General Topology and its Applications 1 (1971) 57–63. North-Holland Publishing Company

A SURVEY OF THE THEORY OF σ-SPACES

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Received 14 July 1970

AMS Subj. Class.: 5435

metric spaces metrizability

1. In general topology a metric space is one of the most fundamental and important spaces, and as its topological characterization, the Nagata-Smirnov's metrization theorem plays an important role. Their theorem is as follows:

1.1. Theorem (Nagata [16], Smirnov [20]). In order that a regular T_1 space X be metrizable it is necessary and sufficient that X has a σ-locally finite base; that is, X has a base which is expressed as the countable union of locally finite subcollections.

Hereafter, we assume that all spaces are regular T_1 spaces unless specified.

In view of the point that we consider generalizations of metric spaces, there may be several directions and as one of them it seems natural to weaken the conception of a base dropping the openness of its members.

1.2. Definition. Let $\mathfrak{B}$ be a collection of subsets (not necessary open) of a Hausdorff space X. If, for each point $x \in X$ and open subset G of X with $x \in G$, there exists $B \in \mathfrak{B}$ such that $x \in B \subset G$, then we call $\mathfrak{B}$ a *network* of X (cf. [1]). In particular, when a network $\mathfrak{B}$ is σ-locally finite as a collection of subsets of X, it is called a *σ-locally finite network* and then X is called a *σ-space* (cf. [14,15,17]). More specially, when a network $\mathfrak{B}$ is a countable collection, X is called a *cosmic space* (cf. [1, 2,3,10]).

From the above definition we can see that all metric spaces, more generally, all spaces which are expressed as the countable sum of closed metrizable subspaces, and all cosmic spaces are σ-spaces.

A σ-space was also characterized by the existence of networks of the following forms.

1.3. **Theorem** (Siwiec-Nagata [19]). The following are equivalent for a space X:

(i) X is a σ-space;

(ii) X has a σ-discrete network;

(iii) X has a σ-closure-preserving network.

2. The class of σ-spaces, as a generalization of metric spaces, has some properties similar to metric spaces.

I (Hereditary). Any subspace of a σ-space is also a σ-space (cf. [17]).

II (Product). A product of countably many σ-spaces is a σ-space (cf. [17]).

III. In a collectionwise normal σ-space X the following conditions are equivalent (cf. [17]):

(i) X is separable;

(ii) X is a Lindelöf space;

(iii) X satisfies the countable chain condition; that is, X does not contain uncountably many disjoint, non-empty subsets.

IV. For any normal σ-space X we can consider its completion; that is, there exists a (Hausdorff) σ-space Y which contains X as a dense subspace and is complete in the following sense.

2.1. **Definition.** Let Y be a Hausdorff σ-space which has a σ-locally finite network $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ such that each $\mathcal{B}_n = \{B(\alpha_1, \ldots, \alpha_n) | \alpha_1 \in A_1, \ldots, \alpha_n \in A_n\}$ is locally finite closed covering of Y which is closed under finite intersections and $\mathcal{B}$ satisfies $B(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \subset B(\alpha_1, \ldots \alpha_n)$ for $\alpha_1 \in A_1, \ldots, \alpha_n \in A_n, \alpha_{n+1} \in A_{n+1}$. We call the subcollection $\mathcal{F}$ of $\mathcal{B}$ a *σ-Cauchy filter with respect to* $\mathcal{B}$ if $\mathcal{F}$ is a filter and there is a sequence $(\alpha_1, \alpha_2, \ldots)$ such that $B(\alpha_1, \ldots, \alpha_n) \in \mathcal{F}$ for each n and such that for each n there exists a n_k such that $B(\alpha_1, \ldots, \alpha_{n_k})$ intersects with only finite members of $\mathcal{B}_n$. If any maximal σ-Cauchy filter with respect to $\mathcal{B}$ has non-empty intersection, we call Y *σ-complete with respect to* $\mathcal{B}$.

3. On the other hand, σ-spaces satisfy some properties which are not satisfied in the case of metric spaces.

V. If a space is a countable union of closed subspaces, each of which is a σ-space, it is also a σ-space.

VI. Let X be a space and $\{F_\alpha | \alpha \in A\}$ a closed covering of X. If X is dominated by $\{F_\alpha | \alpha \in A\}$ (in other words, X has the weak topology with respect to $\{F_\alpha | \alpha \in A\}$ in the sense of K. Morita [12]) and if each F_α is a normal σ-space, then X is a σ-space.

Since the proof of this fact is not published, we shall show it here.

3.1. **Proof** of VI. Let $\{F_\alpha | \alpha \in A\}$ be the given covering of X which satisfies the condition in VI, where we can assume that A is a well-ordered set.

Since F_α is a normal σ-space, F_α is perfectly normal, for each $\alpha \in A$. Therefore, X is also perfectly normal (cf. [12]).

