Blow-up for a degenerate parabolic equation with a nonlocal source

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Abstract
In this paper, we investigate the blowup properties of the positive solutions to the following non-local degenerate parabolic equation

\[ v_\tau = x^\alpha (v^m)_{xx} + \int_0^l v^{p_1} \, dx - kv^{q_1} \]

with homogeneous Dirichlet boundary conditions in the interval \((0, l)\), where \(0 < \alpha < 2\), \(p_1 \geq q_1 > m > 1\). We first establish the local existence and uniqueness of its classical solutions. Then we show that the positive solution blows up in finite time if the initial datum is sufficient large. Finally, we prove that the blow-up set is the whole interval and we also obtain the estimates of the blow-up rate.

Keywords: Degenerate parabolic equation; Nonlocal source; Blow-up; Blow-up rate

1. Introduction

In this paper we consider the following nonlocal parabolic problem

\[ v_\tau = x^\alpha (v^m)_{xx} + \int_0^l v^{p_1} \, dx - kv^{q_1}, \quad 0 < x < l, \; \tau > 0, \]

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\[ v(0, \tau) = v(l, \tau) = 0, \quad \tau > 0, \]
\[ v(x, 0) = v_0(x), \quad 0 < x < l, \quad (1) \]

where \(0 < \alpha < 2, p_1 \geq q_1 > m > 1\). In the past two decades, much effort has been devoted to the study of blow-up properties for nonlocal semilinear parabolic equations (see [4,9–11] and references therein). When \(m = 1, \alpha = 0\), the problem (1) has been studied by Wang and Wang (see [11]). They proved that if \(p_1 > q_1 > 1\) or \(p_1 = q_1 > 1\) and \(l > k\) then the solution \(v(x, \tau)\) of (1) with large initial data blows up in finite time. When \(\alpha = 0, m = 1, p_1 > q_1 > 1\), Souplet (see [10]) obtained the asymptotic blow-up behavior of the solution. Recently, Deng et al. ([4]) considered the case where \(\alpha = 0, k = 0\). It is shown that there exist constants \(C_1, C_2\) such that

\[ C_1 (\tau^* - \tau)^{-1/q_1 - 1} \leq \max_{x \in [-l,l]} v(x, \tau) \leq C_2 (\tau^* - \tau)^{-1/q_1 - 1}, \]

where \(\tau^*\) is the blow-up time. In this paper, we introduce the spatial degeneracy and an absorption term. The difficulties arise when we prove the local existence since there exist two degeneracies in the equation. Moreover, for the case where \(p_1 = q_1 > 1\), it is difficult for their methods to get the blow-up rate estimate. In this paper we use other techniques to prove the global blow-up and to get the blow-up rate for this case.

To gain the blow-up properties of (1), we need some transformation first. Let \(v^m = u, \tau = (1/m) t\) in (1), then it becomes

\[ u_t = u^r (x^\alpha u_{xx} + \int_0^l u^p dx - k u^q), \quad 0 < x < l, \quad t > 0, \]
\[ u(0, t) = u(l, t) = 0, \quad \tau > 0, \]
\[ u(x, 0) = u_0(x), \quad 0 < x < l, \quad (2) \]

where \(0 < r = (m - 1)/m < 1, p = p_1/m, q = q_1/m, u_0 = v_0^m\) and \(p \geq q > 1\).

In this paper, the blowup means that there exists a \(T^* < \infty\) such that \(\|u(\cdot, t)\|_\infty < \infty\) for \(t \in (0, T^*)\) and \(\lim_{t \to T^*} \|u(\cdot, t)\|_\infty = \infty\).

**Definition 1.1.** A point \(x_0 \in [0, l]\) is a blow-up point of \(u(x, t)\) if there exists a sequence \(\{(x_n, t_n)\}\) such that \(t_n \to T^*, x_n \to x_0\) as \(n \to +\infty\) and

\[ \lim_{n \to \infty} u(x_n, t_n) = \infty. \]

We call the set of all blow-up points to be the blow-up set, which is denoted by \(S\). If \(S = [0, l]\), we say that the solutions of (2) blow up in finite time globally.

Before stating our main results, we make some assumptions on the initial values \(u_0(x)\).

- **(H1)** \(u_0(x) > 0\) in \((0, l), u_0(0) = u_0(l) = 0\) and \(u_{0x}(0) > 0, u_{0x}(l) < 0\);
- **(H2)** \(u_0(x) \in C^{2+\beta}((0, l)) \cap C[0, l]\) for some \(0 < \beta < 1\);
- **(H3)** \(x^\alpha u_{0xx} + \int_0^l u_0^p dx - k u_0^q \geq 0\) for \(x \in (0, l)\),
where \( l > k \) for \( p = q > 1 \).

In this paper we first give a proof of the local existence for (2) by a two-steps regularization procedure, since the degeneracy of (2) is introduced by two factors: \( u(0,t) = u(l,t) = 0 \) and \( x^q |_{x=0} = 0 \). Then, under appropriate hypotheses, we prove that the solution blows up in finite time. Finally, we show that the blow-up set is the whole interval and obtain the estimates of blow-up rate,

(i) if \( p > q > 1 \), then

\[
 u(x,t) \sim \left((p + r - 1)l(T^* - t)\right)^{-\frac{1}{p+r-1}}, \quad \text{a.e. in } (0,l) \text{ as } t \to T^*,
\]

that is,

\[
 \lim_{t \to T^*} u(x,t)(T^* - t)^{-\frac{1}{p+r-1}} = \left((p + r - 1)l\right)^{-\frac{1}{p+r-1}}, \quad \text{a.e. in } (0,l);
\] (3)

(ii) if \( p = q > 1 \) and \( l > k \), then there exist positive numbers \( c \) and \( c' \) such that

\[
 c(T^* - t)^{-\frac{1}{p_1-1}} \leq \|u(\cdot,t)\|_{\infty} \leq c'(T^* - t)^{-\frac{1}{p_1-1}},
\] (4)

as \( t \) close enough to \( T^* \).

Remark 1.1. For problem (1), letting \( r = (m-1)/m, \) \( p = p_1/m, \) \( q = q_1/m, \) \( t = m\tau, \) and \( u = v^m \) in (3), for the case \( p_1 > q_1 \), we obtain

\[
 \lim_{\tau \to \tau^*} v(x,\tau)(\tau^* - \tau)^{-\frac{1}{p_1-1}} = \left((p_1 - 1)l\right)^{-\frac{1}{p_1-1}}, \quad \text{a.e. in } (0,l),
\] (5)

where \( \tau^* \) is the blow-up time. It is clear that the estimate of the blow-up rate herein depend only on \( p_1 \), not of \( \alpha, m, k, q_1 \). For the case \( p_1 = q_1 \), we obtain from (4) that

\[
 c_0(\tau^* - \tau)^{-\frac{1}{p_1-1}} \leq \|v(\cdot,\tau)\|_{\infty} \leq c'_0(\tau^* - \tau)^{-\frac{1}{p_1-1}},
\] (6)

as \( \tau \) close enough to \( \tau^* \), where \( c_0 \) and \( c'_0 \) depend only on \( p_1 \) and \( k \) (see Remark 4.1 below).

