Uniqueness properties of solutions of Schrödinger equations

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Received 31 May 2005; accepted 1 June 2005
Communicated by J. Bourgain
Available online 18 August 2005

Abstract

Under suitable assumptions on the potentials $V$ and $a$, we prove that if $u \in C([0, 1], H^1)$ is a solution of the linear Schrödinger equation

$$(i\partial_t + \Delta_x)u = Vu + a \cdot \nabla_x u \quad \text{on } \mathbb{R}^d \times (0, 1)$$

and if $u \equiv 0$ in $\{|x| > R\} \times (0, 1)$ for some $R \geq 0$, then $u \equiv 0$ in $\mathbb{R}^d \times [0, 1]$. As a consequence, we obtain uniqueness properties of solutions of nonlinear Schrödinger equations of the form

$$(i\partial_t + \Delta_x)u = G(x, t, u, \nabla_x u, \nabla_x v) \quad \text{on } \mathbb{R}^d \times (0, 1),$$

where $G$ is a suitable nonlinear term. The main ingredient in our proof is a Carleman inequality of the form

$$\|e^{\beta \phi_j(x_1)} v\|_{L^2_t L^2_x} + \|e^{\beta \phi_j(x_1)} |\nabla_x v|\|_{B^\infty_{2,2} L^2_t} \leq C \|e^{\beta \phi_j(x_1)} (i\partial_t + \Delta_x) v\|_{B^{1,2}_{2,2} L^2_t}$$

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1 Supported in part by an NSF grant and an Alfred P. Sloan research fellowship.

2 Supported in part by an NSF grant.

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doi:10.1016/j.jfa.2005.06.005
for any $v \in C(\mathbb{R} : H^1)$ with $v(.,t) \equiv 0$ for $t \notin [0, 1]$. In this inequality, $B^\infty_{x,2}$ and $B^1_{x,2}$ are Banach spaces of functions on $\mathbb{R}^d$, and $e^{\beta_0(x_1)}$ is a suitable weight.

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MSC: 35B37

Keywords: Carleman inequalities; Local smoothing; Uniqueness of solutions; Parametrices

1. Introduction

In this paper we study uniqueness properties of solutions of certain nonlinear Schrödinger equations of the form

$$(i\partial_t + \Delta_x)u = G(x, t, u, \bar{u}, \nabla_x u, \nabla_x \bar{u}) \text{ on } \mathbb{R}^d \times (0, 1),$$

(1.1)

where $G$ is a nonlinear term. We are concerned with the following type of uniqueness question:

**Question Q.** Assume that $u_1$ and $u_2$ are solutions of (1.1) in a suitable function space, and $u_1(x,t) = u_2(x,t)$ for $t \in \{0, 1\}$ and $|x| \geq R$, for some $R \geq 0$. Can we then conclude that $u_1 \equiv u_2$ in $\mathbb{R}^d \times [0, 1]$?

This type of uniqueness question seems to originate in control theory and was answered in the affirmative under various assumptions on the nonlinear term $G(x, t, u, \bar{u}, \nabla_x u, \nabla_x \bar{u})$ (no dependence on the gradient terms $\nabla_x u$ and $\nabla_x \bar{u}$). Zhang [16] used the inverse scattering theory to answer the question Q in the affirmative in the special case $d = 1$, $G = z|u|^2 u$, $z \in \mathbb{R}$, $u_2 \equiv 0$. Bourgain [1] proved uniqueness under analyticity assumptions on the nonlinear term $G = G(u, \bar{u})$, with $u_2 \equiv 0$, and the stronger assumption that $u_1$ is compactly supported for all $t \in [0, 1]$. Kenig et al. [9] answered question Q in the affirmative for sufficiently smooth (in particular, bounded) solutions $u_1$ and $u_2$, when $G = G(u, \bar{u})$ satisfies estimates of the form

$$|\nabla G(u, \bar{u})| \leq C(|u|^{p_1-1} + |u|^{p_2-1}), \quad 1 < p_1, p_2.$$  

The boundedness requirement on the solutions $u_1$ and $u_2$ was relaxed to optimal $L^p$ conditions by Ionescu and Kenig [4].

A similar uniqueness question was also considered in the setting of the generalized KdV equation

$$(i^{s-1} \partial_t + \partial_x^s)u = G(x, t, u, \partial_x u, \ldots, \partial_x^{s-1} u) \text{ on } \mathbb{R} \times (0, 1), \quad s \geq 2$$

by Zhang [15], Bourgain [1] and Kenig et al. [8,10,11], under various assumptions on the nonlinear term $G$ and the solutions $u_1$ and $u_2$. A brief description of these results can be found in the introduction of [4].
The main contribution of this paper is that we allow gradient terms \( \nabla_x u \) and \( \nabla_x \overline{u} \) in the nonlinear term on the right-hand side of (1.1), in all dimensions \( d \geq 1 \). Our proof is based on the method of Carleman inequalities. In view of the well-known difficulties in dealing with gradient potentials in the case of the time-independent Schrödinger operator \( \Delta_x \) on \( \mathbb{R}^d, d \geq 3 \) (i.e. failure of “natural” Carleman inequalities, see, for example, the discussion in [6]), it is somewhat surprising that we can still use Carleman inequalities to obtain essentially optimal results (at least for the gradient term) in the case of the time-dependent Schrödinger operator \( i \partial_t + \Delta_x \) on \( \mathbb{R}^d \times (0, 1) \). By linearizing (1.1), we are led to considering solutions of the linear Schrödinger equation

\[
(i \partial_t + \Delta_x)u = V_1 u + V_2 \overline{u} + a_1 \cdot \nabla_x u + a_2 \cdot \nabla_x \overline{u} \quad \text{on} \quad \mathbb{R}^d \times (0, 1),
\]

where \( V_1 \) and \( V_2 \) are complex-valued potentials, and \( a_1 \) and \( a_2 \) are vector-valued potentials. Let \( H \) denote the operator \( i \partial_t + \Delta_x \) acting on the space of distributions \( S'(\mathbb{R}^d \times \mathbb{R}) \).

For any \( k \in \mathbb{Z}^d \) let \( Q_k \) denote the cube \( \{ x \in \mathbb{R}^d : x_\ell \in [k_\ell - 1/2, k_\ell + 1/2], \ell = 1, \ldots, d \} \). For \( p, q \in [1, \infty] \) we define the Banach space \( B^{p,q} = B^{p,q}(\mathbb{R}^d) \) using the norm

\[
\| f \|_{B^{p,q}} := \left( \sum_{k \in \mathbb{Z}^d} \| f \|_{L^q(Q_k)}^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad \text{and} \quad \| f \|_{B^{\infty,q}} := \sup_{k \in \mathbb{Z}^d} \| f \|_{L^q(Q_k)}.
\]

It is easy to see that \( B^{p,p} = L^p, 1 \leq p \leq \infty, \) and

\[
B^{p_1,q_1} \hookrightarrow B^{p_2,q_2} \quad \text{if} \quad q_1 \geq q_2 \quad \text{and} \quad p_1 \leq p_2.
\]

Our main theorem is the following:

**Theorem 1.1.** Assume that \( u \in C([0, 1] : H^1) \) is a solution of the equation

\[
H u = V u + a \cdot \nabla_x u \quad \text{on} \quad \mathbb{R}^d \times (0, 1),
\]

where

\[
V \in B^{2,\infty}_x L^{\infty}_t(\mathbb{R}^d \times [0, 1]) \quad \text{and} \quad |a| \in B^{1,\infty}_x L^{\infty}_t(\mathbb{R}^d \times [0, 1]).
\]

If \( u \equiv 0 \) in \( \{|x| \geq R\} \times [0, 1] \) for some \( R \geq 0 \), then \( u \equiv 0 \) in \( \mathbb{R}^d \times [0, 1] \).

**Remark.** It is not clear to us whether assumption (1.4) can be improved significantly. In the case \( a \equiv 0 \), the conclusion of Theorem 1.1 holds for a much larger class of potentials \( V \), namely

\[
V \in L^{p_1}_t L^{q_1}_x(\mathbb{R}^d \times [0, 1]) + L^{p_2}_t L^{q_2}_x(\mathbb{R}^d \times [0, 1])
\]
for some $p_1, q_1, p_2, q_2 \in [1, \infty)$ with $2/p_1 + d/q_1 \leq 2$ and $2/p_2 + d/q_2 \leq 2$ (see [4, Theorem 2.5]). The assumption $|a| \in B^{1,\infty}_x L^\infty_t (\mathbb{R}^d \times [0, 1])$ is natural (and, possibly, close to optimal), in view of the local well-posedness theory (1.10). By going through the proof of Theorem 1.3 below, it is clear that the assumption $V \in B^{1,\infty}_x L^\infty_t (\mathbb{R}^d \times [0, 1])$ can be relaxed somewhat; however, we do not know whether the conclusion of Theorem 1.1 holds for, say, $V$ as in (1.5) and $|a| \in B^{1,\infty}_x L^\infty_t (\mathbb{R}^d \times [0, 1])$. One of the difficulties in using the proofs in [4] when there are gradient potentials is the failure of the “natural” Carleman inequality (1.11).

