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Homotopy groups of Hom complexes of graphs

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A R T I C L E   I N F O

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The notion of $\times$-homotopy from [Anton Dochtermann, Hom complexes and homotopy theory in the category of graphs, European J. Combin., in press] is investigated in the context of the category of pointed graphs. The main result is a long exact sequence that relates the higher homotopy groups of the space $\mathrm{Hom}_*(G, H)$ with the homotopy groups of $\mathrm{Hom}_*(G, H^I)$. Here $\mathrm{Hom}_*(G, H)$ is a space which parameterizes pointed graph maps from G to H (a pointed version of the usual Hom complex), and H^I is the graph of based paths in H. As a corollary it is shown that $\pi_i(\mathrm{Hom}_*(G, H)) \cong [G, \Omega^i H]_\times$, where ΩH is the graph of based closed paths in H and $[G, K]_\times$ is the set of $\times$-homotopy classes of pointed graph maps from G to K. This is similar in spirit to the results of [Eric Babson, Hélène Barcelo, Mark de Longueville, Reinhard Laubenbacher, Homotopy theory of graphs, J. Algebraic Combin. 24 (1) (2006) 31–44], where the authors seek a space whose homotopy groups encode a similarly defined homotopy theory for graphs. The categorical connections to those constructions are discussed.

1. Introduction

In several recent papers (see for instance [2,6]), a homotopy theory of reflexive graphs termed A-theory has been developed in which graph theoretic homotopy groups are defined to measure 'combinatorial holes' in simplicial complexes. In [2] the authors construct a cubical complex X_G (associated to the graph G), and a homomorphism from the homotopy groups of the geometric realization of X_G to the A-theory groups of G; modulo a (yet unproved) version of cubical approximation they show that this map is in fact an isomorphism.

In the paper [7], a similar homotopy theory for general graphs called $\times$-homotopy is developed. Both theories are discussed in the common framework of exponential graph constructions associated

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to the relevant product (cartesian for A-theory, categorical for $\times$-homotopy). There it is shown that $\times$-homotopy is characterized by topological properties of the so-called Hom complex, a functorial way to assign a poset (and hence topological space) to a pair of graphs, first introduced to provide lower bounds on the chromatic number of graphs. In particular, the $\times$-homotopy class of maps from graphs G to H are seen to coincide with the path components of the space $\mathrm{Hom}(G, H)$.

In this paper, we consider the graph theoretic notions of homotopy groups that arise in the context of $\times$-homotopy. In order to give a topological interpretation of these constructions it is necessary to restrict our attention to the category of pointed graphs. We show that these combinatorially defined groups are isomorphic to the usual homotopy groups of a pointed version of the Hom complex (denoted Hom_*). Our method for proving this is to construct a 'path graph' G^I associated to a (pointed) graph G and to show that the natural endpoint map induces a long exact sequence of homotopy groups of the Hom_* complexes.

The paper is organized as follows. In Section 2, we describe the category of pointed graphs and recall the notions of both A-homotopy and $\times$-homotopy. In Section 3, we introduce the Hom_* functors and establish some basic facts about them, including interaction with the relevant adjunction and also a graph operation known as folding. In Section 4, we introduce the notion of a path graph, and state and prove our main result regarding the long exact sequence of the homotopy groups of the Hom_* complexes. In Section 5, we end with a brief discussion regarding other aspects of our construction.

2. Basic objects of study

We begin with the basic definitions. For us, a *graph* $G = (V(G), E(G))$ consists of a vertex set $V(G)$ and an edge set $E(G) \subseteq V(G) \times V(G)$ such that if $(v, w) \in E(G)$ then $(w, v) \in E(G)$. Hence our graphs are undirected and do not have multiple edges, but may have loops (if $(v, v) \in E(G)$). If $(v, w) \in E(G)$ we will say that v and w are *adjacent* and denote this as $v \sim w$. Given a pair of graphs G and H, a *graph homomorphism* (or *graph map*) is a mapping of the vertex set $f : V(G) \to V(H)$ that preserves adjacency: if $v \sim w$ in G, then $f(v) \sim f(w)$ in H. With these as our objects and morphisms we obtain a category of graphs which we denote $\mathcal{G}$. If v is a vertex of a graph G, then $N(v) := \{w \in V(G): (v, w) \in E(G)\}$ is called the *neighborhood* of v. A *reflexive* graph is a graph G with loops on all the vertices, so that $v \sim v$ for all $v \in V(G)$. The category of reflexive graphs is denoted $\mathcal{G}^\circ$. For more about graph homomorphisms and the category of graphs, see [10].

There are two relevant monoidal structures on the category of graphs. The first is the *categorical product* $G \times H$ of graphs G and H, defined to be the graph with vertex set $V(G) \times V(H)$ and adjacency given by $(g, h) \sim (g', h')$ if $g \sim g'$ and $h \sim h'$. The other is the *cartesian product* $G \square H$, defined to be the graph with the same vertex set $V(G) \times V(H)$ with adjacency given by $(g, h) \sim (g', h')$ if either $g \sim g'$ and $h = h'$, or $h \sim h'$ and $g = g'$. Each of these products has a right adjoint given by versions of an internal hom construction (see [7] for further discussion).

In this paper, we work primarily in the category of pointed graphs. A *pointed graph* $G = (G, x)$ is a graph G together with a specified looped vertex x. A *map of pointed graphs* $f : (G, x) \to (H, y)$ is a map of graphs such that $f(x) = y$. The resulting category of pointed graphs will be denoted $\mathcal{G}_*$. The category $\mathcal{G}_*$ enjoys some useful properties which we discuss next.

Definition 2.1. For pointed graphs $G = (G, x)$ and $H = (H, y)$, the *smash product* $G \wedge H$ is the pointed graph with vertex set given by the quotient of $(V(G) \times V(H))$ under the identifications $(x, h) = (g, y)$ for all $g \in V(G)$, $h \in V(H)$. Adjacency is given by $[(g, h)] \sim [(g', h')]$ if $g \sim g'$ and $h \sim h'$ for some representatives. The graph $G \wedge H$ is pointed by the vertex $[(x, y)]$ (see Fig. 1).

Definition 2.2. For pointed graphs $G = (G, x)$ and $H = (H, y)$, the *pointed* internal hom graph, denoted H^G, is the pointed graph with vertices given by all *set* maps $\{f : V(G) \to V(H): \ f(x) = y\}$. Adjacency is given by $f \sim g$ if $f(v) \sim g(v')$ in H for all $v \sim v'$ in G. The graph H^G is pointed by the graph map that sends every vertex of G to the vertex $y \in H$.

These last two constructions are adjoint to one another, as described by the following lemma. We let $\mathcal{G}_*(G, H)$ denote the set of pointed maps between the pointed graphs G and H.

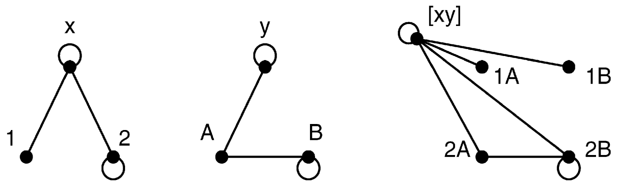

Fig. 1. The graphs $G = (G, x)$, $H = (H, y)$, and $G \wedge H = (G \wedge H, [(x, y)])$.

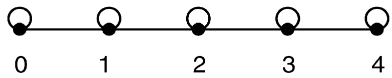

Fig. 2. The graph I_4.

Lemma 2.3. *For pointed graphs $A = (A, x)$, $B = (B, y)$, and $C = (C, z)$ we have a natural bijection of sets $\varphi : \mathcal{G}_*(A \wedge B, C) \to \mathcal{G}_*(A, C^B)$, given by $\varphi(f)(a)(b) = f[(a, b)]$ for $a \in V(A)$ and $b \in V(B)$.*

Proof. First note that $\varphi(f)$ is well defined since $f(x, b) = f(a, y) = z$ for all $a \in V(A)$, $b \in V(B)$. Next we see that $\varphi(f)(a) \in C^B$ since $\varphi(f)(a)(y) = f(a, y) = z$ and $\varphi(f)(x)(b) = f(x, b) = z$. To see that $\varphi(f)$ is a graph map, suppose $a \sim a'$; we need to check that $\varphi(f)(a) \sim \varphi(f)(a')$. If $b \sim b'$ then we have $\varphi(f)(a)(b) = f(a, b)$ and $\varphi(f)(a')(b') = f(a', b')$, and hence (the equivalence classes) are adjacent in $A \wedge B$.