Before leaving this section, we should remark that degenerate parabolic equations (especially, porous medium equations) without nonlocal terms were studied extensively by many other authors (for example, see [13–17] and references therein).

This paper is organized as follows. In Section 2, we establish the local existence and uniqueness of the solutions of the problem (2). Results relating to blow-up in finite time are present in Section 3. And in Section 4, results regarding the estimates of the blow-up rate for the problem (2) are established. In Appendix A we give an example of the initial value \( u_0(x) \) which satisfies the assumptions which we need in this paper.

2. Local existence

Since the problem (2) is doubly degenerate, the standard parabolic theory cannot be used directly to obtain the local existence of its classical solutions. We shall prove the local
existence by two steps. First we prove the local existence of the solution of the following problem

\[
\frac{\partial w}{\partial t} = (w + \delta)^r \left( x^\alpha w_{xx} + \int_0^l w_P^p \, dx - kw^q \right), \quad 0 < x < l, \quad t > 0,
\]

\[
w(0, t) = w(l, t) = 0, \quad t > 0,
\]

\[
w(x, 0) = u_0(x), \quad 0 < x < l.
\]

(7)

where \( \delta < 1 \) is a small positive constant. Let \( \varepsilon < l \) be a small positive constant. We consider the following regularized problem

\[
\frac{\partial w_\varepsilon}{\partial t} = (w_\varepsilon + \delta)^r \left( x^\alpha w_{\varepsilon xx} + \int_\varepsilon^l w_\varepsilon^P \, dx - kw_\varepsilon^q \right), \quad \varepsilon < x < l, \quad t > 0,
\]

\[
w_\varepsilon(0, t) = w_\varepsilon(l, t) = 0, \quad t > 0,
\]

\[
w_\varepsilon(x, 0) = u_{0\varepsilon}(x), \quad \varepsilon < x < l.
\]

(8)

To show the existence of the classical solution \( w_\varepsilon(x, t) \) of (8), let us introduce a cut off function \( \rho(x) \). By [5, p. 1640], there exists a nondecreasing function \( \rho(x) \in C^3(R) \) such that \( \rho(x) = 0 \) if \( x \leq 0 \) and \( \rho(x) = 1 \) if \( x \geq 1 \). Let

\[
\rho_\varepsilon(x) = \begin{cases} 
0, & x \leq \varepsilon, \\
\rho\left(\frac{x}{\varepsilon} - 1\right), & \varepsilon < x < 2\varepsilon, \\
1, & x \geq 2\varepsilon,
\end{cases}
\]

and let \( u_{0\varepsilon}(x) = \rho_\varepsilon(x)u_0(x) \). We note that

\[
\frac{\partial}{\partial \varepsilon} u_{0\varepsilon}(x) = \begin{cases} 
0, & x \leq \varepsilon, \\
-\frac{x}{\varepsilon^2} \rho'\left(\frac{x}{\varepsilon} - 1\right)u_0(x), & \varepsilon < x < 2\varepsilon, \\
0, & x \geq 2\varepsilon.
\end{cases}
\]

Since \( \rho(x) \) is nondecreasing, we have \( (\partial/\partial \varepsilon)u_{0\varepsilon} \leq 0 \). From \( 0 \leq \rho(x) \leq 1 \), we have \( u_0(x) \geq u_{0\varepsilon}(x) \) and \( \lim_{\varepsilon \to 0} u_{0\varepsilon}(x) = u_0(x) \). Next we consider the following regularized problem

\[
\frac{\partial w_\varepsilon}{\partial t} = (w_\varepsilon + \delta)^r \left( x^\alpha w_{\varepsilon xx} + \int_\varepsilon^l w_\varepsilon^P \, dx - kw_\varepsilon^q \right), \quad \varepsilon < x < l, \quad t > 0,
\]

\[
w_\varepsilon(0, t) = w_\varepsilon(l, t) = 0, \quad t > 0,
\]

\[
w_\varepsilon(x, 0) = u_{0\varepsilon}(x), \quad \varepsilon < x < l.
\]

(9)

**Lemma 2.1.** There exist \( t_0 \) and an a priori supersolution \( h \in C^1[0, t_0] \) depending only on \( u_0 \) and \( p \) such that for all \( \varepsilon > 0 \) there exists a unique classical positive solution \( w_\varepsilon \) of (9) in \((\varepsilon, l) \times (0, t_0)\) with \( 0 \leq w_\varepsilon \leq h \).

**Proof.** Consider the following ordinary differential problem
\[ h'(t) = lh^p(t)(h(t) + 1)\gamma, \quad t > 0, \]
\[ h(0) = \max_{x \in [0, l]} u_0(x). \]

From the theory of the ordinary differential equations, we know that there exists a positive constant \( t_0 \) such that the above problem admits a unique positive solution \( h(t) \) on \([0, t_0]\). It is easily to verify that \( h(t) \) is a supersolution of (9) for all \( \epsilon \). It follows from Theorem 4.2.2 in [8] that there exists a unique classical positive solution \( w_\epsilon \) of (9) in \((\epsilon, l) \times (0, t_0)\) with \( 0 \leq w_\epsilon \leq h \) for all \( \epsilon \). The proof is complete. \( \square \)

**Lemma 2.2.** Let \( \epsilon_0 > \epsilon_1 > \epsilon_2 > 0 \) and suppose that \( w_{\epsilon_1} \) and \( w_{\epsilon_2} \) are solutions of (9) on \((0, t_0)\), where \( \epsilon_0 < l \) is a small positive constant. Then \( w_{\epsilon_2}(x, t) \geq w_{\epsilon_1}(x, t) \) for all \( (x, t) \in (\epsilon_1, l) \times (0, t_0) \).