Theorem 1.1 applies to solutions of nonlinear Schrödinger equations. For $\sigma \in \mathbb{R}$ let $J^\sigma : S' (\mathbb{R}^d) \to S' (\mathbb{R}^d)$ denote the operator defined by the Fourier multiplier $\xi \mapsto (1 + |\xi|^2)^{\sigma/2}$. Let $H^\sigma$ denote the standard Sobolev space on $\mathbb{R}^d$ with $\| f \|_{H^\sigma} := \| J^\sigma f \|_{L^2}$. Assume that $P : C \times C \times C \times C \to C$ is a polynomial, and consider the nonlinear Schrödinger equation with derivatives

$$Hu = P(u, \bar{u}, \nabla_x u, \nabla_x \bar{u}) \text{ on } \mathbb{R}^d \times (0, 1).$$

(1.6)

**Corollary 1.2.** (a) Assume that $u_1, u_2$ are solutions of (1.6), and

$$\begin{align*}
J^\sigma u_1, J^\sigma u_2 & \in C([0, 1] : L^2), \quad \sigma > d/2 + 3, \\
|P(z_1, z_2, w_1, w_2)| \leq & |z_1|^2 + |z_2|^2 + |w_1|^3 + |w_2|^3, \quad |(z_1, z_2, w_1, w_2)| \leq 1.
\end{align*}$$

(1.7)

If $u_1 \equiv u_2$ in $\{|x| \geq R\} \times \{0, 1\}$ for some $R \geq 0$, then $u_1 \equiv u_2$ in $\mathbb{R}^d \times [0, 1]$.

(b) Assume that $u_1, u_2$ are solutions of (1.6), and

$$\begin{align*}
J^\sigma u_1, J^\sigma u_2 & \in C([0, 1] : B^{1,2}), \quad \sigma > d/2 + 3, \\
|P(z_1, z_2, w_1, w_2)| \leq & |z_1|^2 + |z_2|^2 + |w_1|^2 + |w_2|^2, \quad |(z_1, z_2, w_1, w_2)| \leq 1.
\end{align*}$$

(1.8)

If $u_1 \equiv u_2$ in $\{|x| \geq R\} \times \{0, 1\}$ for some $R \geq 0$, then $u_1 \equiv u_2$ in $\mathbb{R}^d \times [0, 1]$.

The proof of Corollary 1.2 can be adapted to include more general nonlinear terms. If $G(x, t, u, \nabla_x u, \nabla_x \bar{u})$ is the nonlinear term (as in (1.1)), then we only need to assume that

$$G(x, t, u_1, \nabla_x u_1, \nabla_x \bar{u}_1) - G(x, t, u_2, \nabla_x u_2, \nabla_x \bar{u}_2) = Vu + a \cdot \nabla_x u,$$

where $u := u_1 - u_2$, for some $V$ and $a$ as in (1.4). We have chosen to state Corollary 1.2 in its present form because of the local well-posedness theory of the initial value problem

$$\begin{align*}
Hu = P(u, \bar{u}, \nabla_x u, \nabla_x \bar{u}) \text{ on } \mathbb{R}^d \times (0, \infty), \\
u(., 0) = u_0,
\end{align*}$$

in suitable $H^\sigma$ and weighted $H^\sigma$ spaces (see [7, Theorems 1.1 and 1.2]). More general local well-posedness theorems were recently proved by Kenig et al. in [12].
Our main tool in the proof of Theorem 1.1 and Corollary 1.2 is a Carleman inequality. As in [9,4], let \( \phi \) denote a fixed smooth function on \( \mathbb{R} \) with the following properties: \( \phi(0) = 0 \), \( \phi' \) nonincreasing, \( \phi'(r) = 1 \) if \( r \leq 1 \) and \( \phi'(r) = 0 \) if \( r \geq 2 \). For any \( \lambda \geq 1 \) let \( \phi_{\lambda}(r) = \lambda \phi(r/\lambda) \). Clearly, \( \phi_{\lambda}(r) = r \) if \( r \leq \lambda \) and the function \( r \to \phi_{\lambda}(r) \) is increasing and bounded.

**Theorem 1.3.** There are constants \( C_0, m, \) and \( C \) with the property that

\[
\| e^{\beta \phi_{\lambda}(x_1)} v \|_{L^2_t L^2_x} + \| e^{\beta \phi_{\lambda}(x_1)} |\nabla v| \|_{B^{\infty,2}_{t} L^2_x} \leq C \| e^{\beta \phi_{\lambda}(x_1)} H v \|_{B^{1,2}_{t} L^2_x} \tag{1.9}
\]

for any \( v \in C(\mathbb{R} : H^1) \) with \( v(., t) \equiv 0 \) for \( t \notin [0, 1] \), any \( \beta \in [1, \infty) \), and any \( \lambda \geq C_0 \beta^m \).

See Lemma 3.1 for a stronger variant of this Carleman inequality. As in [9,4], we emphasize that it is critical to prove an inequality like (1.9) with \( C \) independent of \( \beta \) and \( \lambda \), and with the bounded weight \( e^{\beta \phi_{\lambda}(x_1)} \). The Carleman inequality (1.9) with the weight \( e^{\beta x_1} \) replacing \( e^{\beta \phi_{\lambda}(x_1)} \) (which corresponds to \( \lambda = \infty \)) is significantly easier, but can only be applied to functions \( v \) that have faster than exponential decay as \( x_1 \to \infty \).

The choice of the Banach spaces \( B^{\infty,2}_{t} L^2_x \) and \( B^{1,2}_{t} L^2_x \) in (1.9) is motivated by the inhomogeneous local smoothing estimate of Kenig et al. [7]

\[
\| |\nabla v| \|_{B^{\infty,2}_{t} L^2_x} \leq C \| H v \|_{B^{1,2}_{t} L^2_x} \tag{1.10}
\]

for any \( v \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}) \) supported in \( \mathbb{R}^d \times [0, 1] \). It is somewhat surprising, however, that the corresponding Carleman inequality

\[
\| e^{\beta x_1} |\nabla v| \|_{B^{\infty,2}_{t} L^2_x} \leq C \| e^{\beta x_1} H v \|_{B^{1,2}_{t} L^2_x}
\]

also holds, for any \( v \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}) \) supported in \( \mathbb{R}^d \times [0, 1] \) (this is a special case of (1.9) with \( \lambda = \infty \)). In contrast, on Euclidean spaces \( \mathbb{R}^d, d \geq 3 \), the inequality

\[
\| |\nabla v| \|_{L^2} \leq C \| \Delta v \|_{L^{2d/(d+2)}}
\]

holds for any \( v \in C_0^\infty(\mathbb{R}^d) \) supported in the ball \( \{ x : |x| \leq 1 \} \) (by the Sobolev imbedding theorem), while the corresponding "natural" Carleman inequality

\[
\| e^{\beta x_1} |\nabla v| \|_{L^2} \leq C \| e^{\beta x_1} \Delta v \|_{L^{2d/(d+2)}}, \ \beta \gg 1
\]

fails (see, for instance, [6] for counterexamples to inequalities of this type). In Section 11 we show that there is a similar phenomenon in the case of the Schrödinger
operator $H$ on $\mathbb{R}^d \times \mathbb{R}$: the inequality

$$\| |\nabla_x v| \|_{L^2_t L^2_x} \leq C \| Hv \|_{L^1_t H^{1/2}_x}$$

holds for any $v \in C_0^\infty(\mathbb{R}^d \times \mathbb{R})$ supported in $\mathbb{R}^d \times [0, 1]$ (due to the local smoothing estimate of Constantin and Saut [3], Sjölin [13], and Vega [14]), while the corresponding “natural” Carleman inequality

$$\| e^{\beta x_1} |\nabla_x v| \|_{L^\infty_t L^2_x} \leq C \| e^{\beta x_1} Hv \|_{L^1_t H^{1/2}_x}, \quad \beta \gg 1 \quad (1.11)$$

fails. The failure of (1.11) is related to the difficulty explained in the remark after the statement of Theorem 1.1.

The rest of this paper is organized as follows: in Section 2 we show how to use the Carleman inequality (1.9) and a local uniqueness theorem of Isakov [5] to prove Theorem 1.1 and Corollary 1.2. In Section 3 we reduce the proof of Theorem 1.3 to the a priori Carleman inequality in Lemma 3.1, construct suitable parametrices, and reduce the proof of Lemma 3.1 to proving boundedness of six operators between suitable Banach spaces. We then prove the boundedness of these six operators in Sections 4–10. In Section 11, we show that the Carleman inequality (1.11) fails.

