Journal of Differential Equations **180**, 51–64 (2002)
doi:10.1006/jdeq.2001.4056, available online at http://www.idealibrary.com on IDEAL®

Upper and Lower Solution Methods for Fully Nonlinear Boundary Value Problems

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Received April 25, 2001; revised January 2, 2001

Sufficient conditions are given for the existence of a solution of a fourth order nonlinear boundary value problem with nonlinear boundary conditions. The conditions assume the existence of a strong upper solution–lower solution pair, a concept that is defined in the paper. The differential equation has nonlinear dependence on all lower order derivatives of the unknown; in particular, appropriate Nagumo conditions are obtained and employed.

Key Words: nonlinear boundary value problems; Nagumo condition; upper and lower solution.

1. INTRODUCTION

In this paper, we shall employ the method of upper and lower solutions to study the existence of solutions of a boundary value problem (BVP) for a fourth order ordinary differential equation (ODE) with some quite general nonlinear boundary conditions. In order to emphasize the general nature of the boundary conditions, we shall follow the lead of Thompson [28, 29] and refer to the problem we consider as a fully nonlinear BVP.

0022-0396/02 $35.00

The method of upper and lower solutions is extensively developed for lower order equations; see Ako [2, 3, 4], Gaines [14], Jackson [18, 19], Mawhin [23], and Nagumo [25]. More recently, Thompson [28, 29] has continued the development of these methods with applications to fully nonlinear BVPs. Recently, Henderson and Thompson [17] have initiated the application of these methods to discrete problems that have been obtained by discretizing continuous problems. They go on to show that the solutions of the discrete problems provide accurate estimates of the solutions of the continuous problems.

Although the methods have been particularly fruitful for low order ordinary differential equations, Kelly [21] and Klaasen [22] did obtain early applications to higher order ODEs. Recently, Ehme *et al.* [9] employed truncations analogous to those of Kelly [21] or Klaasen [22] and extended the application of the method of upper and lower solutions to $2m$th order ODEs where there is no dependence on odd order derivatives. This paper then provides a generalization of the results obtained by Ehme *et al.*, [9]. The results are valid for a $2m$th order problem. For the sake of exposition, we shall develop the results for a fourth order problem and then indicate the extension of the results and methods to a $2m$th order problem.

We point out that the method of upper and lower solutions, coupled with monotone methods, has been very useful in the study of BVPs for higher order functional equations. We refer the reader to [8, Chap. III] for an elegant exposition of the method and [27] and [10] for applications to nth order problems. There have been numerous applications of the monotone method to a problem similar to the fourth order problem we consider; the recent work by Ma *et al.* [13] provides an excellent account of the method with a good set of references; in a closely related work, Bai [5] first applies a new form of a maximum principle and then relaxes some monotonicity hypotheses. Eloe and Islam [12] have applied the monotone method to a very closely related impulse problem and Pao [26] has recently applied the monotone method to a related BVP for a fourth order elliptic partial differential equation. In each of these works [5, 12, 13, 26], there is no dependence on odd order derivatives.

The methods we use in this paper differ from monotone methods in two fundamental ways. First, we do not seek iterative improvement. Second, the Schauder fixed point theorem is typically employed to obtain the existence of solutions when monotone methods are employed. In the methods we employ, the Schauder fixed point theorem is applied to truncated problems, but the Kamke existence of solutions of the initial value problems theorem is then employed to obtain that a limit point of the solutions of the family of truncated problems is a solution of the original problem.

The paper is organized as follows. In Section 2 we define precisely the fourth order problem we consider. The concept of a strong upper solution–lower solution pair is defined. And for the sake of completeness, we state a version of the Kamke existence of solutions of initial value problems theorem. In Section 3, we state and prove the main theorem for the fourth order problem. In Section 4, we define a general $2m$th order problem and indicate how the results in Section 3 carry over.

2. PRELIMINARIES

In this section, we shall first define the general BVP we consider. We shall then impose Lidstone boundary conditions [1] on the BVP. Once the imposed boundary conditions are in place, we shall define a strong upper solution–lower solution pair for a Lidstone BVP. Finally, so that this work is self-contained, we shall state a version of the Kamke convergence theorem for solutions of initial value problems (IVPs) for ODEs.

