A Topological Characterization of End Sets of a Twinning of a Tree

CURTIS D. BENNETT

We give two topological characterizations of which subsets of ends of a tree $T$ can be the twinned ends of a twinning of $T$. We then use this characterization to prove that in a semi-homogeneous tree with countable degrees, a subset $S$ of ends of $T$ can be the set of twinned ends if and only if $S$ is countable and dense in the topology on the set of ends of $T$.

© 2001 Academic Press

Twin trees were first defined by Mark Ronan and Jacques Tits in 1994 [5]. They gave an important example arising from the group $\text{SL}_2(k[t, t^{-1}])$ and mentioned other examples arising from Kac–Moody groups of rank 2.

A twin tree is a pair of trees, $(T, T')$, together with a codistance function, $\delta$, between vertices of the two trees. This codistance function is a non-negative integer-valued function satisfying the property that if $\delta(x, y) = n$ and if $y'$ is adjacent to $y$ then $\delta(x, y') = \delta(x, y) \pm 1$; furthermore if $n > 0$, then $+1$ occurs for a unique $y'$ adjacent to $y$.

Given a tree $T$, the set of ends, $\mathcal{E}(T)$ of $T$ (the equivalence classes of infinite paths $x_0, x_1, \ldots$) can be topologized as in [7, p. 21] (see Section 1 for more precise definitions). In [5] and [6], Ronan and Tits show that the codistance function for a twin tree $(T, T', \delta)$ picks out dense subsets $E_1 \subset \mathcal{E}(T)$ and $E_2 \subset \mathcal{E}(T')$ of each tree and pairs them up. (We call the subset $E_1$ the \textit{ends of the twinning} $(T, T', \delta)$ for $T$. These subsets are clearly acted upon by the group of automorphisms of the twin tree, and it can be shown that the action on the ends determines the action on the tree. Currently, however, little is known about what properties these end sets must satisfy.

The purpose of this paper is to give two distinct characterizations of when a subset $E \subset \mathcal{E}(T)$ can be the set of ends of a twinning of $T$. We then use these characterizations to prove a surprising characterization of end sets for twinnings of trees having every vertex of countable or finite degree.

In the first characterization, we show that $E \subset \mathcal{E}(T)$ is the set of ends of a twinning of $T$ if and only if the following conditions hold.

1. $E$ is dense in $\mathcal{E}(T)$.
2. There is a ‘rooted tree-like basis’ $B$ (see Section 1.2) for $E$ and an injective map $\phi : E \rightarrow B$ such that $e \in \phi(e)$ for all $e \in E$.

The second characterization involves $E$ being dense and having a special type of well-ordering called a ‘separating well-ordering’ (see Section 1.3), a well-ordering in which no element of $E$ is in the closure (under the tree topology) of its set of predecessors (under the well-order). In particular, in Section 2 we prove the following.

\textbf{THEOREM 1.} \textit{Let $T$ be a semi-homogeneous (see Section 1.1) tree without leaves, and let $E \subset \mathcal{E}(T)$. The following are equivalent:}

1. The set $E$ is a twin end set of $T$ under some twinning $(T, T', \delta)$;
2. The set $E$ is dense in $\mathcal{E}(T)$ in the topology of ends of $T$ and there exists a one-to-one map $\psi : E \rightarrow B$ for some rooted tree-like basis $B$ for the topology of $\mathcal{E}(T)$ with $e \in \phi(e)$ for all $e \in E$; and
3. The set $E$ is dense in $\mathcal{E}(T)$ (in the topology of ends) and $E$ is SWO.
We then use the theorem to prove the following.

**Theorem 2.** If $T$ is a semi-homogeneous tree with every vertex having countable or finite degree, then a subset $E \subset \mathcal{E}(T)$ is the end set of some twinning of $T$ if and only if $E$ is a countable dense subset of $\mathcal{E}(T)$.

For any graph theory terms not explicitly defined in the paper, we refer the reader to [2], and for terms from topology, we refer the reader to [4]. At this point we would like to thank M. Abramson, C. Holland, and M. Rubin for insightful conversations about this paper. We especially thank M. Ronan for his many helpful comments and his aid in cleaning up several of the proofs in the paper.

