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Convergence of the Time-Invariant Riccati Differential Equation towards Its Strong Solution for Stabilizable Systems*

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We prove a necessary and sufficient condition for the solution of the time-invariant Riccati differential equation to converge towards the strong solution of the corresponding algebraic Riccati equation, when the system is stabilizable, without assuming that the Hamiltonian matrix has no eigenvalues on the imaginary axis (or equivalently that there are no critical unobservable modes). The condition is a generalization of an earlier one established by Callier and Willems. Our proof revises an earlier one by Faurre, Clerget, and Germain, leading to additional information of what can be assumed without loss of generality in our context. It is also shown that the convergence is not always exponential and that the presence of critical unobservable modes may slow down but does not prevent the convergence of the solution of the Riccati differential equation. The impact of the condition on linear–quadratic optimal control is briefly discussed. © 1995 Academic Press, Inc.

1. INTRODUCTION

A necessary and sufficient condition for the solution of the time-invariant Riccati differential equation to converge towards the strong solution of the corresponding algebraic Riccati equation is derived when the system is stabilizable and the Hamiltonian matrix may have eigenvalues on the imaginary axis. The rigorous presentation of the main steps of a complete proof makes precise what may be assumed without loss of generality

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in our context and reveals that the presence of unobservable modes on the imaginary axis may slow down but does not prevent the convergence of the solution of the Riccati differential equation. As a byproduct of this analysis the impact of the convergence condition on linear–quadratic (LQ) optimal control is briefly discussed.

We study the time-invariant Riccati differential equation (RDE) arising from LQ-optimal control, i.e.,

$$\dot{P}(\tau) = A^*P(\tau) + P(\tau)A - P(\tau)BB^*P(\tau) + C^*C \tag{1a}$$

$$P(0) = S, \tag{1b}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$, and $\tau = t_1 - t \geq 0$ (implying that $t \in [0, t_1]$ with horizon $t_1 \in \mathbb{R}$ and time $t \in \mathbb{R}$); furthermore, $S \in \mathbb{R}^{n\times n}$ is a symmetric positive semi-definite matrix and so is $P(\tau)$ for all $\tau \geq 0$. We assume now that

$$(A, B) \text{ is stabilizable.} \tag{2}$$

It is known that, under this condition, the corresponding algebraic Riccati equation (ARE)

$$A^*P + PA - PBB^*P + C^*C = 0 \tag{3}$$

has a unique strong solution $P = P_+$, where $P_+ = P_+^* \geq 0$, such that all the eigenvalues of $A_+ := A - BB^*P_+$ have nonpositive real parts; see e.g. [6] and the references therein.

We investigate under which condition on the penalty matrix S will $P(\tau)$ converge to P_+ as τ tends to infinity. This question is important in LQ-optimal control and (using duality) in LQG-estimation problems: it establishes the existence of a limit cost (or a limit estimation error variance) associated with the best possible stability properties for the closed-loop state (or estimation error) dynamics. Here the optimal state trajectories are not always exponentially stable since they may contain critical modes (not necessarily bounded); these critical modes may be unwanted and then the problem of finding a near optimal stabilizing feedback law becomes important. It is expected that our analysis should be useful for choosing a well-motivated tradeoff between performance, i.e. optimality, and stability.

The attraction of $P(\tau)$ towards P_+ has already been studied under more restrictive conditions than (2) in, e.g., [1, Sect. IV; 2, Theorems 4.1 and 4.2; 3, Theorem 3; 4, p. 111; and 5, Proposition 5.3, pp. 100–103] and the references therein. It is the objective of this paper to present a necessary and sufficient condition for attraction when (2) only holds. For understanding this condition the following abbreviations, notions, and notations are used throughout: RDE, ARE, w.l.g., resp., p.s.d., p.d., n.s.d., exp., and $\dot{+}$, which mean respectively the Riccati differential equation, the algebraic Riccati equation, without loss of generality, respectively, positive semi-definite, positive definite, negative semi-definite, exponentially, and the direct sum of subspaces.

For any real matrix P, P^* denotes its transpose, while $P = P^* \geq 0$ means that P is symmetric p.s.d. For any square matrix $A \in \mathbb{R}^{n \times n}$, $\mathcal{L}^-(A)$, $\mathcal{L}^0(A)$, and $\mathcal{L}^+(A)$ denote resp. the A-invariant subspaces of $\mathbb{R}^n$ spanned by the (generalized) eigenvectors of $\mathbb{R}^n$ corresponding to the eigenvalues λ of A such that $\operatorname{Re} \lambda < 0$, $\operatorname{Re} \lambda = 0$, and $\operatorname{Re} \lambda > 0$ (these are the so-called exp. stable, critical, and exp. antistable subspaces of $\mathbb{R}^n$). Furthermore, we use $\mathcal{L}^{0-}(A) := \mathcal{L}^0(A) \dot{+} \mathcal{L}^-(A)$ and $\mathcal{L}^{0+}(A) := \mathcal{L}^0(A) \dot{+} \mathcal{L}^+(A)$; moreover, the property $\mathcal{L}^-(A) = \mathbb{R}^n$ is denoted $\operatorname{Re} \lambda(A) < 0$ (meaning that all eigenvalues of A have negative real parts); $\operatorname{Re} \lambda(A) \leq 0$, $\operatorname{Re} \lambda(A) = 0$, $\operatorname{Re} \lambda(A) \geq 0$, and $\operatorname{Re} \lambda(A) > 0$ have an obvious similar meaning. For any matrix U, we denote by $R(U)$ and $N(U)$ its range and null space. For any constant matrix pair (A, B) where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, we denote by $C(A, B)$ and $S(A, B)$ resp. the (A, B)-controllable and -stabilizable subspaces of $\mathbb{R}^n$, where

$$C(A, B) := \mathcal{R}([B; AB; \ldots; A^{n-1}B]) \tag{4}$$

and

$$S(A, B) := \mathcal{L}^-(A) + C(A, B). \tag{5}$$

Moreover, (A, B) is said to be *controllable*, resp. *stabilizable*, iff $C(A, B) = \mathbb{R}^n$, resp. $S(A, B) = \mathbb{R}^n$. For any constant matrix pair (C, A) where $C \in \mathbb{R}^{p \times n}$ and $A \in \mathbb{R}^{n \times n}$, we denote by $NO(C, A)$ and $NO^{0+}(C, A)$ resp. the (C, A)-unobservable and -undetectable subspaces of $\mathbb{R}^n$, where

$$NO(C, A) := \bigcap_{i=0}^{n-1} \mathcal{N}(CA^i) \tag{6}$$

and

$$NO^{0+}(C, A) := NO(C, A) \cap \mathcal{L}^{0+}(A). \tag{7}$$

Moreover, (C, A) is said to be *observable*, resp. *detectable*, iff $NO(C, A) = \{0\}$, resp. $NO^{0+}(C, A) = \{0\}$. In the following we shall also be involved with the subspaces

$$NO^{+}(C, A),\ NO^{0}(C, A), \text{ and } NO^{0-}(C, A) \tag{8}$$

obtained from (7) by replacing $\mathcal{L}^{0+}(A)$ resp. by $\mathcal{L}^{+}(A)$, $\mathcal{L}^{0}(A)$, and $\mathcal{L}^{0-}(A)$. We assume that the reader is familiar with the properties of $S(A, B)$ and $NO^{0+}(C, A)$ as in e.g. [7] and [1].

