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A NON-INVOLUTORY SELFDUALITY

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Two polyhedra P_1 and P_2 are said to be duals of each other provided there exists a bijection δ from the family of vertices and faces of P_1 to the family of vertices and faces of P_2 which reverses inclusion. If there exists a duality map δ from a polyhedron P to itself, we say that P is selfdual (see, for example, [1–5]). In this case the selfduality map is a permutation on the set of vertices and faces of P. The rank $r(\delta)$ of a selfduality map δ is defined as the smallest positive integer n such that δ^n is the identity. The rank $r(P)$ of a selfdual polyhedron P is the minimum value of $r(\delta)$ over all selfduality maps δ of P (see [2]).

Grünbaum and Shephard have asked (in [2, Problem 1]) whether every selfdual convex polyhedron P has rank 2 or, equivalently, whether every selfdual P admits an involutory selfduality map.

In this note we give an example of a polyhedron which provides the negative answer to the above problem. We assert:

Theorem. *There exists a selfdual convex polyhedron P with rank $r(P) = 4$.*

Proof. A graph of such a polyhedron P is given by the Schlegel diagram shown in Fig. 1. A selfduality map of P is given by the permutation $\delta = (AaBb)(CcDd)(EeFf)(GgHh)(JjKk)(LlMm)(NnOo)$, and clearly $r(\delta) = 4$. To prove that $r(P) = 4$ we shall show that the assumption that P admits a selfduality ρ with $r(\rho) = 2$ leads to a contradiction. Indeed such a permutation ρ would have to be the product of 14 pairs, each consisting of a vertex and a face. It is easy to see that the vertex A could occur only in either (Aa) or (Ab).

In the former case, by considering that the permutation ρ maps the set of faces $\{f, g, j\}$ to the set of vertices $\{E, H, K\}$ we see that, because of the valences of the vertices and faces adjacent to these, the pair (Ef) has to occur in ρ. Then, by similar arguments, the pair (Dc) is forced to appear, and after that the pair (Ab)—which contradicts the assumption. On the other hand, if (Ab) is assumed to occur, the same reasoning in reverse order leads first to (Dc), then to (Ef) and finally to (Aa)—hence again to a contradiction.

This completes the proof of the theorem. □

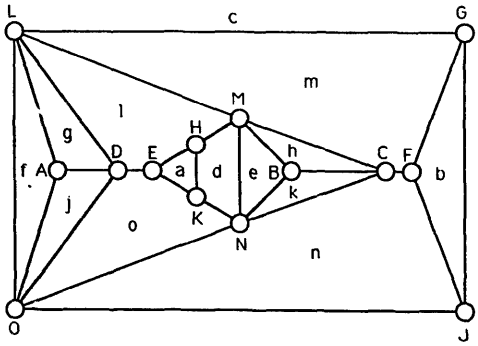

Fig. 1.

Remarks. 1. The graph in Fig. 1 can be realized by a polyhedron which has a plane of reflective symmetry (containing the vertices A, B, C, D, E, F) as well as an axis of two-fold rotational symmetry (bisecting the edges LM and NO).

2. As mentioned in [2], if a polyhedron P has a selfduality δ such that $r(\delta) = 2^i k$ where k is odd, then $r(P) \leq 2^i$. It is not known whether for every $n = 2^i$ there exists selfdual polyhedron P with $r(P) = n$.

3. For other properties of selfdual convex polyhedra see [1, 3, 4, 5].

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