**Proof.** By the comparison principle, \( w_{\epsilon_2} \) and \( w_{\epsilon_1} \) are positive in their respective domains. In particular, \( w_{\epsilon_2}(x, t) > 0 \) when \( x = \epsilon_1 \), \( w_{\epsilon_2}(l, t) = w_{\epsilon_1}(l, t) = 0 \), \( u_{0\epsilon_1}(x) \leq u_{0\epsilon_2}(x) \) for all \( x \in (\epsilon_1, l) \). Then

\[
(\epsilon_2, l) \quad \text{for } t > 0. \text{ By the comparison principle, } w_{\epsilon_2} \geq w_{\epsilon_1} \text{ for all } (x, t) \in (\epsilon_1, l) \times (0, t_0).
\]

The proof is completed. \( \square \)

**Proposition 2.1.** The solution \( w(x, t) \) of (7) defined in (10) is unique.

**Proof.** The proof is similar to [2, Lemma 10 and Theorem 12]. \( \square \)

Next we consider the following parabolic problem
\[
u_{\delta t} = (u_{\delta} + \delta)^r \left( x^a u_{\delta xx} + \int_0^l u_\delta^p \, dx - ku_\delta^q \right), \quad 0 < x < l, \ t > 0,
\]
\[
u_\delta(0, t) = u_\delta(l, t) = 0, \quad t > 0,
\]
\[
u_\delta(x, 0) = u_0(x), \quad 0 < x < l, \quad (11)
\]

where \(u_0\) satisfies (H1)–(H3), it follows from Theorem 2.1 and Proposition 2.1 that there exists a classical solution of (11). Again by (H3) and the maximum principle, similar to the proof of [9, Lemma 5.4.1], we can prove that there exists a small positive constant \(\delta_0\) such that the solution \(u_\delta\) of (11) in \((0, t_0)\) satisfies that \(u_\delta t \geq 0\) for all \(\delta \in (0, \delta_0)\). Then \(u_0(x) \leq u_\delta(x, t) \leq h(t)\), \((x, t) \in [0, l] \times [0, t_0]\), where \(h(t)\) is given in Lemma 2.1.

**Lemma 2.3.** Let \(0 < \delta_2 < \delta_1 < \delta_0\) and suppose that \(u_{\delta_1}\) and \(u_{\delta_2}\) are solutions of (11) in \((0, t_0)\). Then \(u_{\delta_1} \geq u_{\delta_2}\).

**Proof.** From (H3) it follows that \(u_\delta t \geq 0\) for all \(\delta \in (0, \delta_0)\). Thus, we have
\[
u_{\delta t} = (u_{\delta_1} + \delta_1)^r \left( x^a u_{\delta_1 xx} + \int_0^l u_{\delta_1}^p \, dx - ku_{\delta_1}^q \right) \geq (u_{\delta_1} + \delta_2)^r \left( x^a u_{\delta_1 xx} + \int_0^l u_{\delta_1}^p \, dx - ku_{\delta_1}^q \right), \quad 0 < x < l, \ t > 0,
\]
\[
u_{\delta_1}(x, t)|_{x=0,l} = u_{\delta_2}(x, t)|_{x=0,l} = 0, \quad t > 0,
\]
\[
u_{\delta_1}(x, 0) = u_{\delta_2}(x, 0) = u_0(x), \quad 0 < x < l.
\]

By the comparison principle, we get the result. \(\square\)

Lemma 2.3 implies that a function \(u\) can be constructed as
\[
u(x, t) = \lim_{\delta \to 0} u_{\delta}(x, t), \quad \forall \quad (x, t) \in [0, l] \times [0, t_0]. \quad (12)
\]

**Theorem 2.2.** Assume that (H1)–(H3) hold. Then the function \(u(x, t)\) given in (12) is a classical solution of (2) in \((0, l) \times (0, t_0)\).

**Proof.** The proof is similar to [4, Lemma 2.7]. \(\square\)

**Proposition 2.2.** The solution \(u(x, t)\) of (2) defined by (12) is unique.

**Proof.** The proof is similar to [2, Lemma 10 and Theorem 12]. \(\square\)

3. Blow-up in finite time

We give a comparison theorem.
Lemma 3.1. Let \( u(x, t) \) be the solution of the problem (2) and assume that a nonnegative function \( w(x, t) \in C^{2,1}((0, l) \times (0, T)) \cap C([0, l] \times [0, T]) \) satisfies
\[
\begin{align*}
& w_t \geq (\leq) w^p (x^p w_{xx} + \int_0^l w^p \, dx - kw^q), \quad 0 < x < l, \ 0 < t < T, \\
& w(0, t), w(l, t) \geq (\leq) 0, \quad 0 < t < T, \\
& w(x, 0) \geq (\leq) u_0(x), \quad 0 < x < l.
\end{align*}
\]
Then \( w(x, t) \geq (\leq) u(x, t) \) on \([0, l] \times [0, T]\).

**Proof.** The proof is similar to [1, Theorems 2.2, 2.3]. \( \square \)

First we consider the case of \( p = q > 1 \) and \( l > k \). We need the following lemma.

Lemma 3.2. (See [11, Lemma 3.2].) If \( p = q > 1 \) and \( l > k \), then there exist \( 0 < \delta < 1 \) and a function \( w(x) \in C_0^\infty((0, l)) \) such that \( \int_0^l w(x) \, dx \equiv 1 \) and \( \int_0^l w^p \, dx - kw^p \geq \delta \) in \((0, l)\).

Theorem 3.1. Assume that \( p = q > 1 \), \( l > k \) and \( u_0(x) \) satisfies (H1)–(H3). Then there exists a constant \( a_0 > 0 \) such that the solution \( u(x, t) \) of (2) blows up in finite time if \( u_0(x) \geq a_0 w(x) \).

**Proof.** Let \( v_0(x) = a_0 w(x) \). From Lemma 3.2, we know that \( w(x) \in C_0^\infty((0, l)) \), i.e., \( \text{supp} (w(x)) \subset (0, l) \). Hence there exists a positive constant \( \lambda_0 \) such that \(-x^p w_{xx} \leq \lambda_0 \) and \(-x^q w_{xx} \leq \lambda_0 \) on \([0, l]\). Choose \( a_0 > 0 \) such that
\[
a_0 \lambda_0 \leq \delta a_0^p \quad \text{and} \quad \delta a_0^{p-1} \left( \int_0^l w^p \, dx \right)^{\frac{p-1}{p}} \geq 2 \lambda_0 l^{\frac{p-1}{p}},
\]
where \( \delta \) is given in Lemma 3.2. Combining with Lemma 3.2, we have
\[
-x^p v_{xx} \leq a_0 \lambda_0 \leq \delta a_0^p \leq a_0^p \left( \int_0^l w^p \, dx - kw^p \right) = \int_0^l v_0^p \, dx - kv_0^p.
\]