Next, to show that φ is a bijection we define a map $\psi : \mathcal{G}_*(A, C^B) \to \mathcal{G}_*(A \wedge B, C)$ via $\psi(g)[(a, b)] = g(a)(b)$. We note that $\psi(g)(x, b) = g(x)(b) = z$ and $\psi(g)(a, y) = g(a)(y) = z$ for $a \in V(A)$ and $b \in V(B)$, and hence $\psi(g)$ is well defined on the vertices of $A \wedge B$. Similarly, one checks that $\psi(g)$ is a pointed graph map. It is clear that ψ is the inverse to φ and the result follows. □

We next turn to the definition of 'homotopy' in the context of the pointed category. The construction runs along the same lines of the $\times$-homotopy introduced in [7]. Although the constructions for the pointed category are straightforward modifications of the notions for the unpointed category, we will record all the definitions here for convenience.

If G is a graph we define G_* to be the pointed graph obtained by adding a distinguished disjoint looped vertex (denoted $*$) to the graph G.

Definition 2.4. The graph I_n is the (reflexive) graph with vertices $\{0, 1, \ldots, n\}$ and with adjacency given by $i \sim j$ if $|i - j| \leqslant 1$ (see Fig. 2).

For our definition of homotopy, we will want to consider path components of the exponential graph H^G. To capture this notion in the category of pointed graphs, the relevant graph to map from will be the pointed graph I_{n*}, since a pointed map $I_{n*} \to (G, x)$ will send $*$ to x, and 0 and n to the endpoints of a path in G.

Definition 2.5. A pair of maps $f, g : G \to H$ between pointed graphs is called $\times$-*homotopic* if there is an integer n and a (pointed) graph map $F : I_{n*} \to H^G$ such that $F(0) = f$ and $F(n) = g$.

This defines an equivalence relation on $\mathcal{G}_*(G, H)$, and the set of $\times$-homotopy classes of pointed maps between G and H will be denoted $[G, H]_\times$ (or simply $[G, H]$ if the context is clear). Applying the adjunction of Lemma 2.3 we note that a homotopy between f and g is the same as a pointed map $\tilde{F} : G \wedge I_{n*} \to H$ with $\tilde{F}(?, 0) = f$ and $\tilde{F}(?, n) = g$.

In [7], we relate the construction of our $\times$-homotopy to the A-theory of [2]. We briefly recall the discussion here.

Definition 2.6. A pair of pointed graph maps $f, g : (G, x) \to (H, y)$ between reflexive graphs is called *A-homotopic*, denoted $f \simeq_A g$, if there is an integer n and a graph map

$$\phi : G \,\square\, I_n \to H,$$

such that $\phi(?, 0) = f$ and $\phi(?, n) = g$, and such that $\phi(x, i) = y$ for all i.

One can check that f and g are A-homotopic if and only if there is a pointed graph map $\tilde{\phi} : I_{n*} \to H^G$ such that $\tilde{\phi}(0) = f$ and $\tilde{\phi}(n) = g$; here H^G is the internal hom graph that is right adjoint to the *cartesian* product. In other words, our definition of $\times$-homotopy is the 'same' as that of A-homotopy, with the categorical product playing the role of the cartesian product.

In the paper [2], the authors consider the so-called *A-homotopy groups* of a reflexive pointed graph (G, x), which they denote $A_n(G, x)$. By definition, $A_n(G, x)$ is the set of A-homotopy classes of graph maps

$$f : \left(I^m, \partial I^m\right) \to (G, x).$$

Here $I^m = I_n \,\square\, I_n \,\square\, \cdots \,\square\, I_n$ is the m-fold cartesian product of I_n, and ∂I^m is the subgraph of I^m consisting of vertices with at least one coordinate equal to 0 or n.

The authors of [2] seek to construct a topological space whose homotopy groups encode the A-homotopy groups of the graph G; this is seen as a generalization of a result from [6], where it is shown that $A_1(G, x)$ is isomorphic to the fundamental group of the space obtained by attaching 2-cells to all 3- and 4-cycles of the graph G. The main result of [2] is the construction of a cubical complex $M_*(G)$ associated to the reflexive graph G, and a homomorphism from the geometric realization $X_G := |M_*(G)|$ to the A-homotopy groups of G. Here the i-cube $M_i(G)$ is defined to be the set $\mathcal{G}(I_1^m, G)$ of all graph maps from I_1^m to G.

In this paper we consider the analogous questions in the context of $\times$-homotopy. One can follow the procedure of [2] and construct a cubical complex built from the sets $\mathcal{G}(I_1^m, G)$, where this time I_1^m denotes the m-fold *categorical* product. In this way one obtains a map from the realization of this space to the graph-theoretically defined '$\times$-homotopy groups' of G. However, it turns out that we can follow a somewhat different route to obtain a space $\mathrm{Hom}_*(T, G)$ whose homotopy groups do in fact coincide with what we will call the 'T-homotopy groups of G.' We turn to a discussion of these spaces in the next section.

3. Hom$_*$ complexes

The Hom complex is a functorial way to assign a poset (and hence topological space) to a pair of graphs. Spaces of this sort were first introduced by Lovász in [14], and have more recently been studied by various people in a variety of contexts (see for example [4,5,8,11,13,16,18,19]). The connection to $\times$-homotopy of ordinary (unpointed) graphs is explored in the paper [7]. There is a natural notion of the Hom complex in the pointed setting (which we will denote as Hom$_*$). As in the unpointed setting, this construction interacts well with the adjunction of Lemma 2.3, and the path components of this space characterize the set of $\times$-homotopy of pointed maps. We collect these facts next.

Definition 3.1. For pointed graphs $G = (G, x)$, $H = (H, y)$, we define $\mathrm{Hom}_*(G, H) \subseteq \mathrm{Hom}(G, H)$ to be the (pointed) poset whose elements are given by all functions $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, such that $\eta(x) = \{y\}$, and if $(v, w) \in E(G)$ then for all $\tilde{v} \in \eta(v)$ and $\tilde{w} \in \eta(w)$ we have $(\tilde{v}, \tilde{w}) \in E(H)$. The relation is given by containment, so that $\eta \leqslant \eta'$ if $\eta(v) \subseteq \eta'(v)$ for all $v \in V(G)$.

Example 3.2. As an example, we consider the pointed graphs $G = (G, x)$ and $H = (H, y)$ in Fig. 3.

Each element of $\mathrm{Hom}_*(G, H)$ consists of certain functions η from the vertex set of G to nonempty subsets of the vertex set of H. In particular, the pointed vertex $x \in V(G)$ must be sent to $\{y\} \subset V(H)$ for every η. The realization of $\mathrm{Hom}_*(G, H)$ is the barycentric subdivision of the complex pictured in Fig. 4, where the atoms of the poset are labeled with the images of the nonpointed vertices of G. In this case we see that $\mathrm{Hom}_*(G, H) \simeq \mathbb{S}^1$.

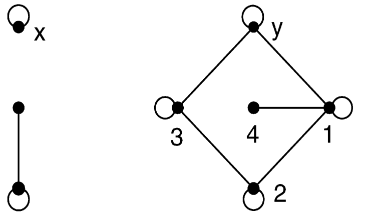

Fig. 3. The graphs (G, x) and (H, y).

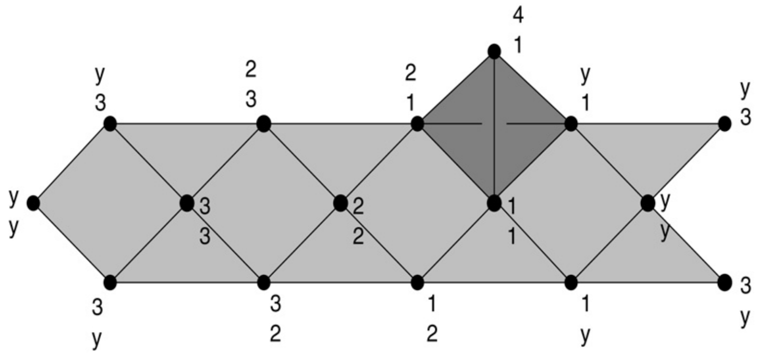

Fig. 4. The realization of $\mathrm{Hom}_*(G, H)$, up to barycentric subdivision.

