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Almost every graph is vertex-oblique

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Abstract

The type of a vertex v of a graph G is the ordered degree sequence of the vertices adjacent to v. The graph G is called vertex-oblique if it contains no two vertices of the same type. We will show that almost every graph $G \in \mathscr{G}(n, p)$ is vertex-oblique, if the probability p for each edge to appear in G is within certain bounds.

Keywords: Oblique graphs; Asymmetric graphs

1. Introduction

The concept of vertex-oblique graphs was introduced in [4] and originated in the investigation of asymmetric polyhedral graphs. For every graph G one can define a type for each vertex v by assigning a type-vector $t(v)$ consisting of the ordered degree sequence of its neighbors. A graph is called vertex-oblique if there are no two vertices u and v with $t(u) = t(v)$. While the question of the number of such graphs in the class of polyhedral graphs remains open, some results about the general case are known. In [4] an infinite class of vertex-oblique graphs of arbitrarily high connectivity is given, where moreover the set of vertex-types for each graph has an empty intersection with the corresponding set of the complement graph. Another infinite class of so-called dually vertex-oblique graphs, where each graph has the same set of vertex-types as its complement is given in [3]. Since a graph automorphism can only map vertices of the same type onto each other, every vertex-oblique graph has automatically only the trivial automorphism group, i.e. it is an asymmetric graph. For these asymmetric graphs it is known from a theorem of Wright from 1971 (cf. [2]) that the probability of a random graph to be asymmetric tends to one, if the average vertex-degree is at least logarithmic in the number of vertices. So the natural question arises whether the same is true for the more restrictive property of a graph to be vertex-oblique.

2. Basic definitions and denotations

Let $G = (V, E)$ be a graph.

- The *type-vector* $t(v)$ of a vertex $v \in V(G)$ is the sequence of degrees of vertices adjacent to v in non-decreasing order.

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- A graph is said to be *vertex-oblique* if there are no two vertices $u, v \in V(G)$ such that $t(u) = t(v)$.
- $\mathcal{G}(n, p)$ denotes the usual random graph model as it is used for instance in [2], where we consider all graphs on the vertex set $\{x_1, \ldots, x_n\}$[1] and for each possible edge the probability of its appearance is p, which may be a function of n. All edges are chosen independently and $q := 1 - p$.
- $\mathrm{o}(f) = \{g : \mathbb{R} \to \mathbb{R} | \lim_{x\to\infty} g(x)/f(x) = 0\}$ denotes the usual little o-notation.
- $b(k; n, p) := \binom{n}{k} \cdot p^k \cdot q^{n-k}$ denotes the probability $P(X = k)$ where X is a binomially distributed random variable with parameters n and p.

Two inequalities which are used in some places in the paper are:

1. $\binom{n}{k} \leqslant \binom{n}{\lfloor n/2 \rfloor} < (1/\sqrt{n})2^n$.
2. $b(k; n, p) \leqslant b(\lfloor np + p \rfloor, n, p) < 1/\sqrt{npq}$.

Both inequalities can be derived from the Stirling formula.

3. Main result

In order to point out the idea of the proof we will state the main result first and prove some necessary lemmas later on.

Theorem 1. *Let $G \in \mathcal{G}(n, p)$ be a random graph, with $p = p(n)$ and $c_1 \cdot n^{\alpha-1} < p(n) < 1 - c_2 \cdot n^{\alpha-1}$, where $c_1, c_2 > 0$ are constants and $\frac{1}{2} < \alpha < 1$. Then*

$$\lim_{n\to\infty} P(G \text{ is vertex oblique}) = 1.$$

Proof. For technical reasons, that will become clear later, we prove the statement for a graph $\tilde{G} \in \mathcal{G}(n + 2, p(n + 2))$. As was shown in [4], a graph is vertex-oblique, if and only if its complement is vertex-oblique. Thus, we can assume without loss of generality that $p \leqslant q = 1 - p$ and therefore $q \geqslant \frac{1}{2}$. Furthermore, if $p(n + 2)$ is within the given bounds for some constants c_1, c_2 and α, then $p(n + 2) > n^{\alpha'-1}$ for an α' with $\frac{1}{2} < \alpha' < \alpha$ for large enough n. Now consider the following events:

O: $\tilde{G}$ is vertex-oblique;
T: $t(u) = t(v)$ where $u := x_{n+1}$ and $v := x_{n+2}$.