Let \( v(x, t) \) be the solution of the problem (2) with initial value \( v_0(x) \). It follows from Lemma 3.1 that \( v(x, t) \) is nondecreasing in \( t \). And from (13) we have
\[
\begin{align*}
& \frac{\delta}{2} \left( \int_0^l v_0^p \, dx \right)^{\frac{p-1}{p}} - \lambda_0 l^{\frac{p-1}{p}} \geq \frac{\delta}{2} \left( \int_0^l v_0^p \, dx \right)^{\frac{p-1}{p}} - \lambda_0 l^{\frac{p-1}{p}} \\
& = \frac{\delta}{2} a_0^{p-1} \left( \int_0^l w^p \, dx \right)^{\frac{p-1}{p}} - \lambda_0 l^{\frac{p-1}{p}} > 0.
\end{align*}
\]
Let $J(t) = \frac{1}{1-r} \int_0^l v^{1-r} w \, dx$, then we have

$$J'(t) = \int_0^l v^{-r} v_t w \, dx = \int_0^l \left( x^a v_{xx} + \int_0^l v^p \, dx - k v^p \right) w \, dx$$

$$= \int_0^l (x^a w)_{xx} v \, dx + \int_0^l v^p \, dx - k \int_0^l v^p w \, dx$$

$$\geq -\lambda_0 \int_0^l v \, dx + \int_0^l v^p \, dx - (1-\delta) \int_0^l v^p \, dx = -\lambda_0 \int_0^l v \, dx + \delta \int_0^l v^p \, dx$$

$$\geq -\lambda_0 \left(\int_0^l v^p \, dx\right)^{\frac{p}{p-1}} + \delta \int_0^l v^p \, dx$$

$$= \frac{\delta}{2} \int_0^l v^p \, dx + \left( \int_0^l v^p \, dx \right)^{\frac{1}{p}} \left( \frac{\delta}{2} \left(\int_0^l v^p \, dx\right)^{\frac{p-1}{p}} - \lambda_0 \right) \geq \frac{\delta}{2} \int_0^l v^p \, dx.$$

Combining with the following inequality,

$$\int_0^l v^{1-r} w \, dx \leq \left( \int_0^l v^p \, dx \right)^{\frac{1-p}{p}} \left( \int_0^l w^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}},$$

it follows that

$$J'(t) \geq c \left( J(t) \right)^{\frac{p}{p-1}},$$

for some constant $c > 0$. By $p > 1$ and $J(0) > 0$, we yield that $v(x, t)$ blows up in finite time. By Lemma 3.1, $u(x, t)$ becomes infinite in a finite time if $u_0(x) \geq a_0 w(x)$. The proof is completed. $\square$

Now we consider the case of $p > q > 1$.

**Theorem 3.2.** Assume that $p > q > 1$. Then there exists $a_2 > 0$ such that the solution $u(x, t)$ of the problem (2) blows up in finite time if $u_0(x) \geq a_2 \theta(x)$, where $\theta(x)$ is the first eigenfunction of the following problem

$$-x^a \theta_{xx} = \lambda_0 \theta, \quad x \in (0, l), \quad \theta(0) = \theta(l) = 0,$$

with $\int_0^l x^2 \theta(x) \, dx = 1$.

**Proof.** It is well known that the eigenvalue problem (15) is solvable if $0 < a < 2$; see [3]. Let $a = a_2/l^2$, $\psi(x) = x^2 \theta(x)$ and $v_0(x) = a \psi(x)$, where $a_2$ is a positive constant to be fixed later. Clearly there exists $\lambda_0 > 0$ such that $-(x^{2+a} \psi)_{xx} \leq \lambda_0$ and $-x^a (x^2 \theta)_{xx} \leq \lambda_0$. Choose $a_2$ sufficiently large such that
\[ -x^q v_{0xx} \leq a_\lambda v_0 \leq a_\lambda \int_0^I \psi^p \, dx - k a^q \psi^q = \int_0^I v^p_0 \, dx - k v^q_0, \quad (16) \]

\[ a^{q(p-q)} A^{q} \left( \int_0^I \psi^{q+1} \, dx \right)^{p-q} \geq (2k)^q, \quad (17) \]

\[ a^{p-1} \left( \int_0^I \psi^p \, dx \right)^{\frac{p-1}{p}} \geq 4 \lambda_0 \frac{\alpha^{q-1}}{p}, \quad (18) \]

where \( A = \left( \int_0^I \psi^p \, dx \right)^{\frac{q}{q-p}}. \)

Let \( v(x,t) \) be the solution of (2) with initial datum \( v_0(x) = a \psi(x) \). From (12) and Lemma 3.1, it follows that \( v(x,t) \geq v_0(x) = a \psi(x) \) and \( v(x,t) \) is nondecreasing in \( t \).

Set \( J(t) = \frac{1}{1-t} \int_0^I v^{1-r} \psi \, dx \), then we have

\[
J'(t) = \int_0^I v^{-r} v \psi \, dx = \int_0^I \left( x^q v_{xx} + \int_0^I \psi^p \, dx - k \psi^q \right) \psi \, dx
\]

\[
\geq -\lambda_0 \int_0^I v \, dx + \int_0^I v^p \, dx - k \int_0^I \psi^q \, dx.
\]

Since \( \int_0^I \psi^q \, dx \leq \left( \int_0^I \psi^p \, dx \right)^{\frac{q}{p}} \left( \int_0^I \psi^{\frac{q}{q-p}} \, dx \right)^{1-\frac{q}{p}}, \)

\[
J'(t) \geq -\lambda_0 \int_0^I v \, dx + \frac{1}{2} \int_0^I v^p \, dx + \frac{A}{2} \left( \int_0^I \psi^q \, dx \right)^{\frac{q}{p}} - k \int_0^I \psi^q \, dx
\]

\[
= -\lambda_0 \int_0^I v \, dx + \frac{1}{2} \int_0^I v^p \, dx + \int_0^I \psi^q \, dx \left( \frac{A}{2} \left( \int_0^I \psi^q \, dx \right)^{\frac{q}{p}} - k \right)
\]

\[
\geq -\lambda_0 \int_0^I v \, dx + \frac{1}{2} \int_0^I v^p \, dx + \int_0^I \psi^q \, dx \left( \frac{A}{2} \left( \int_0^I \psi^q \, dx \right)^{\frac{q}{p}} - k \right)
\]

\[
\geq -\lambda_0 \int_0^I v \, dx + \frac{1}{2} \int_0^I v^p \, dx
\]
\[
\begin{align*}
\geq -\lambda_0 l^{\frac{p-1}{p}} \left( \int_0^l v^p \, dx \right)^{\frac{1}{p}} + \frac{1}{2} \int_0^l v^p \, dx \\
= \frac{1}{4} \int_0^l v^p \, dx + \left( \int_0^l v^p \, dx \right)^{\frac{1}{p}} \left( \frac{1}{4} \left( \int_0^l v^p \, dx \right)^{\frac{p-1}{p}} - \lambda_0 l^{\frac{p-1}{p}} \right) \\
\geq \frac{1}{4} \int_0^l v^p \, dx + \left( \int_0^l v^p \, dx \right)^{\frac{1}{p}} \left( \frac{1}{4} \left( \int_0^l v_0^p \, dx \right)^{\frac{p-1}{p}} - \lambda_0 l^{\frac{p-1}{p}} \right) \\
\geq \frac{1}{4} \int_0^l v^p \, dx.
\end{align*}
\]