Let $f\colon [0,1]\times\mathbb{R}^4\to\mathbb{R}$ be continuous, and let $k_j\colon \mathbb{R}^6\to\mathbb{R}$, $l_j\colon \mathbb{R}^6\to\mathbb{R}$, $j=1,2$, be continuous. We consider the fully nonlinear BVP

$$x^{(iv)}(t)=f(t,x(t),x'(t),x''(t),x'''(t)),\quad 0<t<1, \tag{1}$$

$$\begin{aligned} k_1(\bar{x})&=0, \qquad & l_1(\bar{x})&=0\\ k_2(\bar{x})&=0, \qquad & l_2(\bar{x})&=0, \end{aligned} \tag{2}$$

where $\bar{x}=(x(0),x(1),x'(0),x'(1),x''(0),x''(1))$.

We remark that far more general boundary conditions can be considered. For example, one might consider multipoint boundary conditions analogous to those considered by Gupta *et al.* ([15], for example) and let

$$\bar{x}=(x(0),x(1/2),x(1),x'(0),x'(1/2),x'(1),x''(0),x''(1/2),x''(1)).$$

We consider the specific conditions (2) for the sake of exposition.

In order to analyze the BVP (1), (2), we shall construct an equivalent set of boundary conditions. In particular, we shall force Lidstone boundary conditions (see [1]). So, consider the equivalent set of boundary conditions

$$\begin{aligned} x(0)&=h_1(\bar{x})=k_1(\bar{x})+x(0), \qquad & x(1)&=i_1(\bar{x})=l_1(\bar{x})+x(1),\\ x''(0)&=h_2(\bar{x})=k_2(\bar{x})+x''(0), \qquad & x''(1)&=i_2(\bar{x})=l_2(\bar{x})+x''(1). \end{aligned} \tag{3}$$

Clearly, the BVPs (1), (2) and (1), (3) are equivalent; we shall address the BVP (1), (3).

DEFINITION 2.1. Let $\alpha, \beta \in C^4[0, 1]$. Set

$$A = \max\{\|\alpha'\|, \|\beta'\|\}, \qquad B = \max\{|\beta(0) - \alpha(1)|, |\beta(1) - \alpha(0)|\},$$

where $\|\cdot\|$ denotes the supremum norm on $C[0, 1]$. Set $C = 2A + B$ and set

$$\bar{\alpha} = (\alpha(0), \alpha(1), -C, -C, \alpha''(0), \alpha''(1)),$$
$$\bar{\beta} = (\beta(0), \beta(1), C, C, \beta''(0), \beta''(1)).$$

Then α, β are said to be a *strong upper solution–lower solution pair* for the BVP (1), (3) if

$$(-1)^j \alpha^{(2j)}(t) \leqslant (-1)^j \beta^{(2j)}(t), \qquad 0 \leqslant t \leqslant 1, \qquad j = 0, 1, \tag{4}$$

$$\begin{aligned} &\alpha^{(iv)}(t) \leqslant f(t, \alpha(t), -C, \alpha''(t), \alpha'''(t)), \\ &\beta^{(iv)}(t) \geqslant f(t, \beta(t), C, \beta''(t), \beta'''(t)), \qquad 0 < t < 1, \end{aligned} \tag{5}$$

$$\begin{aligned} \alpha(0) &\leqslant h_1(\bar{\alpha}), & \alpha(1) &\leqslant i_1(\bar{\alpha}), \\ \beta(0) &\geqslant h_1(\bar{\beta}), & \beta(1) &\geqslant i_1(\bar{\beta}), \\ \alpha''(0) &\geqslant h_2(\bar{\alpha}), & \alpha''(1) &\geqslant i_2(\bar{\alpha}), \\ \beta''(0) &\leqslant h_2(\bar{\beta}), & \beta''(1) &\leqslant i_2(\bar{\beta}). \end{aligned} \tag{6}$$

We close this section with a version of the Kamke convergence theorem for solutions of IVPs for ODEs. We state the theorem for a general nth order equation. We refer the reader to [18] for the particular version we state, and we refer the reader to [16, p. 14] for a general discussion.