Consider now symmetric solutions $P = P^* \in \mathbb{R}^{n\times n}$ of the ARE (3). Among these, two solutions are very important. We call the *strong solution* of the ARE (3) the solution $P_+ = P_+^* \geq 0$ such that, with

$$A_+ := A - BB^*P_+, \tag{9}$$

$$\operatorname{Re} \lambda(A_+) \leq 0. \tag{10}$$

Similarly, we call the *antistrong solution* of the ARE (3) the solution $P_- = P_-^* \leq 0$ such that with

$$A_- := A - BB^*P_-, \tag{11}$$

$$\operatorname{Re} \lambda(A_-) \geq 0. \tag{12}$$

We consider also the Hamiltonian matrix $H \in \mathbb{R}^{2n\times 2n}$ given by

$$H = \begin{bmatrix} A & -BB^* \\ -C^*C & -A^* \end{bmatrix}. \tag{13}$$

We have then (see e.g. [6] and the references therein)

Fact 1. Consider the ARE (3). Then the following holds:

(a) There exists a unique strong solution P_+ iff (A, B) is stabilizable.

(b) There exists a unique antistrong solution P_- iff $(-A, B)$ is stabilizable.

(c) P_+ and P_- exist uniquely iff (A, B) is controllable.

(d) If P_+ exists, then

$$\mathcal{N}(P_+) = NO^{0-}(C, A). \tag{14}$$

(e) If P_- exists, then

$$\mathcal{N}(P_-) = NO^{0+}(C, A). \tag{15}$$

(f) If (A, B) is stabilizable, then

$$NO^{0}(C, A) := NO(C, A) \cap \mathcal{L}^{0}(A) = \mathcal{L}^{0}(A_+) \tag{16}$$

and

$$NO^0(C, A) = \{ 0 \} \tag{17}$$

iff

$$\mathcal{L}^0(H) = \{ 0 \}. \tag{18}$$

(g) If (A, B) is controllable, then

$$NO^0(C, A) = \mathcal{L}^0(A_+) = \mathcal{L}^0(A_-) = \mathcal{N}(P_+ - P_-), \tag{19}$$

where $P_+ - P_-$ is the so-called gap. ▮

Observe that, in Fact 1(f) above, P_+ is stabilizing, i.e., Re $\lambda(A_+) < 0$, iff $NO^0(C, A) = \{ 0 \}$. It is the purpose of this paper to prove

THEOREM 1. *Consider the* RDE (1) *where*

$$(A, B) \textit{ is stabilizable.} \tag{20}$$

Then, for a given $P(0) = S = S^* \geq 0$,

$$\lim_{\tau \to \infty} P(\tau) = P_+ \tag{21}$$

iff

$$\mathcal{N}(S) \cap NO^+(C, A) = \{ 0 \}. \tag{22}$$

Comments. 1 (a) Theorem 1 is a generalization of Callier and Willems [1, condition (32)] whereby the same condition holds when (2) holds and $NO^0(C, A) = \{ 0 \}$, i.e., $\mathcal{L}^0(H) = \{ 0 \}$.

(b) An incomplete proof of Theorem 1 was presented in [5, pp. 101–103] when (A, B) is controllable; moreover, their condition reads (see below)

$$\mathcal{N}(S) \cap NO^{0+}(C, A) \subseteq NO^0(C, A),$$

which is equivalent to (22) if $\mathcal{N}(S) \cap NO^{0+}(C, A)$ is A-invariant but not otherwise.

(c) The essential contribution of this paper is the rigorous presentation of the main steps of a complete proof making precise what may be assumed without loss of generality under (2) and (22), namely that (A, B)

is controllable and the rank of $S - P_-$ is minimal; i.e., $\dim(\mathcal{N}(S) \cap NO^{0+}(C, A)) = \dim(NO^0(C, A))$. See below.

The paper is organized as follows. Section 1 is the present introduction. In Section 2 it is shown that the condition of Theorem 1 is necessary. In Section 3 we show that w.l.g. in Theorem 1 the assumption "(A, B) is stabilizable" may be replaced by "(A, B) is controllable." In Section 4 it is shown that the condition of Theorem 1 is sufficient. Section 5 contains conclusions with the impact of Theorem 1 on LQ-optimal control theory.

2. Necessity of Condition (22)

THEOREM 2. *Consider the* RDE (1) *and let assumption* (2) *hold. Then, for a given* $P(0) = S = S^* \geq 0$,

$$\lim_{\tau \to \infty} P(\tau) = P_+ \tag{21}$$

implies

$$\mathcal{N}(S) \cap NO^+(C, A) = \{ 0 \}. \tag{22}$$

Proof. Suppose for a contradiction that there exists a nonzero x_1 in $\mathcal{N}(S) \cap NO^+(C, A)$. We know that, for all $\tau \geq 0$,

$$x_0^* P(\tau) x_0 = \inf_{u(\cdot)} \left\{ \int_0^\tau (\|Cx(t)\|^2 + \|u(t)\|^2)\, dt + x(\tau)^* S x(\tau) \right\}, \tag{23}$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$$
$$u(\cdot) \text{ is continuous.}$$

For all $\tau \geq 0$ consider now normalized backtraced initial conditions

$$x(0, \tau) := \exp(-A\tau)x_1 \cdot (\|\exp(-A\tau)x_1\|)^{-1}. \tag{24}$$

Since $NO^+(C, A)$ is A-invariant, one has

$$x(0, \tau) \in NO^+(C, A) \qquad \text{for all } \tau \geq 0. \tag{25}$$

Set $u(t) \equiv 0$ and observe that the solution of $\dot{x} = Ax + Bu$ due to $x(0) = x(0, \tau)$ is $x(t) = \exp(At) \cdot x(0, \tau)$. It is clear that $x(t) \in NO(C, A)$, whence

$Cx(t) \equiv 0$ and $Sx(\tau) = Sx_1 \cdot (\|\exp(-A\tau)x_1\|)^{-1} = 0$. It follows that, for $x(0) = x(0, \tau)$, the cost above is zero, hence minimal, such that

$$x(0, \tau)^* P(\tau) x(0, \tau) = 0 \qquad \text{for all } \tau \geq 0.$$

Now consider a sequence $(\tau_n) \subseteq \mathbb{R}_+$ such that $\tau_n \to \infty$ and observe that, by (24), $\|x(0, \tau_n)\| = 1$ for all n. Now, using the compactness of the unit sphere in $\mathbb{R}^n$, we may assume w.l.g. that the sequence $(x(0, \tau_n))$ converges to $x_0 \in \mathbb{R}^n$ with $\|x_0\| = 1$. Moreover, by (25), $x_0 \in NO^+(C, A)$. On the other hand, since $P(\tau) \to P_+$, we have also

$$\lim_{\tau \to \infty} x(0, \tau_n)^* P(\tau_n) x(0, \tau_n) = x_0^* P_+ x_0 = 0.$$

So, using Fact 1, $x_0 \in \mathcal{N}(P_+) = NO^{0-}(C, A)$. Hence, finally x_0 is in $NO^+(C, A) \cap NO^{0-}(C, A) = \{ 0 \}$; i.e., $x_0 = 0$, which contradicts $\|x_0\| = 1$. ∎

Remark 1. The idea of the proof above stems from [5, Necessity Proof of Proposition 5.3, p. 103].