Let us put $P_\alpha = \bigcup_{\beta<\alpha} F_\alpha$, for each $\alpha \in A$. Then, by the assumption, P_α is closed in X and, therefore, we have $P_\alpha = \bigcap_{n=1}^{\infty} G_{\alpha n}$ with open subsets $G_{\alpha n}$ of X, for each $\alpha \in A$.

Now, let us put $H_{\alpha n} = F_\alpha - G_{\alpha n}$, for each $\alpha \in A$ and $n = 1,2, \ldots$. Then $\{H_{\alpha n} | \alpha \in A\}$ is discrete in X. Because, for any $x \in X$ there exists the first $\alpha \in A$ with $x \in F_\alpha$. If we set $U = (X - P_\alpha) \cap G_{\alpha+1,n}$, then U is an open subset of X containing x such that $U \cap H_{\beta n} \neq \emptyset$ implies $\beta = \alpha$. This shows that $\{H_{\alpha n} | \alpha \in A\}$ is locally finite in X, and since it is clearly disjoint, it is discrete in X. Furthermore, $\{H_{\alpha n} | \alpha \in A; n = 1,2, \ldots\}$ covers X. Because, for any $x \in X$, we take $\alpha \in A$ as the first element in A with $x \in F_\alpha$, again. Since $x \notin P_\alpha$, $x \notin G_{\alpha n}$ for some n. This shows that $x \in F_\alpha - G_{\alpha n} = H_{\alpha n}$.

Consequently, X has a σ-locally finite closed covering $\{H_{\alpha n} | \alpha \in A; n = 1,2, \ldots\}$, each of which is a σ-space, and so X is a σ-space (cf. [17]).

VII. Let A be a closed subset of a topological space X and f a continuous map from A into a topological space Y, and let $X \cup_f Y$ be an adjunction space of X and Y by f (cf. [9]). Then, if X and Y are normal σ-spaces, so is $X \cup_f Y$ (cf. [17]).

VIII. Let f be a closed continuous map from a space X onto a space Y. If X is a normal σ-space, then Y is also a normal σ-space such that the set $\{y \in Y | \text{boundary of } f^{-1}(y) \text{ is not countably compact}\}$ is σ-discrete in Y (cf. [18,19]).

4. As for the relationship between σ-spaces and metric spaces we have the following two metrization theorems.

IX (Metrization theorems). (A). If X is a collectionwise normal σ-space and an M-space (cf. [13]), then X is metrizable (cf. [17]).

(B). If X is a collectionwise normal σ-space with a point-countable base, then X is metrizable (cf. [19]).

Finally, concerning the product space, we have

X. Let X be a paracompact σ-space and Y a paracompact, perfectly normal T_1 space. Then $X \times Y$ is paracompact and perfectly normal (cf. [17]).

5. The following two conceptions are generalizations of σ-spaces.

5.1. **Definition.** A space X is called *semi-stratifiable* if for each open subset U of X there exists a sequence $\{U_n | n = 1,2, \ldots\}$ of closed subsets of X such that (i) $\bigcup_{n=1}^{\infty} U_n = U$ and (ii) if U and V are open with $U \subset V$, then $U_n \subset V_n$ for $n = 1,2, \ldots$.

This is clearly a modification of a stratifiable space (cf. [4]).

5.2. **Definition** (Nagami [15]). A space X is called a *Σ-space* if there exists a sequence $\{\mathcal{F}_n | n = 1,2, \ldots\}$ of locally finite closed coverings of X satisfying the following condition:

If $K_1 \supset K_2 \supset \ldots$ is a sequence of non-empty closed subsets of X such that $K_n \subset C(x, \mathcal{F}_n) = \cap \{F \in \mathcal{F}_n | x \in F\}$ for some $x \in X$ and for each n, then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Firstly, as for a semi-stratifiable space it satisfies the properties I_s, II_s, III_s, V_s, VI_s, VII_s and $VIII_s$. Among them we can see I_s and V_s from the definition, II_s in [8], III_s in [21] and the proof of VII_s is similar to VII. Here we put the suffix 's' to denote the property of the same type replacing a σ-space by a semi-stratifiable space.

As for VI_s we have the following:

VI_s. Let X be a space and $\{F_\alpha | \alpha \in A\}$ a closed covering of X. If X is dominated by $\{F_\alpha | \alpha \in A\}$ and if each F_α is a normal semistratifiable space, then so is X.

5.3. **Proof** of VI_s. Using the same notations as in 3.1, for any open subset U of X there exists a semi-stratification $\{U(\alpha,n,k) | k = 1,2, \ldots\}$ for $U \cap H_{\alpha n}$ in $H_{\alpha n}$ for each $\alpha \in A$ and for $n = 1,2, \ldots$. Now, let us put

$U_m = \bigcup_{n+k<m} \bigcup_{\alpha \in A} U(\alpha, n, k)$ for $m = 1, 2, \ldots$. Then it is easily shown that the sequence $\{U_m | m = 1,2, \ldots\}$ forms a semi-stratification for U in X.