Combining with the following inequality,
\[
\int_0^l v^{1-r} \psi \, dx \leq \left( \int_0^l v^p \, dx \right)^{\frac{1-r}{r}} \left( \int_0^l \psi^{\frac{p}{1-r}} \, dx \right)^{\frac{r}{1-r}},
\]
we have
\[
J'(t) \geq c \left( J(t) \right)^{\frac{r}{1-r}},
\]
for some constant \( c > 0 \), which implies that \( v(x,t) \) blows up in finite time. By Lemma 3.1, \( u(x,t) \) becomes infinite in a finite time if \( u_0(x) \geq a_2 \theta(x) \geq a \psi(x) = a x^2 \theta(x)/l^2 \). The proof is completed. \( \square \)

4. Global blow-up and blow-up rate

In this section, we assume that the solution \( u(x,t) \) of (2) blows up in finite time and the blow-up time is \( T^* \). Throughout this section, we will assume the initial data \( u_0(x) \) satisfy (H1)–(H3) and the following assumptions:

(H4) (compatibility condition) \( \lim_{x \to 0^+} x^a u_{0xx}(x) = l^a u_{0xx}(l) = -\int_0^l u_{0x}^p \, dx, \)

(H5) \( u_{0xx} \leq 0 \) in \( (0,l) \).

From (H3)–(H5), we know that there exist a very small positive constant \( \varepsilon_0 \) and a function \( v_0(x) \) \( \varepsilon \in (0,\varepsilon_0) \) such that \( v_0(x) \in C^2[e,l-\varepsilon] \cap C[e,l-\varepsilon] \) for some \( \beta \in (0,1) \), \( v_0(x) = v_0(l-\varepsilon) = 0, v_0(x) < u_0(x), x \in (\varepsilon,2\varepsilon) \cup (l-2\varepsilon,l-\varepsilon) \), \( v_0(x) = u_0(x), x \in [2\varepsilon,l-2\varepsilon], (v_0(x))_{xx}(x) \leq 0, x \in (\varepsilon,l-\varepsilon) \). Moreover, \( v_{0x} \) is nonincreasing in \( \epsilon \) in \( (0,\varepsilon_0] \), \( \epsilon^a (v_{0x})_{xx}(\varepsilon) = (l-\varepsilon)^a (v_{0x})_{xx}(l-\varepsilon) = -\int_{\varepsilon}^{l-\varepsilon} v_{0x}^p \, dx, x^a (v_{0x})_{xx} + f_{\varepsilon}^{l-\varepsilon} v_{0x}^p \, dx = k t_{q_0} \geq 0, \epsilon \in (0,\varepsilon_0], x \in (\varepsilon,l-\varepsilon) \). It is clear that \( \lim_{\varepsilon \to 0} v_{0x} = u_0(x) \). Now we consider the following regularized problem

\[
\[ v_{et} = (v_e + \delta)^r \left( x^\alpha v_{exx} + \int_\varepsilon^{l-\varepsilon} v_t^p \, dx - kv_t^q \right), \quad x \in (\varepsilon, l - \varepsilon), \ t > 0, \]

\[ v_e(\varepsilon, t) = v_e(l - \varepsilon, t) = 0, \quad t > 0, \]

\[ v_e(x, 0) = v_{0e}(x), \quad x \in (\varepsilon, l - \varepsilon). \] (19)

It is clear that there exists a unique positive solution \( v_e(x, t) \) to the above problem. And in the same way as before, we can prove that

\[ \lim_{\delta \to 0, \varepsilon \to 0} v_e(x, t) = u(x, t) \]

where \( u(x, t) \) is the solution of (2).

**Lemma 4.1.** Assume that (H1)–(H5) hold. Then the solution \( u(x, t) \) of the problem (2) satisfies

\[ u_{xx} \leq 0, \quad \text{in } (0, l) \times (0, T^*). \]

**Proof.** It follows from \( x^\alpha (v_{0e})_{xx} + \int_\varepsilon^{l-\varepsilon} v_{0e}^p \, dx - kv_{0e}^q \geq 0, \ x \in (\varepsilon, l - \varepsilon) \) and Lemma 3.1 that \( v_e \) is nondecreasing in \( t \). Set \( w = v_{exx} \). Differentiating the differential equation in (19) with respect to \( x \) twice, we get

\[ w_t - x^\alpha (v_e + \delta)^r w_{xx} - 2x^\alpha (v_e + \delta)^r (rv_{exx}(v_e + \delta)^{-1} + \alpha x^{-1})w_x \]

\[ - (r(v_e + \delta)^{-1}v_{et} + 2r\alpha x^\alpha - 1(v_e + \delta)^{-1}v_{xx} \]

\[ + \alpha(\alpha - 1)x^{\alpha - 2}(v_e + \delta)^r - kv_e^{q-1}(v_e + \delta)^r \]

\[ = (r(r - 1)(v_e + \delta)^{-2}v_{xx} - 2r \alpha x^\alpha - 1(v_e + \delta)^{r-1}v_{xx}^{-1} \]

\[ - kq(q - 1)(v_e + \delta)^{r}v_{xx}^{q-2})v_{xx} \]

\[ \leq 0, \quad \text{in } (\varepsilon, l - \varepsilon) \text{ for } t > 0. \]

From \( v_{xx} \geq 0 \), we have

\[ w(\varepsilon, t) = -\frac{1}{\varepsilon^\alpha} \int_\varepsilon^{l-\varepsilon} v_t^p \, dx < 0 \]

and

\[ w(l - \varepsilon, t) = -\frac{1}{(l - \varepsilon)^\alpha} \int_\varepsilon^{l-\varepsilon} v_t^p \, dx < 0. \]

Since \( (v_{0e})_{xx} \leq 0 \text{ in } (\varepsilon, l - \varepsilon) \), by the maximum principle, we have \( w(x, t) \leq 0 \text{ for all } x \in (\varepsilon, l - \varepsilon), t > 0. \) From the arbitrariness of \( \varepsilon \) and \( \delta \), we know that \( u_{xx} \leq 0 \text{ for all } (x, t) \in (0, l) \times (0, T^*). \) The proof is completed. \( \square \)

Now we turn to the estimates of the blow-up rate, which are obtained by several lemmata. Set \( g(t) = \int_0^t u^p \, dx \) and \( G(t) = \int_0^t g(s) \, ds. \) First we claim the following lemma.
Lemma 4.2. Assume that \((H1)-(H5)\) hold and \(u(x,t)\) blows up in finite time. Then \(\lim_{t \to T^*} g(t) = \infty\) and \(\lim_{t \to T^*} G(t) = \infty\).

