Hypergraphical \( t \)-designs

Ngo Dac Tuan

Département de Mathématiques, CNRS-Université de Paris Nord, 99 avenue Jean-Baptiste Clement, 93430 Villetaneuse, France

Received 11 September 2002; received in revised form 8 May 2004; accepted 24 October 2005

Available online 5 April 2006

Abstract

We introduce a theory of hypergraphical \( t \)-designs. We show the existence of these designs and prove a finiteness theorem on these designs for infinitely many parameter sets. We also give effective bounds on the number of points in these cases. These results generalize some results on graphical \( t \)-designs of Alltop, Chee and Betten–Klin–Laue–Wassermann.

© 2006 Elsevier B.V. All rights reserved.

Keywords: Block designs; Graphical designs; Hypergraphical designs

0. Introduction

The paper describes a new type of designs, called hypergraphical \( t \)-designs, which is a generalization of graphical designs which have been studied before by Alltop, Chee and Betten–Klin–Laue–Wassermann.

For positive integers \( t, v, k \) and \( \lambda \) satisfying \( t < k < v \), a \( t \)-design \( D \) with parameter \((v, k, \lambda)\) is a set \( X \) of \( v \) points, together with a collection \( B \) of subsets of \( X \), called blocks, such that each block contains exactly \( k \) points and each set of \( t \) points is contained in exactly \( \lambda \) blocks. A \( t - (v, k, \lambda) \) design is simple if repeated blocks are not allowed and is non-trivial if not all \( k \)-subsets of points are blocks. In this paper, all the designs are supposed to be simple and non-trivial.

A graphical design is a design whose points are the edges of a complete graph with \( n \) vertices, and whose blocks are full orbits of \( S_n \) on graphs, such that the design condition is satisfied. As the block set consists of full orbits of \( S_n \), it suffices to list the unlabelled graphs representing those block orbits in order to describe the design. The presence of a lot of symmetry makes these structures nice and interesting. Some graphical designs have been constructed in the literature, see [5, 7, 8, 11–13]. For general results on graphical designs, see [2, 9, 10].

A hypergraphical design is a natural generalization of this concept to hypergraphs. Let \( X \) be a set of \( n \) elements. Let \( X(m) \) denote the set of \( m \)-subsets of \( X \). Then the natural action of the symmetric group induces an action on \( X^{(m)} \).

We consider orbits of \( k \)-subsets of \( X^{(m)} \) under the action of \( S_n \). If a collection of these orbits forms a (simple) \( t \)-design (with point set \( X^{(m)} \)) then this design is called \( m \)-hypergraphical design on \( n \) points. Any block of such a design is a collection of \( km \)-subsets of \( X \). Obviously, this is a \( t - \left( \binom{n}{m} , k, \lambda \right) \) design for some \( \lambda \).

We will prove three fundamental results on hypergraphical designs.

(i) These designs exist (see Theorem 2.2).
(ii) Fix two integers $m \geq 2$ and $t \geq 2$, there are only finitely many non-trivial $m$-hypergraphical $t - \left( \binom{m+2}{2}, m + 3, \lambda \right)$-designs under a certain restriction between $k$ and $t$ (see Theorem 3.4).

(iii) With the notations above, effective bounds of $n$ are given (see Theorem 3.5).

The first result is simple and relies on the idea of Alltop [1]. The second result is a natural generalization of a result of Chee [10] for graphical designs. Only the third part may contain some new ideas.

1. Basic definitions

For a finite set $X$, the set of $m$-subsets of $X$ is denoted by $X^{(m)}$, i.e., $X^{(m)} = \{ Y \mid Y \subseteq X, |Y| = m \}$. A $m$-uniform hypergraph $\mathcal{H}$ is an ordered pair $(V, E)$ such that $V \neq \emptyset$ and $E \subseteq V^{(m)}$. The set $V$ is the set of vertices of $\mathcal{H}$ and the set $E$ is the set of edges of $\mathcal{H}$.

