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Parameterized complexity of the weighted independent set problem beyond graphs of bounded clique number [☆,☆☆]

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ABSTRACT

The maximum independent set problem is known to be NP-hard for graphs in general, but is solvable in polynomial time for graphs in many special classes. It is also known that the problem is generally intractable from a parameterized point of view. A simple Ramsey argument implies the fixed-parameter tractability of the maximum independent set problem in classes of graphs of bounded clique number. Beyond this observation very little is known about the parameterized complexity of the problem in restricted graph families. In the present paper we develop fpt-algorithms for graphs in some classes extending graphs of bounded clique number.

1. Introduction

We study simple undirected graphs without loops or multiple edges. In a graph, an independent set is a subset of vertices no two of which are adjacent and a clique is a subset of pairwise adjacent vertices. The maximum size of an independent set in a graph G is called the *independence number* of G and is denoted $\alpha(G)$, while the maximum size of a clique is called the *clique number* of G and is denoted $\omega(G)$.

The MAXIMUM INDEPENDENT SET problem is that of finding an independent set of maximum size in a given graph. From a computational point of view this is a difficult problem, i.e. it is NP-hard. Moreover, it remains NP-hard under substantial restrictions, for instance, for triangle-free graphs [21] and for planar cubic graphs [1]. On the other hand, in many special graph classes the problem admits polynomial-time algorithms, which is the case for perfect graphs [13], claw-free graphs [19], and graphs of bounded clique-width [6].

A practical approach to deal with NP-hard problems is based on the notion of fixed-parameter tractability (fpt), which is a relaxation of classical polynomial-time solvability. A parameterized problem is said to be *fixed-parameter tractable* if it can be solved in time $f(k)p(n)$ on instances of input size n, where $f(k)$ is a computable function depending only on the value of the parameter k and $p(n)$ is a polynomial independent of k. Unfortunately, the MAXIMUM INDEPENDENT SET problem remains difficult even under this relaxation. More formally, it is W[1]-hard [9]. However, for graphs in some restricted families the

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problem becomes fixed-parameter tractable. In particular, this is true for graphs without large cliques, which follows from a simple Ramsey argument (see e.g. [25]). This argument alone implies fixed-parameter tractability of the problem for graphs of bounded degree, of bounded degeneracy, of bounded chromatic number, in all proper minor-closed graph classes (which includes, in particular, classes of graphs excluding single-crossing graphs as minors [8]) and all proper classes closed under taking subgraphs (not necessarily induced). Beyond this argument, very little is known on the parameterized complexity of the problem in restricted graph families. Other classes where the problem is known to be fixed-parameter tractable are the complements of t-multiple-interval graphs [11] and segment intersection graphs with a bounded number of directions [15].

We develop fpt-algorithms that solve the MAXIMUM INDEPENDENT SET problem in several new classes of graphs, generalising some of the previously known results. In fact, our results apply to a natural generalisation of the problem for weighted graphs. We say that a graph G is a weighted graph if each vertex of G is assigned a real number $\geqslant 1$, the weight of the vertex. The MAXIMUM WEIGHT INDEPENDENT SET problem is that of finding an independent set of maximum weight in a weighted graph, where the weight of a set of vertices is the sum of the weights of its elements. This maximum weight is denoted $\alpha_w(G)$. We study the following parameterization of the MAXIMUM WEIGHT INDEPENDENT SET problem:

WEIGHTED INDEPENDENT SET
Instance: A weighted graph G with weight function $w : V(G) \to \mathbb{R}$ and a positive real number W.
Parameter: W.
Problem: Decide whether G has an independent set of weight at least W and find such a set if it exists. If no such set exists, find an independent set of weight $\alpha_w(G)$ instead.

