JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **120**, 547–556 (1986)

On Opial-Type Integral Inequalities

B. G. PACHPATTE

Department of Mathematics and Statistics, Marathwada University, Aurangabad 431004, Maharashtra, India

Submitted by C. L. Dolph

Received May 15, 1985

The aim of the paper is to establish some new integral inequalities involving two functions and their first order and higher order derivatives. Our results in the special cases yield the well-known Opial inequality and some of its generalizations. © 1986 Academic Press, Inc.

1. INTRODUCTION

In 1960, Opial [6] proved the following interesting inequality:

Let $f(x)$ be of class C^1 on $0 \leqslant x \leqslant h$, and satisfy $f(0)=f(h)=0$, $f(x)>0$ on $(0, h)$. Then

$$\int_0^h |f(x) f'(x)|\, dx \leqslant \frac{h}{4} \int_0^h f'^2(x)\, dx. \tag{1}$$

The constant $h/4$ is best possible.

Integral inequalities of the form (1) have an interest in their own right and also have important applications in the theory of ordinary differential equations and boundary value problems (see [5, 8, 9]). Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations, see, for example [7; 4, pp. 154–162] and the references given therein. In [10] Yang has established some interesting generalizations of the Opial's inequality by using the Mallows method [3] of the proof of inequality (1). In the present paper we establish a number of new integral inequalities involving two functions and their first order and higher order derivatives which in the special cases yield the well known Opial inequality (1) and some of its generalizations given by Das [2] and Yang [10]. The method employed in the proofs of our results depends on the modification of the method used by Yang in [10].

0022-247X/86 $3.00

2. Statement of Results

In this section we state our main results on the integral inequalities to be proved in this paper. Theorems 1–4 deal with the integral inequalities involving two functions and their first order derivatives.

THEOREM 1. *Let $p(x)$ be positive and continuous function on a finite or infinite interval $a<x<b$ such that $\int_a^b p^{-1}(x)\,dx<\infty$. If $u(x)$ and $v(x)$ are absolutely continuous functions on (a, b) and $u(a)=u(b)=0$, $v(a)=v(b)=0$, then*

$$\int_a^b [|u(x)\,v'(x)|+|v(x)\,u'(x)|]\,dx$$

$$\leqslant \tfrac{1}{2}A\int_a^b p(x)[|u'(x)|^2+|v'(x)|^2]\,dx, \tag{2}$$

where

$$A=\int_a^c p^{-1}(x)\,dx=\int_c^b p^{-1}(x)\,dx, \qquad a\leqslant c\leqslant b. \tag{3}$$

Equality holds in (2) if and only if

$$u(x)=v(x)=M\int_a^x p^{-1}(t)\,dt, \qquad a\leqslant x\leqslant c,$$

$$u(x)=v(x)=M\int_x^b p^{-1}(t)\,dt, \qquad c\leqslant x\leqslant b,$$

where M is a constant.

Remark 1. In the special case when $u(x)=v(x)$, Theorem 1 reduces to the inequality established by Yang [10, Theorem 1] which in turn contains as a special case the Opial inequality (1).

THEOREM 2. *Let $p(x)$ be positive and continuous function on an interval $a\leqslant x\leqslant c$, with $\int_a^c p^{-1}(x)\,dx<\infty$, and let $q(x)$ be bounded positive, continuous, and nonincreasing function on $a\leqslant x\leqslant c$. If $u(x)$ and $v(x)$ are absolutely continuous functions on $a\leqslant x\leqslant c$ and $u(a)=v(a)=0$, then*

$$\int_a^c q(x)[|u(x)\,v'(x)|+|v(x)\,u'(x)|]\,dx$$

$$\leqslant \tfrac{1}{2}\int_a^c p^{-1}(x)\,dx\int_a^c p(x)\,q(x)[|u'(x)|^2+|v'(x)|^2]\,dx, \tag{4}$$

with equality if and only if $q(x)=$ constant and $u(x)=v(x)=M\int_a^x p^{-1}(t)\,dt$ for $a\leqslant x\leqslant c$, where M is a constant.