Let $\mathcal{H} = (V, E)$ be a $m$-uniform hypergraph. Another $m$-uniform hypergraph $\mathcal{H}' = (V', E')$ is said to be subgraph of $\mathcal{H}$ if $V' \subseteq V$ and $E' \subseteq E$. Two $m$-uniform hypergraphs $\mathcal{H} = (V, E)$ and $\mathcal{H}' = (V', E')$ are isomorphic if there exists a one-one correspondence $\sigma$ between the vertex sets $V$ and $V'$ and the image $\sigma(E)$ of the edge set $E$ is exactly $E'$. A $m$-uniform hypergraph $\mathcal{H} = (V, E)$ is the union of two $m$-uniform hypergraphs $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ if

$$V = V_1 \cup V_2,$$

$$E = E_1 \cup E_2,$$

and we write $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$.

For a positive integer $n$, we denote by $I(n) = (X, E)$ the following $m$-uniform hypergraph. We take

$$X = \{1, 2, \ldots, mn\},$$

$$E = \{km + 1, km + 2, \ldots, km + m \mid 0 \leq k \leq n - 1\} \subseteq X^{(m)}.$$

Let $\mathcal{H} = (V, E)$ be an $m$-uniform hypergraph. We say that the vertex $x \in V$ and the edge $Y \in E$ are incident if $x \in Y$. Two edges $Y_1$ and $Y_2$ are adjacent if $Y_1 \cap Y_2 \neq \emptyset$ and are independent if $Y_1 \cap Y_2 = \emptyset$. An edge $Y$ of $\mathcal{H}$ is independent if for every edge $Y' \neq Y$ and $Y'$ are independent. It is easy to see that $\mathcal{H}$ contains $n$ independent edges if and only if $\mathcal{H}$ is the union of two $m$-uniform hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_2$ is isomorphic to $I(n)$.

2. Existence of $m$-hypergraphical designs for all $m \geq 2$

First, we prove the following result for graphical designs:

**Proposition 2.1.** For all $m \geq 2$, there exists a graphical $2 - \left( \binom{m+2}{2}, m + 3, \lambda \right)$ design for some $\lambda$.

**Proof.** Let $X$ be a set of $m + 2$ points. Let $T$ be the set of $2$-subsets of $X^{(2)}$, i.e., the set of pairs of pairs of elements of $X$. The action of $S_n$ on $T$ decomposes $T$ into two orbits $T_1$ and $T_2$. The elements in $T_1$ (respectively, $T_2$) are isomorphic to $\{\{a, b\}, \{a, c\}\}$ (respectively, $\{\{a, b\}, \{c, d\}\}$) where $a, b, c, d$ are four distinct points of $X$.

Let $B$ be an $(m + 3)$-subset of $X^{(2)}$. Let $u_i$ $(i = 1, 2)$ denote the number of members of $T_i$ contained in $B$. Alltop [1] showed that the single $S_n$-orbit containing $B$ forms a graphical $2 - \left( \binom{m+2}{2}, m + 3, \lambda \right)$ design if and only if

$$4u_2/u_1 = m - 1.$$  

(1)
Depending on the residue class of $m$ modulo 6, we will now explicit an $(m + 3)$-subset $B$ of $X^{(2)}$ with $u_1 = 2m + 4$ and $u_2 = (m - 1)(m + 2)/2$. We distinguish six cases

1. $m = 6l - 3$ ($l \geq 1$).

2. $m = 6l - 2$ ($l \geq 1$).

3. $m = 6l - 1$ ($l \geq 1$).

4. $m = 6l$ ($l \geq 1$).

5. $m = 6l + 1$ ($l \geq 1$).

6. $m = 6l + 2$ ($l \geq 0$).