1. Definitions and Preliminaries

1.1. Trees and twin trees. Let $T$ be a tree with vertex set $V$. For a vertex $v \in V$, the degree of $v$, written $\text{deg}(v)$, is the number of edges of $T$ on which $v$ lies. Throughout we will assume that the degree of every vertex of the tree $T$ is at least 2. We say that the tree $T$ is *semi-homogeneous* if $V$ can be partitioned into subsets $U$ and $V$ such there exist cardinalities $m$ and $n$ with $m, n \geq 2$ with:

1. for all $u \in U$, $\text{deg}(u) = m$,
2. for all $v \in V$, $\text{deg}(v) = n$, and
3. if $x$ and $y$ are adjacent vertices of $T$, then one of them is in $U$ and one of them is in $V$.

A $v$-ray of the tree $T$ is an infinite path of $T$ with initial vertex $v$. If $r$ is a $v$-ray, we say that $r$ is based at $v$. A ray of $T$ is a $v$-ray of $T$ for some vertex $v$. We then define a natural equivalence relation on the set of rays of $T$ by

$$r_1 \sim r_2 \iff r_1 \cap r_2$$

is a ray.

We denote the equivalence class of a ray $r$ by $[r]$ and define an end of $T$ to be an equivalence class of rays of $T$. Since $T$ is connected, each end of $T$ has a unique representative based at each vertex $v \in V$. Denote the set of ends of $T$ by $\mathcal{E}(T)$.

Following Serre [7, p. 21], define a topology on the set of ends of the tree $T$ by defining a basis for the topology in the following manner. Fix a vertex $v \in V$. For all $x, y \in V$, define

$$U_x = \{ [r] \mid r \text{ is a } v\text{-ray containing } x \}.$$ 

Define a topology on $\mathcal{E}(T)$ by letting $\mathcal{B}(v) = \{ U_x \mid x \in V \}$ be the set of basic open neighborhoods of $\mathcal{E}(T)$. That is, let the topology be generated by the basis $\mathcal{B}(v)$ (see [4, p. 78]). Note, while the basis $\mathcal{B}(v)$ of the topology is dependent on the choice of root vertex $v$, the topology itself is independent.

There are several nice properties of this topology which we list here as remarks without proof.

**Remark 1.** The topology defined on $\mathcal{E}(T)$ is independent of the choice of $v \in V$. Moreover, $\mathcal{E}(T)$ is a Hausdorff space under this topology.

**Remark 2.** Given any $x, y \in V$, then either $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$ or one set contains the other. Moreover, for any $x \in V$, let

$$\Gamma(x; v) = \{ y \in V \mid y \text{ is adjacent to } x \text{ and } d(v, y) = d(x, y) + 1 \}.$$ 

Then $\mathcal{U}_x$ is the disjoint union of the sets $\mathcal{U}_y$ where $y$ runs over $\Gamma(x; v)$. 

Remark 3. Given a finite set \( \{ e_1, \ldots, e_n \} \) of ends of \( T \), then there exist disjoint basic open neighborhoods \( U_{e_1}, \ldots, U_{e_n} \) such that \( e_i \in U_{e_i} \) for \( i = 1, \ldots, n \).

Remark 4. For all \( e \in \mathcal{E}(T) \) and rays \( r \) such that \([r] = e\), we have \( e = \bigcap_{x \in r} U_x \).

**VC-Functions and Twin Trees:** We follow the work of Ronan and Tits [5, 6]. A VC-function (or vertex codistance function) \( \phi \) for \( T \) is a map \( \phi : \mathcal{V} \to \mathbb{Z} \) such that:

1. For all \( v \in \mathcal{V} \), we have \( \phi(v) \geq 0 \).
2. If \( v \) is adjacent to \( v' \), then \( \phi(v) = \phi(v') \pm 1 \) for all \( v, v' \in \mathcal{V} \), and
3. If \( \phi(v) > 0 \), then there exists a unique \( u \in \mathcal{V} \) adjacent to \( v \) such that \( \phi(u) = \phi(v) + 1 \) (these appear as \( f_x \) in [6]).