Note that $\mathrm{Hom}_*(G, H)$ is itself a pointed poset, with a distinguished element given by the map that sends all vertices of G to the vertex $y \in V(H)$. One can also check that if G is a connected graph with at least one edge, we have isomorphisms of posets

$$\mathrm{Hom}_*(G_*, H_*) = \mathrm{Hom}(G, H_*) = \mathrm{Hom}(G, H) \amalg \mathrm{Hom}(G, *) = \big(\mathrm{Hom}(G, H)\big)_*,$$

where the last poset is obtained by adding a disjoint element to $\mathrm{Hom}(G, H)$.

For pointed graphs $A = (A, x)$, $B = (B, y)$, and $C = (C, z)$, we have seen that the exponential graph construction provides the adjunction $\mathcal{G}_*(A \wedge B, C) = \mathcal{G}_*(A, C^B)$, an isomorphism of sets. As is the case in the unpointed context (see [7] and also [12]), this extends to a homotopy equivalence of the analogous Hom_* complexes.

Proposition 3.3. *Let $A = (A, x)$, $B = (B, y)$, and $C = (C, z)$ be pointed graphs. The complex $\mathrm{Hom}_*(A \wedge B, C)$ can be included in $\mathrm{Hom}_*(A, C^B)$ so that $\mathrm{Hom}_*(A \wedge B, C)$ is a strong deformation retract of $\mathrm{Hom}_*(A, C^B)$. In particular, we have $\mathrm{Hom}_*(A \wedge B, C) \simeq \mathrm{Hom}_*(A, C^B)$.*

Proof. We follow the proof of the analogous statement in the unpointed context (see [7]). For convenience we let $P = \mathrm{Hom}_*(A \wedge B, C)$ and $Q = \mathrm{Hom}_*(A, C^B)$ denote the respective posets. We define a map of posets $j : P \to Q$ according to

$$j(\alpha)(a) = \big\{f : \big(V(B), y\big) \to \big(V(C), z\big) \;\big|\; f(b) \in \alpha(a, b),\ \forall b \in B\big\},$$

for every $\alpha \in P$ and $a \in A$. Note that $z \in \alpha(x, b)$ for all $b \in V(B)$ and hence the constant function $f_z : V(B) \to V(C)$, given by $b \mapsto z$ for all b, is an element of $j(\alpha)(x)$. Also, if $(a, a') \in E(A)$ then we have $(f, f') \in E(C^B)$ for all $f \in j(\alpha)(a)$, $f' \in j(\alpha)(a')$. Hence $j(\alpha)$ is indeed an element of $\mathrm{Hom}_*(A, C^B)$.

To see that j is injective, suppose $\alpha \neq \alpha' \in \mathrm{Hom}_*(A \wedge B, C)$ with $\alpha(a, b) \neq \alpha'(a, b)$. Then we have $\{f(b) \mid f \in j(\alpha)(a)\} \neq \{g(b) \mid g \in j(\alpha')(a)\}$, so that indeed $j(\alpha) \neq j(\alpha')$.

Next we define a closure operator $c : Q \to Q$. If $\gamma : V(A) \to 2^{V(C^B)} \setminus \{\emptyset\}$ is an element of Q, we define $c(\gamma) \in \mathrm{Hom}_*(A, C^B)$ as follows. Fix some $a \in V(A)$ and for every $b \in V(B)$ let $C^{\gamma}_{ab} = \{f(b) \in V(C) \mid$

$f \in \gamma(a)\}$; define $c(\gamma)(a)$ to be the collection of functions $g : V(B) \to V(C)$ where $g(b)$ varies over all $x \in C^{\gamma}_{ab}$. One can verify that $c(\gamma) \in \mathrm{Hom}_*(A, C^B)$ and also that $c(p) \geqslant p$ and $(c \circ c)(p) = c(p)$ for all $p \in P$.

Next we claim that $c(Q) \subseteq j(P)$. To see this, suppose $\gamma \in Q = \mathrm{Hom}_*(A, C^B)$ so that $c(\gamma) \in c(Q)$. We define $\alpha : V(A \wedge B) \to 2^{V(C)} \setminus \{\emptyset\}$ by $\alpha(a,b) = C^{\gamma}_{ab}$, where $C^{\gamma}_{ab} \subseteq V(C)$ is as above. One can verify that $\alpha \in \mathrm{Hom}_*(A \wedge B, C)$.

Finally $j(P) \subseteq c(Q)$ since $j(P) \subseteq Q$ and $c(j(P)) = j(P)$. We conclude $j(P) = c(Q)$, so that $P \cong j(P)$ is the image of a closure operator on Q. Closure operators of posets induce strong deformation retracts of their order complexes (see for instance [3]) and so the result follows. □

We let $\mathbf{1}_*$ denote the graph consisting of a pair of disjoint, looped vertices, and note that $G \wedge \mathbf{1}_* = G$ for every pointed graph $G = (G, x)$. The above proposition gives us $\mathrm{Hom}_*(G, H) = \mathrm{Hom}_*(\mathbf{1}_* \wedge G, H) \simeq \mathrm{Hom}_*(\mathbf{1}_*, H^G)$, for pointed graphs $G = (G, x)$ and $H = (H, y)$. The last of these posets is simply the face poset of the *clique complex* of H^G, denoted $\Delta(H^G)$, which is by definition the simplicial complex whose faces are given by complete subgraphs on the looped vertices of H^G. Recall that the looped vertices in H^G are the (pointed) graph homomorphisms $(G, x) \to (H, y)$. Hence, for pointed graphs G and H, the complex $\mathrm{Hom}_*(G, H)$ can be realized up to homotopy type as the clique complex of the subgraph of H^G induced by the (pointed) graph homomorphisms.

With this observation we obtain the following characterization of $\times$-homotopy.

Lemma 3.4. *Suppose $G = (G, x)$ and $H = (H, y)$ are pointed graphs, and $f, g : G \to H$ are pointed graph maps. Then f and g are $\times$-homotopic* (*as pointed maps*) *if and only if they are in the same path-connected component of* $\mathrm{Hom}_*(G, H)$.

Proof. A $\times$-homotopy $F : I_{n*} \to H^G$ from f to g is a path in the 1-skeleton of $\Delta(H^G) \simeq \mathrm{Hom}_*(G, H)$. Conversely, a (topological) path $I \to |\mathrm{Hom}_*(G, H)|$ can be approximated as a simplicial map from some finite subdivision of I into $\mathrm{Hom}_*(G, H) \simeq \Delta(H^G)$. □

For a (not necessarily pointed) graph map $f : G \to H$, and a fixed graph T, the $\mathrm{Hom}(T, ?)$ and $\mathrm{Hom}(?, T)$ functors provide maps f_T and f^T in the category of topological spaces. In Theorem 5.1 of [7] it is shown that $\times$-homotopy of graph maps is characterized by the homotopy properties of these induced maps. For our purposes, we will only need the following implication.

Lemma 3.5. *Let $f, g : G \to H$ be maps of graphs. Then f and g are $\times$-homotopic if and only if the induced maps of posets $f_T, g_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ are homotopic for every graph T.*

The pointed Hom_* complexes also interact well with a graph operation known as *folding*. We review this construction next.

Definition 3.6. Let u and v be vertices of a pointed graph $G = (G, x)$ with $v \neq x$ such that $N(v) \subseteq N(u)$. Then we have a (pointed) map $f : G \to G \setminus v$ given by $f(y) = y$, $y \neq v$, and $f(v) = u$. We call the map f a *folding* of G at the vertex v. The inclusion $i : G \setminus v \to G$ is called an *unfolding*.