THEOREM 2.1. *Assume for $k = 0, 1, \ldots,$ the functions g_k are continuous on $I \times \mathbb{R}^n$ where I is an interval of the real line. Assume*

$$\lim_{k \to \infty} g_k(t, x, x', \ldots, x^{(n-1)}) = g(t, x, x', \ldots, x^{(n-1)})$$

uniformly on each compact subset of $I \times \mathbb{R}^n$. Assume that $\{t_k\} \subset I$ and $\lim_{k \to \infty} t_k = t_0$ and that for each k, x_k is a solution of

$$x^{(n)} = g_k(t, x, x', \ldots, x^{(n-1)}),$$

which is defined on a maximal interval, $I_k \subset I$ with $t_k \in I_k$. Further, assume that

$$\lim_{k \to \infty} x_k^{(j-1)}(t_k) = a_j, \qquad j = 1, \ldots, n,$$

for $a_j \in \mathbb{R}$, $j = 1, \ldots, n$. *Then there is a subsequence*, $\{x_{k_j}\}$, *of* $\{x_k\}$, *and there is a solution*, x, *of*

$$x^{(n)} = g(t, x, x', \ldots, x^{(n-1)}),$$

defined on a maximal interval $I_0 \subset I$ *such that* $t_0 \in I_0$, $x^{(j-1)}(t_0) = a_j$, $j = 1, \ldots, n$, *and such that for any compact interval contained in* I_0 *it follows that the compact interval is contained in* I_{k_j} *for sufficiently large* j *and*

$$\lim x_{k_j}^{(i-1)} = x^{(i-1)}$$

uniformly on that compact interval, for each $i = 1, \ldots, n$.

3. EXISTENCE OF SOLUTIONS

We first introduce a fixed point operator upon which we apply the Schauder fixed point theorem. Since we forced Lidstone boundary conditions in (3), we consider a fixed point operator associated with a Lidstone BVP. In particular, let

$$G(t, s) = \begin{cases} t(s-1), & 0 \leqslant t < s \leqslant 1, \\ s(t-1), & 0 \leqslant s < t \leqslant 1. \end{cases} \tag{7}$$

Further, define

$$H(t, s) = \int_0^1 G(t, r)\, G(r, s)\, dr,$$

where G is given by (7). Define

$$K_1 x(t) = i_1(\bar{x})\, t + h_1(\bar{x})(1-t) + \int_0^1 G(t, s)(i_2(\bar{x})\, s + h_2(\bar{x})(1-s))\, ds,$$

and define

$$K_2 x(t) = \int_0^1 H(t, s)\, f(s, x(s), x'(s), x''(s), x'''(s))\, ds.$$

LEMMA 3.1. *x is a solution of the BVP* (1), (3) *if, and only if,* $x \in C^3[0, 1]$ *and*

$$x(t) = K_1 x(t) + K_2 x(t), \qquad 0 \leqslant t \leqslant 1.$$

We refer the reader to [11] for a proof of this lemma for $2m$th order Lidstone problems; for $2m=4$, the proof employs a straightforward calculation.

We now proceed as motivated by Kelly [21] or Klaasen [22] and define truncations with respect to a strong upper solution–lower solution pair, α, β. For $x \in C^3[0, 1]$, define

$$T_0x(t) = \max\{\alpha(t), \min\{x(t), \beta(t)\}\}.$$

Note $\alpha(t) \leqslant T_0x(t) \leqslant \beta(t)$, $0 \leqslant t \leqslant 1$. Define

$$T_1x'(t) = \max\{-C, \min\{x'(t), C\}\}.$$

Note $|T_1x'(t)| \leqslant C$, $0 \leqslant t \leqslant 1$. Define

$$T_2x''(t) = \max\{\beta''(t), \min\{x''(t), \alpha''(t)\}\}.$$

Note $\beta''(t) \leqslant T_2x''(t) \leqslant \alpha''(t)$, $0 \leqslant t \leqslant 1$. Now define a truncation, F, by

$$\begin{aligned} F(t, x, x', x'', x''') &= f(t, T_0x, T_1(x'), T_2(x''), x''') \\ &\quad + (x'' - T_2(x''(t)))/(1 + |x'' - T_2(x''(t))|) \end{aligned}$$

and truncations H_j, I_j, $j = 1, 2$, by

$$\begin{aligned} H_1(\bar{x}) &= h_1(\bar{T}x), \qquad & I_1(\bar{x}) &= i_1(\bar{T}x), \\ H_2(\bar{x}) &= h_2(\bar{T}x), \qquad & I_2(\bar{x}) &= i_2(\bar{T}x), \end{aligned}$$

where $\bar{T}x = (T_0x(0), T_0x(1), T_1x'(0), T_1x'(1), T_2x''(0), T_2x''(1))$. Define further truncations, F_n, by

$$F_n(t, x, x', x'', x''') = \begin{cases} F(t, x, x', x'', N+n), & \text{if } \; x''' > N+n, \\ F(t, x, x', x'', x'''), & \text{if } \; |x'''| \leqslant N+n, \\ F(t, x, x', x'', -(N+n)), & \text{if } \; x''' < -(N+n), \end{cases} \tag{8}$$

where $N = \max\{\|\alpha'''\|, \|\beta'''\|\}$.