2. The uniqueness theorems

In this section, we show how to use the Carleman inequality in Theorem 1.3 to prove Theorem 1.1 and Corollary 1.2. We start by showing that Corollary 1.2 is a consequence of Theorem 1.1. Let

$$u := u_1 - u_2. \quad (2.1)$$

Then

$$Hu = V_1 u + V_2 \bar{u} + a_1 \cdot \nabla_x u + a_2 \cdot \nabla_x \bar{u} \quad \text{on} \quad \mathbb{R}^d \times (0, 1), \quad (2.2)$$

where

$$V_1 = [P(u_1, \bar{u}_1, \nabla_x u_1, \nabla_x \bar{u}_1) - P(u_2, \bar{u}_1, \nabla_x u_1, \nabla_x \bar{u}_1)]/(u_1 - u_2),$$

$$V_2 = [P(u_2, \bar{u}_1, \nabla_x u_1, \nabla_x \bar{u}_1) - P(u_2, \bar{u}_2, \nabla_x u_1, \nabla_x \bar{u}_1)]/(\bar{u}_1 - \bar{u}_2),$$

$$a_1^\ell = [P(u_2, \bar{u}_2, \partial_{x_1} u_2, \ldots, \partial_{x_{\ell-1}} u_2, \partial_{x_{\ell+1}} u_1, \ldots, \partial_{x_d} u_1, \nabla_x \bar{u}_1)$$

$$- P(u_2, \bar{u}_2, \partial_{x_1} u_2, \ldots, \partial_{x_{\ell-1}} u_2, \partial_{x_{\ell+1}} u_1, \ldots, \partial_{x_d} u_1, \nabla_x \bar{u}_1)]/(\partial_{x_\ell} u_1 - \partial_{x_\ell} u_2)$$
and

\[ a_2^f = [P(u_2, \bar{u}_2, \nabla_x u_2, \partial_x \bar{u}_2, \ldots, \partial_{x_{L-1}} \bar{u}_2, \partial_{x_{L+1}} \bar{u}_1, \ldots, \partial_{x_d} \bar{u}_1) \\
- P(u_2, \bar{u}_2, \nabla_x u_2, \partial_x \bar{u}_2, \ldots, \partial_{x_{L-1}} \bar{u}_2, \partial_{x_{L+1}} \bar{u}_1, \ldots, \partial_{x_d} \bar{u}_1)]/(\partial_{x_1} \bar{u}_1 - \partial_{x_1} \bar{u}_2). \]

We prove now that if (1.7) or (1.8) holds, then

\[
\begin{cases}
V_1, V_2 \in B_x^{2, \infty} L_t^{\infty}, \\
|a_1|, |a_2| \in B_x^{1, \infty} L_t^{\infty}.
\end{cases}
\]

(2.3)

For this we use the simple observation that for \( \sigma' > d/2 \)

\[ J^{-\sigma'}(L^2) \hookrightarrow B^{2, \infty}, \quad J^{-\sigma'}(B^{1,2}) \hookrightarrow B^{1, \infty} \quad \text{and} \quad B^{2, \infty} \cdot B^{2, \infty} \subseteq B^{1, \infty}. \]

(2.4)

If (1.7) holds then \( u_1, \bar{u}_1, \nabla_x u_1, \nabla_x \bar{u}_1, u_2, \bar{u}_2, \nabla_x u_2, \nabla_x \bar{u}_2 \in C([0,1] : H^{\sigma-1}). \) Since \( H^{d/2+} \) is an algebra under multiplication and \( P \) is a polynomial,

\[ P(u_1, \bar{u}_1, \nabla_x u_1, \nabla_x \bar{u}_1), P(u_2, \bar{u}_2, \nabla_x u_2, \nabla_x \bar{u}_2) \in C([0,1] : H^{\sigma-1}). \]

Since \( \Delta u_1, \Delta u_2 \in C([0,1] : H^{\sigma-2}) \), it follows that \( u_1, u_2 \in C([0,1] : H^{\sigma-2}) \) (using (1.6)). Thus \( u_1, \bar{u}_1, \nabla_x u_1, \nabla_x \bar{u}_1, u_2, \bar{u}_2, \nabla_x u_2, \nabla_x \bar{u}_2 \in C([0,1] : H^{\sigma-3}). \) By (2.4) and the fact that \( \sigma - 3 > d/2 \), we have \( V_1, V_2 \in C([0,1] : B^{2, \infty}) \) and \( a_1, a_2 \in C([0,1] : B^{1, \infty}) \), which is better than (2.3). The proof of (2.3) is similar if (1.8) holds. Thus Corollary 1.2 follows from Theorem 1.1.

**Proof of Theorem 1.1.** We apply (1.9) to

\[ v(x,t) = \begin{cases}
\Phi_{R'}(x) u(x,t) & \text{if } t \in [0,1], \\
0 & \text{if } t \notin [0,1],
\end{cases} \]

where \( \Phi : \mathbb{R}^d \rightarrow [0,1] \) is a smooth function equal to 0 if \( |x| \leq 1 \) and equal to 1 if \( |x| \geq 2 \), \( \Phi_{R'}(x) := \Phi(x/R') \), and \( R' \geq R + 1 \) will be fixed sufficiently large. Eq. (1.3) gives

\[ Hv = Vv + a \cdot \nabla_x v + E \quad \text{on} \quad \mathbb{R}^d \times \mathbb{R}, \]

(2.5)

where

\[ E = \begin{cases}
2 \nabla_x \Phi_{R'} \cdot \nabla_x u + (\Delta \Phi_{R'} - a \cdot \nabla_x \Phi_{R'}) u & \text{if } t \in [0,1], \\
0 & \text{if } t \notin [0,1].
\end{cases} \]
For any set $A$ let $1_A$ denote its characteristic function. Using (1.9) and (2.5)
\[ \|e^{\beta \varphi_z(x)} v\|_{L^2_t L^2_x} + \|e^{\beta \varphi_z(x)} |\nabla_x v|\|_{B_t^{\infty,2} L^2_x} \]
\[ \leq C \left[ \|e^{\beta \varphi_z(x)} V v\|_{B_t^{1,2} L^2_x} + \|e^{\beta \varphi_z(x)} a \cdot \nabla_x v\|_{B_t^{1,2} L^2_x} + \|e^{\beta \varphi_z(x)} E\|_{B_t^{1,2} L^2_x} \right] \]
\[ \leq C \left[ \|e^{\beta \varphi_z(x)} v\|_{L^2_t L^2_x} \cdot \|1_{|x| \geq R'} V\|_{B_t^{\infty,\infty} L^2_x} + \|e^{\beta \varphi_z(x)} \|\|_{B_t^{1,\infty} L^\infty_t} \right. \]
\[ + \|e^{\beta \varphi_z(x)} |\nabla_x v|\|_{B_t^{\infty,2} L^2_x} \cdot \|1_{|x| \geq R'} |a|\|_{B_t^{1,\infty} L^\infty_t} + \|e^{\beta \varphi_z(x)} E\|_{B_t^{1,2} L^2_x}. \] (2.6)

We now fix $R' = R + 1$ with the property that
\[ \|1_{|x| \geq R'} V\|_{B_t^{\infty,\infty} L^2_x}, \|1_{|x| \geq R'} |a|\|_{B_t^{1,\infty} L^\infty_t} \leq 1/(2C). \]

Since the weight $e^{\beta \varphi_z(x)}$ is bounded, the left-hand side of (2.6) is finite. Thus we can absorb the first two terms in the right-hand side of (2.6) into the left-hand side. In addition, we notice that $E$ is supported in the set $\{(x, t) : |x| \leq 2R', t \in [0, 1]\}$, $E \in L^2_t L^2_x$, and $e^{\beta \varphi_z(x)} \leq e^{2\beta R'}$ on the support of $E$. By letting $\beta, \lambda \to \infty$, it follows that
\[ u \equiv 0 \text{ in } \{(x, t) : x_1 \geq 2R'\}. \]

By rotation invariance, we conclude that $u(x, t) = 0$ if $\max_{\ell=1,\ldots,d} |x_\ell| \geq 2R'$. Thus it suffices to prove Lemma (2.1) below. \hfill \Box

**Lemma 2.1.** Assume that $u \in C([0, 1] : H^1)$ is a solution of the equation
\[ Hu = Vu + a \cdot \nabla u \text{ on } \mathbb{R}^d \times (0, 1), \] (2.7)
where $V, |a| \in L^\infty_t L^\infty_x (\mathbb{R}^d \times [0, 1])$. If $u \equiv 0$ in $\{|x| \geq R'\} \times [0, 1]$ for some $R' \geq 1$, then $u \equiv 0$ in $\mathbb{R}^d \times [0, 1]$.

**Proof.** This is a direct consequence of [5, Corollary 6.3]. We provide the proof here for the sake of self-containment. We use the following Carleman inequality: for any $R \geq 1$ there is a constant $C_R$ with the property that
\[ \beta^{3/2} \|e^{\beta |x|^2} v\|_{L^2_{x,t}} + \beta^{1/2} \|e^{\beta |x|^2} |\nabla_x v|\|_{L^2_{x,t}} \leq C_R \|e^{\beta |x|^2} H v\|_{L^2_{x,t}} \] (2.8)
for any $v \in C_0^\infty (\mathbb{R}^d \times \mathbb{R})$ supported in $\{|x| \in [1, R]\} \times [0, 1]$, and any $\beta \in [1, \infty)$.