Proposition 3.7. *Suppose $G = (G, x)$ and $H = (H, y)$ are pointed graphs, and u and v are vertices of G with $v \neq x$ such that $N(v) \subseteq N(u)$. Then $i^H : \mathrm{Hom}_*(G, H) \to \mathrm{Hom}_*(G \setminus v, H)$ and $f_H : \mathrm{Hom}_*(H, G) \to \mathrm{Hom}_*(H, G \setminus v)$ are both strong deformation retracts, where i^H and f_H are the poset maps induced by the graph unfolding and folding maps.*

Proof. We mimic the proof given in [12] of the analogous statement in the unpointed setting. For the first deformation retract, we identify $\mathrm{Hom}_*(G \setminus v, H)$ with the subposet of $\mathrm{Hom}_*(G, H)$ consisting of all α such that $\alpha(v) = \alpha(u)$. We define X to be the subposet of $\mathrm{Hom}_*(G, H)$ given by $X = \{\alpha \in \mathrm{Hom}_*(G, H)\colon\ \alpha(u) \subseteq \alpha(v)\}$. Next, we define poset maps $\varphi : \mathrm{Hom}_*(G, H) \to X$ and $\psi : X \to \mathrm{Hom}_*(G \setminus v, H)$ according to

$$\varphi(\alpha)(w) = \begin{cases} \alpha(u) \cup \alpha(v) & \text{if } w = v, \\ \alpha(w) & \text{otherwise,} \end{cases}$$
$$\psi(\alpha)(w) = \begin{cases} \alpha(u) & \text{if } w = v, \\ \alpha(w) & \text{otherwise.} \end{cases}$$

We see that both ψ and φ are closure maps, and that $i^H = \psi\varphi$. As above, closure maps of posets induce strong deformation retracts of their order complexes, and since $\mathrm{im}(i^H) = \mathrm{Hom}_*(G \setminus v, H)$ we obtain the result for i^H.

For the other statement, we define Y to be the subposet of $\mathrm{Hom}_*(H, G)$ given by $Y = \{\beta \in \mathrm{Hom}_*(H, G)\colon\ \beta(w) \cap \{u, v\} \neq \{v\}$ for all $w \in V(H)\}$. Define poset maps $\rho : \mathrm{Hom}_*(H, G) \to Y$ and $\sigma : Y \to \mathrm{Hom}_*(H, G \setminus v)$ according to

$$\rho(\beta)(w) = \begin{cases} \beta(w) \cup \{u\} & \text{if } v \in \beta(w), \\ \beta(w) & \text{otherwise,} \end{cases}$$
$$\sigma(\beta)(w) = \begin{cases} \beta(w) \setminus \{v\} & \text{if } v \in \beta(w), \\ \beta(w) & \text{otherwise.} \end{cases}$$

Once again, we see that both ρ and σ are closure maps, with $\sigma\rho = f_H$. Since $\mathrm{im}(f_H) = \mathrm{Hom}_*(H, G \setminus v)$, the result follows. $\square$

Remark 3.8. In [12], Kozlov has shown that a closure map $c : P \to P$ induces a collapsing of the order complex of P onto the order complex of $c(P)$. Accordingly, one can strengthen the conclusions of Propositions 3.3 and 3.7 to obtain a *simple* homotopy equivalence between the relevant spaces. We have only stated them in the form sufficient for our purposes.

For each $n \geqslant 0$, the graph I_n is pointed by the vertex 0. We can use Lemma 3.7 to show that all Hom_* complexes involving the pointed graph I_n are contractible.

Lemma 3.9. *For every pointed graph $G = (G, x)$ and for every integer n, the complex* $\mathrm{Hom}_*(I_n, G)$ *is contractible.*

Proof. If $n = 0$, then I_0 is a single looped vertex, and $\mathrm{Hom}_*(I_0, G)$ is a vertex. For $n > 0$, we have $N(n) \subset N(n-1)$, and hence the unfolding map $i : I_{n-1} \to I_n$ induces a homotopy equivalence $f^G : \mathrm{Hom}_*(I_n, G) \to \mathrm{Hom}_*(I_{n-1}, G)$. The claim follows by induction. $\square$

4. *T*-homotopy groups and the main result

We have seen that the path components of $\mathrm{Hom}_*(G, H)$ characterize the $\times$-homotopy groups of maps from G to H. To relate the higher homotopy groups to graph theoretical constructions we will want to work with the 'path graph' of a given pointed graph G, which we define next. For this and subsequent constructions we will make use of the notion of a *colimit of a diagram* (of graphs or topological spaces). We refer to [15] for a thorough discussion of this concept, but point out that in our context all such colimits are obtained from sequences of inclusions and hence can be thought of simply as unions. For our first such construction, recall that the graph I_n is pointed by the vertex 0.

Definition 4.1. For a pointed graph $G = (G, x)$, we define G^I to be the pointed graph obtained as the colimit (union) of the diagram

$$\begin{tikzcd} G^{I_0} \arrow[r] \arrow[drr] & \cdots \arrow[r] & G^{I_n} \arrow[r] \arrow[d] & G^{I_{n+1}} \arrow[r] \arrow[dl] & \cdots \\ & & G^I & & \end{tikzcd}$$

where the maps $j_n : G^{I_n} \to G^{I_{n+1}}$ are induced by the maps $I_{n+1} \to I_n$ given by $i \mapsto i$ $(i \neq n+1)$, and $n+1 \mapsto n$ (see Fig. 5).

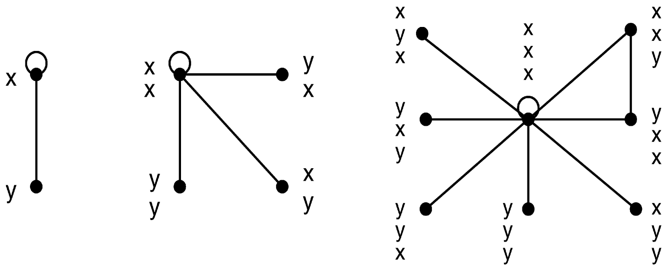

Fig. 5. The graphs G, G^{I_1} (with the images of 0, 1), and G^{I_2} (with the images of 0, 1, 2).

Note that a vertex of G^I is a (set) map $f : \mathbb{N} \to V(G)$ from the nonnegative integers into the vertices of G with $f(0) = x$ which is *eventually constant*; there exists some integer N_f such that $f(i) = f(j)$ for all $i, j \geqslant N_f$. Adjacency is given by $f \sim g$ if $f(i) \sim g(j)$, for all i, j with $|i - j| \leqslant 1$. We think of G^I as the graph that parameterizes the collection of paths in $G = (G, x)$ which begin at the vertex $x \in G$. The looped vertices of G^I are those paths which involve only looped vertices of G. We have the *endpoint map* $\varphi : G^I \to G$ given by $\varphi(f) = f(N_f)$.

Definition 4.2. For a pointed graph $G = (G, x)$, we define the *loop space graph* ΩG to be the (pointed) subgraph of G^I induced by elements that are eventually constant on the vertex $x \in G$.

Hence ΩG is the graph whose vertices are given by closed paths in G that start at x and eventually end (and stabilize) at x. The looped vertices of ΩG are those closed paths that involve only looped vertices of G. The path and loop space graph functors commute with exponentials of finite graphs, as described by the following observation.

Lemma 4.3. *If $G = (G, x)$ and $T = (T, y)$ are pointed graphs, with T finite, then we have graph isomorphisms $(G^I)^T = (G^T)^I$ and $(\Omega G)^T = \Omega(G^T)$.*

Proof. Both isomorphisms follow from identical arguments, and so we prove only the second of these claims. Define a map $\alpha : (\Omega G)^T \to \Omega(G^T)$ by $\alpha(f)(i)(t) = f(t)(i)$, for $f \in (\Omega G)^T$. To show that $\alpha(f) \in \Omega(G^T)$, pick an integer j such that each element of $\{f(t)\}_{t \in V(T)}$ stabilizes at j (this is possible since $V(T)$ is finite). So we have $f(t)(k) = f(t)(k') = x$ for all $t \in V(T)$ and $k, k' \geqslant j$. Hence $\alpha(f)(k) = \alpha(f)(k')$ so that $\alpha(f)$ stabilizes at j. It is easy to check that α is a pointed graph map. To show that it is an isomorphism, define a map $\beta : \Omega(G^T) \to (\Omega G)^T$ by $\beta(g)(t)(i) = g(i)(t)$. Suppose $g \in \Omega(G^T)$ stabilizes at the integer j. Then we have $g(k) = g(k')$ as elements of G^T, for all $k, k' \geqslant j$. Hence for every t, we have that $\beta(g)(t)(k) = \beta(g)(t)(k')$ so that in fact β maps to $(\Omega G)^T$. Again one can check that β is a pointed graph map and the inverse to α. The result follows. □

Note that ΩG is pointed by the closed path that is constant on the vertex $x \in G$. More generally, there is a natural group structure on the set $\pi_0(\mathrm{Hom}(T, \Omega H))$ for any graph H and finite graph T. To describe this structure, we let $G := H^T$ and appeal to Lemma 4.3 and Proposition 3.3 to obtain the string of bijections $\pi_0(\mathrm{Hom}_*(T, \Omega H)) \cong \pi_0(\mathrm{Hom}_*(\mathbf{1}_*, (\Omega H)^T)) \cong \pi_0(\mathrm{Hom}_*(\mathbf{1}_*, \Omega(H^T))) \cong [\mathbf{1}_*, \Omega G]_\times$. Hence for our purposes it is sufficient to describe a group structure on the set $[\mathbf{1}_*, \Omega G]_\times$, whose elements we can identify with connected components of ΩG.