Equivalent to the proposition in the theorem is $\lim_{n\to\infty} P(\overline{O}) = 0$. We estimate:

$$\begin{aligned} P(\overline{O}) &= P(\exists i, j \in \{1, \ldots, n + 2\}, i \neq j : t(x_i) = t(x_j)) \\ &= P\left(\bigcup_{i,j\in\{1,\ldots,n+2\}, i\neq j} t(x_i) = t(x_j)\right) \\ &\leqslant \sum_{i,j\in\{1,\ldots,n+2\}, i\neq j} P(t(x_i) = t(x_j)) \\ &= \binom{n+2}{2} P(T). \end{aligned}$$

That means in order to prove the theorem we only have to show that $P(T) \in \mathrm{o}(n^{-2})$. Let $G = \tilde{G}\backslash\{u, v\}$. Obviously G is a random graph in $\mathcal{G}(n, p)$. Let $d_1 > \cdots > d_r$ be the different vertex degrees occurring in G and $V_i := \{x \in V(G) | d_G(x) = i\}$. Then, for any fixed $m \in \mathbb{N}$, G must have at least one of the following properties:

A: G contains vertices of at least $\sqrt{n}$ different degrees ($r \geqslant \sqrt{n}$).
B_m: There is a class V_{d_i} $i \in \{1, \ldots, r\}$ of vertices, such that $|V_{d_i}| \geqslant (1/m)n$.

[1] In the literature the vertex set is often chosen to be the set $\{1, \ldots, n\}$.

C_m: There are at least m classes $V_{d_{i_1}}, \ldots, V_{d_{i_m}}$ of vertices of $V(G)$ each of which is of cardinality greater than $(1/m)\sqrt{n}$.

To prove this, we suppose that A and B_m are not true. Then for the number r of classes we have $m < r < \sqrt{n}$. Let $V_{d_{i_1}}, \ldots, V_{d_{i_m}}$ be the classes of largest cardinality, where $|V_{d_{i_1}}| \geqslant \cdots \geqslant |V_{d_{i_m}}|$. Then clearly:

$$\begin{array}{ccccccc}
\frac{1}{m}n & > & |V_{d_{i_1}}| & \geqslant & \frac{n}{r} & > & \sqrt{n} \\
\frac{1}{m}n & > & |V_{d_{i_2}}| & \geqslant & \dfrac{\frac{m-1}{m}n}{r-1} & > & \frac{m-1}{m}\sqrt{n} \\
 & \vdots & & \vdots & & \vdots & \\
\frac{1}{m}n & > & |V_{d_{i_m}}| & \geqslant & \dfrac{\frac{1}{m}n}{r-m+1} & > & \frac{1}{m}\sqrt{n}.
\end{array}$$

Thus C_m is true. Therefore, for the random graph G we have $A \cup B_m \cup C_m = \Omega$ is the certain event for arbitrary m. That is why:

$$\begin{aligned}
P(T) &= P(T \cap \Omega) \\
&= P((T \cap A) \cup (T \cap B_m) \cup (T \cap C_m)) \\
&\leqslant P(T \cap A) + P(T \cap B_m) + P(T \cap C_m) \\
&= \mathbf{P(T|A)} \cdot P(A) + P(T|B_m) \cdot \mathbf{P(B_m)} + \mathbf{P(T|C_m)} \cdot P(C_m).
\end{aligned}$$

We will show in the following lemmas that the three probabilities in bold are in $\mathrm{o}(n^{-2})$. Since the other probabilities are smaller than 1, this suffices to show that $P(T) \in \mathrm{o}(n^{-2})$, and therefore, the proof is complete. □

First we will prove that $P(B_m) \in \mathrm{o}(n^{-2})$. We will do this by proving the following somewhat stronger lemma:

Lemma 2. *Let $G \in \mathcal{G}(n, p)$ be a random graph, where p is an arbitrary probability such that $pqn \to \infty$. Then $P(G$ contains a class of βn vertices of the same degree$) < (\frac{1}{2})^n$ for large enough n, where β is a rational number such that $\beta n \in \mathbb{N}$.*

