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Iterative solution of simultaneous equations

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Abstract: Stationary linear iteration methods are used to obtain generalized solutions for simultaneous equations. In particular the case is considered that the iterates terminate after finitely many steps. On the other side also the case is pointed out that the iterates converge, but not to a generalized solution. The well known method of Kaczmarz arises as a special case. Finally, the connection to matching rules is set up, which appear in the asymptotic theory of singular perturbations.

Keywords: Generalized inverses, projectors, linear equations, iterations, maching rules.

1. Preliminaries

Let us consider an arbitrary linear equation

$$Ax = f \tag{1}$$

with given linear operator $A \in (\mathscr{D} \to \mathscr{R})$, given right-hand side $f \in \mathscr{R}$ and with unknown $x \in \mathscr{D}$, where $\mathscr{D}$ and $\mathscr{R}$ are linear vector spaces. We choose a linear Operator $B \in (\mathscr{R} \to \mathscr{D})$ and obtain from (1) the fixed point equation

$$x = Tx + Bf \tag{2}$$

with

$$T = I - BA. \tag{3}$$

Every linear one-step iteration method arises from (2) for suitable B, cf. Maess [11]. We restrict ourselves to stationary iteration methods, according to nonstationary methods cf. [15] and [5]. By iteration we obtain from (2) and (3)

$$x = T^k x + Y_k Bf \tag{4}$$

with

$$Y_k = \sum_{j=0}^{k-1} T^j, \qquad Y_k BA = Y_k(I - T) = I - T^k. \tag{5}$$

Let $X \in (\mathscr{R} \to \mathscr{D})$ be an inner inverse of A, i.e. according to Nashed and Votruba 1976

$$AXA = A, \tag{6}$$

and define the projectors

$$P = I - XA, \qquad Q = I - AX, \tag{7}$$

where I as before denotes suitable unit operators. Then

$$x = Px + Xf \tag{8}$$

is called the generalized solution of (1) with respect to X. This means that (7) with an arbitrary $x \in \mathscr{D}$ on the right-hand side is the general solution of (7), so far as the necessary and sufficient solvability condition

$$Qf = 0 \tag{9}$$

is satisfied.

The question arises, whether (4) coincides with (8), i.e. whether $T^k = P$ and $Y_k B = X$. Comparing (5) with (6) we find the answer in the following lemma.

Lemma 1. *For a fixed k equation* (4) *gives the generalized solution of* (1), *if and only if*

$$AT^k = 0. \tag{10}$$

2. Simultaneous equations

Now we consider the case of a linear system of simultaneous equations

$$A_i x = f_i, \tag{11}$$

$i = 1, \ldots, n$, with $A_i \in (\mathscr{D} \to \mathscr{R}_i)$, $f_i \in \mathscr{R}_i$. Introducing the block vectors

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \tag{12}$$

cf. [7], the system (11) can be reduced to (1) with $\mathscr{R} = \mathscr{R}_1 \times \cdots \times \mathscr{R}_n$. Each A_i shall have an inner inverse X_i, and using analogous notations as in (7) we see from (8) and (9) that every solution x of (11) satisfies

$$x = P_i x + X_i f_i \tag{13}$$

for $i = 1, \ldots, n$, and that

$$Q_i f_i = 0 \tag{14}$$

are necessary solvability conditions for (11). Vice versa, in case of (14) every solution x of (13) satisfies also (11), but since in general the conditions (14) are not sufficient for the solvability of the whole system, it can be possible that also (13) is not solvable.