We define a multiplication on the components of Ω in the following way. Given a pair of elements f and g in ΩG, pick a number N_g such that $g(n) = x$ for all $n \geqslant N_g$. Define $[f] \cdot [g] := [(f \cdot_{N_g} g)]$, where

$$(f \cdot_{N_g} g)(i) = \begin{cases} g(i) & \text{if } i < N_g, \\ f(i - N_g) & \text{otherwise.} \end{cases}$$

The identity element is given by $[c_x]$, where c_x is the path that is constant on the vertex $x \in G$. The inverse of $[f]$ is given by (the equivalence class of) the path f traversed in the opposite direction, so that

$$[f]^{-1}(i) = \begin{cases} f(N_f - i) & \text{if } i \leqslant N_f, \\ x & \text{otherwise.} \end{cases}$$

First we show that our construction does not depend on the choice of N_g. For this, suppose we pick $N'_g > N_g$ in our construction of $[f] \cdot [g]$; we need to show that $[f] \cdot_{N'} [g]$ and $[f] \cdot_N [g]$ are in the same component of ΩG. By induction it is enough to assume that $N'_g = N_g + 1$, and in this case we build a path $(f \cdot_{N'} g = f \cdot_{N+1} g := h_0, h_1, \ldots, h_{k-1}, h_k := f \cdot_N g)$ in ΩG according to

$$h_j(i) = \begin{cases} g(i) & \text{if } i < N_g, \\ f(i - N_g) & \text{if } N_g \leqslant i < N_g + j, \\ f(i - N_g - 1) & \text{if } i \geqslant N_g + j. \end{cases}$$

This path is perhaps best understood with the help of the following diagram, where each row represents an element h_j, and each column represents the image of i under h_j:

N_g	N_g+1	N_g+2	N_g+3		$\cdots$	N_g+N_f-1	N_g+N_f	N_g+N_f+1
x	x	$f(1)$	$f(2)$	$f(3)$	$\cdots$	$f(N_f-2)$	$f(N_f-1)$	$f(N_f)=x$
x	$f(1)$	$f(1)$	$f(2)$	$f(3)$	$\cdots$	$f(N_f-2)$	$f(N_f-1)$	x
x	$f(1)$	$f(2)$	$f(2)$	$f(3)$	$\cdots$	$f(N_f-2)$	$f(N_f-1)$	x
x	$f(1)$	$f(2)$	$f(3)$	$f(3)$	$\cdots$	$f(N_f-2)$	$f(N_f-1)$	x
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x	$f(1)$	$f(2)$	$f(3)$	$f(4)$	$\cdots$	$f(N_f-1)$	$f(N_f-1)$	x
x	x	$f(1)$	$f(2)$	$f(3)$	$\cdots$	$f(N_f-1)$	$f(N_f)=x$	x

We conclude that $f \cdot_N g$ and $f \cdot_{N'} g$ are in the same component of ΩG, and hence our product does not depend on the choice of N_G (up to $\times$-homotopy). Similarly, if $f \simeq_\times f'$ and $g \simeq_\times g'$ in $[\mathbf{1}_*, \Omega G]_\times$ (so that f and f', respectively g and g', are in the same component of ΩG), one can check that $f \cdot g$ and $f' \cdot g'$ are in the same component of ΩG, and hence the product is well defined on $\times$-homotopy classes of $[\mathbf{1}_*, \Omega G]_\times$. It is clear that $[c_x] \cdot [f] = [f] \cdot [c_x] = [f]$, and it is not hard to see that $[f] \cdot [f]^{-1} = [c_x] = [f]^{-1} \cdot [f]$.

As an example of the latter claim, consider the graph (H, y) in Fig. 3, and let $f = (y, 1, 2, 3, y, y, \ldots)$ be an element of ΩH. Taking $N_f = 4$, we obtain $(y, 3, 2, 1, y, y, \ldots)$ as an element of the equivalence class $[f]^{-1}$, and $(y, 1, 2, 3, y, y, 3, 2, 1, y, y, \ldots)$ as an element of $[f]^{-1} \cdot [f]$. We get a homotopy from this product path to c_x according to

0	1	2	3	4	5	6	7	8	9	$\cdots$
y	1	2	3	y	y	3	2	1	y	y
y	1	2	3	3	3	3	2	1	y	y
y	1	2	2	2	2	2	2	1	y	y
y	1	1	1	1	1	1	1	1	y	y
y	y	y	y	y	y	y	y	y	y	y

As one might expect, the group structure described above coincides with the fundamental group of the complex $\mathrm{Hom}_*(T, G)$.

Lemma 4.4. *Let $G = (G, x)$ and $T = (T, y)$ be pointed graphs, with T finite. Then we have an isomorphism of groups*

$$\pi_1\big(\mathrm{Hom}_*(T, G)\big) \cong \pi_0\big(\mathrm{Hom}_*(T, \Omega G)\big),$$

where the basepoint of $\mathrm{Hom}_*(T, G)$ is taken to be the element which sends each vertex of T to $\{x\}$, the set containing the pointed vertex of G.

Proof. Each element $\alpha \in \pi_1(\mathrm{Hom}_*(T, G)) \cong \pi_1(\mathrm{Hom}_*(\mathbf{1}_*, G^T))$ can be represented up to homotopy by a simplicial map from a subdivision of the circle (say with n vertices) into the clique complex of the graph G^T. This map can be thought of as a pointed graph map $\tilde{\alpha} : C'_n \to G^T$, where C'_n is a reflexive (loops on all vertices) cycle of length n (the graph I_n with the endpoints identified) and where the basepoint of G^T is the vertex set map which sends all vertices of T to $x \in G$. Hence α can be considered to be an element of $\Omega(G^T)$. Similarly, if α and β are homotopic as pointed maps $\mathbb{S}^1 \to G^T$, then for sufficiently large n these can be represented as graph maps $\tilde{\alpha}, \tilde{\beta} : C'_n \to G^T$. The maps α and β are recovered (up to homotopy) by the induced $\tilde{\alpha}_{\mathbf{1}_*}, \tilde{\beta}_{\mathbf{1}_*} : \mathrm{Hom}_*(\mathbf{1}_*, C'_n) \to \mathrm{Hom}_*(\mathbf{1}_*, G^T)$. These maps are homotopic by assumption, and hence by Lemma 3.5 we have that $\tilde{\alpha}$ and $\tilde{\beta}$ are $\times$-homotopic as graph maps. By definition this means that we have a path along looped vertices in $(G^T)^{C'_n}$ between (the looped vertices) $\tilde{\alpha}$ and $\tilde{\beta}$. But $(G^T)^{C'} \subseteq \Omega(G^T)$ and hence this provides a path in $\Omega(G^T)$. This in turn implies that $\tilde{\alpha}$ and $\tilde{\beta}$ are in the same component of $\Omega(G^T)$ and hence the same as elements of $\pi_0(\mathrm{Hom}_*(\mathbf{1}_*, \Omega(G^T))) \cong \pi_0(\mathrm{Hom}_*(T, \Omega G))$.

This assignment is a homomorphism of groups since concatenation of loops corresponds to the group structure on $\Omega(G^T)$ described above. Hence we obtain a map $\varphi : \pi_1(\mathrm{Hom}_*(\mathbf{1}_*, G^T)) \to \pi_0(\mathrm{Hom}_*(T, \Omega G))$. As above, any element of $\pi_0(\mathrm{Hom}_*(T, \Omega G))$ can be represented by a graph map from some finite reflexive cycle C'_n into G^T, and hence φ is surjective. For injectivity of φ, suppose α and β are elements in the same $\times$-homotopy class of $[\mathbf{1}_*, \Omega(G^T)]_\times$. We can choose n large enough so that α and β are realized as $\times$-homotopic elements of $[C'_n, G^T]_\times$. Applying $\mathrm{Hom}_*(\mathbf{1}_*, ?)$ to these maps then gives representives in $\pi_1(\mathrm{Hom}_*(\mathbf{1}_*, G^T))$ which are homotopic by Lemma 3.5. □

We next turn to a consideration of the higher homotopy groups of Hom_* complexes. For this we will consider iterations of the loop space construction, and will use $\Omega^n(G)$ to denote the graph $\Omega(\Omega(\ldots(\Omega(G))\ldots))$ (n times). We will also need the following observation.