All classes of graphs considered in this paper are hereditary, in the sense that for any graph G in such a class, all induced subgraphs of G are also in the class. It is known that a class of graphs is hereditary if and only if it can be characterised by a set of forbidden induced subgraphs. We say that a graph is M-free if it contains no induced subgraphs from a set M of graphs. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. We denote the number of vertices of G by n. If v is a vertex of G, then $N_G(v)$ is the neighbourhood of v in G and $N_G[v] = N_G(v) \cup \{v\}$ is the closed neighbourhood of v in G. For a subset $U \subseteq V(G)$, we let $G[U]$ be the subgraph of G induced by U. The complement of a graph G is denoted $\overline{G}$. We use $R(r,s)$ to denote the Ramsey number, i.e. the minimum number n such that every graph with at least n vertices has either an independent set of size r or a clique of size s. As usual, K_n, C_n, and P_n denote the complete graph, the chordless cycle and the chordless path on n vertices, respectively. We denote the graph obtained from K_n by deleting an edge by $K_n - e$ and the disjoint union of r complete graphs of order 2 by rK_2. For a real number x, $\lceil x \rceil$ denotes the smallest integer $\geqslant x$.

2. $(K_r - e)$-free graphs

As we have noted above, a simple Ramsey argument implies the fixed-parameter tractability of MAXIMUM INDEPENDENT SET in K_r-free graphs. We first extend this result to the weighted case.

Theorem 1. *For $r \in \mathbb{N}$, the* WEIGHTED INDEPENDENT SET *problem is fixed-parameter tractable in the class of K_r-free graphs.*

Proof. Let (G, W) be an instance of the WEIGHTED INDEPENDENT SET problem, with G being a K_r-free graph on n vertices. Since the weight of each vertex is $\geqslant 1$, the weight of every independent set is at least its size. Therefore, if G has at least $R(\lceil W \rceil, r)$ vertices, then it necessarily has an independent set of size (and therefore of weight) at least W. If the number of vertices of G is at least $R(\lceil W \rceil, r)$, we can delete any $n - R(\lceil W \rceil, r)$ vertices from G, since the remaining graph still necessarily has an independent set of weight at least W. Now the number of vertices of G is at most $R(\lceil W \rceil, r)$, so the problem can be solved in time independent of n. This clearly implies the fixed-parameter tractability of WEIGHTED INDEPENDENT SET for K_r-free graphs. □

Since K_{r-1} is an induced subgraph of $K_r - e$, our next result generalises Theorem 1.

Theorem 2. *For $r \in \mathbb{N}$, $r \geqslant 2$, the* WEIGHTED INDEPENDENT SET *problem is fixed-parameter tractable in the class of $(K_r - e)$-free graphs.*

Proof. Let (G, W) be an instance of the WEIGHTED INDEPENDENT SET problem, with G being a $(K_r - e)$-free graph on n vertices. Let I be an independent set of G such that I is maximal with respect to set-inclusion and there are no two non-adjacent vertices u and v in $V(G) \setminus I$ for which $(N_G(u) \cup N_G(v))$ contains exactly one vertex of I (i.e. I admits no so-called *augmenting* K_1 or *augmenting* P_3). Clearly, if one of these two conditions fails, one can immediately construct a larger independent set. This implies that a set with these properties can be found in time polynomial in n. Since the vertices of the graph have weights $\geqslant 1$, if we find an independent set of size $\geqslant W$, then returning this set correctly solves WEIGHTED INDEPENDENT SET. Hence we suppose $|I| < W$. (If this happens, the procedure actually solves the MAXIMUM WEIGHT INDEPENDENT SET problem.)

We partition the vertices in $V(G) \setminus I$ into *classes* according to their neighbourhood in I, i.e. two vertices of $V(G) \setminus I$ belong to the same class if and only if they have the same neighbours in I. A class is *light* if its elements have exactly one neighbour in I and *heavy* otherwise.

By the choice of I, each light class is a clique and hence any independent set in G contains at most one vertex from each of the $|I|$ light classes. Furthermore, no vertex u from a light class has $r-2$ neighbours in another light class, since otherwise a $K_r - e$ arises using u, some $r-2$ neighbours of u in another light class, and their unique neighbour in I.

Since G is $(K_r - e)$-free, every heavy class C induces a K_{r-2}-free graph, since otherwise a clique K of order $r-2$ in C together with two neighbours in I of the vertices in K would form a $K_r - e$. Hence, if some heavy class contains at least $R(\lceil W\rceil, r-2)$ vertices, we can find an independent set of size at least W as explained in the proof of Theorem 1. Therefore, we suppose that each heavy class contains less than $R(\lceil W\rceil, r-2)$ vertices, which implies that the union H of I and all the heavy classes contains at most $(W-1)+2^W R(\lceil W\rceil, r-2)$ vertices, which is bounded in terms of W and r.