THEOREM 3. *Let $p(x)$ be positive and continuous function on an interval $c \leqslant x \leqslant b$, with $\int_c^b p^{-1}(x)\,dx < \infty$, and let $q(x)$ be bounded positive, continuous and nondecreasing function on $c \leqslant x \leqslant b$. If $u(x)$ and $v(x)$ are absolutely continuous functions on $c \leqslant x \leqslant b$ and $u(b) = v(b) = 0$, then*

$$\int_c^b q(x)[|u(x)\,v'(x)| + |v(x)\,u'(x)|]\,dx$$

$$\leqslant \tfrac{1}{2} \int_c^b p^{-1}(x)\,dx \int_c^b p(x)\,q(x)[|u'(x)|^2 + |v'(x)|^2], \tag{5}$$

with equality if and only if $q(x) = \text{constant}$ and $u(x) = v(x) = M \int_x^b p^{-1}(t)\,dt$ for $c \leqslant x \leqslant b$, where M is a constant.

Remark 2. We note that Theorem 1 is a special case of the combination of Theorems 2 and 3 when $q = \text{constant}$. In the special case when $u(x) = v(x)$, Theorems 2 and 3 reduces to Theorems 3 and 3′ given in [10] which deals with the simplified proofs of the results established by Beesack [1] and at the same time generalizations of the results established in [1].

THEOREM 4. *If $u(x)$ and $v(x)$ are absolutely continuous functions on $a \leqslant x \leqslant b$ with $u(a) = u(b) = 0$, $v(a) = v(b) = 0$, then*

$$\int_a^b |u(x)\,v(x)|^m[|u(x)\,v'(x)| + |v(x)\,u'(x)|]\,dx$$

$$\leqslant \frac{(b-a)^{2m+1}}{2^{2(m+1)}(m+1)} \int_a^b [|u'(x)|^{2(m+1)} + |v'(x)|^{2(m+1)}]\,dx, \tag{6}$$

where $m \geqslant 0$ is a constant. Equality holds in (6) if and only if

$$u(x) = v(x) = M(x-a), \qquad a \leqslant x \leqslant c,$$

$$u(x) = v(x) = M(b-x), \qquad c \leqslant x \leqslant b,$$

where M is a constant.

Remark 3. It is interesting to note that in the special case when $u(x) = v(x)$ and $2m + 1 = n$, our Theorem 4 reduces to the inequality established by Yang in [10, Theorem 4] which in itself contains as a special case the Opial inequality when $n = 1$, $a = 0$, and $b = h$.

Our next two theorems deals with integral inequalities involving pair of functions and their nth order derivatives.

THEOREM 5. *Let $u, v \in C^{(n-1)}[a, b]$ be such that $u^{(i)}(a) = v^{(i)}(a) = 0$ for $i = 0, 1, \ldots, n-1$, where $n \geqslant 1$. Let $u^{(n-1)}$, $v^{(n-1)}$ be absolutely continuous and $\int_a^b |u^{(n)}(x)|^2 \, dx < \infty$, $\int_a^b |v^{(n)}(x)|^2 \, dx < \infty$. Then*

$$\int_a^b [|u(x) \, v^{(n)}(x)| + |v(x) \, u^{(n)}(x)|] \, dx$$

$$\leqslant B(b-a)^n \int_a^b [|u^{(n)}(x)|^2 + |v^{(n)}(x)|^2] \, dx, \tag{7}$$

where

$$B = \frac{1}{2(n!)} \left(\frac{n}{2n-1} \right)^{1/2}. \tag{8}$$

Equality holds in (7) if and only if $n = 1$ and $u^{(n)}(x) = v^{(n)}(x) = M$, where M is a constant.