Lemma 4.5. *For a pointed graph G and a finite pointed graph T, $\mathrm{Hom}_*(T, G^I)$ is contractible.*

Proof. We prove the claim for $T = \mathbf{1}_*$, from which the result follows from the homotopy equivalence $\mathrm{Hom}_*(T, G^I) \simeq \mathrm{Hom}_*(\mathbf{1}_*, (G^I)^T) = \mathrm{Hom}_*(\mathbf{1}_*, (G^T)^I)$.

We first show that $\mathrm{Hom}_*(\mathbf{1}_*, G^{I_n})$ is contractible for any integer n. By Proposition 3.3 we have that $\mathrm{Hom}_*(\mathbf{1}_*, G^{I_n}) \simeq \mathrm{Hom}_*(\mathbf{1}_* \wedge I_n, G) = \mathrm{Hom}_*(I_n, G)$ (where I_n is pointed by the vertex 0). The latter space is contractible by Lemma 3.9.

Finally we prove that $\mathrm{Hom}_*(\mathbf{1}_*, G^I)$ is contractible. We have seen that $\mathrm{Hom}_*(\mathbf{1}_*, X) = \mathrm{Hom}(\mathbf{1}, X)$ is the clique complex on the looped vertices of the graph X. Hence, as a functor, $\mathrm{Hom}_*(\mathbf{1}_*, ?)$ preserves colimits. By definition, G^I is the colimit of the sequence of maps $\cdots \to G^{I_n} \to G^{I_{n+1}} \to \cdots$, and hence $\mathrm{Hom}_*(\mathbf{1}_*, G^I) = \mathrm{colim}(\mathrm{Hom}_*(\mathbf{1}_*, G^{I_n}))$. We have seen that the sequence defining the last of these spaces is composed of all contractible spaces. Hence the colimit is contractible, and the result follows. □

Next we state and prove our main result, a long exact sequence in the homotopy groups of the Hom_* complexes induced by the endpoint map $\varphi : G^I \to G$. Our main tool will be the so-called Quillen fiber Lemma B (see [17]). If $\psi : P \to Q$ is a map of posets and $q \in Q$, we let $\psi^{-1}(\leqslant q) := \{p \in P \mid \psi(p) \leqslant q\}$ denote the *Quillen fiber* of q.

Theorem 4.6 *(Quillen). Let $\psi : P \to Q$ be a map of posets such that for all $q \leqslant q'$ the induced map $\psi^{-1}(\leqslant q) \to \psi^{-1}(\leqslant q')$ is a homotopy equivalence. Then for all $q \in Q$ and all $p \in \psi^{-1}(\leqslant q)$ there exists a connecting homomorphism $\delta : \pi_{i+1}(Q, q) \to \pi_i(\psi^{-1}(\leqslant q), p)$ that fits into the long exact sequence*

$$\cdots \longrightarrow \pi_{i+1}(Q, q) \xrightarrow{\delta} \pi_i(\psi^{-1}(\leqslant q), p) \xrightarrow{\iota_*} \pi_i(P, p) \xrightarrow{\psi_*} \pi_i(Q, q) \longrightarrow \cdots,$$

where ψ_* is the map induced by ψ and ι_* is induced by the inclusion $\iota : (\psi^{-1}(\leqslant q), p) \to (P, p)$.

Theorem 4.7. *Let G be a pointed graph, let T be a finite pointed graph, and let $\varphi_T : \mathrm{Hom}_*(T, G^I) \to \mathrm{Hom}_*(T, G)$ be the map induced by the endpoint map. Then for all $\gamma \in \mathrm{Hom}_*(T, G)$, and all $\beta \in \varphi_T^{-1}(\leqslant \gamma)$ we*

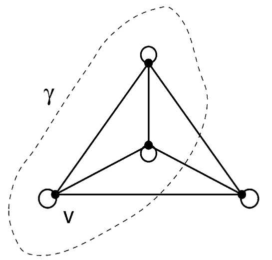

Fig. 6. $v \in \gamma \subseteq \gamma'$.

have a connecting homomorphism $\delta : \pi_{i+1}(\mathrm{Hom}_*(T,G),\gamma) \to \pi_i(\varphi_T^{-1}(\leqslant \gamma),\beta)$ *that fits into the following long exact sequence*:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{i+1}(\mathrm{Hom}_*(T,G),\gamma) \arrow[r, "\delta"] & \pi_i(\varphi_T^{-1}(\leqslant \gamma),\beta) \arrow[dll, "\iota_*"'] & \\ \pi_i(\mathrm{Hom}_*(T,G^I),\beta) \arrow[r, "\varphi_*"] & \pi_i(\mathrm{Hom}_*(T,G),\gamma) \arrow[r] & \cdots. \end{tikzcd}$$

Here φ_* *is the map induced by* φ_T *and* ι_* *is induced by the inclusion* $\iota : (\varphi_T^{-1}(\leqslant \gamma),\beta) \to (\mathrm{Hom}_*(T,G^I),\beta)$.

Proof. We first prove the claim for the case $T = \mathbf{1}_*$, and the map $\varphi_{\mathbf{1}_*} : \mathrm{Hom}(\mathbf{1}_*, G^I) \to \mathrm{Hom}(\mathbf{1}_*, G)$.

We use the Quillen fiber Lemma B applied to the poset map $\varphi_{\mathbf{1}_*}$. Suppose $\gamma \leqslant \gamma' \in \mathrm{Hom}_*(\mathbf{1}_*, G)$. As elements of $\mathrm{Hom}_*(\mathbf{1}_*, G)$, γ and γ' can each be identified with a collection of looped vertices of the graph G, each of which determines a clique (complete subgraph) of G. We will also use γ and γ' to denote these collections of vertices, and will distinguish an element $v \in \gamma \subseteq \gamma' \subseteq V(G)$. Note that v is adjacent to all other elements of γ' (see Fig. 6).

Next, we let $Y = \varphi_{\mathbf{1}_*}^{-1}(\leqslant \gamma)$ and $Y' = \varphi_{\mathbf{1}_*}^{-1}(\leqslant \gamma')$ denote the respective Quillen fibers. Here Y and Y' are both subposets of $\mathrm{Hom}_*(\mathbf{1}_*, G^I)$. We wish to show that the induced map $k : Y \to Y'$ is a homotopy equivalence, from which the result would follow. For this, we consider finite approximations of these spaces, in the following sense. We let $\varphi_n : G^{I_n} \to G$ denote the endpoint map of the finite path graph G^{I_n}, given by $\varphi_n(f) = f(n)$. We let $H_n \subseteq G^{I_n}$ denote the induced subgraph on the vertices $\varphi_n^{-1}(\gamma)$, and similarly H'_n on $\varphi_n^{-1}(\gamma')$. We can think of H_n (respectively H'_n) as the subgraph of G^{I_n} induced by maps from I_n that end at some vertex of γ (respectively γ'). We have the obvious inclusions $k_n : H_n \to H'_n$ and also the inclusions $i_n : H'_n \to H'_{n+1}$ and $j_n : H_n \to H_{n+1}$ given by $i_n(f)(n+1) = f(n)$ (and similarly for j_n).