Now consider the sequence of truncated BVPs,

$$x^{(iv)}(t) = F_n(t, x(t), x'(t), x''(t), x'''(t)), \qquad 0 < t < 1, \tag{9}$$

$$\begin{aligned} x(0) &= H_1(\bar{x}), \qquad & x(1) &= I_1(\bar{x}), \\ x''(0) &= H_2(\bar{x}), \qquad & x''(1) &= I_2(\bar{x}). \end{aligned} \tag{10}$$

THEOREM 3.1. *Assume that each solution of* (1) *either extends to* $[0, 1]$ *or one of the solution, derivative, or second derivative of that solution* *becomes unbounded on its maximal interval of existence. In addition to assuming the* $f, h_j, i_j, j=1,2$ *are continuous, assume that* $f(t, x_1, x_2, x_3, x_4)$ *is increasing in each of* x_1 *and* x_2. *Moreover, assume that* $(-1)^{j-1}$ $h_j(x_1, ..., x_6)$ *is increasing in each* x_l, $l=1, ..., 4$ *and is decreasing in each of* x_5 *and* x_6, $j=1,2$. *Assume* $(-1)^{j-1}\, i_j$ *satisfies the same monotonicity conditions as the corresponding* $(-1)^{j-1}\, h_j$. *Assume there exists a strong upper solution–lower solution pair,* α, β, *for the BVP* (1), (3). *Then, there exists a solution,* x, *of the BVP,* (1), (3) *such that*

$$\beta''(t) \leqslant x''(t) \leqslant \alpha''(t), \qquad 0 \leqslant t \leqslant 1, \tag{11}$$

$$|x'(t)| \leqslant C, \qquad 0 \leqslant t \leqslant 1, \tag{12}$$

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \qquad 0 \leqslant t \leqslant 1. \tag{13}$$

Proof. For each n consider the truncated BVP (9), (10). Since each F_n, H_1, H_2, I_1, I_2 is bounded and continuous, it follows from Lemma 3.1 and the Schauder fixed point theorem that there exists a solution, x_n, of the BVP (9), (10) for each $n=1, 2, ...$. We first show that each x_n satisfies (11), (12), and (13).

We begin with (11). First, note that

$$x_n''(0) = H_2(\bar{x}_n) = h_2(\bar{T}x_n) \leqslant h_2(\bar{\alpha}) \leqslant \alpha''(0).$$

The first inequality follows from the monotonicity assumptions on h_2 and the second inequality follows from the definition of a strong lower solution. The inequality $x_n''(1) \leqslant \alpha''(1)$ follows similarly. Assume for the sake of contradiction that $x_n''(t) \leqslant \alpha''(t)$, $0 < t < 1$, is not true. Suppose $x_n'' - \alpha''$ attains a positive maximum at $t_0 \in (0,1)$. Then $(x_n^{(iv)} - \alpha^{(iv)})(t_0) \leqslant 0, T_2 x_n''(t_0) = \alpha''(t_0)$, and $x_n'''(t_0) = \alpha'''(t_0)$. However,

$$\begin{aligned}(x_n^{(iv)} - \alpha^{(iv)})(t_0) &\geqslant f(t_0, T_0 x_n(t_0), T_1 x_n'(t_0), T_2 x_n''(t_0), \alpha'''(t_0)) \\ &\quad - f(t_0, \alpha(t_0), -C, \alpha''(t_0), \alpha'''(t_0)) \\ &\quad + (x_n''(t_0) - T_2 x_n''(t_0))/(1 + |x_n''(t_0) - T_2 x_n''(t_0)|) \\ &\geqslant (x_n''(t_0) - T_2 x_n''(t_0))/(1 + |x_n''(t_0) - T_2 x_n''(t_0)|) > 0.\end{aligned}$$

The first inequality follows from the definition of a strong lower solution. The second inequality follows from the monotonicity assumptions on f. This contradiction completes the argument that

$$x_n''(t) \leqslant \alpha''(t), \qquad 0 \leqslant t \leqslant 1.$$

The argument that

$$x_n''(t) \geqslant \beta''(t), \qquad 0 \leqslant t \leqslant 1,$$

is similar and the argument that x_n satisfies (11) is complete.