Given a VC-function \( \phi \), define the set \( \mathcal{E}(\phi) \) of ends of \( T \) associated to \( \phi \) by

\[
\mathcal{E}(\phi) = \{ [r] \mid \text{there exists } r_1 \in [r] \text{ such that } \phi|_{r_1} \text{ is one-to-one} \}.
\]

VC-functions will play a special role in our study of twin trees and topological properties. We shall need the following Proposition.

**Proposition 1.1 ([5, Section 3, Remark 3]).** For any VC-function on \( T \), the set \( \mathcal{E}(\phi) \) is dense in \( \mathcal{E}(T) \).

**Proof.** Fix a vertex \( v \in \mathcal{V}(T) \). We will show that every basic neighborhood of \( B(v) \) contains an element of \( \mathcal{E}(\phi) \). Let \( U_{x_0} \in B(v) \). Suppose \( r = x_0, x_1, x_2, \ldots \) is a ray based at \( x_0 \) with \( x_1 \) not on the path from \( v \) to \( x \). If \( \phi|_r \) is injective we are done. Otherwise, there exists \( i \) such that \( \phi(x_i) > \phi(x_{i+1}) \). By conditions 2 and 3, it follows that for \( j = i + \phi(x_i) \), we have \( \phi(x_j) = 0 \). Since \( \deg(x_j) \geq 2 \), by conditions 1 and 3 there exists an \( x_j \)-ray \( r' \) not through \( x_{j-1} \) with \( \phi|_{r'} \) one-to-one, and \([r'] \in U_{x_j} \cap \mathcal{E}(\phi) \subset U_{x} \cap \mathcal{E}(\phi) \).

We shall need two final elementary remarks. The first is a consequence of Remark 1 in Section 3 of [5].

**Remark 5.** Let \( \phi \) be a VC-function on a tree \( T \). If \( \phi(x) > 0 \), there exists a unique \( x \)-ray \( r \) such that \( \phi|_r \) is injective.

**Remark 6 ([6, Proposition 1]).** Let \( \phi \) be a VC-function on a tree \( T \). If \( \phi(x) = 0 \) and \( s \) is an \( x \)-ray of \( T \), then there exists a unique subray \( r \subset s \) such that \( \phi|_r : r \to \mathbb{Z}_{\geq 0} \) is bijective.

**Definition.** A twin tree is a triple \((T_+, T_-, \delta)\) such that \( T_+ \) and \( T_- \) are (combinatorial) trees, and

\[
\delta : (T_+ \times T_-) \cup (T_- \times T_+) \to \mathbb{Z}_{\geq 0}
\]

is a symmetric function satisfying the following conditions.

1. \( \delta(x, y) \geq 0 \) for all \((x, y)\) in the domain of \( \delta \).
2. If \( \delta(x, y) = m \) and \( y \sim y' \) (where \( y \sim y' \) means that \( y, y' \) is an edge of one of the trees), then \( \delta(x, y') = m \pm 1 \).
3. If \( \delta(x, y) = m > 0 \), then there is a unique \( y' \) adjacent to \( y \) (in \( T_+ \) or \( T_- \) as \( y \) is) with \( \delta(x, y') = m + 1 \).
We term $\delta(x, y)$ the codistance of $x$ and $y$. The twin tree $(T_+, T_-, \delta)$ is thick if all vertices of $T_+$ and all the vertices of $T_-$ have degree at least 3. Define vertices $x \in T_+$ and $y \in T_-$ to be opposite if $\delta(x, y) = 0$.

Note that if $(T_+, T_-, \delta)$ is a twin tree, and $u$ is a vertex of $T_-$, then the function $\phi_u : \mathcal{V}(T_+) \to \mathbb{Z}$ defined by $\phi_u(v) = \delta(u, v)$ is a VC-distance function on $T_+$. It follows from the definition of the codistance function that the set of ends of $\phi_u$ is independent of the choice of vertex $u$ of $T_-$. 

**Proposition 1.2 ([6]).** Suppose that $(T_+, T_-, \delta)$ is a twin tree. If $u, u'$ are any two vertices of $T_-$, then $\mathcal{E}(\phi_u) = \mathcal{E}(\phi_{u'})$.