Proof. Suppose that \(\lim_{t \to T^*} g(t) < \infty\). Let \(x_0 \in [0, l]\) be a blow-up point, then there exists a sequence \((x_n, t_n) \to (x_0, T^*)\) such that \(\lim_{n \to \infty} u(x_n, t_n) = \infty\). Then
\[
\limsup_{n \to \infty} u_t(x_n, t_n)
= \limsup_{n \to \infty} u'(x_n, t_n)
\left( x_n u_{xx}(x_n, t_n) + \int_0^l u^p(x, t_n) \, dx - ku^q(x_n, t_n) \right)
= -\infty,
\]
which is a contradiction to the nondecreasing of \(u\) in \(t\). Next we show that \(\lim_{t \to T^*} G(t) = \infty\). We still assume that \(x_0 \in [0, l]\) is a blow-up point.

Lemma 4.3. Assume that \((H1)-(H5)\) hold and \(u(x,t)\) blows up in finite time. Then
\[
\lim_{t \to T^*} \frac{1}{u_{xx}} \frac{1}{g(t)} = 0 \quad \text{a.e. in } (0, l).
\]

Proof. Let \((a, b) \subset (0, l)\). Set \(\inf_{x \in [a,b]} \psi(x) = m\), where \(\psi(x)\) is the solution of the following elliptic problem
\[-\psi_{xx} = 1 \quad \text{in } (0, l), \quad \psi(0) = \psi(l) = 0.
\]

From Lemma 1, we have \(u_{xx} \leq 0\) in \((0, l) \times (0, T^*)\), then
\[
\int_0^l u \, dx = - \int_0^l u \psi_{xx} \, dx = - \int_0^l u_{xx} \psi \, dx \geq - \int_a^b u_{xx} \psi \, dx \geq -m \int_a^b u_{xx} \, dx.
\]

Since \(\lim_{t \to T^*} \int_a^b u \, dx / \int_0^l u^p \, dx = 0\), we have
\[
0 \leq \lim_{t \to T^*} -m \int_a^b \frac{u_{xx}}{g(t)} \, dx \leq \lim_{t \to T^*} \int_a^b \frac{u_{xx} \, dx}{u^p} = 0,
\]
that is,
\[
\lim_{t \to T^*} \int_a^b \frac{u_{xx}}{g(t)} \, dx = 0. \quad (20)
\]
Denote $\Omega_1 = (a, b)$. We now show that
\[ \lim_{t \to T^*} \frac{\partial^2 u}{\partial x^2} g(t) = 0, \quad \text{a.e. in } \Omega_1. \]

It then follows from (20) that for every $0 < \varepsilon < 1$, there exists a positive constant $\delta$ satisfies
\[ \frac{\delta}{2} \leq T^* - t \leq \delta \] such that
\[ \int_{\Omega_1} \frac{-\partial^2 u}{g(t)} dx \leq \varepsilon. \] (21)

Choose a fixed $t_1$ such that $\delta/2 \leq T^* - t_1 \leq \delta$.

Let
\[ \Omega_2 = \{ x : \frac{-\partial^2 u(x, t_1)}{g(t_1)} \geq \sqrt{\varepsilon}, x \in \Omega_1 \}, \]
then it follows from (21) that
\[ |\Omega_2| \leq \sqrt{\varepsilon}. \]

It is clear that
\[ \delta \to 0, \quad t_1 \to T^*, \quad |\Omega_2| \to 0 \quad \text{as } \varepsilon \to 0. \]

Therefore,
\[ \lim_{t \to T^*} \frac{\partial^2 u}{g(t)} = 0, \quad \text{a.e. in } \Omega_1. \]

By the arbitrariness of $\Omega_1$, we get
\[ \lim_{t \to T^*} \frac{\partial^2 u}{g(t)} = 0, \quad \text{a.e. in } (0, l). \] (22)

Set
\[ \Omega_3 = \{ x : \lim_{t \to T^*} \frac{\partial^2 u}{g(t)} = 0, x \in (0, l) \}. \] (23)

Using (22), we have
\[ |\Omega_3| = l. \]

The proof is complete.

Lemma 4.4. Assume that (H1)–(H5) hold and $u(x, t)$ blows up in finite time. Then $u(x, t)$ blows up in the whole interval.

Proof. First, we show that $\Omega_3 \subset S$, where $\Omega_3$ is given by (23) and $S$ is the blow-up set of $u(x, t)$. Assume by contradiction that $x_0 \in \Omega_3$ is not a blow-up point. Integrating the differential equation in (2) over $(0, t)$, we have
\[ \frac{1}{1-r} (u^{1-r}(x_0, t) - u^{1-r}_0(x_0)) = G(t) + \int_0^t (x_0^{\alpha} u_{xx}(x_0, s) - ku^q(x_0, s)) ds. \] (24)
By \( \lim_{t \to T^*} G(t) = \infty \) and \( \lim_{t \to T^*} u_{xx}/g(t) = 0 \) in \( \Omega_3 \), we have
\[
\lim_{t \to T^*} \frac{\int_0^l \alpha u_{xx}(x_0, s) ds}{G(t)} = 0.
\]
Comparing the two sides of (24), we obtain a contradiction, then \( \Omega_3 \subset S \). Since \( |\Omega_3| = l \), it is easy to demonstrate that \( S = [0, l] \). The proof is completed.

After establishing the above preliminary lemmata, we can determine the blow-up rate. First we consider the case of \( p > q > 1 \).

**Theorem 4.1.** Assume that (H1)–(H5) hold and \( u(x, t) \) blows up in finite time. If \( p > q > 1 \), then
\[
\lim_{t \to T^*} \frac{u}{g(t)} = 0, \quad \text{a.e. in } (0, l),
\]
as \( t \to T^* \).