THEOREM 6. *Let $u, v \in C[a, b]$ and $u', \ldots, u^{(n-1)}$, $v', \ldots, v^{(n-1)}$ are piecewise continuous, $u^{(n-1)}$, $v^{(n-1)}$ are absolutely continuous with $\int_a^b |u^{(n)}(x)|^2 \, dx < \infty$, $\int_a^b |v^{(n)}(x)|^2 \, dx < \infty$, $u^{(i)}(a) = u^{(i)}(b) = 0$, $v^{(i)}(a) = v^{(i)}(b) = 0$ for $i = 0, 1, \ldots, n-1$, where $n \geqslant 1$, then*

$$\int_a^b [|u(x) \, v^{(n)}(x)| + |v(x) \, u^{(n)}(x)|] \, dx$$

$$\leqslant B \left(\frac{b-a}{2} \right)^n \int_a^b [|u^{(n)}(x)|^2 + |v^{(n)}(x)|^2] \, dx, \tag{9}$$

where B is as given in (8). Equality holds in (9) if and only if $n = 1$ and

$$u(x) = v(x) = M(x-a)^n, \qquad a \leqslant x \leqslant \frac{a+b}{2},$$

$$u(x) = v(x) = M(b-x)^n, \qquad \frac{a+b}{2} \leqslant x \leqslant b,$$

where M is a constant.

Remark 4. We note that in the special case when we take $u(x) = v(x)$, in our Theorems 5 and 6 we get the integral inequalities established by Das in [2, Theorem 1 and Remark on p. 259] which in turn contains as a special case the Opial inequality (1) and the sharper version of the inequality established by Willet in [9].

3. Proofs of Theorems 1–4

Let $c \in [a, b]$ and define

$$y(x) = \int_a^x |u'(t)|\, dt, \qquad z(x) = \int_a^x |v'(t)|\, dt, \tag{10}$$

for $a \leqslant x \leqslant c$ and

$$r(x) = -\int_x^b |u'(t)|\, dt, \qquad w(x) = -\int_x^b |v'(t)|\, dt, \tag{11}$$

for $c \leqslant x \leqslant b$, then we have

$$y'(x) = |u'(x)|, \qquad z'(x) = |v'(x)|, \tag{12}$$

for $a \leqslant x \leqslant c$ and

$$r'(x) = |u'(x)|, \qquad w'(x) = |v'(x)| \tag{13}$$

for $c \leqslant x \leqslant b$. We note that

$$u(x) = \int_a^x u'(t)\, dt, \qquad v(x) = \int_a^x v'(t)\, dt, \tag{14}$$

for $a \leqslant x \leqslant c$ and

$$u(x) = -\int_x^b u'(t)\, dt, \qquad v(x) = -\int_x^b v'(t)\, dt, \tag{15}$$

for $c \leqslant x \leqslant b$. From (14) and (10) and (15) and (11) we observe that

$$|u(x)| \leqslant y(x), \qquad |v(x)| \leqslant z(x), \tag{16}$$

for $a \leqslant x \leqslant c$ and

$$|u(x)| \leqslant -r(x), \qquad |v(x)| \leqslant -w(x), \tag{17}$$

for $c \leqslant x \leqslant b$. Now from (16), (12), and upon using the elementary inequality

$$\alpha\beta \leqslant \tfrac{1}{2}[\alpha^2 + \beta^2] \qquad \text{for} \quad \alpha, \beta \text{ reals}, \tag{18}$$

the definitions of $y(x)$ and $z(x)$ given in (10) and Schwarz inequality we have

$$\begin{aligned}
\int_a^c & [|u(x)\,v'(x)| + |v(x)\,u'(x)|]\,dx \\
&\leqslant \int_a^c [y(x)\,z'(x) + z(x)\,y'(x)]\,dx \\
&= \int_a^c \frac{d}{dx}[y(x)\,z(x)]\,dx \\
&= y(c)\,z(c) \\
&\leqslant \tfrac{1}{2}[y^2(c) + z^2(c)] \\
&= \tfrac{1}{2}\left[\left(\int_a^c (1/\sqrt{p(x)})\,\sqrt{p(x)}\,|u'(x)|\,dx\right)^2\right. \\
&\qquad \left. + \left(\int_a^c (1/\sqrt{p(x)})\,\sqrt{p(x)}\,|v'(x)|\,dx\right)^2\right] \\
&\leqslant \tfrac{1}{2}\int_a^c p^{-1}(x)\,dx \int_a^c p(x)[|u'(x)|^2 + |v'(x)|^2]\,dx. \qquad (19)
\end{aligned}$$