Note that Section 3 of [6] proves an equivalent statement to this theorem. The proof is not hard, and we leave it to the interested reader. Define the twin end set of $T$ under the twinning $(T, T', \delta)$ to be the set $\mathcal{E}(\phi_u)$ for some vertex $u$ of $T'$. We say that a subset $E \subset \mathcal{E}(T)$ is a twin end set of $T$ if $E$ is the twin end set of $T$ under some twinning $(T, T', \delta)$ of $T$.

One consequence of the definition of a twin tree is that if $(T, T', \delta)$ is a twin tree, then the trees $T$ and $T'$ must be semi-homogeneous. An important result of Ronan and Tits [6, Section 5] shows that for any VC-function $\phi$ on $\mathcal{V}(T)$, there exists a twinning $(T, T', \delta)$ of $T$ with a vertex $u$ of $T'$ such that $\phi = \phi_u$. More precisely, 

**Theorem 1.3 ([6]).** Let $T$ be a semi-homogeneous tree. A subset $E$ of $\mathcal{E}(T)$ is a twin end set of $T$ if and only if there exists a VC-function $\phi : T \to \mathbb{Z}$ with $E = \mathcal{E}(\phi)$.

### 1.2. Tree-like bases.

Let $S$ be a topological Hausdorff space. A basis, $B$, for the topology on $S$ is tree-like if the semi-lattice of elements of $B$ ordered under inclusion is a combinatorial tree. More precisely, that

1. any two elements of $B$ either intersect trivially or one contains the other, and
2. the union of any two elements of $B$ is contained in some element of $B$.

Given a tree-like basis, $B$, for a Hausdorff space $S$, we associate a tree to $B$ by letting the vertex set be $B$ and letting there be an edge between two vertices $U$ and $V$ if one is a proper subset of the other and maximal such in $B$ with that property.

If $T$ is a tree with all vertices of degree at least 2, then for any vertex $v \in \mathcal{V}(T)$, the set $B(v)$ is a tree-like basis for the topology on $\mathcal{E}(T)$. Moreover, the tree associated to the basis $B(v)$ is isomorphic to $T$.

For a more interesting example, consider the Cantor middle thirds set $C$, and define

$$B = \left\{ \left[ \frac{m}{3^n}, \frac{m+1}{3^n} \right] \cap C \mid m, n \in \mathbb{N}, 0 \leq m \leq n, \text{ where the base 3 expansion of } m \text{ has no 1's} \right\}.$$ 

An easy check shows $B$ is a tree-like basis for the usual topology on $C$. In fact, if one constructs the lattice associated to $B$, one sees that this example arises from a rooted tree with root vertex degree 2 and all other vertices having degree 3 (the binary tree).

In general there will be many tree-like bases for a given Hausdorff space. It is not the case, however, that the associated trees are isomorphic. It is easy to construct tree-like bases for the topology of ends of the binary tree in which every vertex in the associated tree has degree $n$ for any finite integer $n \geq 3$.

Define a tree-like basis $B$ for a Hausdorff space $S$ to be rooted if $S \in B$ and for all $U \in B$, the number of elements of $B$ containing $U$ is finite. In other words, the tree associated to $B$ forms a combinatorial rooted tree with root vertex $S$. In the Cantor set example the basis constructed is a rooted tree-like basis. In the tree example, the basis $B(v)$ is rooted, and the root vertex in the associated tree naturally corresponds to $v$. 

1.3. Separating well-orderable sets. Recall that a well-ordering of a set $S$ is a total ordering $\leq$ such that if $S' \subset S$, then $S'$ has a minimal element under the ordering $\leq$. If $(S, \leq)$ is a well-ordered set and $s \in S$, the set of predecessors of $s$ (under $\leq$) is

$$P(s) = \{ t \in S \mid t < s \}.$$ 

We define a Hausdorff space $S$ to be separating well-orderable (or SWO) if there exists a well-ordering $\leq$ of $S$ such that no point of $S$ is an accumulation point of its predecessors.

We caution the reader that while the set $S$ may come with an order initially, the SWO property merely implies the existence of some well-ordering with this property. Moreover, the topology on $S$ given by the well-ordering plays no role in the definition.