3. Without Loss of Generality (A, B) Is Controllable

The purpose of this section is to show that in Theorem 1, assumption (2), i.e. "(A, B) is stabilizable," may w.l.g. be replaced by "(A, B) is controllable." In words, we mean that when using a decomposition of the state space in controllable and uncontrollable states, then, under assumption (2), the convergence (21) and condition (22) hold iff these conditions hold for the controllable part of the problem under investigation. More precisely, use the decomposition

$$\mathbb{R}^n = C(A, B) \dot{+} NO(B^*, A^*), \tag{26}$$

where, by assumption (2), $NO(B^*, A^*) \subseteq \mathcal{L}^-(A^*)$. We get w.l.g., with controllable items indexed by 1 and $\dim C(A, B) = k$,

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \qquad C = [C_1 \; C_2], \tag{27a}$$

$$P = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{bmatrix}, \qquad S = \begin{bmatrix} S_1 & S_{12} \\ S_{12}^* & S_2 \end{bmatrix}. \tag{27b}$$

Here $A_1 \in \mathbb{R}^{k \times k}$, $A_1 = A|_{C(A, B)}$, $\operatorname{Re} \lambda(A_2) < 0$, and $P_1(\tau) \in \mathbb{R}^{k \times k}$ satisfies a *reduced controllable RDE* under $P_1(0) = S_1 = S_1^* \geq 0$ (obtained from (1) by indexing all symbols by 1). We note hereby that w.l.g. any state $x \in \mathbb{R}^n$ is decomposed according to

$$x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, \tag{27c}$$

where

$$C(A, B) = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R}^k \right\}. \tag{27d}$$

We have then that, in Theorem 1, w.l.g. (A, B) is controllable if, for a given $P(0) = S = S^* \geq 0$,

(a) $\mathcal{N}(S) \cap NO^+(C, A) = \{0\}$ iff $\mathcal{N}(S_1) \cap NO^+(C_1, A_1) = \{0\}$, and

(b) $\lim_{\tau\to\infty} P(\tau) = P_+$ iff $\lim_{\tau\to\infty} P_1(\tau) = P_{+1}$,

where P_{+1} is the strong solution of a reduced ARE with (A_1, B_1) controllable. Assertions (a) and (b) above are proved in Lemmas 1 and 3 below. These lemmas are established under the assumptions of Theorem 1 and the relations (27).

LEMMA 1. *There holds*

$$\mathcal{N}(S) \cap NO^+(C, A) = \{0\} \quad \textit{iff}\ \mathcal{N}(S_1) \cap NO^+(C_1, A_1) = \{0\}. \tag{28}$$

Proof. Note that (A, B) is stabilizable iff $L^{0+}(A) \subseteq C(A, B)$. Hence, under (2),

$$NO^{0+}(C, A) \subseteq C(A, B), \tag{29}$$

whence

$$NO^+(C, A) \cap C(A, B) = NO^+(C, A). \tag{30}$$

Claim (28) follows now easily using (30) and (27d). ∎

The following result is well known from LQ-theory (see e.g. [7, p. 336]) and used in Lemma 3. Here $P(\tau)$ means the solution of the RDE (1), which is also the optimal quadratic cost matrix of a finite horizon LQ-optimal control problem.

LEMMA 2. *If (A, B) is stabilizable and $P(0) = S = S^* \geq 0$, then the symmetric p.s.d. matrix function $P(\tau)$ is bounded on $\tau \geq 0$; more precisely, there exists $k > 0$ such that, for all $x_0 \in \mathbb{R}^n$,*

$$x_0^* P(\tau) x_0 \leq k \cdot \|x_0\|^2 \qquad \textit{on}\ \tau \geq 0.$$

LEMMA 3. *For a given* $P(0) = S = S^* \geq 0$ *one has*

$$\lim_{\tau\to\infty} P(\tau) = P_+ \qquad \textit{iff} \lim_{\tau\to\infty} P_1(\tau) = P_{+1}. \tag{31}$$

Proof. We handle only sufficiency since necessity is obvious. For this purpose we use the decomposition (26) leading to the relations (27) and study the difference

$$D := \begin{bmatrix} D_1 & D_{12} \\ D_{12}^* & D_2 \end{bmatrix} := P - P_+. \tag{32}$$

Note that we have $\lim_{\tau\to\infty} D_1(\tau) = 0$ and we are done if $D_{12}(\tau)$ and $D_2(\tau)$ tend to zero as τ tends to infinity. Now standard manipulations using the RDE (1) and an ARE solved by P_+ give, with

$$A_+ := A - BB^*P_+, \tag{9}$$

$$\dot{D}(\tau) = A_+^*D(\tau) + D(\tau)A_+ - D(\tau)BB^*D(\tau), \tag{33a}$$

$$D(0) = S - P_+. \tag{33b}$$

Moreover, using the ARE and relations (27) one gets, with

$$P_+ = \begin{bmatrix} P_{+1} & P_{+12} \\ P_{+12}^* & P_{+2} \end{bmatrix},$$

$$A_+ = \begin{bmatrix} A_1 - B_1B_1^*P_{+1} & A_{12} - B_1B_1^*P_{+12} \\ 0 & A_2 \end{bmatrix} =: \begin{bmatrix} A_{+1} & A_{+12} \\ 0 & A_2 \end{bmatrix}, \tag{33c}$$

where $\text{Re}\,\lambda(A_{+1}) \leq 0$ and $\text{Re}\,\lambda(A_2) < 0$. Note here that these relations imply (see e.g. [7, p. 185]):

(a) for any $\varepsilon > 0$ there exists a constant $M_1 > 0$ such that

$$\|\exp(A_{+1}\,\tau)\| \leq M_1 \cdot \exp(\varepsilon\tau) \qquad \text{on } \tau \geq 0; \tag{33d}$$

(b) there exist constants $\sigma_2 > 0$ and $M_2 > 0$ such that

$$\|\exp(A_2\tau)\| \leq M_2 \cdot \exp(-\sigma_2\tau) \qquad \text{on } \tau \geq 0. \tag{33e}$$

Now, using (32), (33a), and (33c), we have (omitting the dependence upon τ)

$$\dot{D}_{12} = A_{+1}^*D_{12} + D_{12}A_2 + F, \tag{33f}$$

where

$$F := D_1 \cdot (A_{+12} - B_1 B_1^* D_{12}), \tag{33g}$$

and

$$\dot{D}_2 = A_2^* D_2 + D_2 A_2 + G \tag{33h}$$

where

$$G := A_{+12}^* D_{12} + D_{12}^* A_{+12} - D_{12}^* B_1 B_1^* D_{12}. \tag{33i}$$

We show now that $D_{12}(\tau)$ tends to zero as $\tau \to \infty$. From (33f) and (33b) we get

$$\begin{aligned} D_{12}(\tau) &= \exp(A_{+1}^* \tau) \cdot (S - P_+)_{12} \cdot \exp(A_2 \tau) \\ &+ \int_o^\tau \exp(A_{+1}^*(\tau - \xi))\, F(\xi) \exp(A_2(\tau - \xi))\, d\xi. \end{aligned} \tag{34a}$$