Similarly, from (17), (13) and on using the elementary inequality (18), the definitions of $r(x)$ and $w(x)$ given in (11) and Schwarz inequality we have

$$\begin{aligned}
\int_c^b & [|u(x)\,v'(x)| + |v(x)\,u'(x)|]\,dx \\
&\leqslant \tfrac{1}{2}\int_c^b p^{-1}(x)\,dx \int_c^b p(x)[|u'(x)|^2 + |v'(x)|^2]\,dx. \qquad (20)
\end{aligned}$$

From (19), (20) and the definition of A given in (3), the desired inequality in (2) follows. This completes the proof of Theorem 1.

Let $c \in [a, b]$ and define

$$y(x) = \int_a^x \sqrt{q(t)}\,|u'(t)|\,dt, \qquad z(x) = \int_a^x \sqrt{q(t)}\,|v'(t)|\,dt, \tag{21}$$

for $a \leqslant x \leqslant c$ and

$$r(x) = -\int_x^b \sqrt{q(t)}\,|u'(t)|\,dt, \qquad w(x) = -\int_x^b \sqrt{q(t)}\,|v'(t)|\,dt, \tag{22}$$

for $c \leqslant x \leqslant b$, then we have

$$y'(x) = \sqrt{q(x)}\,|u'(x)|, \qquad z'(x) = \sqrt{q(x)}\,|v'(x)|, \tag{23}$$

for $a \leqslant x \leqslant c$ and

$$r'(x) = \sqrt{q(x)}\,|u'(x)|, \qquad w'(x) = \sqrt{q(x)}\,|v'(x)|, \tag{24}$$

for $c \leqslant x \leqslant b$. Now from (14), (21), nonincreasing character of $q(x)$ on $a \leqslant x \leqslant c$ and (15), (22), nondecreasing character of $q(x)$ on $c \leqslant x \leqslant b$ we observe that

$$|u(x)| \leqslant (1/\sqrt{q(x)})\, y(x), \qquad |v(x)| \leqslant (1/\sqrt{q(x)})\, z(x), \tag{25}$$

for $a \leqslant x \leqslant c$ and

$$|u(x)| \leqslant -(1/\sqrt{q(x)})\, r(x), \qquad |v(x)| \leqslant -(1/\sqrt{q(x)})\, w(x), \tag{26}$$

for $c \leqslant x \leqslant b$, respectively. Now the proofs of Theorems 2 and 3 follows by closely looking at the proof of Theorem 1 given above with suitable modifications. We omit the further details of the proofs of Theorems 2 and 3.

From (12), (16), and upon using the elementary inequality (18), the definitions of $y(x)$ and $z(x)$ given in (10), Schwarz inequality and Hölder's inequality we have

$$\begin{aligned}
&\int_a^c |u(x)\, v(x)|^m [\,|u(x)\, v'(x)| + |v(x)\, u'(x)|\,]\, dx \\
&\leqslant \int_a^c y^m(x)\, z^m(x) [y(x)\, z'(x) + z(x)\, y'(x)]\, dx \\
&= \int_a^c \frac{d}{dx}\left(\frac{1}{m+1} y^{m+1}(x)\, z^{m+1}(x)\right) dx \\
&= \frac{1}{m+1} y^{m+1}(c)\, z^{m+1}(c) \\
&\leqslant \frac{1}{2(m+1)} [(y^{m+1}(c))^2 + (z^{m+1}(c))^2] \\
&= \frac{1}{2(m+1)}\left[\left\{\left(\int_a^c |u'(x)|\, dx\right)^2\right\}^{m+1} + \left\{\left(\int_a^c |v'(x)|\, dx\right)^2\right\}^{m+1}\right] \\
&\leqslant \frac{(c-a)^{2m+1}}{2(m+1)} \int_a^c [\,|u'(x)|^{2(m+1)} + |v'(x)|^{2(m+1)}]\, dx.
\end{aligned} \tag{27}$$