**Example 1.** Any countable set with a Hausdorff topology is SWO. Let $S$ be a countable Hausdorff space, and let $\phi : S \to \mathbb{N}$ be an enumeration of $S$. For $a, b \in S$, define $a \leq_{\phi} b$ if and only if $\phi(a) \leq \phi(b)$. It follows that $P_{\leq_{\phi}}(s)$ is finite for all $s \in S$. Hence, no element of $S$ can be an accumulation point of its predecessors under the order $\leq_{\phi}$.

**Example 2.** Any uncountable subset $S$ of the Cantor set $C$ is not SWO under the usual topology. To see this, suppose $\leq'$ is a well-ordering of $S$. The idea here is to find a dense (in $S$) countable subset $S' \subset S$ such that any point of $S \setminus S'$ is an accumulation point of its predecessors. The construction of $S'$ proceeds inductively using the construction of the Cantor set. Let $S_0$ be the singleton set containing the least element of $S$ under the order $\leq'$. Let $S_1$ be the subset of $S$ containing $S_0$ and the least elements if they exist (under $\leq'$) of $S \setminus S_0$ in the intervals remaining after the first middle third is removed. Given $S_n$, we inductively define $S_{n+1}$ to be the set containing $S_n$ and the least elements of $S \setminus S_n$ of all of the remaining intervals of $[0, 1]$ after removing the $n$th set of middle thirds. Letting $S' = \bigcup_{n=0}^{\infty} S_n$, we see that $S'$ is countable. If $x \in S \setminus S'$, then at every stage at which $x$ could have been chosen it was not. That is to say that for every interval containing $x$ that remained after a deletion of middle thirds, there is a predecessor of $x$ in $S'$. Hence, every open neighborhood of $x$ contains a predecessor of $x$. Thus, $x$ is a limit point of its predecessors.

The second example is a basic example, at least in the case of a subset $E$ of the ends of a tree $T$ where the maximum degree of a vertex of $T$ is finite. In this case, $E$ is SWO if and only if $E$ is countable and dense (the proof of this will be in the next section). Since an easy check shows that $E$ is also homeomorphic to a subset of the Cantor set, we see in the finite degree case that the Cantor set example is key. There do, however, exist examples of Hausdorff spaces which are not SWO and which do not contain a subset homeomorphic to an uncountable subset of the Cantor set. We have been unable to prove that any of these can be realized as a subset of ends of a (combinatorial) tree $T$. This suggests the following conjecture.

**Conjecture 1.4.** Let $T$ be a (combinatorial) tree, and let $E \subset \mathcal{E}(T)$. Then $E$ is separating well-orderable if and only if $E$ contains no uncountable subset homeomorphic to a subset of the Cantor set.

2. **Main Results**

The purpose of this section is to give two topological characterizations of when a subset $E$ of ends of a tree $T$ can be the ends of a VC-function on $T$. The first characterization is that $E$ is dense in $\mathcal{E}(T)$ and there exists an injective function from $E$ to a rooted tree-like basis for $\mathcal{E}(T)$. The second characterization is that $E$ is dense and SWO in the subset topology of $\mathcal{E}(T)$.
THEOREM 2.1. Let $T$ be a tree with all vertices of degree at least 3. Let $E$ be a subset of the set $E(T)$ of ends of $T$. Then the following are equivalent.

1. There exists a VC-function $\phi$ on $V(T)$ such that $E(\phi) = E$.
2. The set $E$ is dense in $E(T)$ (in the topology of ends of $T$) and there exists an injective map $\psi : E \rightarrow B$ for some rooted tree-like basis $B$ for the topology of $E(T)$ with $e \in \psi(e)$ for all $e \in E$.
3. The set $E$ is dense in $E(T)$ (in the topology of ends) and $E$ is SWO.

Theorem 2.1 will follow from Lemmas 2.3–2.5 which will show respectively that condition 1 implies condition 2, condition 2 implies condition 3, and condition 3 implies condition 1.

As an immediate corollary to this theorem using Theorem 1.3 we obtain the following

COROLLARY 2.2. Let $T$ be a semi-homogeneous tree without leaves, and let $E$ be a subset of the set $E(T)$ of ends of $T$. The the following are equivalent.