Counting $u_1$ and $u_2$ in each of the six cases shows that $B$ satisfies condition (1) and hence defines a graphical $2 - \left( \binom{m+2}{2}, m + 3, \lambda \right)$-design. This finishes the proof. □

**Theorem 2.2.** For every $m \geq 2$, $m$-hypergraphical designs exist.

**Proof.** Let $X$ be a set of $m + 2$ points. By taking the complement on $X$, we establish a canonical bijection between $X^{(2)}$ and $X^{(m)}$. Further, this bijection is compatible with the action of $S_{m+2}$. Thus a graphical $t - \left( \binom{m+2}{2}, k, \lambda \right)$ design
induces an \( m \)-hypergraphical \( t - \left( \binom{m+2}{m}, k, \lambda \right) \) design. Hence Theorem 2.2 follows easily from Proposition 2.1 above. □

In recent years, many new \( t \)-designs have been constructed prescribing groups of automorphisms with the help of DISCRETA, a program written by Betten et al. [4], see for example [3,2]. In [16], the author considers the question whether, for a given couple \((t, G)\) where \( t \) is a positive integer and \( G \) is a finite group, there exists a simple non-trivial \( t \)-design having \( G \) as a group of automorphisms. More information on this problem can be found in [15,16].

Since every finite group \( G \) is a subgroup of a symmetric group \( S_m \) for some large \( m \), Proposition 2.1 implies

**Corollary 2.3.** For every finite group \( G \), there exists a simple non-trivial \( 2 \)-design having \( G \) as a group of automorphisms.

**Remarks.**

(i) It remains an open problem to decide whether for every finite group \( G \) there exists a simple non-trivial \( 2 \)-design whose automorphism group is exactly \( G \).

(ii) For all \( k \), Alltop [1] constructed a graphical \( 2 - \left( \binom{2k-3}{2}, k, \lambda \right) \) design where the block set \( B \) is the set of edges of a cycle of length \( k \). This gives us another construction of \( m \)-hypergraphical designs when \( m \) is odd.

### 3. Some results on \( m \)-hypergraphical \( t \)-designs

Let \( \mathcal{D} \) be an \( m \)-hypergraphical \( t - \left( \binom{n}{m}, k, \lambda \right) \) design on a set \( X \) of \( n \) points. Any subset \( E \) of \( X^{(m)} \) can be seen as the set of edges of the associated \( m \)-uniform hypergraph \( \mathcal{H} = (X, E) \). Let \( T \) be a \( t \)-subset of \( X^{(m)} \) and \( B \) be a \( k \)-subset of \( X^{(m)} \). Assume that \( T \) is contained in containing \( B \). Then for any element \( g \) in \( S_n \), the set \( T_g \) is also contained in exactly \( m(T, B^{S_n}) \) elements of \( B^{S_n} \). Therefore it suffices need to test the design condition for a set of representatives of \( t \)-subsets of \( X^{(m)} \) under the action of \( S_n \). Kramer and Mesner [14] proved

**Theorem 3.1 (Kramer–Mesner [14]).** An \( m \)-hypergraphical \( t - \left( \binom{n}{m}, k, \lambda \right) \) design exists if and only if there is a \( \{0, 1\} \)-solution vector \( u \) to the diophantine system of equations

\[
\sum_j m(T_i, B_j^{S_n})u_j = \lambda,
\]

where the \( T_i \) and \( B_j \) run through a system of representatives of the \( t \)-subsets and \( k \)-subsets of \( X^{(m)} \) under the action of \( S_n \) and where \( u_j \) denotes the \( j \)th component of the vector \( u \).

**Remark.** Note that \( u_j = 1 \) if and only if \( B_j \) (hence every element \( B \) in \( B_j^{S_n} \)) is a block of \( \mathcal{D} \).

For a subset \( B \) of \( X^{(m)} \), we define the support of \( B \) by

\[
\text{supp}(B) = \{x \in X | \exists Y \in B \text{ such that } x \in Y\}.
\]

Alltop [1] proved:

**Lemma 3.2 (Alltop [1]).** The term \( m(T, B^{S_n}) \) is a polynomial in \( n \) whose degree is the difference between the size of the support of \( B \) and the support of \( T \).