We let $Y_n = \mathrm{Hom}(\mathbf{1}, H_n)$ and $Y'_n = \mathrm{Hom}(\mathbf{1}, H'_n)$ denote the respective posets. The i_n and j_n maps determine directed systems for which

$$Y = \varphi^{-1}(\leqslant \gamma) = \mathrm{Hom}(\mathbf{1}, \operatorname{colim} H_n) = \operatorname{colim} Y_n,$$

$$Y' = \varphi^{-1}(\leqslant \gamma') = \mathrm{Hom}(\mathbf{1}, \operatorname{colim} H'_n) = \operatorname{colim} Y'_n.$$

The poset map $k : Y \to Y'$ is given by $\operatorname{colim}(k_n : H_n \to H'_n)$. We also need the graph map $h_n : H'_n \to H_{n+1}$ given by

$$h_n(f)(i) = \begin{cases} f(i) & \text{if } i \leqslant n, \\ v & \text{if } i = n+1. \end{cases}$$

These maps all fit into the following diagram of graphs:

$$\begin{tikzcd}
\cdots \arrow[r] & H'_n \arrow[r, "i_n"] \arrow[dr, "h_n"] & H'_{n+1} \arrow[r, "i_{n+1}"] \arrow[dr, "h_{n+1}"] & H'_{n+2} \arrow[r] & \cdots \\
\cdots \arrow[r] & H_n \arrow[r, "j_n"'] \arrow[u, "k_n"] & H_{n+1} \arrow[r, "j_{n+1}"'] \arrow[u, "k_{n+1}"] & H_{n+2} \arrow[r] \arrow[u, "k_{n+2}"] & \cdots .
\end{tikzcd}$$

We claim that this diagram commutes up to (graph) $\times$-homotopy. In particular, we have $k_{n+1}h_n \simeq_\times i_n$ and $h_{n+1}k_{n+1} \simeq_\times j_{n+1}$. For the first homotopy, define a map $A : H'_n \times I_1 \to H'_{n+1}$ according to

$$A(f,i)(j) = \begin{cases} f(j) & \text{if } i = 0 \text{ and } j \leqslant n, \\ f(n) & \text{if } i = 0 \text{ and } j = n+1, \\ v & \text{if } i = 1. \end{cases}$$

Recall that $v \in \alpha \subseteq V(G)$ is our distinguished vertex. Now, it is easy to see that $A(?,0) = i_n$ and $A(?,1) = k_{n+1}h_n$:

$$\begin{tikzcd}
H'_n \arrow[d, "\iota_0"'] \arrow[dr, "i_n"] & \\
H'_n \times I_1 \arrow[r, "A"] & H'_{n+1}. \\
H'_n \arrow[u, "\iota_1"] \arrow[ur, "k_{n+1}h_n"'] &
\end{tikzcd}$$

To check that A is a graph map, suppose f and f' are adjacent vertices in H'_n. We need $A(f,0)$ and $A(f',1)$ to be adjacent in X'_{n+1}. Note that $A(f,0)(n+1) = f(n) \in \gamma'$ and $A(f',1)(n+1) = v$. The element v is adjacent to all elements in the clique γ', and hence adjacent to $f(n)$. Also, $A(f,0)(n+1) = f(n)$ and $A(f',1)(n) = f'(n)$, which are adjacent since $f \sim f'$ in H'_n. Finally, we have $A(f,0)(n) = f(n)$ and $A(f,1)(n+1) = x$. Once again, these are adjacent since $f(n) \in \gamma'$.

To check the homotopy $h_{n+1}k_{n+1} \simeq_\times j_{n+1}$, we similarly define a map $B : H_n \times I_1 \to H_{n+1}$ according to

$$B(f,i)(j) = \begin{cases} f(n) & \text{if } i = 0 \text{ and } j = n+1, \\ v & \text{if } i = 1 \text{ and } j = n+1, \\ f(j) & \text{if } j \leqslant n. \end{cases}$$

Again one can check that B is indeed a graph map and that $B(?,0) = j_{n+1}$ and $B(?,1) = h_{n+1}k_{n+1}$. We conclude that the diagram under consideration commutes up to $\times$-homotopy, and hence by Lemma 3.5 any diagram of posets induced by a $\mathrm{Hom}(T,?)$ functor also commutes up to (topological) homotopy.

Next, to show that $k : Y \to Y'$ is a homotopy equivalence we show that k induces an isomorphism on homotopy groups. Suppose $\rho, \sigma : \mathbb{S}^m \to Y$ are pointed maps from the m-sphere into Y, and let $\rho' = k\rho$ and $\sigma' = k\sigma$ be the induced maps $\mathbb{S}^m \to Y'$. Suppose that $\rho' \simeq \sigma'$ are homotopic as maps into Y' via a homotopy $\Psi : \mathbb{S}^m \times I \to Y'$. We claim that in fact $\rho \simeq \sigma$ are also homotopic, so that k is injective on all homotopy groups. To see this, pick n big enough so that the image of Ψ sits inside the subcomplex $Y'_n := \mathrm{Hom}(\mathbf{1}, H_n) \subseteq Y'$ (this is possible since $\mathbb{S}^m \times I$ is compact). Now, the composition $h_{n_1}\Psi : \mathbb{S}^m \times I \to Y_{n+1}$ is a homotopy from $h_{n_1}\rho' = h_{n_1}k_{n_1}\rho$ to $h_{n_1}\sigma' = h_{n_1}k_{n_1}\sigma$. But $h_{n_1}k_{n_1} \simeq j_{n_1}$ and hence ρ and σ are homotopic as maps into Y_{n+1}, as desired:

$$
\begin{tikzcd}
\mathbb{S}^m \arrow[d] \arrow[dr, "\rho"'] & Y'_n \arrow[r, "i_{n_1}"] \arrow[dr, "h_{n_1}"] & Y'_{n+1} \\
\mathbb{S}^m \times I \arrow[r, "\Psi"'] & Y_n \arrow[r, "j_{n_1}"'] \arrow[u, "k_{n_1}"] & Y_{n+1} \arrow[u, "k_{n_1}"'] . \\
\mathbb{S}^m \arrow[u] \arrow[ur, "\sigma"'] & &
\end{tikzcd}
$$

Next, we claim that k induces a surjection on each homotopy group. To see this, suppose $\rho' : \mathbb{S}^m \to Y'$ is a pointed map of the m-sphere into Y'. We wish to find a map $\rho : \mathbb{S}^m \to Y$ such that $k\rho \simeq \rho'$. As above, choose n large enough so that the image of the map ρ' is contained in Y'_n, and let $\rho = h_{n_1}\rho' : \mathbb{S}^m \to Y_{n+1}$. Then we have $k\rho = k_{(n+1)_1} h_{n_1}\rho' \simeq i_{n_1}\rho' \simeq \rho'$, as desired. We conclude that k induces an isomorphism on each homotopy group, so that k is a homotopy equivalence. Hence the conditions of the Quillen Lemma B are satisfied, and we get a long exact sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & \pi_{i+1}(\mathrm{Hom}_*(\mathbf{1}_*, G), \gamma) \arrow[r] & \pi_i(\varphi_{\mathbf{1}_*}^{-1}(\leqslant \gamma), \beta) \arrow[dl] & \\
& \pi_i(\mathrm{Hom}_*(\mathbf{1}_*, G), \beta) \arrow[r] & \pi_i(\mathrm{Hom}_*(\mathbf{1}_*, G), \gamma) \arrow[r] & \cdots .
\end{tikzcd}
$$

It remains for us to prove the claim for general finite T. For this we note that $\mathrm{Hom}_*(T, G) \simeq \mathrm{Hom}_*(\mathbf{1}_*, G^T)$, and also $\mathrm{Hom}_*(T, G^I) \simeq \mathrm{Hom}_*(\mathbf{1}_*, (G^I)^T) = \mathrm{Hom}_*(\mathbf{1}_*, (G^T)^I)$. Hence we take $G = G^T$ and make the appropriate substitutions in the above sequence. $\square$

Corollary 4.8. *For pointed graphs G and T, with T finite, we have $\pi_i(\mathrm{Hom}_*(T, G), \gamma) \simeq [T, \Omega^i(G)]_\times$.*

Proof. Suppose $T = (T, y)$ and $G = (G, x)$ are pointed graphs, with T finite. Let $\varphi : \mathrm{Hom}_*(T, G^I) \to \mathrm{Hom}_*(T, G)$ be the map of posets induced by the (pointed) endpoint map $G^I \to G$. We apply Theorem 4.7 and in the long exact sequence choose $\gamma \in \mathrm{Hom}_*(T, G)$ to be the basepoint (where γ is given by $\gamma(t) = x$ for all $t \in V(T)$). Hence γ is an atom in the poset $\mathrm{Hom}_*(T, G)$, so that $\varphi_T^{-1}(\leqslant \gamma) = \varphi_T^{-1}(\gamma) = \mathrm{Hom}_*(T, \Omega G)$. We choose β to be the basepoint in $\mathrm{Hom}_*(T, \Omega G)$. Hence our sequence becomes

$$
\begin{tikzcd}
\cdots \arrow[r] & \pi_{i+1}(\mathrm{Hom}_*(T, G), \gamma) \arrow[r, "\delta"] & \pi_i(\mathrm{Hom}_*(T, \Omega G), \beta) \arrow[dl, "\iota_*"'] & \\
& \pi_i(\mathrm{Hom}_*(T, G^I), \beta) \arrow[r, "\varphi_*"] & \pi_i(\mathrm{Hom}_*(T, G), \gamma) \arrow[r] & \cdots .
\end{tikzcd}
$$