1. The set $E$ is a twin end set of $T$ under some twinning $(T', T, \delta)$.
2. The set $E$ is dense in $E(T)$ in the topology of ends of $T$ and there exists a one-to-one map $\psi : E \rightarrow B$ for some rooted tree-like basis $B$ for the topology of $E(T)$ with $e \in \psi(e)$ for all $e \in E$.
3. The set $E$ is dense in $E(T)$ (in the topology of ends) and $E$ is SWO.

The corollary gives a topological characterization for end sets of twinnings for a particular tree. Moreover, since the proof of the theorem is constructive, it gives a way of constructing a twinning with end set $E$ once you have a separating well-order for $E$. Finding such an ordering, however, remains a difficult problem.

LEMMA 2.3. Let $T$ be a tree with every vertex having degree at least 3 and let $\phi$ be a VC-function on $V(T)$. Then $E(\phi)$ is dense in $E(T)$ and there exists a rooted tree-like basis $B$ for $E(T)$ and an injective map $\psi : E(\phi) \rightarrow B$ such that $e \in \psi(e)$ for all $e \in E(\phi)$.

PROOF. Let $E = E(\phi)$ and $V = V(T)$. Fix $v \in V$ with $\phi(v) = 0$, and let $B = B(v)$. From Proposition 1.1 we already have that $E$ is dense in $E(T)$.

To define the function $\psi : E \rightarrow B(v)$, we will work in two steps. We first define a function $\eta : E \rightarrow V$ in such a way that if $r$ is the ray based at $\eta(e)$ in the equivalence class of $e$ then:

1. $\phi|_r : r \rightarrow \mathbb{Z}_{>0}$ is bijective, and
2. $r$ is a subray of the ray in $e$ based at $v$.

Once we have defined $\eta$, we let $\psi(e) = U_{\eta(e)}$. The first condition on $\eta$ and Remark 5 will imply that $\psi$ is injective, and the second condition will imply that $e \in \psi(e)$.

Given $e \in E$, let $s$ denote the ray based at $v$ in the equivalence class of $e$. By Remark 6 there exists a unique subray $r$ of $s$ such that $\phi|_r : r \rightarrow \mathbb{Z}_{>0}$ is one-to-one and onto. In particular, if $x$ is the basepoint of $r$, then $\phi(x) = 1$. Define $\eta(e) = x$ where $x$ is the point given above. (In the language of [6], $\eta(e)$ is the element of $\sigma(e)$ closest to $v$ where $f_\sigma = \phi$.) We note that by Remark 5 there is a unique $x$-ray for which $\phi$ is injective. In particular, this means that $\eta$ is injective. Next define $\psi : E \rightarrow B(v)$ by $\psi(e) = U_{\eta(e)}$. By definition of $\eta$, the function $\psi$ is injective and satisfies the condition $e \in \psi(e)$ for all $e \in E$.

LEMMA 2.4. Suppose $X$ is a Hausdorff space, $E \subset X$, $B$ is a rooted tree-like basis for $X$, and $\psi : E \rightarrow B$ is an injective map with $e \in \psi(e)$ for all $e \in E$. Then, $E$ is SWO (under the subspace topology).
PROOF. Let $\widetilde{T}$ be the tree associated to the basis $B$, and let $\tilde{v}$ be the root vertex of $\widetilde{T}$. Define

$$\widetilde{V}_n = \{ U \in \widetilde{T} \mid d(\tilde{v}, U) = n \}$$

where $d$ is the usual distance function on the tree $\widetilde{T}$. We have $\mathcal{V}(\widetilde{T}) = \bigcup_{n=0}^{\infty} \widetilde{V}_n$ since every open set of $B$ has only finitely many sets of $B$ containing it (since $B$ is rooted). For each $n \in \{0, 1, \ldots\}$ choose a well-ordering $\leq_n$ of $V_n$. Define a well-ordering $\leq_E$ on $E$ by $e_1 \leq_E e_2$ if and only if either

1. $d(\tilde{v}, \psi(e_1)) < d(\tilde{v}, \psi(e_2))$, or
2. $d(\tilde{v}, \psi(e_1)) = d(\tilde{v}, \psi(e_2))$, and $\psi(e_1) \leq_n \psi(e_2)$.