For graphical designs, Chee [10] proved:

**Theorem 3.3 (Chee [10]).** Let \( t \) be a natural number greater than 2. Then there exist only finitely many non-trivial graphical \( t - \left( \binom{n}{2}, k, \lambda \right) \) designs when \( k \leq 4t/3 \).

Now we will generalize this result to hypergraphical designs.
\textbf{Theorem 3.4.} Let \( t \) and \( m \) be natural numbers greater than 2. Then there exist only finitely many non-trivial \( m \)-hypergraphical \( t - \left( \begin{array}{c} n \\ m \end{array} \right), k, \lambda \) designs with \( k \leq (m + 2)/(m + 1)t \).

\textbf{Proof.} We will follow closely the idea used by Chee [10]. We observe that it is sufficient to prove that for each natural number \( k \) such that \( t < k \leq (m + 2)/(m + 1)t \), if \( n \) is sufficiently large then, every \( m \)-hypergraphical \( t - \left( \begin{array}{c} n \\ m \end{array} \right), k, \lambda \) design is trivial.

Let \( k \) be a natural number with \( t < k \leq (m + 2)/(m + 1)t \). Let \( B \) be a \( k \)-subset of \( X^{(m)} \) and \( T \) be a \( t \)-subset of \( X^{(m)} \). By definition, if \( m(T, B^{Sn}) \neq 0 \), then there exists an element \( g \in S_n \) such that \( T \subseteq B^{g} \). Remark that

\[ |\text{supp}(B)| = |\text{supp}(B^g)|. \]

Hence Lemma 3.2 implies that \( m(T, B^{Sn}) \) is a polynomial in \( n \) whose degree is

\[ |\text{supp}(B^g)| - |\text{supp}(T)| \leq |\text{supp}(B^g \setminus T)| \leq m(k - t) \]

because \( B^g \setminus T \) is a \((k - t)\)-subset of \( X^{(m)} \). Equality occurs if and only if in the \( m \)-uniform hypergraph \( \mathcal{H} = (X, B^g) \), the edges in \( B^g \setminus T \) are independent.

Let \( T_i \) and \( B_j \) be a system of representatives of the \( t \)-subsets and the \( k \)-subsets of \( X^{(m)} \) under the action of \( S_n \). We denote by \( \mathcal{H} \) the set of all \( m \)-uniform hypergraphs \( \mathcal{H} = (X, B) \) such that

(i) \( B \) is a \( k \)-subset of \( X^{(m)} \),
(ii) \( \mathcal{H} \) contains a sub-\( m \)-uniform hypergraph isomorphic to \( I(t) \).

Let \( \mathcal{D} \) be an \( m \)-hypergraphical \( t - \left( \begin{array}{c} n \\ m \end{array} \right), k, \lambda \) design. If \( I(k) \) is not a block of \( \mathcal{D} \), then we will consider the complement design \( \mathcal{D}' \) of \( \mathcal{D} \) instead of \( \mathcal{D} \). This design is also an \( m \)-hypergraphical \( t - \left( \begin{array}{c} n \\ m \end{array} \right), k, \lambda \) design and \( I(k) \) is a block of \( \mathcal{D}' \).

So we can suppose that \( I(k) \) is a block of \( \mathcal{D} \). Then

\[ \lambda \geq m(I(t), I(k)^{S_n}) \]

which is a polynomial of \( n \) of degree \( m(k - t) \).