We now obtain (13) for each x_n. Note that

$$x_n(0) = H_1(\bar{x}_n) = h_1(\bar{T}x) \geqslant h_1(\bar{\alpha}) \geqslant \alpha(0).$$

Similarly, $x_n(1) \geqslant \alpha(1)$. We now employ a Green's function representation to argue that $x_n(t) \geqslant \alpha(t)$, $0 \leqslant t \leqslant 1$. Let $G(t, s)$ be given by (7). Then

$$x_n(t) - \alpha(t) = (x_n(1) - \alpha(1))\, t + (x_n(0) - \alpha(0))(1-t)$$
$$+ \int_0^1 G(t, s)(x_n''(s) - \alpha''(s))\, ds.$$

Since, $G(t, s) < 0$ on $(0, 1) \times (0, 1)$ and we have already argued that $x_n''(t) \leqslant \alpha''(t)$, $0 \leqslant t \leqslant 1$, it readily follows that $x_n(t) \geqslant \alpha(t)$, $0 \leqslant t \leqslant 1$. The argument that $x_n(t) \leqslant \beta(t)$, $0 \leqslant t \leqslant 1$, is analogous and the argument that x_n satisfies (13) is complete.

Next, we obtain (12) for each x_n. By the mean value theorem, there exists $t_n \in (0, 1)$ such that

$$|x_n'(t_n)| = |x_n(1) - x_n(0)| \leqslant \max\{|\beta(0) - \alpha(1)|, |\beta(1) - \alpha(0)|\} = B.$$

Employ (11) to obtain (for the sake of exposition, assume $t > t_n$)

$$\int_{t_n}^t \beta''(s)\, ds \leqslant x_n'(t) - x_n'(t_n) \leqslant \int_{t_n}^t \alpha''(s)\, ds.$$

A similar inequality follows for $t < t_n$ and so x_n satisfies (12).

To complete the proof we will apply Kamke's theorem. Again by the mean value theorem, there exists $t_n \in (0, 1)$ such that

$$|x_n'''(t_n)| = |x_n''(1) - x_n''(0)| \leqslant \max\{|\beta''(0) - \alpha''(1)|, |\beta''(1) - \alpha''(0)|\}.$$

Thus, each of the sequences,

$$\{x_n^{(j)}(t_n)\}, \qquad j = 0, 1, 2, 3,$$

is bounded. Choose a subsequence of $\{t_n\}$ which we relabel as $\{t_n\}$ such that

$$\lim t_n = t_0, \qquad \lim x_n^{(j)}(t_n) = x_0^{(j)}, \qquad j = 0, 1, 2, 3.$$

Apply Kamke's theorem and there exists a solution x of

$$x^{(iv)} = F(t, x, x', x'', x''')$$

on a maximal subinterval $J \subset [0, 1]$ and a subsequence of $\{x_n\}$ that converges to x in $C^3(D)$ for any compact subinterval D of J. Thus, x

satisfies (11), (12), (13) on J. Since, x extends to all of $[0, 1]$ or one of x, x', or x'' become unbounded, $J = [0, 1]$ and the proof of the theorem is complete. ∎

Remark 3.1. We will say that $f(t, x_1, x_2, x_3, x_4)$ satisfies a *Nagumo condition* on $[0, 1] \times \mathbb{R}^4$ if f is continuous and given any $M > 0$ there is a positive continuous function $h_M(s)$ on $[0, \infty)$ such that

$$|f(t, x_1, x_2, x_3, x_4)| \leqslant h_M(|x_4|)$$

for all $(t, x_1, x_2, x_3, x_4) \in [0, 1] \times [-M, M]^2 \times \mathbb{R}$ and such that

$$\int_0^\infty (s/h_M(s))\, ds = \infty.$$

THEOREM 3.2. *If f satisfies a Nagumo condition on* $[0, 1] \times \mathbb{R}^4$, *then* (1) *has the property that each solution either extends to all of* $[0, 1]$ *or one of the solution, derivative, or second derivative of that solution becomes unbounded on its maximal interval of existence.*

We refer the reader to [16, p. 428] for the technical details. Intuitively, if $x^{(4)} = f$ then

$$|x^{(4)}/h_M(|x'''|) \leqslant 1.$$

One integrates with the appropriate change of variable, applies the Nagumo condition, notes that $[0, 1]$ is bounded, and obtains that $|x'''|$ is bounded.