Substituting x from the first equation of (13) into the second and afterwards the result into the third equation we obtain

$$x = P_2 P_1 x + P_2 X_1 f_1 + X_2 f_2,$$
$$x = P_3 P_2 P_1 x + P_3 P_2 X_1 f_1 + P_3 X_2 f_2 + X_3 f_3$$

and finally after $n - 1$ similar steps

$$x = P_n \cdots P_1 x + P_n \cdots P_2 X_1 f_1 + \cdots + P_n X_{n-1} f_{n-1} + X_n f_n. \tag{15}$$

This is an equation of type (2) with

$$T = P_n \cdots P_1,\ B = (P_n \cdots P_2 X_1 \cdots P_n X_{n-1} X_n). \tag{16}$$

Lemma 2. *In case of*

$$A_i X_j = 0 \tag{17}$$

for $1 \leqslant i < j \leqslant n$, *condition* (10) *of Lemma* 1 *is satisfied for* $k = 1$ *and we have*

$$Q = \operatorname{diag}(Q_1\ Q_2\ \dots\ Q_n). \tag{18}$$

The short proof is contained in [1], cf. also [2]. Since inner inverses are not uniquely determined, one can try to satisfy the additional conditions (17).

In the general case condition (10) means $A_i T^k = 0$, and this is equivalent to

$$P_i T^k = T^k \tag{19}$$

for $i = 1, \dots, n$, where the last equation with $i = n$ is satisfied automatically in view of (16).

Example 1. Let be $n = 2$,

$$P_1 = \begin{pmatrix} C \\ I - DC \end{pmatrix}(D\ I), \qquad P_2 = \begin{pmatrix} I \\ 0 \end{pmatrix}(I\ 0)$$

and, let us say, $A_i = X_i = I - P_i$ for $i = 1, 2$, then we have for $T = P_2 P_1$

$$T^k = \begin{pmatrix} I \\ 0 \end{pmatrix}(CD)^{k-1} C (D\ I), \qquad P_1 T^k = \begin{pmatrix} C \\ I - DC \end{pmatrix}(DC)^k (D\ I),$$

so that (19) is satisfied in case of $(CD)^{k-1} C = (CD)^k C$. The example

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \\ & & & & 0 \\ 0 & & & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & 1 \\ 0 & & & 0 \end{pmatrix},$$

$$CD = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \\ & & & \ddots & 0 \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}$$

with matrices C, D of sizes $(k-1) \times k$ and $k \times (k-1)$, respectively, shows the possibility that (19) is satisfied for a fixed k, but not for smaller k, since $(CD)^{k-2}$ possesses only one entry 1 in the position $(1, k-1)$ and $(CD)^{k-1} = 0$.

3. The limit case

If the condition (10) is not satisfied for any k, we can try to go to the limit $k \to \infty$. We assume that $\mathscr{R}$ is a limit space satisfying together with the appearing operators the conditions of Berg [3][1], then we have, following Maess [10], the following lemma.

[1] Note that the quoted paper contains the letter R in two different meanings, at the beginning as a single operator and later on after Lemma 1 as a ring of operators.

Lemma 3. *If the limits*

$$T^{\infty} = \lim_{k \to \infty} T^k, \tag{20}$$

$$Z = \lim_{k \to \infty} Y_k B \tag{21}$$

exist and if

$$AT^{\infty} = 0 \tag{22}$$

is satisfied, then (8) *with* $P = T^{\infty}$ *and* $X = Z$ *gives the generalized solution of* (1).

Now we specialize ourselves to the case (16) belonging to (12).

Proposition 1. *For orthogonal projectors* P_i, $i = 1, \ldots, n$, *into a finite dimensional Hilbert space* $\mathscr{H}$ *the conditions* (20) *and* (22) *are satisfied.*

Proof. It suffices to restrict all operators to $\mathscr{H}$. Then we have $\| P_i \| \leqslant 1$ for all i and therefore $\| T \| \leqslant 1$ as well as $\| T^k \| \leqslant 1$ for all k. The Pythagoras theorem $\| x \|^2 = \| P_i x \|^2 + \| (I - P_i) x \|^2$ shows that $\| x \| = \| P_i x \|$ if and only if $x = P_i x$. Hence $\| Tx \| = \| x \|$ implies $x = P_i x$ for all i and therefore $x = Tx$. This means that for the eigenvalues λ of T we only can have either $\lambda = 1$ or $|\lambda| < 1$. All Jordan blocks belonging to $\lambda = 1$ must be simple, because otherwise T^k cannot be bounded. This shows the convergence of (20). The equivalence of $x = Tx$ and $x = P_i x$ for $i = 1, \ldots, n$ together with $A_i P_i = 0$ shows that (22) is satisfied, too. □