For a \( k \)-subset \( B \) of \( X^{(m)} \), we denote by \( \mathcal{H}_B = (X, B) \) the associated \( m \)-uniform hypergraph to \( B \). For every \( k \)-subset \( B \) of \( X^{(m)} \) such that \( \mathcal{H}_B \in \mathcal{H} \), \( \mathcal{H}_B \) contains at least \( t - (k - t)m \geq t/(m + 1) \geq k - t \) independent edges. Hence the entries \( m(T_i, B^{Sn}) \) are polynomials of \( n \) whose degree are less than \( m(k - t) \). Further, there is exactly one entry \( m(T_i, B^{Sn}) \) which is a polynomial of \( n \) whose degree is \( m(k - t) \): we remove \( k - t \) independent edges from \( \mathcal{H}_B \) to obtain a new \( m \)-uniform hypergraph \( (X, T(B)) \) with \( t \) edges and \( T_i \) represents the \( S_n \)-orbit of \( T(B) \).

Suppose that \( B \in B^{Sn}_i \). By the discussion above, we deduce that

(i) \( \sum_{j \neq i} m(T_i, B^{Sn}_j)u_j \) is a polynomial of \( n \) of degree strictly less than \( m(k - t) \),
(ii) \( m(T_i, B^{Sn}_i) \) is a polynomial of \( n \) of degree \( m(k - t) \).

For \( n \) sufficiently large, the relation

\[ \sum_{j} m(T_i, B^{Sn}_j)u_j = \lambda \]

forces that \( u_i = 1 \), i.e. \( B_i \) is a block of \( \mathcal{D} \) and hence \( B \) is also a block of \( \mathcal{D} \). Thus, all elements in \( \mathcal{H} \) are blocks. Therefore, our design is trivial. The proof is complete. \( \square \)

In Theorem 3.4 above, we have proved the following fact. For fixed \( m, t \) and \( k \) with \( t < k \leq (m + 2)/(m + 1)t \), if there exists a non-trivial \( m \)-hypergraphical \( t - \left( \begin{array}{c} n \\ m \end{array} \right), k, \lambda \) design, then \( n \) is bounded from above. However, we do not know how big the value \( n \) can be. The following result makes this bound more precise.

\textbf{Theorem 3.5.} Let \( m \) and \( t \) be natural integers greater than 2. Let \( l \) be a positive integer such that \( l \leq t/(m + 1) \). Suppose that \( \mathcal{D} \) is a non-trivial \( m \)-hypergraphical \( t - \left( \begin{array}{c} n \\ m \end{array} \right), t + l, \lambda \) design. Then there exist positive real constants \( z, \gamma, C_1 \).
and \( C_2 \) such that
\[
x \sqrt{\gamma} + C_1 \leq n \leq (\gamma + x) \sqrt{\gamma} + C_2,
\]
here the constants \( x, \gamma, C_1 \) and \( C_2 \) depend only on \( m \) and \( l \).

**Proof.** We set
\[
x = \sqrt{\frac{m}{m!}}.
\]

Consider the function
\[
f(x) = \frac{(x + x)^m - m!}{x^m} = \sum_{j=0}^{m-1} \binom{m}{j} \left( \frac{x}{x} \right)^j.
\]

For \( x > 0 \), this function is decreasing and tends to 1 as \( x \) tends to \( \infty \). Thus, we denote by \( \gamma \) the unique positive real number such that \( f(\gamma) = \frac{1}{\sqrt{2}} \). Then we have
\[
\sqrt{2} \gamma^m = (\gamma + x)^m - m!.
\]

By definition of a design, we have
\[
\left( \frac{n}{m} \right) > k = t + l.
\]

This implies that there is a positive real constant \( C_1 \) which depends on \( m \) and \( l \) such that
\[
n \geq x \sqrt{\gamma} + C_1.
\]

Suppose that \( \mathcal{D} \) is a non-trivial \( m \)-hypergraphical \( t - (\left( \frac{n}{m} \right), t + l, \lambda) \) design on a set \( X \) of \( n \) points. We consider the set of all \( m \)-uniform hypergraphs \( \mathcal{H} = (X, T) \) such that

(i) \( T \) is a \( t \)-subset of \( X^{(m)} \).
(ii) \( \mathcal{H} \) contains \( l(m + 1) \) independent edges.