**Proof.** By the Hölder inequality and \( p > q \), we have
\[
\lim_{t \to T^*} \int_0^l \frac{u^q}{g(t)} dx = \lim_{t \to T^*} \int_0^l \frac{u^q}{u^p} dx = 0.
\]
Then we get
\[
\lim_{t \to T^*} \frac{u^q}{g(t)} = 0, \quad \text{a.e. in } (0, l),
\]
Hence we obtain
\[
\lim_{t \to T^*} \frac{u^q}{g(t)} = 0, \quad \text{a.e. in } \Omega_3.
\]
Set
\[
\Omega_4 = \left\{ x : \lim_{t \to T^*} \frac{u^q}{g(t)} = 0, \ x \in \Omega_3 \right\}.
\]
It is clear that \( |\Omega_4| = |\Omega_3| = l \). Therefore, we have
\[
\frac{1}{1 - r} \frac{du^{1-r}}{dt} \sim g(t) = \int_0^l u^p dx \quad \text{in } \Omega_4,
\]
as \( t \to T^* \), from which it follows that
\[
\frac{1}{1 - r} \frac{d\frac{1}{1-r}}{ds} \sim G(t) = \int_0^l \int_0^l u^p dx ds \quad \text{in } \Omega_4,
\]
as \( t \to T^* \), i.e.,
\[
u \sim (1 - r)^{\frac{1}{1-r}} \left( G(t) \right)^{\frac{1}{1-r}} \quad \text{in } \Omega_4
\]
as $t \to T^*$. Therefore $G'(t) = g(t) = \int_0^l u^p \, dx \sim l(1 - r) \frac{p}{1 - r} (G(t))^{\frac{p}{1 - r}}$, as $t \to T^*$. Integrating it over $(t, T^*)$, we have

$$G(t) \sim \frac{1}{1 - r} \left( (r + p - 1)l(T^* - t) \right)^{-\frac{1}{p-1}}, \quad \text{as } t \to T^*.$$ 

Therefore, we get

$$u(x, t) \sim \left( (p + r - 1)l(T^* - t) \right)^{-\frac{1}{p-1}} \text{ in } \Omega_4 \text{ as } t \to T^*.$$ 

Thus the proof is completed. 

Finally, we consider the case of $p = q > 1$.

**Theorem 4.2.** Assume that (H1)–(H5) hold and $u(x, t)$ blows up in finite time. If $p = q > 1$ and $l > k$, then there exist positive constants $c$ and $c'$ such that

$$c(T^* - t)^{-\frac{1}{p-1}} \leq \left\| u(\cdot, t) \right\|_{\infty} \leq c'(T^* - t)^{-\frac{1}{p-1}}, \quad \text{as } t \to T^*.$$ 

**Proof.** Let $U(t) = \max_{x \in \Omega} u(x, t)$. From (2) and [7, Theorem 4.5], we know that

$$\frac{1}{1-r} \frac{dU^{1-r}}{dt} \leq \int_0^l U^p \, dx - kU^p = (l-k)U^p, \quad \text{a.e. in } (0, T^*).$$ 

Hence, we have

$$\frac{1}{1-r-p} \frac{dU^{1-r-p}}{dt} \leq (l-k), \quad \text{a.e. in } (0, T^*).$$ 

Integrating it over $(t, T^*)$ yields the lower bound of estimates of the blow-up rate

$$U(t) \geq \left( (p + r - 1)(l-k)(T^* - t) \right)^{-\frac{1}{p-1}}, \quad \text{in } (0, T^*).$$ 

Now we give an upper bound of estimates of the blow-up rate. Since $u_t \geq 0$, we have

$$\int_0^l u^p(x, t) \, dx \geq kU^p(t), \quad \text{in } (0, T^*). \quad (25)$$ 

Denote by $c_i$ the different constants in what follows. Integrating (2) over $(0, l) \times (0, t)$ and combining with

$$\lim_{t \to T^*} \frac{\int_0^l \int_0^l x^2 u_{xx}(x, s) \, dx \, ds}{G(t)} = 0,$$

$$\frac{1}{1-r} \int_0^l u^{1-r} \, dx - \frac{1}{1-r} \int_0^l u_0^{1-r} \, dx = \int_0^l \int_0^l x^2 u_{xx} \, dx \, ds + (l-k)G(t),$$
we have
\[ \int_0^l u^{1-r} \, dx \sim (1-r)(l-k)G(t), \quad \text{as } t \to T^*, \]
from which it follows that there exists a positive constant \( t_0 < T^* \) such that
\[ G(t) \leq c_1 \int_0^l u^{1-r} \, dx, \quad \text{in } (t_0, T^*). \]
By the H"older inequality, we have
\[ G^{\frac{p}{1+r}}(t) \leq c_2 \int_0^l u^p \, dx, \quad \text{in } (t_0, T^*). \quad (26) \]
On the other hand, integrating (2) over \((0, t)\) and combining with \(u_{xx} \leq 0\), we also have
\[ u^{1-r} \leq c_3 G(t), \quad \text{in } (t_0, T^*), \]
if we choose \( t_0 \) closer enough to \( T^* \), from which we also obtain
\[ \int_0^l u^p \, dx \leq c_4 G^{\frac{p}{1+r}}(t), \quad \text{in } (t_0, T^*). \quad (27) \]
From (26) and noticing \( G'(t) = g(t) = \int_0^l u^p \, dx \), we have
\[ G(t) \leq c_5 (T^* - t)^{-\frac{1}{p+r-1}}, \quad \text{in } (t_0, T^*). \]
From (25), (27), we obtain
\[ U(t) \leq c_6 \left( \int_0^l u^p \, dx \right)^{\frac{1}{p}} \leq c_7 (G(t))^{\frac{1}{1+r}} \leq c'(T^* - t)^{-\frac{1}{p+r-1}}, \quad \text{in } (t_0, T^*). \]
The proof is completed. \( \square \)

**Remark 4.1.** By the same way, we can get the estimate of the blow-up rate for problem (1) in the case of \( p_1 = q_1 > 1, l > k \). A careful calculation shows that the coefficients of the estimate of the blow-up rate depend only on \( p_1 \) and \( k \).