Use now (33d) and (33e) where $\varepsilon > 0$ is chosen sufficiently small such that $\mu := \sigma_2 - \varepsilon > 0$. It follows then that

$$\|D_{12}(\tau)\| \le M_1 M_2 (\|(S - P_+)_{12}\| \exp(-\mu\tau) + \int_0^\tau \|F(\xi)\| \exp(-\mu(\tau - \xi))\, d\xi). \tag{34b}$$

Note here that the function $F(\tau)$ given by (33g) is a continuous bounded function that tends to zero as $\tau \to \infty$ (this holds because $D(\tau)$ is bounded by Lemma 2 and $D_1(\tau)$ tends to zero as $\tau \to \infty$). Now the first term on the right-hand side (RHS) of (34b) tends to zero as $\tau \to \infty$ and so does the second term. Indeed, the latter is the convolution of $\exp(-\mu\tau)$ with the function $\|F(\tau)\|$ (see e.g. [7, Proof of Theorem 7.2.75, p. 192]). Hence

$$\lim_{\tau \to \infty} D_{12}(\tau) = 0. \tag{35}$$

We show now that the same holds for $D_2(\tau)$. Observe that, by (33h) and (33b),

$$\begin{aligned} D_2(\tau) &= \exp(A_2^* \tau) \cdot (S - P_+)_2 \cdot \exp(A_2 \tau) \\ &+ \int_0^\tau \exp(A_2^*(\tau - \xi))\, G(\xi) \exp(A_2(\tau - \xi))\, d\xi. \end{aligned}$$

Hence, by (33e),

$$\|D_2(\tau)\| \leq \|(S - P_+)_2\| \exp(-2\sigma_2\tau) + \int_0^\tau \|G(\xi)\| \exp(-2\sigma_2(\tau - \xi))\, d\xi). \tag{36}$$

Note here that, by (35), the function $G(\tau)$, given by (33i), is a continuous bounded function that tends to zero as $\tau \to \infty$. Note also that $\sigma_2 > 0$. Hence, using observations similar to those above, the RHS of (36) tends to zero as $\tau \to \infty$. So

$$\lim_{\tau\to\infty} D_2(\tau) = 0. \quad \blacksquare \tag{37}$$

Remark 2. It follows from the proof of Lemma 3 that, if $D_1(\tau)$ tends to zero exponentially (exp.) as $\tau \to \infty$, then so do $D_{12}(\tau)$ and $D_2(\tau)$. Thus that $P_1(\tau) \to P_{+1}$ exp. as $\tau \to \infty$ implies $P(\tau) \to P_+$ exp. as $\tau \to \infty$. In addition, the inequalities (34b) and (36) provide quantitative estimates of the speed of convergence of $P(\tau)$ towards P_+, provided that one has previously obtained computable characteristics of the convergence of $P_1(\tau)$ towards P_{+1}.

4. Sufficiency of Condition (22)

We prove now

THEOREM 3. *Consider the* RDE (1) *where*

$$(A, B) \textit{ is controllable.} \tag{38}$$

Let $P(0) = S = S^* \geq 0$ *be given. Then that*

$$\mathcal{N}(S) \cap NO^+(C, A) = \{0\} \tag{22}$$

implies

$$\lim_{\tau\to\infty} P(\tau) = P_+. \tag{21}$$

DISCUSSION. Our method will proceed partially along the lines of the proof of [5, Prop. 5.3, pp. 101–103]. Here, with $P_- \leq 0$ the symmetric n.s.d. antistrong solution of the ARE, we use B, $S - P_-$, and $A_- := A -$

BB^*P_- instead of G, A, and $-F_*$ resp. in [5]. We shall also use differences with respect to $P_- \leq 0$, namely

$$\Delta(\tau) := P(\tau) - P_- \geq 0, \tag{39}$$

where $\Delta(\tau)$ solves the RDE

$$\dot{\Delta}(\tau) = A_-^*\Delta(\tau) + \Delta(\tau)A_- - \Delta(\tau)BB^*\Delta(\tau), \qquad \tau \geq 0, \tag{40a}$$

with

$$\Delta(0) = S - P_- \geq 0. \tag{40b}$$

It follows that $\Delta(\tau)$ satisfies the standard evolution formula

$$\Delta(\tau) = \exp(A_-^*\tau)\ \Delta(0)\ [I + \exp(A_-\tau)\ M(\tau)\ \exp(A_-^*\tau)\ \Delta(0)]^{-1}\ \exp(A_-\tau) \tag{41a}$$

on $\tau \geq 0$, where

$$M(\tau) := \int_0^\tau \exp(-A_-\xi)\ BB^*\ \exp(-A_-^*\xi)\ d\xi \tag{41b}$$

is the p.d. controllability grammian of the pair $(-A_-, B)$, with $\mathrm{Re}\ \lambda(A_-) \geq 0$.

Note that after some manipulations one gets [5, Formula (5.11), pp. 101–102], i.e., on $\tau > 0$,

$$\Delta(\tau) = M(\tau)^{-1} - B(\tau)^{-1},$$

where

$$B(\tau) := M(\tau) + M(\tau)\ \exp(A_-^*\tau)\ (S - P_-)\ \exp(A_-\tau)\ M(\tau).$$

Note also that we are done if

$$\lim_{\tau\to\infty} \Delta(\tau) = P_+ - P_-. \tag{42}$$

The method attempted in [5, pp. 102–103] is to show that

(a) $$\lim_{\tau\to\infty} M(\tau)^{-1} = P_+ - P_-. \tag{43}$$

and

(b) condition (22), or equivalently (52) below, implies

$$\lim_{\tau\to\infty} B(\tau)^{-1} = 0.$$

The latter assertion however is not explained on [5, p. 103] and is not obvious. Hence a more detailed analysis is needed. For this purpose our method will use the well-known *comparison principle* of the RDE (40a); see e.g. [3].

LEMMA 4. *Let L_1 and L_2 be two square $n \times n$ symmetric p.s.d. matrices and let $\Delta(\tau, L_i)$ denote the solution of* (40a) *due to* $\Delta(0) = L_i$, $i = 1, 2$. *Then*

$$L_1 \leq L_2 \tag{44}$$

implies

$$\Delta(\tau, L_1) \leq \Delta(\tau, L_2) \qquad \text{on } \tau \geq 0. \quad \blacksquare \tag{45}$$

Now using this lemma we shall find below that, for any given $\Delta(0) = S - P_- \geq 0$, $\Delta(\tau, S - P_-)$ is bounded above and below by two expressions which tend to $P_+ - P_-$ as $\tau \to \infty$. As a consequence (42) holds and we are done. We are now ready to build up the proof of Theorem 3. This will be the goal of the decomposition and Lemmas 5 to 7 below.

