equation for \( U_2(t) \), the solution at the second stage, as

\[
U_2^\Delta(t) = [B(t) - U_1(t)C(t)]U_2(t) + U_2(\sigma(t))[B^*(t) - C(t)U_1(t)]
\]
\[
+ A(t) + U_1(t)C(t)U_1(t),
\]

with \( U_2(0) = E \). Continuing in this fashion we construct a sequence of matrix approximations \( \{U_n(t)\} \), where

\[
U_{n+1}^\Delta(t) = [B(t) - U_n(t)C(t)]U_{n+1}(t) + U_{n+1}(\sigma(t))[B^*(t) - C(t)U_n(t)]
\]
\[
+ A(t) + U_n(t)C(t)U_n(t)
\]

(5.1)

with \( U_{n+1}(0) = E \). For any \( n \),

\[
[R(\sigma(t)) - U_n(t)]C(t)[R(t) - U_n(t)] \geq 0
\]

or

\[
R(\sigma(t))C(t)R(t) \geq U_n(t)C(t)R(t) + R(\sigma(t))C(t)U_n(t) - U_n(t)C(t)U_n(t).
\]

Using the above inequality in (GMRDE), we have

\[
R^\Delta(t) \leq [B(t) - U_n(t)C(t)]R(t) + R(\sigma(t))[B^*(t) - C(t)U_n(t)] + A(t) + U_n(t)C(t)U_n(t),
\]

(5.2)

with \( R(0) = E \). From (5.1) and (5.2),

\[
[U_{n+1}(t) - R(t)]^\Delta \geq [B(t) - U_n(t)C(t)][U_{n+1}(t) - R(t)] + [U_{n+1}(\sigma(t)) - R(\sigma(t))][B^*(t) - C(t)U_n(t)],
\]

with \( U_{n+1}(0) - R(0) = 0 \). Therefore, we can conclude that

\[
R(t) \leq U_{n+1}(t), \quad n = 1, 2, 3, \ldots
\]

within the interval of existence of solution \( R(t) \). That is, \( R(t) \) is a lower bound of the sequence.
Theorem 5.1. The solutions of the successive approximations of Eq. (3.2) form a monotonically decreasing sequence.

Proof. If we write down the equation for the $n$th and $(n+1)$st approximations to Eq. (3.2), we get

\[ U_n^\Delta(t) = [B(t) - U_{n-1}(t)C(t)]U_n(t) + U_n(\sigma(t))[B^*(t) - C(t)U_{n-1}(t)] + A(t) + U_{n-1}(t)C(t)U_{n-1}(t), \] \tag{5.3} \]

with $U_0(0) = E$, and Eq. (5.1) with $U_{n+1}(0) = E$. Now consider the identity

\[ U_{n-1}(t)C(t)U_{n-1}(t) = [U_n(\sigma(t)) + U_{n-1}(t) - U_n(\sigma(t))]C(t)[U_n(t) + U_{n-1}(t) - U_n(t)], \]
\[ \geq U_n(\sigma(t))C(t)U_{n-1}(t) + U_{n-1}(t)C(t)U_n(t) - U_n(\sigma(t))C(t)U_n(t). \]

Using the above inequality and Eq. (5.3) we obtain

\[ U_n^\Delta(t) \geq [B(t) - U_{n-1}(t)C(t)]U_n(t) + U_n(\sigma(t))[B^*(t) - C(t)U_{n-1}(t)] + A(t) + U_{n-1}(t)C(t)U_{n}(t). \]

Thus we have

\[ U_n^\Delta(t) \geq [B(t) - U_n(t)C(t)]U_n(t) + U_n(\sigma(t))[B^*(t) - C(t)U_n(t)] + A(t) + U_n(t)C(t)U_n(t) \] \tag{5.4} \]

with $U_n(0) = E$. From (5.1) and (5.4), we have

\[ [U_n(t) - U_{n+1}(t)]^\Delta \geq [B(t) - U_n(t)C(t)][U_n(t) - U_{n+1}(t)] \]
\[ + [U_n(\sigma(t)) - U_{n+1}(\sigma(t))][B^*(t) - C(t)U_n(t)] \] \tag{5.5} \]

with $U_n(0) - U_{n+1}(0) = 0$. Letting $Y(t) = U_n(t) - U_{n+1}(t)$, then the inequality (5.5) can be written as

\[ Y^\Delta(t) \geq [B(t) - U_n(t)C(t)]Y(t) + Y(\sigma(t))[B^*(t) - C(t)U_n(t)] \]

with $Y(0) = 0$. This last inequality above can be written as

\[ Y^\Delta(t) = [B(t) - U_n(t)C(t)]Y(t) + Y(\sigma(t))[B^*(t) - C(t)U_n(t)] + Q(t) \] \tag{5.6} \]
with \( Y(0) = 0 \), where \( Q(t) \) is positive definite \( n \times n \) matrix. If \( \Phi_2(t) \) and \( \Psi_2(t) \) are fundamental matrix solutions of

\[
Y^\Delta(t) = [B(t) - U_n(t)C(t)]Y(t)
\]

and

\[
Y^\Delta(t) = [B^*(t) - C(t)U_n(t)]^TY(\sigma(t)),
\]

respectively, then the solution of (5.6) can be written as

\[
Y(t) = \int_0^t \Phi_2(t, \sigma(s))Q(s)\Psi_2^*(t, s) \Delta s.
\]

Therefore, \( U_{n+1}(t) \leq U_n(t) \), for all \( n \). Thus the successive approximations form a monotonically decreasing sequence.

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