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Appendix A. An example of the initial value \( u_0(x) \)

Before we give an example of the initial value, we first consider the eigenvalue problem:

\[
\theta''(x) = -\lambda x^{-\alpha} \theta(x), \quad \theta(0) = \theta(l) = 0, \quad 0 < \alpha < 2. 
\]

(A.1)

Set \( x_1 = \frac{1}{l} \), \( \varphi(x_1) = \theta(lx_1) \) and \( \mu = l^{2-\alpha} \lambda \). We have

\[
\varphi''(x_1) = -\mu x_1^{-\alpha} \varphi(x_1), \quad \varphi(0) = 0 = \varphi(1).
\]

By the transformation \( \varphi(x_1) = x_1^{\frac{1}{2-\alpha}} y(x_1) \), the above differential equation becomes

\[
x_1^2 y''(x_1) + x_1 y'(x_1) + \left( -\frac{1}{4} + \mu x_1^{2-\alpha} \right) y(x_1) = 0.
\]

Let \( x_1 = z^{\frac{2}{2-\alpha}} \), we obtain

\[
z^2 y''(z) + z y'(z) + \frac{-1 + 4\mu z^2}{(2-\alpha)^2} y(z) = 0,
\]

whose general solution is given by

\[
y(z) = AJ_{1/(2-\alpha)}\left( \frac{2\sqrt{\mu}}{2-\alpha} z \right) + BJ_{-1/(2-\alpha)}\left( \frac{2\sqrt{\mu}}{2-\alpha} z \right),
\]

where \( A \) and \( B \) are arbitrary constants, and \( J_{1/(2-\alpha)} \) and \( J_{-1/(2-\alpha)} \) denote Bessel functions of the first kind of orders \( 1/(2-\alpha) \) and \( -1/(2-\alpha) \), respectively. Let \( \gamma \) be the first zero of \( J_{1/(2-\alpha)}(2\sqrt{\mu}/(2-\alpha)) \). By McLachlan [12, pp. 29 and 75], it is positive. It is easy to know that \( \lambda = l^{\alpha-2}\gamma \) is the principle eigenvalue of the above eigenvalue problem (A.1), and its corresponding eigenfunction is given by

\[
\chi(x) = \left( \frac{x}{l} \right)^{\frac{1}{2}} J_{1/(2-\alpha)}\left( \frac{2\sqrt{\mu}}{2-\alpha} \left( \frac{x}{l} \right)^{\frac{2}{2-\alpha}} \right),
\]

which is bounded positive function for \( x \in (0, l) \).

Then, by the properties of Bessel function, we know

\[
\lim_{x \to 0^+} \frac{\chi(x)}{x} = \frac{l^{-1}(\sqrt{\mu}/(2-\alpha))^{1/(2-\alpha)}}{\Gamma(1/(2-\alpha) + 1)}, \quad (A.2)
\]

and

\[
\lim_{x \to l^-} \frac{\chi(x)}{(l-x)^2} = 0, \quad (A.3)
\]

where

\[
\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx, \quad J_n(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k! F(n + k + 1)} \left( \frac{x}{2} \right)^{2k+n}.
\]

Let \( \theta(x) = D\chi(x) \), where \( D \) is a positive constant such that \( \int_0^l x^2 \theta(x) \, dx = 1 \).
In the next part, we give an example of the initial data $u_0(x)$, which satisfies the assumptions (H1)–(H5) and the blow-up conditions that we set in Theorems 3.1 and 3.2. For convenience, we only consider a special case of (2) where $\alpha = 1/2$, $l = 1$, $p = 2$ and $p \geq q > 1$.

We set the function

$$
\Phi(x) = 64(x - x^{3/2}) - cx(1 - x)(9 - 3x - x^2).
$$

By the direct calculation, we know that there exists a constant $c$ such that $0 < c < 1$ and $\int_0^1 \Phi^2(x) \, dx = 48$. Then we can verify that $\Phi(x)$ satisfies (H1)–(H5) for some positive constant $k$. Since $0 < c < 1$, by the simple calculation, we obtain

$$
\lim_{x \to 0^+} \frac{\Phi(x)}{x} = 64 - 9c > 0, \quad \lim_{x \to 1^-} \frac{\Phi(x)}{1 - x} = 32 - 5c > 0.
$$

Let $\Phi_1(x) = a\Phi(x)$, where $a > 1$. It is clear that $\Phi_1(x)$ satisfies (H1), (H2) and (H5).

Since the function $\Phi(x)$ satisfies (H1)–(H5) and $p = 2, p \geq q > 1$, we have

$$
x^{1/2} \Phi_{1xx}(x) + \int_0^1 \Phi_1^2(x) \, dx - k\Phi_1^q(x)
\geq a^2 \left( x^{1/2} \Phi_{1xx}(x) + \int_0^1 \Phi_1^2(x) \, dx - k\Phi_1^q(x) \right) \geq 0, \quad x \in (0, 1)
$$

and

$$
\lim_{x \to 0^+} \left( x^{1/2} \Phi_{1xx}(x) + \int_0^1 \Phi_1^2(x) \, dx - k\Phi_1^q(x) \right) = 48a(a - 1) > 0,
$$

$$
x^{1/2} \Phi_{1xx}(x) + \int_0^1 \Phi_1^2(x) \, dx - k\Phi_1^q(x)|_{x=1} = 48a(a - 1) > 0.
$$

From supp $w(x) \subset (0, 1)$, (A.2)–(A.7), it follows that $\Phi_1(x)$ satisfies (H1)–(H3) and (H5), and $\Phi_1(x) \geq a_0w(x), \Phi_1(x) \geq a_2\theta(x)$ provided the positive constant $a$ is sufficiently large, where $a_0w(x), a_2\theta(x)$ are defined in Theorems 3.1 and 3.2, respectively.

Let $\varepsilon$ be a small positive constant. Choose $\varepsilon_1$ and $b$, which depend on $\varepsilon$ and satisfy $0 < \varepsilon_1 < \varepsilon, b > a$, and $b \to a$ as $\varepsilon \to 0$. We may set the initial data function as

$$
u_0(x) = \begin{cases} 
 b\Phi_1(x), & x \in [0, \varepsilon_1] \cup [1 - \varepsilon_1, 1], \\
 \Phi_2(x), & x \in [\varepsilon_1, \varepsilon] \cup (1 - \varepsilon, 1 - \varepsilon_1), \\
 \Phi_1(x), & x \in [\varepsilon, 1 - \varepsilon],
\end{cases}
$$

where $u_0(x) \in C^{2+\beta}(0, 1) \cap C[0, 1]$ for some $0 < \beta < 1$ and $\Phi_2(x) > \Phi_1(x)$ is a sufficiently smooth function satisfying $\Phi_{2xx}(x) \leq 0$ for $x \in [\varepsilon_1, \varepsilon] \cup (1 - \varepsilon, 1 - \varepsilon_1]$. It is clear that $u_0(0) = u_0(1) = 0, u_0(x) \leq \Phi_1(x) > 0$ and $u_0_{xx} \leq 0$ in $(0, 1)$. Furthermore, in view of (A.5)–(A.7), there must exist appropriate $\varepsilon_1, b$ and the function $\Phi_2(x)$ such that $u_0(x)$ satisfies (H3) and (H4).
References