Proof. We define the following events:

B^1: "G contains a class of βn vertices of the same degree".
B^2: "The vertices $x_1, \ldots, x_{\beta n}$ are of the same degree in G".

$$\text{Obviously } P(B^1) \leqslant \binom{n}{\beta n} P(B^2) \leqslant \frac{1}{\sqrt{n}} 2^n P(B^2). \tag{1}$$

Let the graph G be constructed in two steps by first deciding for the edges between the first βn vertices, whether they appear, and decide it for the rest of the edges in the second step. Now let H be the random subgraph of G induced by $x_1, \ldots, x_{\beta n}$, which is constructed in the first step. Then by the law of total probability:

$$P(B^2) = \sum_{H' \in \mathcal{G}(\beta n, p)} P(B^2 | H = H') \cdot P(H = H'). \tag{2}$$

Now we estimate $P(B^2|H = H')$, where H' is a fixed graph on the vertex set $\{x_1, \ldots, x_{\beta n}\}$. Let Δ, δ denote the maximum resp. minimum vertex degree of H'. Then for the unique degree r of $x_1, \ldots, x_{\beta n}$ in the graph G we have

$\Delta \leqslant r \leqslant n - \beta n + \delta$. Therefore, the following holds:

$$P(B^2|H = H') = \sum_{r=\Delta}^{(1-\beta)n+\delta} P(d_G(x_1) = \cdots = d_G(x_{\beta n}) = r)$$
$$= \sum_{r=\Delta}^{(1-\beta)n+\delta} P\left(\bigcap_{i=1}^{\beta n} d_G(x_i) = r\right).$$

The random variables X_i := number of vertices in $\{x_{\beta n+1}, \ldots, x_n\}$ which are adjacent to x_i for $i = 1, \ldots, \beta n$ are independent and binomially distributed with parameters $(1-\beta)n$ and p. Therefore:

$$P(B^2|H = H') = \sum_{r=\Delta}^{(1-\beta)n+\delta} P\left(\bigcap_{i=1}^{\beta n} d'_H(x_i) + X_i = r\right)$$
$$= \sum_{r=\Delta}^{(1-\beta)n+\delta} \prod_{i=1}^{\beta n} b(r - d_{H'}(x_i); (1-\beta)n, p)$$
$$\leqslant \sum_{r=\Delta}^{(1-\beta)n+\delta} \prod_{i=1}^{\beta n} \frac{1}{\sqrt{(1-\beta)\mathrm{npq}}}$$
$$= \sum_{r=\Delta}^{(1-\beta)n+\delta} \left(\frac{1}{\sqrt{(1-\beta)\mathrm{npq}}}\right)^{\beta n}$$
$$\leqslant n\left(\frac{1}{\sqrt{(1-\beta)\mathrm{npq}}}\right)^{\beta n}. \tag{3}$$

The inequality is independent from H' so we can conclude from (2) and (3)

$$P(B^2) \leqslant n\left(\frac{1}{\sqrt{(1-\beta)\mathrm{npq}}}\right)^{\beta n}$$

and together with (1) we get

$$P(B^1) \leqslant \sqrt{n}2^n\left(\frac{1}{\sqrt{(1-\beta)\mathrm{npq}}}\right)^{\beta n}$$
$$= \sqrt{n}2^n\left(\frac{1}{2}\right)^{\log_2(\sqrt{(1-\beta)\mathrm{npq}})\cdot\beta n}$$
$$= \sqrt{n}2^n\left(\frac{1}{2}\right)^{(\frac{\beta}{2}\log_2((1-\beta)\mathrm{pqn})n}$$
$$< \left(\frac{1}{2}\right)^n \quad \text{since } \left(\frac{\beta}{2}\log_2((1-\beta)\mathrm{pqn})\right) \to \infty. \qquad \square$$

Before we estimate the remaining two probabilities, we will reformulate the property T. Let V_i denote the set of vertices which are of degree i in the graph G. Furthermore, let $N(u)$, $N(v)$ be the set of neighbors of the vertices, u,v resp. in G. Then the following proposition holds:

Proposition. *Property T is equivalent to* $|N(u) \cap V_i| = |N(v) \cap V_i|$ *for* $i = 0, \ldots, n-1$.