As in [5, p. 102] we decompose $\mathbb{R}^n$ as

$$\mathbb{R}^n = \mathcal{L}^+(A_-) \dotplus \mathcal{L}^0(A_-); \qquad x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, \tag{46}$$

where A_- meets (11) and (12). Moreover, by Fact 1(g),

$$\mathcal{L}^0(A_-) = \mathcal{N}(P_+ - P_-) = \mathcal{N}(P_+) \cap \mathcal{N}(P_-) = NO^0(C, A). \tag{47}$$

Hence, in these coordinates,

$$A_- = \begin{bmatrix} A_{-1} & 0 \\ 0 & A_{-2} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{48a}$$

$$P_- = \begin{bmatrix} P_{-1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} S_1 & S_{12} \\ S_{12}^* & S_2 \end{bmatrix}, \tag{48b}$$

with

$$\operatorname{Re} \lambda(A_{-1}) > 0 \qquad \text{and} \qquad \operatorname{Re} \lambda(A_{-2}) = 0 \tag{48c}$$

(A_{-1} is exp. antistable and A_{-2} is critical). Moreover,

$$P_+ - P_- = \begin{bmatrix} (P_+ - P_-)_1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0, \tag{49a}$$

with

$$(P_+ - P_-)_1 > 0. \tag{49b}$$

Finally, the solution $\Delta(\tau)$, (39), and the grammian $M(\tau)$, (41b), read

$$\Delta := \begin{bmatrix} \Delta_1 & \Delta_{12} \\ \Delta_{12}^* & \Delta_2 \end{bmatrix}, \qquad M := \begin{bmatrix} M_1 & M_{12} \\ M_{12}^* & M_2 \end{bmatrix}. \tag{50}$$

Using the considerations above we have now

LEMMA 5. *Let (A, B) be controllable. Then, with $S = S^* \geq 0$,*

$$\mathcal{N}(S) \cap NO^+(C, A) = \{0\} \tag{22}$$

iff

$$S - P_- \textit{ is positive definite on } \mathcal{L}^+(A_-),$$

i.e.,

$$\mathcal{N}(S - P_-) \cap \mathcal{L}^+(A_-) = \{0\}, \tag{51}$$

or equivalently

$$(S - P_-)_1 > 0. \tag{52}$$

Proof. The last equivalence (51)–(52) follows by (46) and (48b) with $S - P_- \geq 0$, because $S \geq 0$ and $P_- \leq 0$. For the first equivalence (22)–(51) observe that, by (15),

$$\mathcal{N}(P_-) = NO^{0+}(C, A)$$

and that

$$\mathcal{N}(P_-) \cap \mathcal{L}^+(A_-) = NO^+(C, A).$$

The latter holds because

$$x \in NO^+(C, A)$$

iff

$$\begin{cases} P_- x = 0, \text{ and} \\ \lim_{t \to -\infty} \exp(At)x = \lim_{t \to -\infty} \exp(A_- t)x = 0, \end{cases}$$

i.e., iff

$$x \in \mathcal{N}(P_-) \cap \mathcal{L}^+(A_-).$$

Hence

$$\begin{aligned} \mathcal{N}(S - P_-) \cap \mathcal{L}^+(A_-) &= \mathcal{N}(S) \cap \mathcal{N}(P_-) \cap \mathcal{L}^+(A_-) \\ &= \mathcal{N}(S) \cap NO^+(C, A). \quad \blacksquare \end{aligned}$$

Remark 3. Faurre *et al.* [5, p. 103] use condition (52) denoted as $A_{11} > 0$ in the proof of their Proposition 5.3. They conclude in their condition (5.9) that this is equivalent to

$$\begin{aligned} \mathcal{N}(S) \cap NO^{0+}(C, A) = \mathcal{N}(S) \cap \mathcal{N}(P_-) = \mathcal{N}(S - P_-) \subseteq \mathcal{N}(P_+ - P_-) \\ = NO^0(C, A). \end{aligned} \tag{53}$$

In view of the decomposition (46)–(47), it is easy to see that this condition is equivalent to condition (22) if $\mathcal{N}(S) \cap NO^{0+}(C, A)$ is A-invariant, but not otherwise. As such their condition implies (52), but the reverse is not necessarily true.

For example, set $A = \operatorname{diag}[1, 0]$, $B = I_2$, and $C = 0$. Then the antistrong solution of the ARE (3) is $P_- = 0$, whence $A_- = A$. Observe also that the decomposition (46)–(47) holds with

$$\mathcal{L}^+(A_-) = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad \mathcal{L}^0(A_-) = NO^0(C, A) = \operatorname{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

Now take

$$S = S - P_- = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix};$$

obviously

$$\mathcal{N}(S - P_-) = \mathcal{N}(S) = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right).$$

Clearly we have that $\mathcal{N}(S - P_-) \cap \mathcal{L}^+(A_-) = \{0\}$, or equivalently, by Lemma 5, (22) holds; however, $\mathcal{N}(S - P_-)$ is not contained in $NO^0(C, A)$.

The following lemma is a sharpening of (43) needed for proving Theorem 3.

LEMMA 6. *Let (A, B) be controllable. Then the $(-A_-, B)$-controllability grammian $M(\tau)$, given by* (41b), *satisfies*:

(a) *$M(\tau)^{-1}$ is a supremal solution on* $(0, \infty)$ *of the* RDE (40a), *in the sense that, for all $\Delta(0) = \Delta(0)^* \geq 0$,*

$$\Delta(\tau, \Delta(0)) \leq M(\tau)^{-1} \qquad \text{on } \tau > 0. \tag{54}$$

(b)
$$\lim_{\tau\to\infty} M(\tau)^{-1} = P_+ - P_- \qquad \text{(gap)}. \tag{43}$$

(c) *Using the decomposition* (46) *with $M(\tau)$ partitioned as in* (50), *we have*

$$\lim_{\tau\to\infty} M_1(\tau) = (P_+ - P_-)_1^{-1} \tag{55a}$$

$$\lim_{\tau\to\infty} M_{12}(\tau) = M_{12} \tag{55b}$$

$$\lim_{\tau\to\infty} M_2(\tau) = \infty \cdot I. \tag{55c}$$

Proof. (a) Observe that, by (41b), $M(\tau)$ meets, on $\tau \geq 0$,

$$\dot{M}(\tau) = -A_-M(\tau) - M(\tau)A_-^* + BB^*, \qquad M(0) = 0.$$

Now (A, B) is controllable, hence so is $(-A_-, B)$. Thus, on $\tau > 0$, $M(\tau)$ is invertible and satisfies

$$\frac{d}{d\tau}[M(\tau)^{-1}] = A_-^*M^{-1} + M^{-1}A_- - M^{-1}BB^*M^{-1} \tag{56}$$

with

$$M(0)^{-1} = \infty \cdot I.$$

Now, for any $\Delta(0) \geq 0$, for $\mu > 0$ sufficiently large,

$$\Delta(0) \leq \mu \cdot I; \tag{57}$$

whence, by Lemma 4,

$$\Delta(\tau, \Delta(0)) \leq \Delta(\tau, \mu I) \qquad \text{on } \tau \geq 0. \tag{58}$$

Moreover, using the explicit formula (41a), on $\tau > 0$,

$$\begin{aligned} \Delta(\tau, \mu I) &= \exp(A_-^*\tau)[\mu^{-1}I + \exp(A_-\tau)M(\tau)\exp(A_-^*\tau)]^{-1}\exp(A_-\tau) \\ &\leq \exp(A_-^*\tau)[\exp(A_-\tau)M(\tau)\exp(A_-^*\tau)]^{-1}\exp(A_-\tau) \qquad (59) \\ &= M(\tau)^{-1}. \end{aligned}$$

Hence, by (56) and (58)–(59), (54) holds.

(b) Note that, by (41b), $M(\tau)$ increases as τ increases, whence $M(\tau)^{-1}$ decreases. Since $M(\tau)^{-1} > 0$, it follows that $M(\tau)^{-1}$ converges as $\tau \to \infty$. Since $M(\tau)^{-1}$ is a supremal solution of the RDE (40a), it must converge to the maximal solution $P_+ - P_-$ of the corresponding ARE; i.e., (43) holds.