Sufficient conditions for the existence of (21) you find in [3] and in [11]. Another condition can be set up according to Pyle [14], if we first substitute (15) into the equations (13) for $i = n - 1$ step -1 to 1, so that we obtain equation (2) with

$$T = P_1 \cdots P_{n-1} P_n P_{n-1} \cdots P_1 \tag{23}$$

and after denoting the old B from (16) by B_0,

$$B = P_1 \cdots P_{n-1} B_0 + (X_1 \; P_1 X_2 \; \cdots \; P_1 \; \cdots \; P_{n-2} X_{n-1} \; 0). \tag{24}$$

We need the notation of a reflexive inverse X_i, that is an inner inverse being at the same time an outer inverse of A_i, i.e.

$$X_i A_i X_i = X_i. \tag{25}$$

Concerning this case we have the following proposition.

Proposition 2. *Let* X_i *be reflexive inverses of* A_i *and let the projectors* P_i *be orthogonal into a finite dimensional Hilbert space* $\mathscr{H}$ *for* $i = 1, \ldots, n$. *Then with respect to* (23) *and* (24) *all three conditions of Lemma* 3 *are satisfied.*

Proof. Since the proofs of (20) and (22) run as before we restrict ourselves to (21). With respect to $\mathscr{H}$ we have $T^* = T$, so that T^{∞} is an orthogonal projector onto the intersection of the ranges im P_i for $i = 1, \ldots, n$. This projector is uniquely defined and does not depend on the order of the projectors P_i, so that we have

$$T^{\infty} P_i = T^{\infty}$$

for all i, but this together with $P_i X_i = 0$ implies $T^\infty B = 0$. On the other hand, denoting by B_1 the operator analogous to B in (24), only with 0 instead of X_1, we have in view of $TX_1 = 0$ for $j \geqslant 1$

$$T^j B = (T^j - T^\infty)B = (T^j - T^\infty)B_1$$

and therefore

$$\| T^j Bf \| \leqslant \| T^j - T^\infty \| \, \| B_1 f \| \leqslant Mq^j,$$

where $q < 1$ is a bound $q \geqslant |\lambda|$ for all eigenvalues λ of T different from $\lambda = 1$. This inequality proves (21) in the sense of strong convergence. □

Remarks. In the case that all A_i are matrices with linearly independent lines and X_i the corresponding right inverses with orthogonal projectors, the iteration method (4) with (16) is well known from Kaczmarz [9], Peters [13], Maess [10], and others. Though the method is convergent, it is numerically unstable, whereas the change-over to (23) and (24) causes a stabilization effect. However, in general the convergence is bad nevertheless, so that for practical purpose it is necessary to use methods, which accelerate the convergence, cf. [14].

4. Further iterations

If the limits (20) and (21) exist, but if (22) is not satisfied, then the limit equation

$$x = T^\infty x + Zf \tag{26}$$

of (4) with (5) and (16) is not the generalized solution of (1) with (12) for every x on the right-hand side. In this case we consider the differences of (3) with (26), i.e. the equations

$$(T^\infty - P_i)x = X_i f_i - Zf \tag{27}$$

for $i = 1, \dots, n-1$, and we have a proposition.

Proposition 3. *If the solvability conditions* (14) *are satisfied and if x is the general solution of the $n-1$ equations* (27), *then, after substituting this solution into the right-hand side of* (26), *the left-hand side is the general solution of* (11).

Proof. From the foregoing considerations it is clear that every solution of (13) solves also (26) and (27). Now let x be a solution of (27). Then (26) goes over into (13) and in view of (14) moreover into (11) for $i = 1, \dots, n-1$. But according to $A_n P_n = 0$ we have $A_n T = A_n T^\infty = 0$ and

$$A_n Z = A_n B = (0 \ \cdots \ 0 \ A_n X_n),$$

so that from (26) and (14) we obtain $A_n x = A_n X_n f_n = f_n$, i.e. (11) also for $i = n$. □

Let us mention that similar as before the necessary solvability conditions (14) for (11) do not guarantee the solvability of the system (27).