Assuming (2.8), it suffices to prove that for any $\delta > 0$, $u \equiv 0$ in $\mathbb{R}^d \times [\delta, 1 - \delta]$. For $\delta > 0$ we fix $\omega = \omega_{\delta, R} : [0, 1] \to \mathbb{R}$ a smooth function such that $\omega(t) = 0$ if
$t \in [0, \delta/4] \cup [1 - \delta/4, 1]$, $\omega(t) = 6R'$ if $t \in [\delta/2, 1 - \delta/2]$, and $|\omega'(t)| \leq C_{\delta, R'}$ on $[0, 1]$. Let

$$v(x_1, x', t) = u(x_1 - \omega(t), x', t) \Phi_{2R'}(x),$$

where $\Phi$ is as in the proof of Theorem 1.1. Clearly,

$$v \in C([0, 1] : H^1) \quad \text{and} \quad v \text{ is supported in } \{(x, t) : |x| \in [2R', 8R'], t \in [\delta/4, 1 - \delta/4]\}. \quad (2.9)$$

In addition, using (2.7)

$$Hv = V'v + a' \cdot \nabla_x v + E, \quad (2.10)$$

where

$$V'(x, t) = V(x_1 - \omega(t), x', t),$$

$$a'(x, t) = a(x_1 - \omega(t), x', t) - (i\omega'(t), 0, \ldots, 0)$$

and, with $p(x, t) = (x_1 - \omega(t), x', t)$,

$$E(x, t) = 2\nabla_x \Phi_{2R'}(x) \cdot \nabla_x u(p(x, t)) + \Delta_x \Phi_{2R'}(x)u(p(x, t))$$

$$- u(p(x, t))a(p(x, t)) \cdot \nabla_x \Phi_{2R'}(x) + i\omega'(t)u(p(x, t)) \partial_{x_1} \Phi_{2R'}(x).$$

Clearly, $V', a' \in L^\infty_{x, t}$. We apply the Carleman inequality (2.8) to the function $v$ (this is possible in view of (2.9)). An estimate similar to (2.6), using (2.10) and the fact that $v$ is compactly supported in $x$, shows that

$$\beta^{3/2}\|e^{|x|^2}\|v\|^2_{L^2_{x,t}} + \beta^{1/2}\|e^{|x|^2}\|\nabla_x v\|^2_{L^2_{x,t}} \leq C_{\delta, R'}\|e^{|x|^2}\|E\|^2_{L^2_{x,t}}$$

for $\beta$ sufficiently large. We notice now that $E$ is supported in the set $\{(x, t) : |x| \leq 4R'$, $t \in [\delta/4, 1 - \delta/4]\}$. By letting $\beta \to \infty$, we get $v(x, t) = 0$ if $|x| > 4R'$. Since $u(x_1, x', t) = u(x_1 + 6R', x', t)$ if $t \in [\delta, 1 - \delta]$, it follows that $u \equiv 0$ in $\mathbb{R}^d \times [\delta, 1 - \delta]$, as desired.

To prove (2.8) let

$$g(x, t) = e^{|x|^2}v(x, t).$$

Then

$$e^{|x|^2}Hv(x, t) = (H - 4\beta x \cdot \nabla_x + 4\beta^2 |x|^2 - 2d)g(x, t).$$
For (2.8) it suffices to prove that
\[
\beta^{3/2}\|g\|_{L^2_{t,x}} + \beta^{1/2}\|\nabla_x g\|_{L^2_{t,x}} \leq C_R \|(H - 4\beta x \cdot \nabla_x + 4\beta^2|x|^2 - 2d\beta)g\|_{L^2_{t,x}} \tag{2.11}
\]
for any \(g \in C^\infty_0(\mathbb{R}^d \times \mathbb{R})\) supported in \(\{|x| \in [1, R]\} \times [0, 1]\), and any \(\beta \in [1, \infty)\). Let \(A := H + 4\beta^2|x|^2\) and \(B := 4\beta x \cdot \nabla_x + 2d\beta\), such that \(A^* = A\) and \(B^* = -B\). Then
\[
\|(H - 4\beta x \cdot \nabla_x + 4\beta|x|^2 - 2d\beta)g\|_{L^2_{t,x}}^2 = \langle (A - B)g, (A - B)g \rangle \geq \langle (BA - AB)g, g \rangle = (32\beta^3|x|^2 - 8\beta\Delta)g, g \rangle
\]
and the bound (2.11) follows easily. \(\square\)

3. Proof of Theorem 1.3: construction of parametrices

We define the Banach space \(X' = X'_{\beta, \lambda}(\mathbb{R}^d \times \mathbb{R})\) using the norm
\[
\|u\|_{X'} := \max\{\|J^{1/2}u\|_{L^\infty_t L^2_x}, \|J^1u\|_{B^{\lambda,2}_{t}L^2_x}, \|(1 + \beta\phi'(x_1))u\|_{B^{\lambda,2}_{t}L^2_x}\}. \tag{3.1}
\]
We also define the Banach space \(X = X_{\lambda}(\mathbb{R}^d \times \mathbb{R})\) using the norm
\[
\|f\|_X := \inf_{f_1, f_2} \{\|f_1\|_{B^{1,2}_{t}L^2_x} + \lambda^3\|J^1f_2\|_{L^1_t L^2_x}\}. \tag{3.2}
\]

Our main a priori estimate is the following:

**Lemma 3.1.** There are constants \(C'_0, m, \text{ and } C'\) with the property that
\[
\|e^{\beta\phi'(x_1)}u\|_{X'} \leq C'|e^{\beta\phi'(x_1)}Hu\|_{X} \tag{3.3}
\]
for any \(u \in C^\infty_0(\mathbb{R}^d \times \mathbb{R})\) supported in \(\mathbb{R}^d \times [0, 1]\), any \(\beta \in [1, \infty)\), and any \(\lambda \geq C'_0\beta^m\).

We first show that Lemma 3.1 implies Theorem 1.3. By rescaling and translation invariance, we may assume that (3.3) holds for any \(u \in C^\infty_0(\mathbb{R}^d \times \mathbb{R})\) supported in \(\mathbb{R}^d \times [-1, 2]\). Let \(\psi : \mathbb{R}^d \times \mathbb{R} \to [0, \infty)\) denote a smooth function supported in the set \(\{(x, t) : |x|, |t| \leq 1\}\) with \(\int_{\mathbb{R}^d \times \mathbb{R}} \psi(x, t) \ dx \ dt = 1\). For any \(\delta \in (0, 1)\) let \(\psi_\delta(x, t) = \delta^{-(d+1)}\psi(x/\delta, t/\delta)\). Let \(\tilde{\psi} : \mathbb{R}^d \to [0, 1]\) denote a smooth function equal to 1 in the set \(\{x : |x| \leq 1\}\) and equal to 0 in the set \(\{x : |x| \geq 2\}\), and for \(R \geq 1\) let \(\psi_R(x) = \tilde{\psi}(x/R)\).
For \( v \) as in Theorem 1.3, let
\[
  u_{\delta, R}(x, t) = (v * \psi_{\delta})(x, t) \tilde{\psi}_R(x).
\]
Clearly, \( u_{\delta, R} \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}) \) is supported in \( \mathbb{R}^d \times [-1, 2] \). We apply (3.3) to the function \( u_{\delta, R} \):
\[
\| e^{\beta \phi_{\delta}(x_1)} u_{\delta, R} \|_{X'} \leq \overline{C}' \| e^{\beta \phi_{\delta}(x_1)} (H v * \psi_{\delta}) \psi_R \|_{B_{x}^{2,1/2} L_t^{2}}
+ \lambda^2 \overline{C'} \| J^1 [e^{\beta \phi_{\delta}(x_1)}] [2 \nabla_x (v * \psi_{\delta}) \cdot \nabla_x \tilde{\psi}_R]
+ (v * \psi_{\delta}) \Delta \tilde{\psi}_R \|_{L^1_t L^2_x}.
\]
(3.4)
We keep \( \delta \) fixed and let \( R \to \infty \). Since the weight \( e^{\beta \phi_{\delta}(x_1)} \) and its derivatives are bounded, the term in the second line of (3.4) converges to 0 as \( R \to \infty \) (this is the main reason we need to define the space \( X \) as in (3.2) rather than simply \( X = B^{1,2}_x L^2_t \)). Thus
\[
\| e^{\beta \phi_{\delta}(x_1)} (v * \psi_{\delta}) \|_{X'} \leq \overline{C}' \| e^{\beta \phi_{\delta}(x_1)} (H v * \psi_{\delta}) \|_{B_{x}^{2,1/2} L_t^{2}}.
\]
The bound (1.9) follows by letting \( \delta \to 0 \).

We now start with the proof of Lemma 3.1. We will use the letters \( C \) and \( c \) to denote various constants in \( (0, \infty) \) that may depend only on the dimension \( d \). The parametrices in this section were constructed by the authors in [4]. We assume from now on that the constant \( m \) in Lemma 3.1 is \( \geq 4 \) (in fact, the proof gives \( m = 12 \)). Let
\[
f = (i \partial_t + \Delta_x) u.
\]
(3.5)
Let \( U = e^{\beta \phi_{\delta}(x_1)} u \) and \( F = e^{\beta \phi_{\delta}(x_1)} f \). Estimate (1.9) is equivalent to
\[
\| U \|_X \leq C \| F \|_X.
\]
(3.6)
Eq. (3.5) is equivalent to
\[
(i \partial_t + \Delta_x - a_{\beta, \lambda}(x_1) \partial_{x_1} + b_{\beta, \lambda}(x_1)) U = F,
\]
(3.7)
where \( a_{\beta, \lambda} = 2 \beta \phi_\lambda' \) and \( b_{\beta, \lambda} = \beta^2 [\phi_\lambda']^2 - \beta \phi_\lambda'' \). Clearly, \( a_{\beta, \lambda} \) is nonincreasing, \( a_{\beta, \lambda}(x_1) \in [0, 2\beta] \), and \( b_{\beta, \lambda}(x_1) \in [-2\beta^2, 2\beta^2] \) (assuming the constant \( C_0' \) in Lemma 3.1 is large enough); more importantly for any integer \( \sigma \geq 0 \) and \( x_1 \in [\lambda, 2\lambda] \):
\[
\beta^{-1} |\sigma^\alpha a_{\beta, \lambda}(x_1)| + \beta^{-2} |\sigma^\alpha b_{\beta, \lambda}(x_1)| \leq C_\sigma \lambda^{-\sigma}.
\]
(3.8)
The term in the left-hand side of (3.8) is equal to 0 if \( x_1 \notin [\lambda, 2\lambda] \) and \( \sigma \geq 1 \).
Let \( \psi, \chi : \mathbb{R} \to [0, 1] \) denote two smooth, even functions supported in the interval \([-2, 2]\) and equal to 1 in the interval \([-1, 1]\). Let \( 1_+ := 1_{[0,\infty)} \) and \( 1_- := 1_{(-\infty,0)} \). For numbers \( \gamma \geq 1 \) let \( \psi_\gamma(r) = \psi(r/\gamma) \). We fix \( \gamma = 10\beta \). We define the operators \( A_1, A_2, \) and \( A_3 \) (acting on \( \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}) \)) by multiplication with Fourier multipliers:

\[
\begin{align*}
A_1 & \text{ defined by the Fourier multiplier } \psi_\gamma(\xi_1), \\
A_2 & \text{ defined by the Fourier multiplier } [1 - \psi_\gamma(\xi_1)][1 - \psi(10(\tau + |\xi|^2)/\gamma)], \\
A_3 & \text{ defined by the Fourier multiplier } [1 - \psi_\gamma(\xi_1)]\psi(10(\tau + |\xi|^2)/\gamma^2).
\end{align*}
\]

Clearly \( A_1 + A_2 + A_3 = Id \). For \( \varepsilon \in (0,1) \) let \( P_\varepsilon \) denote the operator defined by the Fourier multiplier \((\xi, \tau) \mapsto e^{-\varepsilon^2|\xi|^2}\) and \( Q_\varepsilon \) the operator defined by the Fourier multiplier \((\xi, \tau) \mapsto e^{-\varepsilon^2(\tau + |\xi|^2)}\). We will prove the following estimates:

\[
\|P_\varepsilon A_1(U)\|_{X'} \leq C\|F\|_X + C_1(B, \lambda)\|U\|_{X'}, \quad (3.9)
\]

\[
\|1_{[0,1]}(t)Q_\varepsilon P_\varepsilon A_2(U)\|_{X'} \leq C\|F\|_X + C_1(B, \lambda)\|U\|_{X'} \quad (3.10)
\]

and

\[
\|1_{[0,1]}(t)P_\varepsilon A_3(U)\|_{X'} \leq C\|F\|_X, \quad (3.11)
\]

uniformly in \( \varepsilon \in (0,1) \). The constant \( C_1(B, \lambda) \) is small if \( \lambda \geq C_0^m \beta^m \), with \( m', C_0' \) sufficiently large. Thus estimates (3.9)–(3.11) suffice to prove (3.6).

### 3.1. The parametrix for \( A_1 \)

In this case the variable \( \xi_1 \) is much smaller than \( \lambda \). We construct the parametrices starting from Eq. (3.7), as if the functions \( a_{\beta, \lambda} \) and \( b_{\beta, \lambda} \) were constant. Consider the integral

\[
I_1(F)(x, t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dy \, ds \, F(y, s) \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} e^{-i(t-s)|\xi|^2} \psi_\gamma(\xi_1)[1_-(\xi_1)1_+(t-s) - 1_+(\xi_1)1_-(t-s)]e^{-\varepsilon^2|\xi|^2} e^{a_{\beta, \lambda}(y_1)\xi_1(t-s)}e^{ib_{\beta, \lambda}(y_1)(t-s)} d\xi.
\]

Recall that \( F(y, s) = (i\partial_x + D_y)U(y, s) \) where \( D_y = \Delta_y - a_{\beta, \lambda}(y_1)\partial_{y_1} + b_{\beta, \lambda}(y_1). \) We substitute this into the formula of \( I_1(F)(x, t) \) and integrate by parts in \( s \) and \( y \). The result is

\[
I_1(F)(x, t) = \int_{\mathbb{R}^d} dy \, i U(y, t) \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} \psi_\gamma(\xi_1)e^{-e^2|\xi|^2} d\xi
\]

\[
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dy \, ds \, U(y, s)1_+(t-s)[-i\partial_x + D_y^*]
\]
The function $q_1(y_1, \xi_1, w)$ can be written explicitly by inspecting the above identity; the important fact is that when we compute $(-i\partial_x + D_y^*)$ all the terms that are not small cancel out. The remaining terms either have a derivative of $a_{\beta,\lambda}$ or a derivative of $b_{\beta,\lambda}$. By (3.8), if $|w| \leq 2$ and $|\xi_1| \leq C\beta$, we have

$$\beta |\partial_\xi q_1(y_1, \xi_1, w)| + \lambda^\sigma |\partial_{y_1} q_1(y_1, \xi_1, w)| \leq C \beta^3 / \lambda, \quad \sigma \geq 0$$

(3.13)

for $y_1 \in [\lambda, 2\lambda]$, and the left-hand side of (3.13) is equal to 0 if $y_1 \notin [\lambda, 2\lambda]$. From (3.12) we get

$$P_c A_1(U) = c I_1(F) + c \widetilde{R}_1(U).$$

(3.14)

Notice that $\|1_{[0,1]}(t)v\|_{X'} \leq \|v\|_{X'}$ and $\|1_{[0,1]}(t)g\|_X \leq \|g\|_X$. Thus, for (3.9), we have to first prove that the operator

$$T_1(g)(x, t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dy \, ds \, g(y, s) \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} e^{-i(t-s)|\xi|^2} e^{-i\xi|z|^2} \, \mu_1(y_1, \xi_1, t-s) \, d\xi$$

is bounded from $X$ to $X'$, where

$$\mu_1(y_1, \xi_1, w) = i\chi(w)\psi_1(\xi_1) \left[ 1_-(\xi_1) 1_+(w) - 1_+(\xi_1) 1_-(w) \right] e^{a_{\beta,\lambda}(y_1)\xi_1 w} e^{ib_{\beta,\lambda}(y_1)w}$$

(3.15)

(the reason for having the factor $i$ on the right-hand side of (3.15) is to satisfy the symmetry property (3.31) below for $j = 1, 2, 3$). In addition, we have to prove that
We substitute formula (3.7) and integrate by parts. Let the operator
\[ R_1(g)(x, t) = \mathbf{1}_{[0,1]}(t) \int_\mathbb{R}^d \int_\mathbb{R}^d dy \, ds \, g(y, s) \]
\[ \int_\mathbb{R}^d e^{i(x-y) \cdot \xi} e^{-i(t-s) |\xi|^2} e^{-\xi^2 |\xi|^2} v_1(y_1, \xi_1, t-s) \, d\xi \]
is bounded from \( X \) to \( X' \) with a small norm, where
\[ v_1(y_1, \xi_1, w) = \chi(w) \psi_\gamma(\xi_1) \]
\[ [1_-(\xi_1)1_+(w) - 1_+(\xi_1)1_-(w)]e^{a_\beta, \lambda(y_1)\xi_1 w} e^{ib_\beta, \lambda(y_1)w} q_1(y_1, \xi_1, w). \]  
(Note: the role of the signs is clear: because of the exponential term we require that \( a_\beta, \lambda(y_1) \xi_1 w \leq 0 \); this is achieved because \( w \xi_1 \leq 0 \) and \( a_\beta, \lambda(y_1) \geq 0 \).)

3.2. The parametrix for \( A_2 \)

We start from the integral
\[ I_2(F)(x, t) = \int_\mathbb{R}^d \int_\mathbb{R}^d dy \, ds \, F(y, s) \int_\mathbb{R}^d \int_\mathbb{R}^d d\xi \, d\tau \, e^{i(x-y) \cdot \xi} e^{i(t-s) \tau} e^{-\xi^2 |\xi|^2} e^{-\tau^2 (|\xi|^2)^2} \]
\[ [1 - \psi_\gamma(\xi_1)][1 - \psi(10(\tau + |\xi|^2)/\xi_1^2)][-\tau - |\xi|^2 - i \xi_1 a_\beta, \lambda(y_1) + b_\beta, \lambda(y_1)]^{-1}. \]

We substitute formula (3.7) and integrate by parts. Let \( \tilde{q}_2(y_1, \xi, \tau) = [-\tau - |\xi|^2 - i \xi_1 a_\beta, \lambda(y_1) + b_\beta, \lambda(y_1)]^{-1} \). The result is
\[ I_2(F)(x, t) = c \int_\mathbb{R}^d \int_\mathbb{R}^d dy \, ds \, U(y, s) \int_\mathbb{R}^d \int_\mathbb{R}^d d\xi \, d\tau \, e^{i(x-y) \cdot \xi} e^{i(t-s) \tau} e^{-\xi^2 |\xi|^2} \]
\[ \times e^{-\xi^2 (\tau + |\xi|^2)^2} [1 - \psi_\gamma(\xi_1)][1 - \psi(10(\tau + |\xi|^2)/\xi_1^2)] + c \int_\mathbb{R}^d \int_\mathbb{R}^d dy \, ds \, U(y, s) \int_\mathbb{R}^d \int_\mathbb{R}^d d\xi \, d\tau \, e^{i(x-y) \cdot \xi} e^{i(t-s) \tau} e^{-\xi^2 |\xi|^2} e^{-\tau^2 (|\xi|^2)^2} \]
\[ \times [1 - \psi_\gamma(\xi_1)][1 - \psi(10(\tau + |\xi|^2)/\xi_1^2)][\partial_{\xi_1 a_\beta, \lambda(y_1) \tilde{q}_2(y_1, \xi, \tau) + (a_\beta, \lambda(y_1) \tilde{q}_2(y_1, \xi, \tau)] - 2i \xi_1 \partial_{\xi_1} \tilde{q}_2(y_1, \xi, \tau) + \partial_{\xi_1} \tilde{q}_2(y_1, \xi, \tau)] = c Q_\varepsilon P_\varepsilon A_2(U)(x, t) + c \tilde{R}_2(U)(x, t). \]