We set
\[
p = \min_{\mathcal{H} = (X,T) \in \mathcal{F}} |\text{supp}(T)|
\]
and we take an element \( \mathcal{H}_0 = (X, T_0) \in \mathcal{F} \) such that
\[
p = |\text{supp}(T_0)|.
\]

Then we have inequalities
\[
l(m + 1) + \left( \frac{p - m(m + 1)l - 1}{m} \right) < t \leq l(m + 1) + \left( \frac{p - m(m + 1)l}{m} \right).
\]

Thus, there exists a natural integer \( \beta \) which depends on \( m \) and \( l \) such that
\[
p \leq x \sqrt{\gamma} + \beta.
\]

For each \( m \)-uniform hypergraph \( \mathcal{H} = (X, T) \) of \( t \) edges, we denote by \( \mathcal{H}' = (X, B(T)) \) the \( m \)-uniform hypergraph (up to the isomorphisms induced by \( S_n \)) of \( k = t + l \) edges obtained by adding \( l \) independent edges. It is not difficult to see that
\[
m(T, B(T)_{S_n}) = \frac{(n - q)\ldots(n - q - lm + 1)}{l!m!}.
\]

where \( q = |\text{supp}(T)| \).
Up to isomorphism induced by $S_n$, we denote by $\mathcal{H}$ the set of all $m$-uniform hypergraphs $\mathcal{H} = (X, B)$ such that

(i) $B$ is a $k$-subset of $X^{(m)}$,
(ii) $\mathcal{H}$ contains $\mathcal{H}_0$ as a sub-$m$-uniform hypergraph, i.e. $T_0 \subseteq B$.

For every $k$-subset $B$ of $X^{(m)}$ such that $\mathcal{H}_B = (X, B) \in \mathcal{H}$, $\mathcal{H}_B$ contains at least $l(m + 1) - (k-t)m = l$ independent edges. We denote by $(X, T(B))$ the $m$-uniform hypergraph (up to the isomorphisms induced by $S_n$) of $t$ edges obtained from $\mathcal{H}_B$ by removing $l$ independent edges.

Without loss of generality, we suppose that $B(T_0)$ is a block of $D$. Then

$$\lambda \geq m(T_0, B(T_0)^{S_n}) = \frac{(n-p)(n-p-1)\ldots(n-p-lm)}{l!m^t}.$$

Since the design $D$ is non-trivial, there exists an $m$-uniform hypergraph $(X, B)$ in $\mathcal{H}$ such that $B$ is not a block. We set $q = |\text{supp}(T(B))|$. Hence

$$p - lm \leq q \leq p.$$

Since $B$ is not a block, this implies that

$$\lambda \leq \sum_{B_j \neq B} m(T(B), B_j^{S_n})$$

$$\leq \sum m(T(B), B_j^{S_n}) - m(T(B), B^{S_n})$$

$$= \left( \binom{n}{m} - t \right) - \frac{(n-q)(n-q-1)\ldots(n-q-lm)}{l!m^t}.$$

Therefore, we have

$$\frac{(n-p)(n-p-1)\ldots(n-p-lm)}{l!m^t} \leq \lambda \leq \left( \binom{n}{m} - t \right) - \frac{(n-q)(n-q-1)\ldots(n-q-lm)}{l!m^t}.$$

Thus

$$\frac{(n-p)(n-p-1)\ldots(n-p-lm)}{l!m^t} + \frac{(n-q)(n-q-1)\ldots(n-q-lm)}{l!m^t} \leq \left( \binom{n}{m} - t \right).$$