We can now proceed as follows:

Step 1: Generate all independent sets contained in H. Clearly, the number of such sets and the time needed to generate all of them is bounded in terms of W and r. For each independent set I_H found in this step, execute Step 2.

Step 2: Let L denote the set of vertices u in light classes such that u has no neighbour in I_H. Let L_1 denote the set of vertices in L that belong to light classes C with $|C \cap L| < r\lceil W\rceil$. Furthermore, let $L_2 \subseteq L$ contain the $r\lceil W\rceil$ vertices of largest weight (breaking ties arbitrarily) in $C \cap L$ for each light class C with $|C \cap L| \geqslant r\lceil W\rceil$. Note that $L_1 \cup L_2$ contains at most $r\lceil W\rceil^2$ vertices, which is bounded in terms of W and r. Therefore, we can determine an independent set $I_L \subseteq L_1 \cup L_2$ such that $I_H \cup I_L$ is of largest possible weight in time bounded in terms of W and r.

Let J be an independent set of G with $J \cap H = I_H$ such that J has maximum possible weight and, subject to this condition, J has largest possible intersection with I_L. Let $J_L = J \setminus H$. Since $J \cap H = I_H$ and J is independent, we have $J_L = J \cap L$. We claim that $J_L = I_L$. For contradiction, we assume that $J_L \neq I_L$. In this case, the choice of I_L and J implies that J_L must contain a vertex $x \in L \setminus (L_1 \cup L_2)$. Note that x necessarily belongs to a light class C with $|C \cap L| \geqslant r\lceil W\rceil$. Since there are less than W vertices in $J_L \setminus \{x\}$ and every vertex in a light class has less than $r-2$ neighbours in C, the set $C \cap L_2$ contains a vertex x' that is not adjacent to any vertex in $J_L \setminus \{x\}$. By the choice of L_2, the weight of x' is at least the weight of x. Therefore, the set $(J \setminus \{x\}) \cup \{x'\}$ is independent, has at least the weight of J and a larger intersection with I_L than J, which contradicts the choice of J. This proves $J_L = I_L$, which means that the set $I_H \cup I_L$ found in the second step is an independent of maximum weight intersecting H in I_H. Since we execute the second step for all possible choices of I_H, returning a set of the form $I_H \cup I_L$ that is of largest possible weight correctly solves Weighted Independent Set. Clearly, the running time of the sketched procedure is $f_r(W)p(n)$ where, for fixed r, $f_r(W)$ is a computable function depending on W and $p(n)$ is a polynomial independent of W. $\square$

Note that the polynomial $p(n)$ above is independent of r as well as W, so the problem is fixed-parameter tractable even if parameterized by both W and r.

3. Splittable graphs

In this section, we consider graphs that allow a certain type of decomposition; either of its vertex set or of its edge set.

Definition 3. For $r \in \mathbb{N}$ and a graph G, a partition $V(G) = X \cup Y$ of the vertex set of G is an *r-split partition of* G if $\omega(G[X]) < r$ and $\alpha(G[Y]) < r$. If a graph G has an r-split partition, then G is an *r-split graph*.

The notion of r-split graphs generalises K_r-free graphs and many other important hereditary classes. To see the importance of this notion, observe that for every hereditary class X (see e.g. [3]), there is a natural number k (called the *index* for the class) such that the number X_n of n-vertex graphs (also known as the *speed* of X) satisfies $\lim_{n\to\infty} \frac{\log_2 X_n}{\binom{n}{2}} = 1 - \frac{1}{k(X)}$. Furthermore, if $\mathcal{E}^{i,j}$ denotes the class of graphs whose vertices can be partitioned into at most i independent sets and j cliques, then the index $k(X)$ of a class X is the maximum k such that X contains a class $\mathcal{E}^{i,j}$ with $i + j = k$. In other words, the classes $\mathcal{E}^{i,j}$ with $i + j = k$ are the only minimal classes of index k. Therefore, any class X of index > 1 can be approximated by a minimal class $\mathcal{E}^{i,j}$ of the same index, in the sense that $\lim_{n\to\infty} \frac{\log_2 X_n}{\mathcal{E}^{i,j}_n} = 1$. Clearly, $\mathcal{E}^{i,j}$ is a subclass of $\max\{i+1, j+1\}$-split graphs.