Furthermore, the later half of $VIII_s$ was simplified as below:

$VIII_s$ (Stoltenberg [21]). If f is a closed, continuous map from a normal semi-stratifiable space X onto a space Y, then the set $\{y \in Y | f^{-1}(y)$ is not compact$\}$ is σ-discrete in Y.

Secondly, as for a Σ-space it satisfies VI_Σ adding perfect normality, IX_Σ (B) and X_Σ, last two of which are generalizations of IX (B) and X, respectively, where we use the suffix 'Σ' for the same reason as before.

VI_Σ. Let X be a space and $\{F_\alpha | \alpha \in A\}$ a closed covering of X. If X is dominated by $\{F_\alpha | \alpha \in A\}$ and if each F_α is a perfectly normal Σ-space, then X is a Σ-space.

5.4. **Proof of VI_Σ.** We use the same notations as in 3.1. Since every closed subspace of a Σ-space is also a Σ-space (cf. [15]), there exists a sequence $\{\mathcal{F}_{\alpha nk} | k = 1,2, \ldots\}$ of locally finite closed coverings of $H_{\alpha n}$ which satisfies the condition in 5.2, for each $\alpha \in A$ and for $n = 1,2, \ldots$. In addition, we can assume, without loss of generality, that $\mathcal{F}_{\alpha nk} \subset \mathcal{F}_{\alpha nk+1}$ for $k = 1,2, \ldots$.

Now, let us put $\mathcal{F}_m = \bigcup_{n+k<m} \bigcup_{\alpha nk} \mathcal{F}_{\alpha nk} \cup \{X\}$ for $m = 2,3, \ldots$. Then each $\mathcal{F}_m$ is clearly a locally finite closed covering of X. To show that $\{\mathcal{F}_m | m = 2,3, \ldots\}$ satisfies the condition in 5.2, take an arbitrary sequence $K_1 \supset K_2 \ldots$ of non-empty closed subsets of X such that $K_m \subset C(x, \mathcal{F}_m)$ for some $x \in X$ and for each m. Let β be the first element in A such that $x \in F_\beta$ and l the minimum of n's such that $x \in H_{\beta n}$. Then we have $x \notin H_{\alpha n}$ for either $\alpha \neq \beta$ or $n < l$. Hence, we have $C(x, \mathcal{F}_m) = X$ for any $m \leq l$. Since $\mathcal{F}_{\beta lk} \subset \mathcal{F}_{l+k}$ for $k = 1,2, \ldots$, we have $K_{l+k} \subset C(x, \mathcal{F}_{l+k}) \subset C(x, \mathcal{F}_{\beta lk})$ for $k = 1,2, \ldots$. Therefore, by the assumption, we have $\emptyset \neq \bigcap_{k=1}^\infty K_{l+k} = \bigcap_{m=1}^\infty K_m$. This shows that X is a Σ-space.

IX_Σ (B) (Michael [1]). A paracompact Σ-space with a point-countable base is metrizable.

X_Σ (Nagami [15]). If X is paracompact Σ-space and Y is a paracompact T_1 P-space in the sense of K. Morita (cf. [13]), then $X \times Y$ is paracompact.

6. As for a generalization of IX (A) we state the following conception.

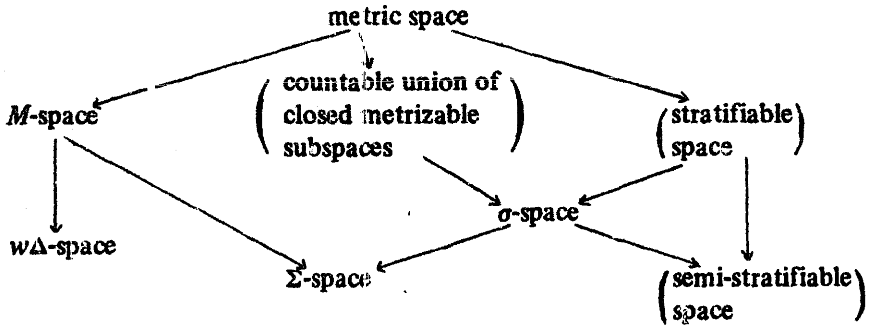
6.1. Definition (Borges [5]). A space X is called a *wΔ-space* if there exists a sequence $\{\mathcal{U}_n \mid n = 1,2,\dots\}$ of open coverings of X satisfying the following condition:

If $\{x_1, x_2, \dots\}$ is a sequence in X such that $x_n \in St(x, \mathcal{U}_n)$ for some $x \in X$ and for each n, then it has a cluster point.

Clearly, an M-space is always a $w\Delta$-space. The property IX (A) was generalized by weakening the assumption of an M-space, as below:

IX′ (A) (Siwiec-Nagata [19]). If X is a collectionwise normal σ-space and a $w\Delta$-space, then it is metrizable.

7. The situation of spaces mentioned above is shown in the following diagram:



In the above diagram, the fact that every stratifiable space is a σ-space was proved by R. Heath.

Besides these facts, some results were obtained relating a developable space (cf. [6,7]).

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