Now we can repeat the whole iteration procedure with respect to the $n-1$ equations (27) instead of the original n equations (11) and so on, until we obtain the solution x or one single equation, for which as in the foregoing cases we assume to have an inner inverse and therefore a generalized solution.

Of course, in practice nobody would proceed in this way. But the following example shows that the assumptions we imposed upon equation (26) are quite well realistic.

Example 2. Let be

$$A_i A_j = A_j$$

for $i,\ j = 1, \dots, n$ as in the case

$$A_i = \begin{pmatrix} I & C_i \\ 0 & 0 \end{pmatrix}.$$

Then $X_i = A_i$ are inner inverses of A_i with $P_i = Q_i = I - A_i$, and we have $P_i A_j = 0$ for all $i,\ j$ as well as

$$P_i P_j = P_i.$$

From (16) we find that $T^{\infty} = T = P_n$ and $Z = B = (0 \ \cdots \ 0\ \mathrm{X}_n)$, so that (26) is nothing else than (13) with $i = n$.

Remarks. By the way, Example 2 shows that the iteration method in single steps

$$x_{nk+i} = P_i x_{nk+i-1} + X_i f_i$$

for $i = 1, \dots, n-1$ and $k = 0,1, \dots$ can be divergent in spite of the convergence of x_{nk}.

On the other side we see that for Example 2 the system (27) can be written in the form

$$(P_n - P_i)x = X_i f_i - X_n f_n \tag{28}$$

for $i = 1, \dots, n-1$. It is easily to see further that also in general Proposition 3 can be transferred to (28) and (13) with $i = n$ instead of (26), but (27) contains more information than (28).

5. Matching rules

In the asymptotic theory of singular perturbations matching rules are of great importance, cf. [6]. An abstract version of matching rules was set up by Felgenhauer [8], using the following notion: A projector P is called a composite projector of the projectors P_i, $i = 1, \dots, n$, if

$$\ker P = \bigcap_{i=1}^{n} \ker P_i, \tag{29}$$

where ker A in general means the null space of the linear operator A. If P is the composite projector of the projectors P_i, then the complementary projector $\overline{P} = I - P$ is the intersection projector of the complementary projectors $\overline{P}_i = I - P_i$ in the sense of Pyle [14] and vice versa. This means that $\overline{P}$ is a projector onto (29), where you have to note that ker P equals to the range im $\overline{P}$ of the complementary projector $\overline{P}$. The following result was shown in [4] for the case $n = 2$, but it is also valid in general.

Proposition 4. *If $A_1, \dots, A_n$ are projectors and if X is an inner inverse of A from* (12), *then XA is a corresponding composite projector.*

Proof. With the notation

$$X=(U_1\ U_2\ \dots\ U_n) \tag{30}$$

we have

$$XA=\sum_{i=1}^{n} U_iA_i.$$

This equation at once implies that the intersection of all ker A_i is contained in ker XA. But the opposite conclusion follows from $AXA=A$ and therefore $A_jXA=A_j$ for all j, what proves the statement, cf. [8]. □

The last equations of the proof can be written in the form

$$A_j=\sum_{i=1}^{n} A_jU_iA_i \tag{31}$$

for $j=1,\dots,n$, and these equations are the already mentioned matching rules. They have the following meaning.

Lemma 4. *The necessary and sufficient solvability conditions for the system* (11) *read*

$$f_j=\sum_{i=1}^{n} A_jU_if_i$$

for $j=1,\dots,n$.

Proof. The necessity follows easily from (11) and (31). The sufficiency follows from (9) with (7), i.e. from $f=AXf$ with (12), in view of (30) and therefore

$$Xf=\sum_{i=1}^{n} U_if_i.$$

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