Note that the class of split graphs (i.e. graphs partitionable into an independent set and a clique) is exactly the class $\mathcal{E}^{1,1}$ and that the graphs in this class are precisely the 2-split graphs. Among various nice properties, split graphs admit polynomial-time recognition. In the next lemma we show that this property extends to r-split graphs for all values of r.

Lemma 4. *For every $r \in \mathbb{N}$, the class of r-split graphs can be recognised in polynomial time, and a certifying r-split partition of the vertex set can be constructed within this time.*

Proof. Let $G = (V, E)$ be a graph and Y an arbitrary subset of its vertices with $\alpha(G[Y]) < r$. It is not difficult to see that in polynomial time one can check if G contains a set Y' such that

$$\left|Y \setminus Y'\right| < R(r,r), \qquad \alpha\left(G\left[Y'\right]\right) < r \quad \text{and} \quad \left|Y'\right| = |Y| + 1. \tag{1}$$

As long as G admits such a set Y' replace Y with Y', i.e. set $Y := Y'$. If no such set can be found, then check if G contains a set Y' such that

$$\left|Y \setminus Y'\right| < R(r,r), \qquad \left|Y' \setminus Y\right| < R(r,r), \qquad \alpha\big(G\big[Y'\big]\big) < r \quad \text{and} \quad \omega\big(G\big[V \setminus Y'\big]\big) < r. \tag{2}$$

If the answer is affirmative, then obviously G is an r-split graph and $Y' \cup (V \setminus Y')$ is a respective partition. Otherwise, G is not an r-split graph. To see this, suppose for contradiction that G admits an r-split partition $V = X_0 \cup Y_0$ with $\omega(G[X_0]) < r$ and $\alpha(G[Y_0]) < r$. By the choice of Y, the graph $G[Y \setminus Y_0]$ is $\overline{K}_r$-free. Also, since $Y \setminus Y_0$ is a subset of X_0, the graph $G[Y \setminus Y_0]$ is K_r-free. Therefore $|Y \setminus Y_0| < R(r,r)$. If additionally $|Y_0 \setminus Y| < R(r,r)$, then $Y' = Y_0$ satisfies (2), contradicting our assumption. If $|Y_0 \setminus Y| \geqslant R(r,r)$, then $|Y_0| > |Y|$ in which case a subset $Y' \subset Y_0$ satisfying (1) can be found. A contradiction in both cases proves correctness of the procedure. The polynomiality follows from the fact that r and $R(r,r)$ are constants independent of the number of vertices in G. □

Now we proceed to algorithms that solve the Weighted Independent Set problem for r-split graphs. For $r = 2$ the problem is known to be solvable in polynomial time, since it is a subclass of perfect graphs. However, for large values of r the problem is NP-hard. In the next theorem we show that the problem is fixed-parameter tractable in the class of r-split graphs for any value of r. Since K_r-free graphs are r-split graphs, our result generalises Theorem 1.

Theorem 5. *For $r \in \mathbb{N}$, the* Weighted Independent Set *problem is fixed-parameter tractable in the class of r-split graphs.*

Proof. Let (G, W) be an instance of the Weighted Independent Set problem with G an r-split graph. First, we apply Lemma 4 in order to find a partition $V(G) = X \cup Y$ such that $G[X]$ is K_r-free and $G[Y]$ is $\overline{K}_r$-free. This takes polynomial time. Since $G[Y]$ is $\overline{K}_r$-free, the graph $G[Y]$ has only polynomially many independent sets. For each such set I_Y of weight $w(I_Y)$, we solve the Weighted Independent Set problem for the instance $(G[X \setminus N_G(I_Y)], W - w(I_Y))$ using Theorem 1, which yields a set $I_X(I_Y)$. Returning an independent set of the form $I_Y \cup I_X(I_Y)$ of maximum weight correctly solves Weighted Independent Set. □