Our intention is to show that $\leq_E$ is a separating well order for $E$.

It is clear that $\leq_E$ is a total order on $E$. Let $S \subset E$. Since the set of natural numbers is well-ordered, there exists a minimal $n$ such that $V_n \cap \psi(S)$ is non-empty. Since $\leq_n$ is a well-ordering of $\widetilde{V}_n$, there exists a minimal element $x$ of $V_n \cap \psi(S)$ with respect to $\leq_n$. Let $y \in S$ be arbitrary, and suppose $\psi(y) \in \overline{V}_m$. By the minimality of $n$, we know that $n < m$. Hence $x \leq_E y$. On the other hand, if $m = n$, then $x \leq_E y$ by choice of $x$, and hence $x \leq_E y$. Hence $\leq_E$ is a well-ordering for $E$. It remains to check the SWO condition. Let $e \in E$. Because of the rooted tree-like structure of $B$, the set $\psi(P_{<E}(e))$ of the image of all predecessors of $e$ has only finitely many basic open neighborhoods intersecting $\psi(e)$ non-trivially. Since $\psi$ is one-to-one, $\psi(e) \cap P_{<E}(e)$ is finite. Since $X$ is a Hausdorff space, $e$ cannot be an accumulation point of a finite set and hence $e$ cannot be an accumulation point of its set of predecessors $P_{<E}(e)$. Thus, $\leq_E$ is a separating well-ordering of $E$. \hfill $\Box$

**Lemma 2.5.** Let $T$ be a tree with all vertices having degree at least 3, let $E$ be a dense subset of $E(T)$ and let $\leq$ be a separating well-ordering of $E$. Then there exists a VC-function $\phi$ on $\mathcal{V}(T)$ such that $\mathcal{E}(\phi) = E$.

**Proof.** To define our VC-function on $\mathcal{V}(T)$, we must first define two associated functions. Let $d$ be the usual distance function on the vertices of the tree $T$. Fix $v \in \mathcal{V}(T)$ and define

$$V_n = \{ x \in \mathcal{V}(T) \mid d(v, x) = n \}.$$ 

For $n > 0$ and $x \in V_n$, let $\tau(x)$ be the unique element of $V_{n-1}$ adjacent to $x$. For $x \in \mathcal{V}(T)$, let $U_x$ denote the basic open neighborhood of $B(v)$ and define $\gamma(x) = \min_{x}(U_x \cap E)$. Since $\leq$ is a well-ordering of $E$, the density of $E$ implies that $\gamma(x)$ is defined for all $x \in \mathcal{V}(T)$. Inductively define functions $\phi_n : V_n \rightarrow \mathbb{Z}$ by:

$$\phi_1(v) = 0$$

$$\phi_1(x) = 1 \quad \text{if } x \in V_1.$$ 

Continue inductively for $n > 1$ by defining $\phi_n : V_n \rightarrow \mathbb{Z}$ by

$$\phi_n(x) =\begin{cases} 1, & \text{if } \phi_{n-1}(\tau(x)) = 0; \\ \phi_{n-1}(\tau(x)) + 1, & \text{if } \gamma(x) = \gamma(\tau(x)), \text{ and } \phi_{n-1}(\tau(x)) = \phi_{n-1}(\tau(x)) + 1; \\ \phi_{n-1}(\tau(x)) - 1, & \text{otherwise}. \end{cases}$$ 