Similarly, from (13), (17) and upon using the elementary inequality (18), the definition of $r(x)$ and $w(x)$ given in (11), Schwarz inequality and Hölder's inequality we obtain

$$\int_c^b |u(x)\, v(x)|^m \,[|u(x)\, v'(x)| + |v(x)\, u'(x)|]\, dx$$
$$\leqslant \frac{(b-c)^{2m+1}}{2(m+1)} \int_c^b [|u'(x)|^{2(m+1)} + |v'(x)|^{2(m+1)}]\, dx. \tag{28}$$

Now taking $c=(a+b)/2$, we obtain the desired inequality in (6) from (27) and (28). This completes the proof of Theorem 4.

4. Proofs of Theorems 5 and 6

First, we note that in view of the assumptions on u, v for any $x \in [a, b]$ we have

$$u(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} u^{(n)}(t)\, dt, \tag{29}$$

$$v(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} v^{(n)}(t)\, dt. \tag{30}$$

Now multiplying (29) and (30) by $v^{(n)}(x)$ and $u^{(n)}(x)$, respectively, and upon using Schwarz inequality we obtain

$$|u(x)\, v^{(n)}(x)| \leqslant \frac{|v^{(n)}(x)|}{(n-1)!} \frac{(x-a)^{n-(1/2)}}{(2n-1)^{1/2}} \left(\int_a^x |u^{(n)}(t)|^2\, dt\right)^{1/2}, \tag{31}$$

$$|v(x)\, u^{(n)}(x)| \leqslant \frac{|u^{(n)}(x)|}{(n-1)!} \frac{(x-a)^{n-(1/2)}}{(2n-1)^{1/2}} \left(\int_a^x |v^{(n)}(t)|^2\, dt\right)^{1/2}. \tag{32}$$

From (31) and (32) we obtain

$$\int_a^b [|u(x)\, v^{(n)}(x)| + |v(x)\, u^{(n)}(x)|]\, dx$$
$$\leqslant \frac{1}{(n-1)!\,(2n-1)^{(1/2)}} \int_a^b (x-a)^{n-(1/2)} \left[|v^{(n)}(x)| \left(\int_a^x |u^{(n)}(t)|^2\, dt\right)^{1/2} \right.$$
$$\left. + |u^{(n)}(x)| \left(\int_a^x |v^{(n)}(t)|^2\, dt\right)^{1/2} \right] dx. \tag{33}$$

Now, first applying Schwarz inequality and then upon using the elementary inequalities $(\alpha+\beta)^2 \leqslant 2(\alpha^2+\beta^2)$ and $\alpha^{1/2}\beta^{1/2} \leqslant \frac{1}{2}(\alpha+\beta)$, $\alpha, \beta \geqslant 0$ (for α, β reals), to the right hand side of (33) we obtain