The notion of r-split graphs admits a further generalisation as follows:

Definition 6. Let $r \in \mathbb{N}$ and $\mathcal{G}$ be a hereditary class of graphs. A partition $E(G) = E_0 \cup E_1$ of the edge set of G is an *$(r, \mathcal{G})$-split* if $G_0 = (V, E_0)$ is rK_2-free and $G_1 = (V, E_1)$ belongs to $\mathcal{G}$. If a graph G has an $(r, \mathcal{G})$-split partition, then G is an *$(r, \mathcal{G})$-split graph.*

It is not difficult to see that any r-split graph is $(r, \mathit{Free}(K_r))$-split, where $\mathit{Free}(K_r)$ stands for the class of K_r-free graphs. Indeed, let $G = (V, E)$ be an r-split graph with an r-split partition $V = X \cup Y$ where $\omega(G[X]) < r$ and $\alpha(G[Y]) < r$, and let $E_0 \cup E_1$ be a partition of E with $E_1 = E(G[X])$ and $E_0 = E \setminus E_1$. Then obviously $G_1 = (V, E_1)$ is K_r-free. To see that $G_0 = (V, E_0)$ is rK_2-free, observe that in this graph the set X is independent and hence every edge contains at least one of its endpoints in the set Y, which means that if G_0 would contain an induced rK_2, then Y would contain an independent set of size r, which is impossible.

As we saw earlier, for any natural r, the class of r-split graphs enjoys the nice property that graphs in this class can be recognised in polynomial time, which in turn implies fixed-parameter tractability of the Weighted Independent Set problem in this class. This is obviously not true for general $(r, \mathcal{G})$-split graphs. However, as we show below, if $\mathcal{G}$ is a class such that the problem is fixed-parameter tractable in it and an $(r, \mathcal{G})$-split partition can be found in polynomial time for any $(r, \mathcal{G})$-split graph, then the problem is also fixed-parameter tractable in the class of $(r, \mathcal{G})$-split graphs.

Theorem 7. *Let $r \in \mathbb{N}$ and $\mathcal{G}$ be a hereditary class of graphs. If*

- *the* Weighted Independent Set *problem is fixed-parameter tractable in $\mathcal{G}$, and*
- *an $(r, \mathcal{G})$-split partition can be found in polynomial time for any $(r, \mathcal{G})$-split graph,*

then the Weighted Independent Set *problem is fixed-parameter tractable in the class of $(r, \mathcal{G})$-split graphs.*

Proof. Given an instance (G, W) of Weighted Independent Set with G being an $(r, \mathcal{G})$-split graph, we first apply the polynomial-time algorithm to find an $(r, \mathcal{G})$-split partition $E(G) = E_0 \cup E_1$ of the edge set of G such that $G_0 = (V, E_0)$ is rK_2-free and $G_1 = (V, E_1)$ belongs to $\mathcal{G}$. Note that the rK_2-free graph G_0 only has a polynomial number of maximal independent sets [2], which can all be generated in polynomial time [27], and that a set of vertices is independent in G if and only if it is independent in G_1 and a subset of some maximal independent set of G_0. Therefore, solving the Weighted Independent Set problem in $G_1[I_0]$ for each of the polynomially many maximal independent sets I_0 of G_0 and returning an independent set of maximum weight obtained in this way, correctly solves Weighted Independent Set on the instance (G, W). Since Weighted Independent Set is fixed-parameter tractable in $\mathcal{G}$, the desired result follows. □

4. Beyond triangle-free graphs

In the search of further results, in this section we study extensions of triangle-free graphs, which is the simplest non-trivial class of graphs of bounded clique number. We start by analysing H-free graphs, where H is a one-vertex extension of a triangle.