Thus for (3.10) we have to first prove that the operator
\[ T_2(g)(x, t) = \int_\mathbb{R}^d \int_\mathbb{R}^d dy \, ds \, g(y, s) \int_\mathbb{R}^d e^{i(x-y) \cdot \xi} e^{-i(t-s) |\xi|^2} e^{-\xi^2 |\xi|^2} \mu_2(y_1, \xi_1, t-s) \, d\xi \]
is bounded from $X$ to $X'$, where

$$
\mu_2(y_1, \zeta_1, w) = \chi(w)[1 - \psi_\gamma(\zeta_1)] \\
\int_{\mathbb{R}} e^{iwr} e^{-e^2r^2} [1 - \psi(10r/\zeta_1^2)][-r - i\zeta_1 a_{\beta, \zeta}(y_1) + b_{\beta, \zeta}(y_1)]^{-1} dr. \quad (3.19)
$$

In addition, we have to prove that the operator

$$
R_2(g)(x, t) = \mathbf{1}_{[0, 1]}(t) \int_{\mathbb{R}^d} \int_{\mathbb{R}} dy ds \, g(y, s) \\
\int_{\mathbb{R}^d} e^{i(x-y)\cdot\zeta} e^{-i(t-s)|\zeta|^2} e^{-e^2|\zeta|^2} v_2(y_1, \zeta_1, t-s) \, d\zeta
$$

is bounded from $X'$ to $X'$ with a small norm, where

$$
v_2(y_1, \zeta_1, w) = \chi(w)[1 - \psi_\gamma(\zeta_1)] \int_{\mathbb{R}} e^{iwr} e^{-e^2r^2} [1 - \psi(10r/\zeta_1^2)] \\
[a_\beta, \zeta'(y_1) q_2(y_1, \zeta_1, r) + (a_{\beta, \zeta}(y_1) - 2i\zeta_1)q_2'(y_1, \zeta_1, r) + q_2''(y_1, \zeta_1, r)] \, dr. \quad (3.20)
$$

The notation in (3.20) is $q_2(y_1, \zeta_1, r) = [-r - i\zeta_1 a_{\beta, \zeta}(y_1) + b_{\beta, \zeta}(y_1)]^{-1}$ and the primes denote differentiation with respect to $y_1$.

### 3.3. The parametrix for $A_3$

This is a more delicate case. We think of the equation as an evolution in $x_1$ rather than $t$ and start from (3.5) rather than (3.7). Let $\tilde{u}(x_1, \zeta', \tau)$, $\tilde{f}(x_1, \zeta', \tau)$, etc denote the partial Fourier transforms of the functions $u, f$ etc in the variables $x'$ and $t$. By taking this partial Fourier transform, Eq. (3.5) becomes

$$
[\dot{\zeta}^2 x_1 - (\tau + |\zeta'|^2)]\tilde{u}(x_1, \zeta', \tau) = \tilde{f}(x_1, \zeta', \tau).
$$

By using this equation and integrating by parts we have

$$
\int_{z_1}^\infty \tilde{f}(y_1, \zeta', \tau) \frac{\sin[(z_1 - y_1)\sqrt{-(\tau + |\zeta'|^2)}]}{\sqrt{-(\tau + |\zeta'|^2)}} \, dy_1 = -\tilde{u}(z_1, \zeta', \tau) \quad (3.21)
$$

whenever $(\tau + |\zeta'|^2) \leq 0$. Let

$$
L(z_1 - y_1, \sqrt{-(\tau + |\zeta'|^2)}) = 1_+(y_1 - z_1) \frac{\sin[(z_1 - y_1)\sqrt{-(\tau + |\zeta'|^2)}]}{\sqrt{-(\tau + |\zeta'|^2)}}. \quad (3.22)
$$
We multiply Eq. (3.21) by $e^{\beta\phi_2(z_1)}$ to obtain

$$
\tilde{U}(z_1, \zeta', \tau) = -\int_{\mathbb{R}} \tilde{F}(y_1, \zeta', \tau) e^{\beta\phi_2(z_1)-\beta\phi_2(y_1)} L(z_1 - y_1, \sqrt{-(\tau + |\zeta'|^2)}) \, dy_1,
$$

and take the Fourier transform in $z_1$ to obtain

$$
\tilde{U}(\xi_1, \zeta', \tau) = -\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-iz_1\xi_1} \tilde{F}(y_1, \zeta', \tau) e^{\beta\phi_2(z_1)-\beta\phi_2(y_1)} L(z_1 - y_1, \sqrt{-(\tau + |\zeta'|^2)}) \, dy_1 \, dz_1.
$$

We multiply this by $[1 - \psi_\gamma(\xi_1)]\psi(10(\tau + |\xi|^2)/\xi_1^2)e^{-\xi^2|\xi|^2}$ and notice that $\psi(10(\tau + |\zeta'|^2)/\xi_1^2) = 0$ unless $(\tau + |\zeta'|^2) \in [-6\xi_1^2/5, -4\xi_1^2/5]$. We use the fact that

$$
\tilde{F}(y_1, \zeta', \tau) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} F(y_1, y', s) e^{-i(y' \cdot \zeta' + \tau s)} \, dy' \, ds
$$

and take the inverse Fourier transform. The result is

$$
P_\varepsilon A_3(U)(x_1, x', t)
= c \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}} F(y_1, y', s) K(x_1, y_1, x', y', t, s) \, dy_1 \, dy' \, ds,
$$

where

$$
K(x_1, y_1, x', y', t, s) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} d\xi_1 d\xi' d\tau d\zeta' e^{i(x_1 - y_1)\xi_1} e^{i(x' - y')\zeta'} e^{i(t-s)\tau} e^{\beta\phi_2(z_1)-\beta\phi_2(y_1)}[1 - \psi_\gamma(\xi_1)]\psi(10(\tau + |\xi|^2)/\xi_1^2)e^{-\xi^2|\xi|^2} L(z_1 - y_1, \sqrt{-(\tau + |\zeta'|^2)}).
$$

We make the changes of variables $z_1 = y_1 - \tau$ and $\tau = -w - |\zeta'|^2$. The integral for $K$ becomes

$$
K(x_1, y_1, x', y', t, s) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} d\xi_1 d\xi' e^{i(x-y)\xi_1} e^{-i(t-s)|\xi'|^2} [1 - \psi_\gamma(\xi_1)] e^{-\xi^2|\xi|^2} \int_{\mathbb{R}} \int_{\mathbb{R}} dz \, dw e^{i\xi_1 z} e^{-i(t-s)w} e^{\beta\phi_2(y_1 - \tau) - \beta\phi_2(y_1)} \psi(10(\xi_1^2 - w)/\xi_1^2) L(-z, \sqrt{w}).
$$
The change of variable \( w = \frac{2}{10}\xi^2 r^2 \) in the inner integral together with the fact that 
\( L(-\alpha, r) = -\mathbf{1}_+(x) \sin(\pi r)/r \) shows that

\[
K(x_1, y_1, x', y', t, s) = -2 \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} d\xi \, d\xi' \, e^{i(x-y)-\xi} e^{-i(t-s)|\xi|^2} [1 - \psi_{\gamma}(\xi)] e^{-\xi^2 |\xi|^2} 
\]

\[
\int_0^\infty \int_0^\infty dx \, dr \, e^{i\xi^2 r^2} e^{-i(t-s)|\xi|^2} e^{\beta \varphi_{\gamma}(y_1-x) - \beta \varphi_{\gamma}(y_1)} \psi(10(1-r^2)) \sin(\xi_1 r x) \xi_1.
\]

For \( r \geq 0 \) let \( \tilde{\psi}(r) = \psi(10(1-r^2)) \); clearly \( \tilde{\psi} \) is smooth and supported in the interval \([(4/5)^{1/2}, (6/5)^{1/2}] \). The formula for \( K \) becomes

\[
K(x_1, y_1, x', y', t, s) = c \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} d\xi \, d\xi' \, e^{i(x-y)-\xi} e^{-i(t-s)|\xi|^2} [1 - \psi_{\gamma}(\xi)] e^{-\xi^2 |\xi|^2} 
\]

\[
\int_0^\infty \int_0^\infty dx \, dr \, e^{i\xi^2 r^2} e^{-i(t-s)|\xi|^2} e^{\beta \varphi_{\gamma}(y_1-x) - \beta \varphi_{\gamma}(y_1)} \tilde{\psi}(r) \sin(\xi_1 r x) \xi_1.
\]