From Lemma 4.5, we have that $\mathrm{Hom}_*(T, G^I)$ is contractible, so that $\pi_i(\mathrm{Hom}_*(T, G^I)) = 0$ for all i. Hence the δ maps are all isomorphisms, and we get $\pi_i(\mathrm{Hom}_*(T, G), \gamma) \cong \pi_{i-1}(\mathrm{Hom}_*(T, \Omega G), \beta)$. Applying this isomorphism $i-1$ times, we get

$$\pi_i\big(\mathrm{Hom}_*(T, G), \gamma\big) \cong \pi_1\big(\mathrm{Hom}_*\big(T, \Omega^{i-1}(G)\big), \tilde{\beta}\big) \cong \pi_0\big(\mathrm{Hom}_*\big(T, \Omega^i(G)\big)\big) \cong \big[T, \Omega^i(G)\big]_\times,$$

where the last two isomorphisms are from Lemmas 4.4 and 3.4, respectively. This proves the claim. $\square$

Since for a (pointed) topological space $\pi_i(X, x) = \pi_{i-1}(\Omega X, \tilde{x})$, and also $\pi_0(\mathrm{Hom}_*(T, G)) = [T, G]_\times$, we see that in some sense the loop space functor Ω commutes with the Hom_* complex, where it becomes the graph theoretic version within the arguments.

5. Concluding remarks

With ΩG as our (pointed) loop space associated to a graph G, we can define a graph theoretic notion of '$\times$-homotopy' groups analogous to the A-homotopy groups from [2] (as discussed in Section 2). For a pointed graph $G = (G, x)$, one can interpret the path connected components of the graph $\Omega^n(G)$ as $\times$-homotopy classes of maps from the "n cube" $I_m \times I_m \times \cdots \times I_m$ (n times) into the graph G such that the boundary is mapped to the pointed vertex $x \in G$. This set is naturally a group under 'stacking' and, by the above result, is isomorphic to the group $\pi_n(\mathrm{Hom}_*(\mathbf{1}_*, G))$. The latter space is the clique complex on the subgraph of G induced by the looped vertices.

In the more general setting, we have a natural notion of the 'T-homotopy groups' of a pointed graph $G = (G, x)$ (for a fixed finite pointed graph T). These are defined according to $\pi_i^T(G, x) := [T, \Omega^i(G)]_\times$; hence the groups described in the previous paragraphs are obtained by setting $T = \mathbf{1}_*$. Note that this definition makes sense in any category 'with a path object' (see [1] for an in-depth discussion). Our results show that $\pi_i^T(G, x) \cong \pi_i(\mathrm{Hom}_*(T, G))$.

We have chosen to restrict our attention to $\mathcal{G}_*$, the category of pointed graphs. This is the natural category to work in when one considers homotopy groups of spaces, which are by definition homotopy classes of certain pointed maps. If one chooses to work in the category of (unpointed) graphs, the statement (and proof) of Theorem 4.7 proceeds almost unchanged, and we obtain a long exact sequence of the relevant homotopy groups of (unpointed) Hom complexes.

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{i+1}(\mathrm{Hom}(T,G),\gamma) \arrow[r, "\delta"] & \pi_i(\varphi_T^{-1}(\leqslant \gamma),\beta) \arrow[dl, "\iota_*"'] & \\ & \pi_i(\mathrm{Hom}(T,G^I),\beta) \arrow[r, "\varphi_*"] & \pi_i(\mathrm{Hom}(T,G),\gamma) \arrow[r] & \cdots. \end{tikzcd}$$

One can check that in this case $\mathrm{Hom}(T, G^I) \simeq \mathrm{Hom}(T, G)$, as expected. However, the $(\varphi_T^{-1}(\leqslant \gamma), \beta)$ term no longer has a natural interpretation as a Hom complex (as in the proof of Corollary 4.8). In focussing on the pointed situation, we can apply the long exact sequence to show that the Hom_* complexes are spaces whose (usual) homotopy groups recover the combinatorically defined homotopy groups obtained from $\times$-homotopy of graph maps.

In conclusion, we wish to point out a certain similarity between our constructions and those of Schultz from [18]. There it is shown that one can recover the space of (equivariant) maps from the circle into $\mathrm{Hom}(K_2, G)$ as a certain colimit (union) of $\mathrm{Hom}(C_{2r+1}, G)$ complexes, where C_{2r+1} is the odd cycle of length $2r + 1$. In the process of proving this, the author implicitly uses the fact that the graph $C_{2r+1}^{K_2}$ is isomorphic to the interval graph I_{4r+2} with the endpoints identified. This suggests a general approach to understanding the topology of the mapping spaces of Hom complexes which is further explored in [9].

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References

[1] Hans Joachim Baues, Algebraic Homotopy, Cambridge Stud. Adv. Math., vol. 15, Cambridge Univ. Press, Cambridge, 1989.
[2] Eric Babson, Hélène Barcelo, Mark de Longueville, Reinhard Laubenbacher, Homotopy theory of graphs, J. Algebraic Combin. 24 (1) (2006) 31–44.
[3] A. Björner, Topological methods, in: Handbook of Combinatorics, vols. 1, 2, Elsevier, Amsterdam, 1995, pp. 1819–1872.
[4] Eric Babson, Dmitry N. Kozlov, Complexes of graph homomorphisms, Israel J. Math. 152 (2006) 285–312.
[5] Eric Babson, Dmitry N. Kozlov, Proof of the Lovász conjecture, Ann. of Math. 165 (3) (2007) 965–1007.
[6] Hélène Barcelo, Reinhard Laubenbacher, Perspectives on A-homotopy theory and its applications, Discrete Math. 298 (1–3) (2005) 39–61.
[7] Anton Dochtermann, Hom complexes and homotopy theory in the category of graphs, European J. Combin., in press.
[8] Anton Dochtermann, The universality of Hom complexes, Combinatorica, in press.

[9] Anton Dochtermann, Carsten Schultz, Constructing test graphs for topological bounds on chromatic number, in preparation.
[10] Pavol Hell, Jaroslav Nešetřil, Graphs and Homomorphisms, Oxford Lecture Ser. Math. Appl., vol. 28, Oxford Univ. Press, Oxford, 2004.
[11] Matthew Kahle, The neighborhood complex of a random graph, J. Combin. Theory Ser. A 114 (2) (2007) 380–387.
[12] Dmitry N. Kozlov, A simple proof for folds on both sides in complexes of graph homomorphisms, Proc. Amer. Math. Soc. 134 (5) (2006) 1265–1270 (electronic).
[13] Dmitry N. Kozlov, Chromatic numbers, morphism complexes, and Stiefel–Whitney characteristic classes, in: Geometric Combinatorics, in: IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2007, pp. 249–315.
[14] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, J. Combin. Theory Ser. A 25 (3) (1978) 319–324.
[15] Saunders Mac Lane, Categories for the Working Mathematician, second ed., Grad. Texts in Math., vol. 5, Springer-Verlag, New York, 1998.
[16] Jiří Matoušek, Günter M. Ziegler, Topological lower bounds for the chromatic number: A hierarchy, Jahresber. Deutsch. Math.-Verein. 106 (2) (2004) 71–90.
[17] Daniel Quillen, Higher algebraic K-theory. I, in: Algebraic K-theory, I: Higher K-theories, Proc. Conf., Battelle Memorial Inst., Seattle, WA, 1972, in: Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin, 1973, pp. 85–147.
[18] Carsten Schultz, Graph colourings, spaces of edges and spaces of circuits, Adv. Math., in press.
[19] Rade T. Živaljević, WI-posets, graph complexes and $\mathbb{Z}^2$-equivalences, J. Combin. Theory Ser. A 111 (2) (2005) 204–223.