Theorem 8. *For each one-vertex extension H of a triangle, the* WEIGHTED INDEPENDENT SET *problem is fixed-parameter tractable in the class of H-free graphs.*

Proof. It is not difficult to see that (up to isomorphism) there are four one-vertex extensions of a triangle: K_4, $K_4 - e$, $K_3 + e$ and $K_3 \cup K_1$, where $K_3 + e$ stands for a triangle plus a pendant edge (also known as a *paw*) and $K_3 \cup K_1$ denotes the union of a triangle and an isolated vertex.

The fixed-parameter tractability of the problem in the classes of K_4-free graphs and $(K_4 - e)$-free graphs follows from Theorems 1 and 2, respectively.

The structure of $(K_3 + e)$-free graphs has been characterised in [22] as follows: A connected $(K_3 + e)$-free graph is either triangle-free or a complete multipartite graph (i.e. the complement of the disjoint union of cliques). Together with the trivial observation that the WEIGHTED INDEPENDENT SET problem can be reduced to connected graphs, this proves the theorem for $(K_3 + e)$-free graphs.

Finally, to derive the same conclusion for $(K_3 \cup K_1)$-free graphs, we invoke the obvious fact that a graph G is $(K_3 \cup K_1)$-free if and only if $G - N_G[u]$ is K_3-free for every vertex $u \in V(G)$. Together with the trivial identity

$$\alpha_w(G) = \max_{u \in V(G)} \bigl\{\omega(u) + \alpha_w\bigl(G - N_G[u]\bigr)\bigr\},$$

the fixed-parameter tractability of the problem in the class of $(K_3 \cup K_1)$-free graphs follows from Theorem 1. □

To further extend one of the classes covered by Theorem 8, we employ the notion of modular decomposition. The idea of modular decomposition was first introduced in the 1960s by Gallai [12], and also appeared in the literature under various other names such as *prime tree decomposition* [10], *X-join decomposition* [14], or *substitution decomposition* [20], and this technique has previously been used to construct fpt-algorithms (see e.g. [24]). To describe this idea, let us fix some terminology.

Given a graph $G = (V, E)$, a subset of vertices $U \subseteq V$ and a vertex $x \in V$ outside U, we say that x *distinguishes* U if x has both a neighbour and a non-neighbour in U. A subset $U \subseteq V$ is called a *module* of G if no vertex in $V \setminus U$ distinguishes U. A module U is *nontrivial* if $1 < |U| < |V|$, otherwise it is *trivial*. A graph is called *prime* if it has only trivial modules.

An important property of maximal modules is that if G and the complement of G are both connected, then the maximal modules of G are pairwise disjoint. Moreover, from the above definition it follows that if U and W are distinct maximal modules, then there are either no edges between them or every vertex in U is adjacent to every vertex in W. Using these properties of maximal modules, we can find a maximum weight independent set in G by

(1) reducing the problem to smaller instances if G or its complement are disconnected,
(2) recursively solving the problem in the subgraphs of G induced by maximal modules,
(3) contracting each maximal module M to a single vertex and assigning to it the weight $\alpha_w(G[M])$, obtaining in this way a new graph G^0,
(4) solving the problem for the graph G^0.

The graph G^0 constructed in step (3) of the outlined procedure is prime. So, the procedure reduces the MAXIMUM WEIGHT INDEPENDENT SET problem for any hereditary class to prime graphs in the class. This reduction can be implemented in polynomial time (see e.g. [18]). Let us show that this is also an fpt-reduction, i.e. it preserves fixed-parameter tractability.

Theorem 9. *Let $\mathcal{X}$ be a hereditary class of graphs and let $\mathcal{X}_0$ denote the class of prime graphs in $\mathcal{X}$. If the* WEIGHTED INDEPENDENT SET *problem is fixed-parameter tractable in $\mathcal{X}_0$, then it is fixed-parameter tractable in $\mathcal{X}$.*

Proof. Let (G, W) be an instance of the WEIGHTED INDEPENDENT SET problem with $G \in \mathcal{X}$. Recall that the modular decomposition tree T of G can be determined in linear time [18,5] and that the set of leaves of T equals the vertex set V of G. To each node v of T we associate the subgraph G_v of G induced by the leaves of the subtree of T rooted at v. Processing the vertices of T in an order of non-increasing height, we will find for each node v of T an independent set I_v of G_v such that the weight $w(I_v)$ of I_v is at least $\min\{W, \alpha_w(G_v)\}$. If the weight of I_v is at least W, we stop the procedure and output I_v. Otherwise, we assign the independent set I_v of weight $\alpha_w(G_v)$ to the node v. The procedure starts by assigning the independent set $I_v = \{v\}$ to each leaf v of T. Now let v be an inner node of T.