Remark 3.2. Jackson [20] obtains a more general Nagumo type condition for nth order ordinary differential equations. In Jackson's setting, all solutions extend or the solution becomes unbounded. Jackson's condition places a more strict growth condition on f as a function of x'''. As we have already established that solutions we seek satisfy (11), (12), and (13), the Nagumo condition given in Remark 3.1 is sufficient.

Remark 3.3. In the proof of Theorem 3.1, (11) implies (13) by the sign property of $G(t, s)$. Hence, the monotonicity of f with respect to x is determined. The monotonicity of f with respect to x' is less determined. In Theorem 3.1, we assume that f, $(-1)^{j-1} h_j$, and $(-1)^{j-1} i_j$ are all increasing in x'. Assume instead, for example, that f, $(-1)^{j-1} h_j$, and $(-1)^{j-1} i_j$ are all decreasing in x'. Modify the definition of a strong upper solution–lower solution pair for the BVP (1), (3) as follows. Set

$$\bar{\alpha} = (\alpha(0), \alpha(1), C, C, \alpha''(0), \alpha''(1)),$$

$$\bar{\beta} = (\beta(0), \beta(1), -C, -C, \beta''(0), \beta''(1)).$$

Then α, β are said to be a strong upper solution–lower solution pair for the BVP (1), (3) if (4) and (6) hold and

$$\alpha^{(iv)}(t) \leqslant f(t, \alpha(t), C, \alpha''(t), \alpha'''(t)),$$
$$\beta^{(iv)}(t) \geqslant f(t, \beta(t), -C, \beta''(t), \beta'''(t)), \qquad 0 < t < 1.$$

An analogue of Theorem 3.1 can be stated and proved in this context.

EXAMPLE 3.1. We now exhibit an example to show that a strong upper solution–lower solution pair can exist naturally. We consider linear, homogeneous, Lidstone boundary conditions and point out that even in this context, the results in this paper are new.

Consider a problem of the form

$$x^{(iv)}(t) = c_1(t)(16x(t)/5)^3 + c_2(t)\, x'(t)^3 + c_3(t)(x''(t)/3)^k + c_4(t)(x'''(t)/12)^2 + c_5(t), \qquad 0 < t < 1, \tag{14}$$

$$x^{(2j)}(0) = x^{(2j)}(1) = 0, \quad j = 0, 1. \tag{15}$$

If $c_j \in C[0, 1]$, $j = 1, \ldots, 5$, $c_1, c_2 > 0$, $k > 0$, and $\|c_1\| + \|c_2\| + \|c_3\| + \|c_4\| + \|c_5\| \leqslant 24$, then there exists a solution x of the BVP (14), (15) satisfying

$$|x(t)| \leqslant t^4 - 2t^3 + t, \qquad 0 \leqslant t \leqslant 1,$$
$$|x'(t)| \leqslant 1, \qquad 0 \leqslant t \leqslant 1,$$
$$|x''(t)| \leqslant 12t - 12t^2, \qquad 0 \leqslant t \leqslant 1.$$

To see this, it is not difficult to show that if

$$\beta(t) = t^4 - 2t^3 + t, \qquad 0 \leqslant t \leqslant 1,$$

then $-\beta$, β form a strong upper solution–lower solution pair for the BVP (14), (15). To verify the details, note that

$$\|\beta\| = 5/16, \qquad \|\beta'\| = 1, \qquad \|\beta''\| = 3, \qquad \|\beta'''\| = 12.$$

EXAMPLE 3.2. If

$$\beta(t) = t^4 - 2t^3 + t, \quad 0 \leqslant t \leqslant 1,$$

then $-\beta$, β also form a strong upper solution–lower solution pair for the BVP

$$x^{(iv)}(t) = (k_1(t)(16x(t)/5)^{c_1} + k_2(t)(x'(t))^{c_2} + k_3(t))(|x''(t)|/3)^{c_3}\,(x'''(t)/12)^2, \quad 0 < t < 1,$$

along with the boundary conditions (15) if $k_1, k_2, k_3 \in C[0, 1]$, $k_1(t)$, $k_2(t) \geqslant 0$, $0 \leqslant t \leqslant 1$, $\|k_1\| + \|k_2\| + \|k_3\| \leqslant 24$, $c_3 \geqslant 0$ $c_j = 0$ or c_j is an odd integer, $j = 1, 2$.