Proof. Let $N'(u) = N(u) \backslash N(v)$, $N'(v) = N(v) \backslash N(u)$ and $M = N(u) \cap N(v)$. $M = \bigcup_{i=0}^{n-1} M_i$, where $M_i = \{x \in M | d_G(x) = i\}$.

Obviously $|N(x) \cap V_i| = |M \cap V_i| + |N'(x) \cap V_i|$ for $x \in \{u, v\}$.

($\Rightarrow$) It remains to show that $|N'(u) \cap V_i| = |N'(v) \cap V_i|$. The type-vectors of u and v contain some entries for their common neighbors. After deleting these entries, the remaining vectors must still be the same. If u and v are adjacent we also delete the entry for $d(u)$, $d(v)$ resp. in both vectors. Now there still must be exactly as many $(i+1)$- entries in both vectors. But the number of $(i+1)$- entries is $|N'(u) \cap V_i|$ or $|N'(v) \cap V_i|$, respectively.

($\Leftarrow$) The type-vector of $x \in \{u, v\}$ contains $|N'(x) \cap V_0|$ entries equal to 1 and $|N'(x) \cap V_{i-1}| + |M_{i-2}|$ entries equal to i for $i = 2, \ldots, n-1$. But since $|N'(u) \cap V_i| = |N'(v) \cap V_i|$ the type-vectors are the same. Both type-vectors contain a further number, if they are adjacent to each other. But in this case, their degrees are equal and therefore these numbers are equal. □

Now we show that $P(T|A) \in \mathrm{o}(n^{-2})$ by the following lemma.

Lemma 3. $P(T|A) \leqslant q^{\sqrt{n}}$.

Proof. Let the graph $\tilde{G}$ be constructed in two steps. First, we decide for all edges between the vertices in $\{x_1, \ldots, x_n\}$ whether they appear in $\tilde{G}$ and construct G in this way. In the second step we choose the edges connecting u and v with G and finally decide whether $uv \in E(\tilde{G})$. Let $d_1, \ldots, d_r$ be the r different vertex degrees occurring in G. Furthermore, let $a_i = |N(u) \cap V_i|$ and $b_i = |N(v) \cap V_i|$. Clearly all random variables $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$ are independent and binomially distributed with parameters $|V_i|$ and p. Then according to the last proposition we can state:

$$\begin{aligned} P(T|A) &= P(\forall i \in \{d_1, \ldots, d_r\} : a_i = b_i | A) \\ &\leqslant P(\forall i \in \{d_1, \ldots, d_r\} : a_i \equiv b_i \pmod 2 | A) \\ &= \prod_{i=1}^{r} P(a_{d_i} \equiv b_{d_i} \pmod 2 | A). \end{aligned}$$

Now let for each i, for all edges connecting u and V_{d_i} and all but one edge connecting v and V_{d_i}, be decided whether they appear in the graph $\tilde{G}$. The probability of making the decision for the last edge right is then p or q depending on the choice for the other edges. But in any case it is less than or equal to $\max(p, q) = q$.

$$P(T|A) \leqslant q^{\sqrt{n}} \text{ because } r \geqslant \sqrt{n}.$$

It remains to show that $q^{\sqrt{n}} \in \mathrm{o}(n^{-2})$.

$$\begin{aligned} q^{\sqrt{n}} &\leqslant \left(1 - \frac{1}{n^{1-\alpha'}}\right)^{\sqrt{n}} \quad \text{for some } \alpha' \text{ with } \frac{1}{2} < \alpha' < 1 \\ &\leqslant \left(\left(1 - \frac{1}{n^{1-\alpha'}}\right)^{n^{1-\alpha'}}\right)^{n^{\alpha'-1/2}}. \end{aligned}$$

The inner part tends to e^{-1}. That's why the whole term tends to zero exponentially fast. □

Finally, we prove that $P(T|C_m) \in \mathrm{o}(n^{-2})$ for a suitable m by the next lemma.

Lemma 4. $P(T|C_m) \in \mathrm{o}(n^{-2})$ *for* $m > \frac{4}{\alpha'-1/2}$.