By (3.23), the bound (3.11) follows if we prove that the operator

\[
T_3(g)(x, t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} dy \, ds \, g(y, s) \int_{\mathbb{R}^d} e^{i(x-y)-\xi} e^{-i(t-s)|\xi|^2} e^{-\xi^2 |\xi|^2} \mu_3(y_1, \xi_1, t-s) \, d\xi
\]

is bounded from \( X \) to \( X' \) where

\[
\mu_3(y_1, \xi_1, w) = \chi(w) [1 - \psi_{\gamma}(\xi_1)] 
\]

\[
\int_0^\infty \int_0^\infty dx \, dr \, e^{i\xi^2 r^2} e^{-i\xi^2 (r^2-1)} e^{\beta \varphi_{\gamma}(y_1-x) - \beta \varphi_{\gamma}(y_1)} \tilde{\psi}(r) \sin(\xi_1 r x) \xi_1. \tag{3.24}
\]

To summarize, it remains to prove that the operators \( T_j, j = 1, 2, 3 \), are bounded from \( X \) to \( X' \) and the operators \( R_j, j = 1, 2 \), are bounded from \( X' \) to \( X' \) with a small norm. By the definitions of the spaces \( X \) and \( X' \), for the operators \( T_j \) it suffices to prove that

\[
J^1 T_j : B_y^{1,2} L_x^2 \to B_x^{\infty,2} L_t^2, \quad j = 1, 2, 3, \tag{3.25}
\]

\[
[1 + \alpha \lambda(x_1)]T_j : B_y^{1,2} L_x^2 \to B_x^{\infty,2} L_t^2, \quad j = 1, 2, 3, \tag{3.26}
\]

\[
J^{1/2} T_j : B_y^{1,2} L_x^2 \to L_x^\infty L_t^2, \quad j = 1, 2, 3 \tag{3.27}
\]

and

\[
\lambda^{-2} J^1 T_j J^{-1} : L_y^1 L_x^2 \to L_x^\infty L_t^2, \quad j = 1, 2, 3. \tag{3.28}
\]
For the operators $R_j$, it suffices to prove that

$$\| J^{1/2} R_j J^{-1/2} \|_{L^\infty_y L^2_x} \rightarrow L^2_y \leq c_1 / \beta, \quad j = 1, 2$$

(3.29)

and

$$\| J^1 R_j J^{-1/2} \|_{L^\infty_y L^2_x} \rightarrow B^\infty_{L^2_y} \leq c_1 / \beta, \quad j = 1, 2,$$

(3.30)

with $c_1 \ll 1$. For later use we record the following symmetry property:

$$\mu_j (y_1, -\xi_1, -w) = \mu_j (y_1, \xi_1, w)$$

(3.31)

for any $y_1, \xi_1, w \in \mathbb{R}$, and $j = 1, 2, 3$.

4. Three lemmas

In the proofs of (3.25)–(3.30) we use three lemmas frequently. The first lemma is our main criterion to establish boundedness on $L^2(\mathbb{R})$ of various operators.

**Lemma 4.1.** For any $h \in S(\mathbb{R})$ and $K \in L^\infty(\mathbb{R} \times \mathbb{R})$,

$$\left\| \int_\mathbb{R} h(r) e^{-ir\tau} K(r, \tau) \, dr \right\|_{L^2_y} \leq C \| h \|_{L^2_y} \sup_{\tau \in \mathbb{R}} \| K(., \tau) \|_{BV_r}.$$

(4.1)

The BV norm is defined by

$$\| m \|_{BV} := \sup_{r \in \mathbb{R}} | m(r) | + \int_\mathbb{R} | m'(r) | \, dr$$

(4.2)

for any $m \in L^\infty(\mathbb{R})$.

**Proof.** We use Carleson’s theorem [2]: the operator

$$C(h)(\tau) = \sup_N \left| \int_{-\infty}^{N} h(r) e^{-ir\tau} \, dr \right|$$

is bounded from $L^2_\tau$ to $L^2_\tau$. Thus, for any $\tau$ we have

$$\left| \int_\mathbb{R} h(r) K(r, \tau) e^{-ir\tau} \, dr \right| = \left| \int_\mathbb{R} \left[ \int_{-\infty}^{r} h(z) e^{-iz\tau} \, dz \right]' K(r, \tau) \, dr \right|$$
\[ \leq |K(\infty, \tau)| \cdot |\hat{h}(\tau)| + \left| \int_{\mathbb{R}} C(h)(\tau) |K'(r, \tau)| \, dr \right| \]

\[ \leq C(h)(\tau) \cdot \|K(\cdot, \tau)\|_{BV_r}. \]

By Carleson’s theorem this proves (4.1). \[\square\]

Our second lemma is the following uniform bound:

**Lemma 4.2.** Assume that \( k \in \mathbb{Z}, m \in C^1([2^k, 2^{k+1}]), \) and \( \Phi \in C^2([2^k, 2^{k+1}]) \) is a real-valued phase function. Assume that

\[
\begin{aligned}
2^k |m(r)| + 2^{2k} |m'(r)| \leq 1 & \text{ for any } r \in [2^k, 2^{k+1}], \\
|\Phi'(r)| \geq \varepsilon & \text{ and } 2^k |\Phi''(r)| \leq C_0 \varepsilon \text{ for any } r \in [2^k, 2^{k+1}]
\end{aligned}
\] (4.3)

for some \( \varepsilon \in [0, \infty) \). Then

\[
\left| \int_{b_1}^{b_2} e^{i\Phi(r)} m(r) \, dr \right| \leq C'[1 + 2^k \varepsilon]^{-1}
\] (4.4)

for any \( b_1, b_2 \in [2^k, 2^{k+1}] \). The constant \( C' \) may depend only on \( C_0 \).

The bound (4.4) follows easily by integration by parts. Our last lemma concerns uniform bounds for oscillatory integrals involving Calderon–Zygmund kernels:

**Lemma 4.3.** Assume that \( m \in C^1(\mathbb{R} \setminus \{0\}) \) is a Calderon–Zygmund kernel, i.e.

\[
\begin{aligned}
|r m(r)| + |r^2 m'(r)| \leq 1 & \text{ for any } r \in \mathbb{R} \setminus \{0\}, \\
\int_{|r| \in [b_1, b_2]} m(r) \, dr \leq 1 & \text{ for any } b_1, b_2 \in (0, \infty).
\end{aligned}
\] (4.5)

Assume also that \( \Phi \in C^2(\mathbb{R}) \) is a real-valued phase function with the property

\( \Phi''(r) \in [C_0^{-1} b_3, C_0 b_3] \) for any \( r \in \mathbb{R} \)

for some \( b_3 \in [0, \infty) \) and \( C_0 \geq 1 \). Then

\[
\left| \int_{|r| \in [b_1, b_2]} e^{\pm i\Phi(r)} m(r) \, dr \right| \leq C',
\] (4.6)

for any \( b_1, b_2 \in (0, \infty) \). The constant \( C' \) may depend only on \( C_0 \).

**Proof.** By dilation invariance, we may assume \( b_3 = 1 \). Let \( A = \Phi'(0) \). Then

\[
|\Phi'(r) - A| \in [C_0^{-1} |r|, C_0 |r|].
\] (4.7)
Thus, for $|r| \not\in [|A|/(10C_0), 10C_0|A|]$,

$$|\Phi'(r)| \geq c_0(|r| + |A|) \quad (4.8)$$

for some constant $c_0 > 0$. We consider two cases: $|A| \leq 1$ and $|A| \geq 1$. If $|A| \leq 1$, then, using (4.7), $|\Phi(r) - \Phi(0)| \leq C|r|$ for $|r| \in [0, 10C_0]$. Thus

$$\left| \int_{|r| \in [b_1, b_2] \cap [0,10C_0]} e^{\pm i\Phi(r)} m(r) \, dr \right| \leq \int_{|r| \in [0,10C_0]} C|r|m(r) \, dr + \left| \int_{|r| \in [b_1, b_2] \cap [0,10C_0]} m(r) \, dr \right| \leq C'. \quad (4.9)$$

In addition, using (4.8) and Lemma 4.2,

$$\left| \int_{|r| \in [b_1, b_2] \cap [2^k,2^{k+1}]} e^{\pm i\Phi(r)} m(r) \, dr \right| \leq C'2^{-2k} \quad (4.10)$$

for $k \geq 0$. The bound (4.6) follows in this case.

The case $|A| \geq 1$ is similar: for the integral over $|r| \in [b_1, b_2] \cap [0, |A|^{-1}]$ we use a bound similar to (4.9). For the integral over $|r| \in [b_1, b_2] \cap [10C_0|A|, \infty)$ we use a bound similar to (4.10). For the integral over $|r| \in [b_1, b_2] \cap [|A|/(10C_0), 10C_0|A|]$, we use assumption (4.5). For the integral over $|r| \in [b_1, b_2] \cap [|A|^{-1}, |A|/(10C_0)]$ we use the bound (4.8) and Lemma 4.2. □

5. Boundedness of the operators $T_j, I$

For $\sigma \in \mathbb{R}$ and $\ell = 1, \ldots, d$, let $J_\ell^\sigma : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ denote the operators defined by the Fourier multipliers $\xi \to (1 + \xi^2_\ell)^{\sigma/2}$. In this section we prove the bounds (3.25). It suffices to prove that

$$J_\ell^1 T_j : B_y^{1,2} L_x^2 \to B_x^{\infty,2} L_y^2, \quad j = 1, 2, 3, \ell = 1, \ldots, d.$$ 

We analyze two cases: $\ell = 1$ and $\ell = 2, \ldots, d$.