(c) This follows using (41b), (48), (43), and (49). ∎

Remark 4. (a) A finer analysis shows that $M_1(\tau)$ and $M_{12}(\tau)$ converge exp. fast to resp. $M_1 := (P_+ - P_-)_1^{-1} > 0$ and M_{12} as $\tau \to \infty$. Moreover, $M_2(\tau)$ diverges at a polynomial rate, i.e.

$$M_2(\tau) = 0(\tau^{2m+1}) \qquad \text{as } \tau \to \infty \tag{60}$$

for some $m \geq 0$.

(b) Another fine analysis shows that $M(\tau)^{-1}$ converges to $P_+ - P_-$ as $\tau \to \infty$, as $O(\tau^{-(2m+1)})$, i.e., *not* exp. fast (because $M_2(\tau) = 0(\tau^{2m+1})$) if $NO^0(C, A) = \mathcal{L}^0(A_-) \neq \{0\}$. This convergence is exponential iff the system has no critical unobservable modes, i.e., $NO^0(C, A) = \{0\}$.

We have now the following crucial result.

LEMMA 7. *Let* (A, B) *be controllable. Let* $(S - P_-)_1$ *be p.d. as in Lemma* 5. *Then there exists a symmetric p.s.d. matrix L such that*

$$L \leq S - P_- \tag{61}$$

for which

$$\lim_{\tau \to \infty} \Delta(\tau, L) = P_+ - P_- \text{ exp. fast}, \tag{62}$$

where $\Delta(\tau, L)$ *is the solution of the* RDE (40a) *due to* $\Delta(0) = L$.

Proof. Consider the decomposition (46) and the relations (47)–(50), where in particular

$$S - P_- =: \begin{bmatrix} \Delta S_1 & \Delta S_{12} \\ \Delta S_{12}^* & \Delta S_2 \end{bmatrix}, \tag{63}$$

where, by assumption, $(S - P_-)_1 = \Delta S_1 > 0$. Define now

$$L := \begin{bmatrix} \Delta S_1 & \Delta S_{12} \\ \Delta S_{12}^* & \Delta S_{12}^*(\Delta S_1)^{-1}\Delta S_{12} \end{bmatrix}. \tag{64}$$

Then, since $S - P_- \geq 0$ with $\Delta S_1 > 0$,

$$L \leq S - P_- \tag{61}$$

(indeed here $S - P_- \geq 0$ iff $\Delta S_2 - \Delta S_{12}^*(\Delta S_1)^{-1}\Delta S_{12} \geq 0$); moreover, there is a square nonsingular matrix D_1 and a matrix D_2 such that

$$\Delta S_1 = D_1^* D_1 \qquad \text{and} \qquad \Delta S_{12} = D_1^* D_2. \tag{65}$$

Hence, with

$$D := [D_1 \; D_2], \tag{67}$$

we get

$$L = D^* D. \tag{68}$$

Now, using the explicit formula (41a) and Eq. (68), we get that $\Delta(\tau, L)$ satisfies

$$\Delta(\tau, L) = \exp(A_-^*\tau)D^*[I + D\exp(A_-\tau)M(\tau)\exp(A_-^*\tau)D^*]^{-1}D\exp(A_-\tau). \tag{69}$$

Consider now the partitioning (50) induced by the decomposition (46); i.e.,

$$\Delta(\tau, L) := \begin{bmatrix} \Delta_1(\tau) & \Delta_{12}(\tau) \\ \Delta_{12}^*(\tau) & \Delta_2(\tau) \end{bmatrix}. \tag{70}$$

Then, using (64)–(70) and (48a), it follows that the properties

$$\Delta_1(0) = \Delta S_1 = (S - P_-)_1 > 0 \tag{71a}$$

and

$$\Delta_2(0) = \Delta S_{12}^* [\Delta S_1]^{-1} \Delta S_{12} = \Delta_{12}^*(0) [\Delta_1(0)]^{-1} \Delta_{12}(0) \tag{71b}$$

hold for all $\tau \geq 0$, i.e.,

$$\Delta_1(\tau) > 0 \qquad \text{on } \tau \geq 0, \tag{72a}$$

and

$$\Delta_2(\tau) = \Delta_{12}^*(\tau) [\Delta_1(\tau)]^{-1} \Delta_{12}(\tau) \qquad \text{on } \tau \geq 0. \tag{72b}$$

Define now

$$J_1(\tau) := \Delta_1(\tau)^{-1} \tag{73}$$

and

$$J_{12}(\tau) := [\Delta_1(\tau)]^{-1} \Delta_{12}(\tau), \tag{74}$$

and apply the relations (48) to the RDE (40a) using (70)–(74). It follows then that this RDE is transformed into a system of two successive linear matrix differential equations and an algebraic relation; viz. on $\tau \geq 0$,

$$\dot{J}_1(\tau) = -J_1(\tau)A_{-1}^* - A_{-1}J_1(\tau) + (B_1 + J_{12}(\tau)B_2)(B_1 + J_{12}(\tau)B_2)^*, \tag{75a}$$

$$\dot{J}_{12}(\tau) = -A_{-1}J_{12}(\tau) + J_{12}(\tau)A_{-2}, \tag{75b}$$

$$\Delta_2(\tau) = J_{12}^*(\tau)[J_1(\tau)]^{-1} J_{12}(\tau), \tag{75c}$$

where

$$J_1(0) = [\Delta S_1]^{-1} = [(S - P_-)_1]^{-1} \tag{75d}$$

$$J_{12}(0) = [\Delta S_1]^{-1} \Delta S_{12} \tag{75e}$$

and

$$\operatorname{Re} \lambda(-A_{-1}) < 0 \qquad \text{and} \qquad \operatorname{Re} \lambda(A_{-2}) = 0. \tag{75f}$$

It follows therefore, by (75b), that

$$J_{12}(\tau) = \exp(-A_{-1}\tau)\, J_{12}(0) \exp(A_{-2}\tau), \tag{76}$$

with $J_{12}(0)$ given by (75e). Hence, by (75f),

$$\lim_{\tau\to\infty} J_{12}(\tau) = 0 \qquad \text{exp. fast.} \tag{77}$$

Consider now (75a) rewritten as

$$\dot{J}(\tau) = -J_1(\tau)A^*_{-1} - A_{-1}J_1(\tau) + B_1B_1^* + F(\tau),$$

with the perturbation term

$$F(\tau) := J_{12}(\tau)B_2B_1^* + B_1B_2^*J_{12}^*(\tau) + J_{12}(\tau)B_2B_2^*J_{12}^*(\tau). \tag{78a}$$

We get then, on $\tau \geq 0$,

$$\begin{aligned} J_1(\tau) &= \exp(-A_{-1}\tau)\, J_1(0) \exp(-A^*_{-1}\tau) \\ &\quad + \int_0^\tau \exp(-A_{-1}\xi)\, B_1B_1^* \exp(-A^*_{-1}\xi)\, d\xi \\ &\quad + \int_0^\tau \exp(-A_{-1}(\tau-\xi))\, F(\xi) \exp(-A^*_{-1}(\tau-\xi))\, d\xi. \end{aligned} \tag{78b}$$