EXAMPLE 3.3. The solution of the BVP

$$x^{(iv)}(t) = -Px''(t) + p(x, t), \qquad 0 \leqslant t \leqslant 1,$$

with boundary conditions (15) represents the deflection of a hinged beam column. P represents the axial loading and p represents the nonconservative force. We refer the reader to monographs [6, 7, 24] for discussions. If $P > 0$ then the axial load is said to be applying compression. The methods developed in [13] apply here since the right-hand side is monotone decreasing as a function of x'' (and assuming p satisfies appropriate properties). If $P < 0$, then the axial force is said to be applying tension. Now the methods developed in [13] do not apply. In terms of Example 3.1, $-\beta$, β form a strong upper solution–lower solution pair if $p(x, t) = c_1 x^3 + \sin \pi t$, $c_1 > 0$, and $c_1 + |P| \leqslant 23$.

Moreover, nonlinear terms can be introduced in P in the following way. The differential equation governing deflection has the more general form

$$x^{(iv)}(t) = -(Px')'(t) + p(x, t), \quad 0 \leqslant t \leqslant 1.$$

In the case of small deflections or for simplicity, P is considered constant. In the case of large deflections, P can depend on x. See [6] for further discussion.

4. THE 2mTH ORDER PROBLEM

In this section, we shall define a general $2m$th order problem and define the appropriate strong upper solution–lower solution pair. We will then state an analogous theorem to Theorem 3.1. We do not exhibit the proof as one only adds inductive features to the proof of Theorem 3.1. The inductive features are developed in [9] and [11].

Let $m \geqslant 1$ denote an integer. Let $f\colon [0, 1] \times \mathbb{R}^{2m} \to \mathbb{R}$ be continuous, and let $k_j\colon \mathbb{R}^{4m-2} \to \mathbb{R}$, $l_j\colon \mathbb{R}^{4m-2} \to \mathbb{R}$, $j = 1, 2, \ldots, 2m-2$, be continuous. Consider the fully nonlinear BVP

$$x^{(2m)}(t) = f(t, x(t), x'(t), \ldots, x^{(2m-1)}), \qquad 0 < t < 1, \tag{16}$$

$$k_j(\bar{x}) = 0, \qquad l_j(\bar{x}) = 0, \tag{17}$$

$j = 1, \ldots, m$, where $\bar{x} = (x(0), x(1), x'(0), x'(1), \ldots, x^{(2m-2)}(0), x^{(2m-2)}(1))$.

Force Lidstone boundary conditions and consider the equivalent set of boundary conditions

$$\begin{aligned} x^{(2(j-1))}(0) &= h_j(\bar{x}) = k_j(\bar{x}) + x^{(2(j-1))}(0), \\ x^{(2(j-1))}(1) &= i_j(\bar{x}) = l_j(\bar{x}) + x^{(2(j-1))}(1), \end{aligned} \tag{18}$$

$j = 1, \ldots, m$.

Definition 4.1. Let $\alpha, \beta \in C^{2m}[0, 1]$. Set

$$\begin{aligned} A_j &= \max\{\|\alpha^{(2j-1)}\|, \|\beta^{(2j-1)}\|\}, \\ B_j &= \max\{|\beta^{(2j-2)}(0) - \alpha^{(2j-2)}(1)|, |\beta^{(2j-2)}(1) - \alpha^{(2j-2)}(0)|\}, \end{aligned}$$

and set $C_j = 2A_j + B_j$, $j = 1, \ldots, m-1$. Set

$$\begin{aligned} \bar{\alpha} = (&\alpha(0), \alpha(1), -C_1, -C_1, \ldots, \alpha^{(2(j-1))}(0), \alpha^{(2(j-1))}(1), \\ &(-1)^j C_j, (-1)^j C_j, \ldots, \alpha^{(2(m-1))}(0), \alpha^{(2(m-1))}(1)), \end{aligned}$$

and

$$\begin{aligned} \bar{\beta} = (&\beta(0), \beta(1), C_1, C_1, \ldots, \beta^{(2(j-1))}(0), \beta^{(2(j-1))}(1), (-1)^{j-1} C_j, \\ &(-1)^{j-1} C_j, \ldots, \beta^{(2(m-1))}(0), \beta^{(2(m-1))}(1)). \end{aligned}$$