$$\begin{aligned}
&\int_a^b [|u(x)\, v^{(n)}(x)| + |v(x)\, u^{(n)}(x)|]\, dx \\
&\quad\leqslant \frac{1}{(n-1)!\,(2n-1)^{1/2}} \left(\int_a^b (x-a)^{2(n-(1/2))}\, dx\right)^{1/2} \\
&\qquad\times\left(\int_a^b \left[|v^{(n)}(x)| \left(\int_a^x |u^{(n)}(t)|^2\, dt\right)^{1/2}\right.\right. \\
&\qquad\left.\left.+ |u^{(n)}(x)| \left(\int_a^x |v^{(n)}(t)|^2\, dt\right)^{1/2}\right]^2 dx\right)^{1/2} \\
&\quad\leqslant \frac{1}{(n-1)!\,(2n-1)^{1/2}} \frac{(b-a)^n}{(2n)^{1/2}} \left(2\int_a^b \left[|v^{(n)}(x)|^2 \left(\int_a^x |u^{(n)}(t)|^2\, dt\right)\right.\right. \\
&\qquad\left.\left.+ |u^{(n)}(x)|^2 \left(\int_a^x |v^{(n)}(t)|^2\, dt\right)\right] dx\right)^{1/2} \\
&\quad= \frac{\sqrt{2}\,(b-a)^n}{(n-1)!\,(2n-1)^{1/2}(2n)^{1/2}} \\
&\qquad\times\left(\int_a^b \frac{d}{dx}\left\{\left(\int_a^x |u^{(n)}(t)|^2\, dt\right)\left(\int_a^x |v^{(n)}(t)|^2\, dt\right)\right\} dx\right)^{1/2} \\
&\quad= \frac{\sqrt{2}\,(b-a)^n}{(n-1)!\,(2n-1)^{1/2}(2n)^{1/2}} \left(\left(\int_a^b |u^{(n)}(t)|^2\, dt\right)\left(\int_a^b |v^{(n)}(t)|^2\, dt\right)\right)^{1/2} \\
&\quad\leqslant \frac{1}{2(n!)} \left(\frac{n}{2n-1}\right)^{1/2} (b-a)^n \int_a^b [|u^{(n)}(x)|^2 + |v^{(n)}(x)|^2]\, dx. \qquad (34)
\end{aligned}$$

This completes the proof of Theorem 5.

The proof of Theorem 6 follows immediately on using (7) once on $[a, (a+b)/2]$ and again on $[(a+b)/2, b]$, where on the latter interval, in view of the assumptions on u, v we use

$$u(x) = \frac{(-1)^n}{(n-1)!} \int_x^b (t-x)^{n-1}\, u^{(n)}(t)\, dt,$$

$$v(x) = \frac{(-1)^n}{(n-1)!} \int_x^b (t-x)^{n-1}\, v^{(n)}(t)\, dt.$$

The details are omitted.

In concluding we note that it does not seem to be possible to extend the analysis of Sections 3 and 4 to obtain the general versions of Theorems 5 and 6 in [10] and Theorem 2 given in [2] in the framework of the general setup of our results involving two functions and their derivatives. The fact that in this case repeated applications of Hölder inequality with suitable indices leads to difficulties which will not overcome by following the analysis in Sections 3 and 4.

References

1. P. R. Beesack, On an integral inequality of Z. Opial, *Trans. Amer. Math. Soc.* **104** (1962), 470–475.
2. K. M. Das, An inequality similar to Opial's inequality, *Proc. Amer. Math. Soc.* **22** (1969), 258–261.
3. C. L. Mallows, An even simpler proof of Opial's inequality, *Proc. Amer. Math. Soc.* **16** (1965), 173.
4. D. S. Mitrinović, "Analytic Inequalities," Springer-Verlag, Berlin/Heidelberg/New York, 1970.
5. J. Myjak, Boundary value problems for nonlinear differential and difference equations of the second order, *Zeszyty Nauk. Uniw. Jagiellon. Prace Mat.* **15** (1971), 113–123.
6. Z. Opial, Sur une inégalité, *Ann. Polon. Math.* **8** (1960), 29–32.
7. D. T. Shum, On a class of new inequalities, *Trans. Amer. Math. Soc.* **204** (1975), 299–341.
8. J. Traple, On a boundary value problem for systems of ordinary differential equations of second order, *Zeszyty Nauk. Uniw. Jagiellon. Prace Mat.* **15** (1971), 159–168.
9. D. Willett, The existence-uniqueness theorem for an nth order linear ordinary differential equation, *Amer. Math. Monthly* **75** (1968), 174–178.
10. G. S. Yang, On a certain result of Z. Opial, *Proc. Japan Acad.* **42** (1966), 78–83.