Note here that, by (77), the perturbation $F(\tau)$ is a continuous bounded function that tends to zero exp. fast as $\tau \to \infty$. Moreover, by (75f), the third term on the RHS of (78b) is bounded by the convolution of the function $\exp(-2\sigma_1\tau)$, with $\sigma_1 > 0$, and the bounded function $\|F(\tau)\|$, which tends to zero exp. fast. Hence the third term on the RHS of (78b) tends to zero exp. fast as $\tau \to \infty$. It follows now also by (75f) that the first term on the RHS of (78b) tends to zero exp. fast as $\tau \to \infty$. Finally, the second term on the RHS of (78b) is (by (41b) and (48a)) equal to $M_1(\tau)$, which, by (75f) and (55a), converges exp. fast to $M_1 = [(P_+ - P_-)_1]^{-1}$. It follows by the results above that

$$\lim_{\tau\to\infty} J_1(\tau) = [(P_+ - P_-)_1]^{-1} \qquad \text{exp. fast.} \tag{79}$$

Observe now that (77) and (79) applied to (73), (74), and (75c) give

$$\lim_{\tau\to\infty} \Delta_1(\tau) = (P_+ - P_-)_1 \qquad \text{exp. fast},$$

$$\lim_{\tau\to\infty} \Delta_{12}(\tau) = 0 \qquad \text{exp. fast},$$

$$\lim_{\tau\to\infty} \Delta_2(\tau) = 0 \qquad \text{exp. fast}.$$

Therefore, by (70) and (49a),

$$\lim_{\tau\to\infty} \Delta(\tau, L) = \begin{bmatrix} (P_+ - P_-)_1 & 0 \\ 0 & 0 \end{bmatrix} = P_+ - P_- \qquad \text{exp. fast}.$$

Hence (62) holds. ∎

Remark 5. By (63)–(64) it follows, under assumption (22) or equivalently $(S - P_-)_1 > 0$, that

$$L = S - P_- \tag{80}$$

iff

$$\dim[\mathcal{N}(S) \cap NO^{0+}(C, A)] = \dim[NO^0(C, A)]. \tag{81}$$

To see this, observe that, when $(S - P_-)_1 > 0$, (80) holds iff $\mathrm{rk}[S - P_-]$ is *minimal*, i.e.

$$\mathrm{rk}[S - P_-] = \mathrm{rk}[(S - P_-)_1] = \dim[\mathcal{L}^+(A_-)],$$

or equivalently

$$\mathrm{null}[S - P_-] = n - \dim[L^+(A_-)] = \dim[\mathrm{L}^0(A_-)];$$

(here $\mathrm{rk}[M]$ and $\mathrm{null}[M]$ denote resp. the rank and the nullity (i.e., the dimension of the null space) of the matrix M). The conclusion (81) follows then by (15) and (47). Note also that (80) means that $S - P_-$ is congruent to

$$\begin{bmatrix} (S - P_-)_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It does not mean that $S - P_-$ is equal to this matrix.

It is easy to conclude this section.

PROOF OF THEOREM 3. By Lemma 5, the assumption (22) is equivalent to $(S - P_-)_I > 0$. Hence, by Lemma 7, there exists a p.s.d. matrix $L \geq 0$ such that $L \leq S - P_-$; whence, by Lemma 4,

$$\Delta(\tau, L) \leq \Delta(\tau, S - P_-) \qquad \text{on } \tau \geq 0.$$

By Lemmas 6 and 7, this can be completed to

$$\Delta(\tau, L) \leq \Delta(\tau, S - P_-) \leq M(\tau)^{-1} \qquad \text{on } \tau > 0,$$

where the left and right expressions tend to $P_+ - P_-$ and $\tau \to \infty$. Hence

$$\lim_{\tau\to\infty} \Delta(\tau, S - P_-) = P_+ - P_-.$$

It follows by (39) that

$$\lim_{\tau\to\infty} P(\tau, S) = P_- + \lim_{\tau\to\infty} \Delta(\tau, S - P_-) = P_+ . \quad \blacksquare$$

Remark 6. (a) It follows by Remark 4(b) and Lemma 7 that $P(\tau, S)$ may not approach P_+ exp. fast when $\mathscr{L}^0(A_-) = NO^0(C, A) \neq \{0\}$. However, if $NO^0(C, A) = \{0\}$, i.e., if the Hamiltonian matrix has no eigenvalues on the imaginary axis, then the speed of convergence is exponential, as shown previously in [10].

(b) A finer analysis based on Remark 4(b) and Lemma 7 and extensions of these ideas shows that the speed of convergence is exponential iff $\text{rk}[S - P_-]$ is minimal.

For example, let A, B, C, and S be as in the example above Lemma 6. Obviously, $\text{rk}[S - P_-] = \text{rk}[S]$ 1 is minimal. Observe also that, by formula (41),

$$P(\tau) = \frac{1}{d(\tau)} \begin{bmatrix} 4 & 2\exp(-\tau) \\ 2\exp(-\tau) & \exp(-2\tau) \end{bmatrix},$$

with

$$d(\tau) := 2 + \exp(-2\tau)\,(\tau - 1).$$

Hence, as $\tau \to \infty$,

$$\lim_{\tau\to\infty} P(\tau) = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = P_+ \qquad \underline{\text{exp.}}\ \text{fast}.$$

Now take $S = S - P_- = I_2$, whence $\mathrm{rk}[S - P_-] = \mathrm{rk}[S] = 2$ is not minimal. Then by (41)

$$P(\tau) = \begin{bmatrix} 2\,[\exp(-2\tau) + 1]^{-1} & 0 \\ 0 & [1 + \tau]^{-1} \end{bmatrix},$$

and $P(\tau)$ tends to P_+ as $\tau \to \infty$. But the convergence is *not* exponential.

Remark 7. The proof of Theorem 3 shows that, under assumption (22), w.l.g. we may assume that $\mathrm{rk}[S - P_-]$ is minimal, or equivalently that (80) or (81) hold, as announced in Comment 1(c). This is also easily illustrated by the example above Lemma 6.

5. Conclusion

The proof of Theorem 1 follows now easily from Theorem 2 (necessity) and Theorem 3 (sufficiency) since w.l.g. (A, B) is controllable.

Comments. 2 (a) Recently De Nicolao and Gevers [3, Theorem 3] proved that the convergence (21) holds under assumption (2) if

$$\mathcal{N}(S) \subseteq NO^0(C, A). \tag{82}$$

This follows easily by Theorem 1. Indeed, (22) is then obviously true. Condition (82) however is not implied by (22) and hence is not necessary, as was rightly observed by De Nicolao and Gevers.

(b) When $NO^0(C, A) = \{0\}$ then obviously criterion (22) reads also

$$\mathcal{N}(S) \cap NO^{0+}(C, A) = \{0\}.$$

Hence criterion (22) generalizes [1, condition (32)] under the sole assumption of (2). The analysis above shows that when $NO^0(C, A) \neq \{0\}$ (or equivalently $L^0(H) \neq \{0\}$), then the presence of unobservable modes on the j-axis may slow down but does not prevent the convergence towards P_+.

In LQ-optimal control the following is important.