If G_v is disconnected, then the children $v_1, v_2, \ldots, v_l$ of v correspond to the connected components of G_v. In this case, we let $I_v = I_{v_1} \cup I_{v_2} \cup \cdots \cup I_{v_l}$.

Fig. 1. The house and the bull graphs.

If the complement of G_v is disconnected, then the children $v_1, v_2, \dots, v_l$ of v correspond to the connected components of the complement of G_v. In this case we let $I_v = I_{v_i}$, where $w(I_{v_i}) = \max\{w(I_{v_1}), w(I_{v_2}), \dots, w(I_{v_l})\}$.

Finally, if both G_v and its complement are connected, then the children $v_1, \dots, v_l$ of v correspond to the subgraphs of G_v induced by the maximal modules $U_1, U_2, \dots, U_l$ of G_v, which partition the vertex set of G_v. Let the graph G_v^0 arise from G_v by contracting each maximal module U_i of G_v into a single vertex denoted i to which we assign the weight $w(i) = w(I_{v_i})$. Since G_v^0 belongs to $\mathcal{X}_0$, there is an algorithm $\mathcal{A}$ that solves Weighted Independent Set on the instance (G_v^0, W) in time $f(W)l^c \leqslant f(W)n^c$, where c is a constant. If I is the output of $\mathcal{A}$, then let $I_v = \bigcup_{i \in I} I_{v_i}$. It is not difficult to see that the set assigned to the root of T correctly solves Weighted Independent Set on the instance (G, W). Since T has $O(n)$ vertices, the overall time complexity is at most $f(W)n^{c+1}$. □

Theorem 9 reduces the Weighted Independent Set problem from general graphs to prime graphs. The corresponding result for the non-parameterized problem is well known.

Now we apply Theorem 9 in order to develop an fpt-algorithm for the weighted independent set problem in the class of $\{house, bull\}$-free graphs. The graphs *house* and *bull* are shown in Fig. 1. Observe that both these graphs contain $K_3 + e$. Therefore, the class of $\{house, bull\}$-free graphs extends the class of $(K_3 + e)$-free graphs for which an fpt solution was shown in Theorem 8.

Theorem 10. *The* Weighted Independent Set *problem is fixed-parameter tractable in the class of* $\{house, bull\}$*-free graphs.*

Proof. To prove the theorem, we use the following characterisation of $\{house, bull\}$-free graphs proposed in [23]: Every prime $\{house, bull\}$-free graph is either triangle-free or the complement of a bipartite chain graph. (A bipartite graph is a bipartite chain graph if the vertices in both parts of the bipartition are linearly ordered by inclusion of neighbourhoods.) Obviously, for the complements of bipartite graphs, the maximum weight independent set problem can be solved in polynomial time, since the size of any independent set in such a graph is at most 2. Also, by Theorem 1, the Weighted Independent Set problem is fixed-parameter tractable in the class of triangle-free graphs. Therefore, by Theorem 9, it is fixed-parameter tractable in the class of $\{house, bull\}$-free graphs. □

5. Concluding remarks and open problems

In this paper, we obtain new results on the parameterized complexity of the weighted independent set problem in hereditary classes of graphs. The new results together with some previously known results allow us to conclude, in particular, that the problem is fixed-parameter tractable in all hereditary classes defined by a single forbidden induced subgraph G with at most 4 vertices, except for $G = C_4$. Finding the parameterized complexity of the problem in the class of C_4-free graphs is a challenging open problem. In addition to the techniques studied in this paper, some other approaches may be useful for finding an answer to the above question, such as graph transformations [17], separating cliques [4], and split decomposition [26].

There has recently been a lot of research on kernel sizes for fpt problems. The kernel sizes given by the algorithms in this paper are quite large. Finding lower bounds for the kernel size is an interesting direction for future research.

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