Then α, β are said to be a *strong upper solution–lower solution pair* for the BVP, (16), (18) if

$$\begin{aligned} &(-1)^j \alpha^{(2j)}(t) \leqslant (-1)^j \beta^{(2j)}(t), \qquad 0 \leqslant t \leqslant 1, \qquad j = 0, 1, \ldots, m-1, \\ &(-1)^m \alpha^{(2m)}(t) \leqslant (-1)^m f(t, \alpha(t), -C, \alpha''(t), C, \ldots, (-1)^{m-1} C, \\ &\qquad\qquad \alpha^{(2(m-1))}(t), \alpha^{(2m-1)}(t)), \\ &(-1)^m \beta^{(2m)}(t) \geqslant (-1)^m f(t, \beta(t), C, \beta''(t), -C, \ldots, (-1)^m C, \\ &\qquad\qquad \beta^{(2(m-1))}(t), \beta^{(2m-1)}(t)), \end{aligned}$$

and

$$\begin{aligned} &(-1)^j \alpha^{(2j)}(0) \leqslant (-1)^j h_{j+1}(\bar{\alpha}), \\ &(-1)^j \alpha^{(2j)}(1) \leqslant (-1)^j i_{j+1}(\bar{\alpha}), \qquad j = 0, \ldots, (m-1), \\ &(-1)^j \beta^{(2j)}(0) \geqslant (-1)^j h_{j+1}(\bar{\beta}), \\ &(-1)^j \beta^{(2j)}(1) \geqslant (-1)^j i_{j+1}(\bar{\beta}), \qquad j = 0, \ldots, (m-1). \end{aligned}$$

Definitions in the literature of lower solutions and upper solutions are not uniform with respect to determining which function is actually the larger of the two. We have provided a definition such that $\beta(t) \geqslant \alpha(t)$, $0 < t < 1$.

THEOREM 4.1. *Assume that each solution of* (16) *either extends to* $[0, 1]$ *or that one of the* j*th derivatives of the solution,* $j = 0, \ldots, 2(m-1)$, *becomes unbounded on its maximal interval of existence. In addition to assuming the* $f, h_j, i_j, j = 0, \ldots, m-1$ *are continuous, assume*

$$(-1)^{m+k-1} f(t, x_{11}, x_{12}, x_{21}, x_{22}, \ldots, x_{m1}, x_{m2})$$

is increasing in each x_{kl}, $k = 1, \ldots, m-1$, $l = 1, 2$. *Moreover, assume that*

$$(-1)^{k+j} h_j(x_{11}, \ldots, x_{14}, x_{21}, \ldots, x_{24}, \ldots, x_{m-1,1}, \ldots x_{m-1,4}, x_{m1}, x_{m2})$$

is increasing in x_{kl}, $k = 1, \ldots, m-1$, $l = 1, \ldots, 4$, *or* $k = m$, $l = 1, 2$. *Assume* i_j *satisfies the same monotonicity conditions as the corresponding* h_j. *Assume there exists a strong upper solution–lower solution pair,* α, β, *for the BVP,* (16), (18). *Then, there exists a solution,* x, *of the BVP* (16), (18) *such that*

$$(-1)^j \alpha^{(2j)}(t) \leqslant (-1)^j x^{(2j)}(t) \leqslant (-1)^j \beta^{(2j)}(t), \qquad 0 \leqslant t \leqslant 1,$$
$$j = 0, 1, \ldots, m-1, \tag{19}$$

and

$$|x^{(2j-1)}(t)| \leqslant C_j, \qquad 0 \leqslant t \leqslant 1, \qquad j = 1, \ldots, m-1. \tag{20}$$

Since each solution satisfies (19) and (20), a Nagumo condition that applies to Theorem 4.1 and that is analogous to that given in Remark 3.1 is readily obtained.

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