On one hand, since $q \leq p$ and $p \leq \sqrt[t]{\alpha} + \beta$, the left term is bounded below by

$$\text{LT} \geq \frac{(n-p)(n-p-1)\ldots(n-p-lm)}{l!m^t} + \frac{(n-p)(n-p-1)\ldots(n-p-lm)}{l!m^t}$$

$$\geq \frac{2(n-p-lm)}{l!m^t}^t$$

$$\geq \frac{2(n-\sqrt[t]{\alpha} - \beta - lm)}{l!m^t}.$$

On the other hand, we have an upper bound for the right term

$$\text{RT} \leq \frac{\binom{n^m - mt}{l}}{l!m^t}.$$

Hence, a simple calculation implies that

$$\sqrt[t]{\lambda (n-\sqrt[t]{\alpha} - \beta - lm)} \leq n^m - m^t.$$
Remark that for all $1 \leq j \leq m$, we have
$$\sqrt{2} \gamma^{m-j} > (\gamma + x)^{m-j}.$$  
In fact, since $x^m = m!$, this implies that $(\gamma + x)^{m-j} [ (\gamma + x)^{j} - \gamma^j ] > m!$. Thus $\sqrt{2} \gamma^m = (\gamma + x)^m - m! > (\gamma + x)^{m-j} \gamma^j$. Hence the desired inequality.

Now choose a positive constant $C_2$ which depends on $m$ and $l$ such that for all $1 \leq j \leq m$, we have
$$\sqrt{2} (C_2 - \beta - lm + 1) \gamma^{m-j} - C_2 \gamma^{m-j-j} > 0.$$  
Then
$$n \leq (\gamma + x)^{\sqrt{2} + C_2}.$$  
Suppose that it is not the case, i.e. we can write
$$n = (\gamma + x)^{\sqrt{2} + y}$$  
with $y > C_2$. Then
$$0 \geq \sqrt{2} (n - x)^{\sqrt{2} + \beta - lm + 1)^m - (n^m - m!t)$$  
$$= \sqrt{2} (\gamma^{\sqrt{2} + y} - \beta - lm + 1)^m - [(\gamma + x)^{\sqrt{2} + y} - m!t]$$  
$$= \sum_{j=1}^{m} \binom{m}{j} (\sqrt{2} (\gamma^{\sqrt{2} + y} - \beta - lm + 1)^j \gamma^{m-j} - y^j (\gamma + x)^{m-j})^{\sqrt{2} \sqrt{m-j}}$$  
$$\geq \sum_{j=1}^{m} \binom{m}{j} (\sqrt{2} (C_2 - \beta - lm + 1)^j \gamma^{m-j} - C_2 \gamma^{m-j})^{\sqrt{2} \sqrt{m-j}}$$  
$$> 0,$$
which is a contradiction. Therefore, we must have
$$n \leq (\gamma + x)^{\sqrt{2} + C_2}.$$  
This finishes the proof. \□

In a particular case where $m = 2, l = 1$, a calculation gives us the following corollary.

**Corollary 3.6.** Suppose that $\mathcal{D}$ is a non-trivial graphical $t - \left( \binom{n}{2}, t + 1, \lambda \right)$ design with $t \geq 3$. Then we have
$$\sqrt{2t} + \frac{1}{2} \leq n \leq 3 \sqrt{2t} + \frac{45}{2}.$$  

In [2], Betten et al. conjectured that there is no graphical $t - \left( \binom{n}{2}, t + 1, \lambda \right)$ design for $t \geq 4$. They proved this conjecture when $t = 5$. When $t = 4$, the conjecture was proved by Chee [9]. For small values of $t$, Corollary 3.6 may allow us to check the conjecture directly by considering all possible values of $n$.

**Acknowledgments**

I am pleased to acknowledge the anonymous referees for numerous helpful remarks and improvements.

**References**

[4] A. Betten, R. Laue, A. Wassermann, DISCRETA, a Program System for the Construction of $t$-Designs with a Prescribed Automorphism Group, University of Bayreuth, ⟨http://www.mathe2.uni-bayreuth.de/discreta/⟩.