Proof. Let $d_{i_1}, \ldots, d_{i_m}$ be m vertex degrees which occur at least $(1/m)\sqrt{n}$ times in G. a_i and b_i are defined as in the previous lemma, and $c_i := |V_i|$.

$$\begin{aligned}
P(T|C_m) &= P(\forall i \in \{1, \ldots, n-1\} : a_i = b_i | C_m) \\
&\leqslant P(\forall j \in \{1, \ldots, m\} : a_{d_{i_j}} = b_{d_{i_j}} | C_m) \\
&= \prod_{j=1}^{m} P(a_{d_{i_j}} = b_{d_{i_j}} | C_m) \\
&= \prod_{j=1}^{m} \sum_{k=0}^{c_{d_{i_j}}} P(a_{d_{i_j}} = b_{d_{i_j}} = k | C_m) \\
&= \prod_{j=1}^{m} \sum_{k=0}^{c_{d_{i_j}}} P(a_{d_{i_j}} = k | C_m) \cdot P(b_{d_{i_j}} = k | C_m) \\
&= \prod_{j=1}^{m} \sum_{k=0}^{c_{d_{i_j}}} b(k; c_{d_{i_j}}, p) \cdot b(k; c_{d_{i_j}}, p) \\
&\leqslant \prod_{j=1}^{m} \sum_{k=0}^{c_{d_{i_j}}} b(k; c_{d_{i_j}}, p) \cdot \frac{1}{\sqrt{c_{d_{i_j}} pq}} \\
&= \prod_{j=1}^{m} \frac{1}{\sqrt{c_{d_{i_j}} pq}} \\
&= \left(\frac{1}{\sqrt{c_{d_{i_j}} pq}} \right)^m \quad \text{and since } c_{d_{i_j}} \geqslant \frac{1}{m}\sqrt{n} \\
&\leqslant \left(\sqrt{\frac{m}{\sqrt{n}\, pq}} \right)^m \\
&\leqslant \left(\frac{m}{n^{\frac{1}{2}} \cdot n^{\alpha' - 1} \cdot \frac{1}{2}} \right)^{m/2} \\
&\leqslant (2m \cdot n^{\frac{1}{2} - \alpha'})^{m/2}.
\end{aligned}$$

The last term is in $o(n^{-2})$ if m is chosen in such a way that $(\alpha' - \frac{1}{2}) \cdot m/2 > 2$. That completes the proof of Theorem 1. □

The bounds for p come from the consideration after Lemma 3 and at the end of Lemma 4. A refinement of the corresponding proofs might lead to a wider range for the probability p. An immediate consequence of the theorem is the following well-known corollary:

Corollary 5. *Let $G \in \mathcal{G}(n, p)$ be a random graph with the same bounds for p as in the theorem, then*:

$$\lim_{n \to \infty} P(G \textit{ is asymmetric}) = 1.$$

The proof is trivial because an automorphism can only map two vertices with the same type-vector onto each other. This result is long known for a larger range of possible probabilities p. The theorem of Wright even holds if $c \cdot \frac{\log(n)}{n} \leqslant p \leqslant 1 - c \cdot \frac{\log(n)}{n}$, where c is any constant.

4. Remarks and open questions

As already stated in [4] the property of being vertex-oblique is equivalent to the existence of a total ordering of the vertex set. A very similar ordering was used by Babai et al. [1] to give a polynomial algorithm for the graph isomorphism problem which almost always works. Given two graphs it is easy to compute the set of vertex-types for each graph. Both sets must be identical, otherwise the graphs are not isomorphic. With probability tending to one there is only one vertex of each occurring type. And since an isomorphism can map only vertices of the same type onto each other, there is only one mapping to be checked. Either it is the isomorphism, or there is no such one. Different from our type definition, the authors only investigated for each vertex the adjacency relations to about $O(\log(n))$ vertices of the highest degrees.

Later, Bollobás [2] improved the bounds for p in that algorithm to all $p < \frac{1}{2}$ such that $p^5 n/(\log(n))^5 \to \infty$. Both proofs can be used to prove, that almost every graph is vertex-oblique within the same bounds for p. Our main result allows a larger variety of possible values for p. It can also be used to show that the probability for the algorithm to work properly also tends to 1 concerning this wider range of p.

Some open questions remain:

- Does the theorem hold for yet a wider range of possible values for p?
- Is there a threshold function for the graph property of being vertex-oblique?

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