Corollary 1. *Consider the* RDE (1) *where*

$$\text{(A, B) } \textit{is stabilizable}. \tag{2}$$

Then

$$\lim_{\tau\to\infty} P(\tau) = P_+ \qquad \text{for all } P(0) = S = S^* \geq 0 \tag{83}$$

iff

$$NO^+(C, A) = \{0\}. \tag{84}$$

Remark 8. Condition (84) is a criterion for the uniqueness of p.s.d. symmetric solutions of the ARE (3) under assumption (2); see [6].

The analysis above handles only the convergence question (21) without considering the solution of an infinite-horizon LQ-optimal control problem. This is now done briefly.

For a given $S = S^* \geq 0$, where $S \in \mathbb{R}^{n\times n}$, we consider the infinite horizon cost

$$J(x_0, u, S) = \int_0^\infty (\|Cx(t)\|^2 + \|u(t)\|^2)\, dt + \lim_{t_1\to\infty} x(t_1)^* Sx(t_1) \tag{85}$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in \mathbb{R}^n,$$

$$u(\cdot) \text{ is continuous.}$$

Let U_{ad} be the set of admissible controls for which the cost exists for all $x_0 \in \mathbb{R}^n$. We are interested in solving the infinite-horizon LQ-optimal control problem $\text{LQ}^\infty(S)$, viz.

$$\operatorname*{Min}_{u(\cdot)\in U_{\text{ad}}} J(x_0, u, S) \tag{86}$$

where $x_0 \in \mathbb{R}^n$ is arbitrary. We assume here that (2) holds and we consider the question under what criterion will the optimal cost be $x_0^* P_+ x_0$ and the latter be realized by the state feedback law

$$\hat{u}(t) = -B^*P_+x(t) \qquad \text{on } t \geq 0. \tag{87}$$

Note also that we solve here the infinite-horizon problem, as the horizon $t_1 \to \infty$, as a limit cost problem realizable by the limit control. Indeed, the latter is known. Standard analysis using the Bellman–Gronwall lemma shows that, under convergence condition (22), the finite-horizon optimal closed-loop transition matrix tends to $\exp(A_+t)$ uniformly on bounded intervals, whence the same happens for the finite horizon optimal control towards $-B^*P_+\exp(A_+t)x_0$. Hence the limit control exists uniformly on bounded intervals and is realizable by state feedback law (87). Therefore the realizability of $x_0^*P_+x_0$ by the limit control is equivalent to its realizability by the feedback law (87).

A first observation is that, with A_+ described by (9)–(10) and (14), (16),

$$\lim_{t\to\infty} P_+\exp(A_+t)x_0 = 0 \qquad \text{for all } x_0 \in \mathbb{R}^n;$$

hence the closed-loop control induced by (87) is exp. stable; i.e., for all $x_0 \in \mathbb{R}^n$,

$$u(t) = -B^*P_+\exp(A_+t)x_0$$

satisfies

$$\lim_{t\to\infty} u(t) = 0 \qquad \text{exp. fast.}$$

However, for a given $S = S^* \geq 0$, it is easy to see that

$$\lim_{t\to\infty} S\exp(A_+t)x_0 = 0 \qquad \text{for all } x_0 \in \mathbb{R}^n$$

iff

$$NO^0(C, A) = \mathcal{L}^0(A_+) \subseteq \mathcal{N}(S). \tag{88}$$

Hence we have by standard arguments

LEMMA 8. *Let* (A, B) *be stabilizable and consider the infinite-horizon cost* (85) *due to the feedback law* (87). *Then, for any given* $S = S^* \geq 0$,

$$J(x_0, \tilde{u}, S) = x_0^*P_+x_0 \qquad \text{for all } x_0 \in \mathbb{R}^n$$

iff

$$NO^0(C, A) \subseteq \mathcal{N}(S). \tag{88}$$

Remark 9. Observe that it takes an additional condition to realize the cost $x_n^*P_+x_0$ by the feedback law (87).

We have now also easily

LEMMA 9. *Let* $S = S^* \geq 0$. *Then one has*

$$\mathcal{N}(S) \cap NO^+(C, A) = \{0\} \tag{22}$$

and

$$NO^0(C, A) \subseteq N(S), \tag{88}$$

iff

$$\mathcal{N}(S) \cap NO^{0+}(C, A) = NO^0(C, A). \tag{89}$$

THEOREM 4. *Consider the infinite-horizon* LQ-*optimal control problem* $\mathrm{LQ}^\infty(S)$ *described by* (85) *and* (86). *Assume that*

$$(A, B) \text{ is stabilizable.} \tag{2}$$

Let P_+ *be the strong solution of the* ARE (3). *Let* $S = S^* \geq 0$ *be given. Let* $P(\tau, S)$ *be the solution of the* RDE (1) *where* $\tau = t_1 - t \geq 0$ *and* $P(0) = S$. *Then, with* $x_0 \in \mathbb{R}^n$ *arbitrary,*

$$\begin{cases} \displaystyle\min_{u(\cdot)\in U_{ad}} J(x_0, u, S) = \lim_{\tau\to\infty} x_0^* P(\tau, S)x_0 = x_0^* P_+ x_0, \\ \text{and this cost is attained by the state feedback law (87)} \end{cases} \tag{90}$$

iff

$$\mathcal{N}(S) \cap NO^{0+}(C, A) = NO^0(C, A). \tag{89}$$

Comments. 3 (a) Condition (89) implies that $\mathcal{N}(S) \cap NO^{0+}(C, A)$ is A-invariant. This property is paramount for getting the solution of an infinite horizon LQ-problem as the limiting case of a receding-horizon LQ-problem; see [8, 9].

(b) If (A, B) is controllable then, by Fact 1, (89) reads equivalently as

$$\mathcal{N}(S - P_-) = \mathcal{N}(P_+ - P_-).$$

(c) In view of Lemmas 8 and 9, criterion (89) is convergence criterion (22) modulo the well-posedness condition (88). It is much more easier to study the convergence of the RDE (1) towards the strong solution P_+ when (88) holds. Indeed, then for all $\tau \geq 0$,

$$NO^0(C, A) \subseteq \mathcal{N}(P(\tau, S))$$

and standard arguments can be used to apply [1, (32)] for obtaining convergence criterion (22).

(d) It should be realized that if $NO^0(C, A) \neq \{0\}$, then the optimal state trajectories of problem $LQ^{\infty}(S)$ are not always exp. stable and may contain critical modes (not necessarily bounded). These critical modes are bounded, or equivalently A_+ is stable in the sense of Lyapunov, iff A restricted to $NO^0(C, A)$ is diagonalizable: this is generically also required. Note that, under this condition, "small unobservable critical initial states produce small stationary ripples." On the other hand, if A restricted to $NO^0(C, A)$ is not diagonalizable, then critical unbounded modes cannot be avoided if there are critical unobservable initial states: they may be unwanted and the problem of finding a near optimal stabilizing feedback law becomes important.

We conclude with a result that was to be expected.

COROLLARY 2. *Claim* (90) *of Theorem* 4 *holds for every* $S = S^* \geq 0$ *iff* (C, A) *is detectable, i.e.*

$$NO^{0+}(C, A) = \{0\}.$$

Remark 10. In Theorem 4, two penalty situations are important, viz. (a) $S = 0$, where (89) holds iff $NO^+(C, A) = \{0\}$, and (b) $S > 0$, where (89) holds iff $NO^0(C, A) = \{0\}$.

Remark 11. Similar results hold for LQG-optimal estimation.

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