We now define $\phi : \mathcal{V}(T) \rightarrow \mathbb{Z}$ by $\phi(x) = \phi_{d(v, x)}(x)$. The function $\phi$ is the desired VC-function.
To see this, note that $\phi(x) \geq 0$ for all $x \in V(T)$, and that by the definitions of the $\phi_n$’s, if $x, y \in V(T)$ are adjacent vertices, it follows that $\phi(x) = \phi(y) \geq 1$. It remains to check condition 3 in the definition of a VC-function. Suppose $x \in V(T)$ with $\phi(x) > 0$. First suppose $\phi(\tau(x)) > \phi(x)$, and $u$ is adjacent to $x$ with $u \neq \tau(x)$, and hence $\tau(u) = x$. In this case, $\phi(\tau(u)) = \phi(x)$, and by assumption $\phi(\tau^2(u)) + 1 \neq \phi(\tau(u))$. Hence by definition, $\phi(u) = \phi(\tau(u)) - 1 = \phi(x) - 1$. In this case, $\tau(x)$ is the unique vertex adjacent to $x$ with $\phi(\tau(x)) = \phi(x) + 1$. In the case where $\phi(\tau(x)) < \phi(x)$, and $d(v, x) = n$, it suffices to show that there exists a unique vertex $u$ of $V_{n+1}$ adjacent to $x$ such that $\gamma(u) = \gamma(x)$. Remark 2 implies that the set of neighborhoods $U_u$ for $u \in V_{n+1}$ adjacent to $x$ partition the set $U_x$. As a result, there exists a unique $u$ in $V_{n+1}$ adjacent to $x$ such that $\gamma(x) \in U_u$, proving uniqueness. Since $U_u \subset U_x$ and $\gamma(x)$ is a minimal element for $U_x$, it follows that $\gamma(x)$ is a minimal element for $U_u$, and hence $\gamma(u) = \gamma(x) = \gamma(\tau(u))$. Thus, $\phi$ is a VC-function for $T$.

It remains to show that $E(\phi) = E$. If $e \in E(\phi)$, then there exists a unique $v$-ray $s$ with $[s] = e$. By Remark 6, there exists a unique subray $r = x_1, x_2, \ldots$ of $s$ with $\phi(x_i) = i$ for $i = 1, 2, \ldots$. This implies $\gamma(x_i) = \gamma(x_{i+1})$ for all $i$. Hence, by Remark 4 $\gamma(x_i) = \cap_{i=1}^\infty U_{x_i}$. But we also have $e = \cap_{i=1}^\infty U_{x_i}$, and hence, $e = \gamma(x_i)$ for all $i$ and $e \in E$. Conversely, suppose $e \in E$. By the SWO property of the ordering $\leq$, there exists an open set $U$ of $E(T)$ such that $U \cap P_{e}(e)$ is empty. Since $B(v)$ is a basis for the topology, there exists a basis open neighborhood $U_v \in B(v)$ contained in $U$. Given one such $x$, the same holds true for any vertex on the ray of $e$ based at $x$. As in the proof of Proposition 1.1, this allows us to assume that $\phi(x) > 0$ and that $\phi(\tau(x)) < \phi(x)$. But in this case, it is clear that $\gamma(x)$ will be an end of $\phi$. Hence, $e \in E(\phi)$ and the lemma is proved completing the proof of the lemma. \hfill \Box

**Proof of Theorem 2.1.** That condition 1 implies condition 2 follows immediately from Lemma 2.3. Letting $E(T) = X$, and noting that $E$ is required to be dense in condition 2, we see from Lemma 2.4 that condition 2 implies condition 3. Finally, Lemma 2.5 completes the proof of the theorem. \hfill \Box

As an elementary corollary of our Theorem 2.1 we obtain the following.

**Corollary 2.6.** If $T$ is a semi-homogeneous tree with every vertex having countable or finite degree, then a subset $E \subset E(T)$ is the end set of some twinning of $T$ if and only if $E$ is a countable dense subset of $E(T)$.

**Proof.** If $T$ has countable or finite valences, then $T$ has a countable number of vertices. In this case, there exists a one-to-one map from the set $E$ to a countable basis $B$ for the topology of $T$. Hence $E$ is countable. For the reverse, note that countable subsets of Hausdorff spaces are always SWO sets by Example 1. \hfill \Box

We conclude with the end set of a twin tree version of our earlier conjecture.

**Conjecture 2.7.** If $T$ is a semi-homogeneous tree, then a subset $E$ of $E(T)$ is an end set of some twinning of $T$ if and only if $E$ is dense in $E(T)$ and $E$ contains no uncountable subset homeomorphic to a subset of the Cantor set.

Readers interested in further work on twin trees should refer to [1, 2, 5 and 6].

**References**

Characterization of twin end sets


Received 1 April 1996 in revised form 24 August 2000

CURTIS D. BENNETT
Bowling Green State University,
Bowling Green,
OH 43403, U.S.A.