The case $\ell = 1$: We prove the following stronger bound:

**Lemma 5.1.** We have

$$\|J_1^1 T_j (1_{[k-1,k+1]}(y_1) g(y, s))(x_1, \ldots)\|_{L_x^{2,j}} \leq C\|g\|_{L_y^{2}} \quad (5.1)$$

for $j = 1, 2, 3, x_1 \in \mathbb{R}$, and $k \in \mathbb{Z}$. \hfill \Box
Proof. The operators $J_1^T_j$ are given by the formula

$$J_1^T_j(g)(x, t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} dy ds \, g(y, s) \int_{\mathbb{R}} e^{i(x-y) \cdot \xi} e^{-i(t-s)|\xi|^2} e^{-\varepsilon_1^2 |\xi|^2} (1 + \varepsilon_1^2)^{1/2} \mu_j(y_1, \xi_1, t-s) d\xi.$$ 

By performing the Fourier transform in $(x', t)$, we have

$$[J_1^T_j(g)](x_1, \zeta_1, \tau) = c e^{\varepsilon_1^2 |\zeta_1|^2} \int_{\mathbb{R}} dy_1 \tilde{g}(y_1, \zeta_1, \tau) \int_{\mathbb{R}} e^{i(x_1-y_1) \zeta_1} e^{-\varepsilon_1^2 \zeta_1^2} H_j(y_1, \xi_1, \tau + |\xi|^2)(1 + \varepsilon_1^2)^{1/2} d\xi_1,$$

where

$$H_j(y_1, \xi_1, \zeta) = \int_{\mathbb{R}} e^{-i\zeta w} \mu_j(y_1, \xi_1, w) dw. \quad (5.2)$$

By the Plancherel theorem and the Minkowski inequality for integrals, for (5.1) it suffices to prove that

$$\left| \int_{\mathbb{R}} e^{i(x_1-y_1) \zeta_1} e^{-\varepsilon_1^2 \zeta_1^2} H_j(y_1, \xi_1, \tau + |\xi|^2)(1 + \varepsilon_1^2)^{1/2} d\xi_1 \right| \leq C(1 + |x_1 - y_1|^{-1/4}) \quad (5.3)$$

for any $x_1, y_1, \zeta \in \mathbb{R}$, and $j = 1, 2, 3$. In fact, we can replace the right-hand side of (5.3) with $C \max[1, -\log |x_1 - y_1|]$ (the same remark applies in many other places). It follows from (3.31) that

$$H_j(y_1, -\zeta_1, \zeta) = H_j(y_1, \zeta_1, \zeta) \quad (5.4)$$

for any $y_1, \zeta_1, \zeta \in \mathbb{R}$, and $j = 1, 2, 3$.

To prove (5.3) for $j = 1$ we use (3.15) and (5.4). It suffices to prove that

$$\left| \int_{\mathbb{R}} 1\!\!\!1(-\zeta_1) e^{i(x_1-y_1) \zeta_1} e^{-\varepsilon_1^2 \zeta_1^2} H_1(y_1, \xi_1, \zeta + \zeta_1^2)(1 + \varepsilon_1^2)^{1/2} d\xi_1 \right| \leq C(1 + |x_1 - y_1|^{-1/4}) \quad (5.5)$$

uniformly in $x_1, y_1$, and $\zeta$. Let

$$G_1(z) = i \int_{\mathbb{R}} \chi(w) 1\!\!\!1_+(w) e^{-i\zeta w} dw \quad (5.6)$$
for \( z \in \mathbb{C}, \Im z \leq 0 \). Then

\[ H_1(y_1, \zeta_1, z + \zeta_1^2) = \psi \zeta_1 G_1(\zeta - b_{\beta, \lambda}(y_1) + \zeta_1^2 + i a_{\beta, \lambda}(y_1) \zeta_1) \]

if \( \zeta_1 \leq 0 \). For the function \( G_1 \) we have the bound

\[ (1 + |z|)|G_1(z)| + (1 + |z|)^2|G_1'(z)| \leq C, \quad \Im z \leq 0 \tag{5.7} \]

and the cancellation property

\[ \int_{|v| \leq b_1} G_1(v - i b_2) \, dv \leq C \tag{5.8} \]

for any \( b_1, b_2 \geq 0 \). Let \( A = -(\zeta - b_{\beta, \lambda}(y_1)) \). If \( A \leq 1 \) then, using (5.7),

\[ |G_1(\zeta_1^2 - A + i a_{\beta, \lambda}(y_1) \zeta_1)| \leq C(1 + |\zeta_1|)^{-2}, \]

\[ |\partial_{\zeta_1}[G_1(\zeta_1^2 - A + i a_{\beta, \lambda}(y_1) \zeta_1)]| \leq C(1 + |\zeta_1|)^{-3}. \tag{5.9} \]

The bound (5.5) follows in this case from Lemma 4.2. If \( A \geq 1 \), the bounds (5.9) still hold if \( \zeta_1 \notin [\sqrt{A/2}, \sqrt{3A/2}] \), so, by Lemma 4.2,

\[ \left| \int_{|\zeta_1| \notin [\sqrt{A/2}, \sqrt{3A/2}]} (1 - (\zeta_1) \psi \zeta_1 e^{i(x_1 - y_1)\zeta_1} e^{-\theta^2 \zeta_1^2} \right| G_1(\zeta_1^2 - A + i a_{\beta, \lambda}(y_1) \zeta_1)(1 + \zeta_1^2)^{1/2} \, d\zeta_1 \leq C(1 + |x_1 - y_1|^{-1/4}). \]

To control the integral over \( |\zeta_1| \in [\sqrt{A/2}, \sqrt{3A/2}] \), we make the change of variables \( \zeta_1 = -(A + v)^{1/2}, \, v \in [-A/2, A/2] \). It suffices to prove that

\[ \left| \int_{-A/2}^{A/2} \psi \zeta_1(\sqrt{A + v}) e^{-i(x_1 - y_1)\sqrt{A + v}} e^{-\theta^2 (A + v)} G_1(v - i a_{\beta, \lambda}(y_1)\sqrt{A + v}) (1 + (A + v)^{-1})^{1/2} \, dv \right| \leq C \]

uniformly in \( x_1, y_1 \in \mathbb{R} \) and \( A \in [1, \infty] \). We may first replace the factor \( 1 + (A + v)^{-1})^{1/2} \) with 1, at the expense of an absolutely integrable error. We may also replace \( \psi \zeta_1(\sqrt{A + v}) \) with \( \psi \zeta_1(\sqrt{A}) \) at the expense of an absolutely integrable error. Using (5.7), the bound (5.10) is clear if \( a_{\beta, \lambda}(y_1) \geq \sqrt{A} \). Assume \( a_{\beta, \lambda}(y_1) \leq \sqrt{A} \). Using (5.7), we may replace the factor \( G_1(v - i a_{\beta, \lambda}(y_1)\sqrt{A + v}) \) with \( G_1(v - i a_{\beta, \lambda}(y_1)\sqrt{A}) \) at
the expense of an error \( \leq Ca_{\beta,\ell}(y_1)|v|/(\sqrt{A}(1+v^2+Aa_{\beta,\ell}(y_1)^2)) \), which is absolutely integrable if \( a_{\beta,\ell}(y_1) \leq \sqrt{A} \). To summarize, it suffices to prove that

\[
\left| \int_{-A/2}^{A/2} e^{-i(x_1-y_1)(\sqrt{A+v}-\sqrt{A})} e^{-v(A+v)} G_1(v - ia_{\beta,\ell}(y_1)\sqrt{A}) \, dv \right| \leq C \quad (5.10)
\]

uniformly in \( x_1, y_1 \in \mathbb{R} \) and \( A \in [1, \infty] \). This follows from Lemma 4.3.

To prove (5.3) in the case \( j = 2 \), we use the formula (3.19) and integrate the variable \( w \) first. The result is

\[
H_2(y_1, \xi_1, \zeta + \xi_1^2) = [1 - \psi_\gamma(\xi_1)] \int_{\mathbb{R}} e^{-\xi_1 r^2} [1 - \psi(10r/\xi_1^2)] [-r - i\xi_1 a_{\beta,\ell}(y_1) + b_{\beta,\ell}(y_1)]^{-1} \hat{\gamma}((\zeta + \xi_1^2 - r) dr.
\]

Clearly, \( |H_2(y_1, \xi_1, \zeta + \xi_1^2)| \leq C(1 + |\xi_1|)^{-2} \) since \( \hat{\gamma} \) is integrable. The change of variables \( r = \xi_1^2 - v \), together with the fact that \( \hat{\gamma} \) is integrable, shows that

\[
|\partial_{\xi_1}[H_2(y_1, \xi_1, \zeta + \xi_1^2)]| \leq C(1 + |\xi_1|)^{-3}.
\]

Estimate (5.3) then follows from Lemma 4.2.

To prove (5.3) in the case \( j = 3 \), we use formula (3.24) and integrate the variable \( w \) first. Then we make the change of variable \( r = v/\xi_1 \). The result is

\[
H_3(y_1, \xi_1, \zeta + \xi_1^2) = [1 - \psi_\gamma(\xi_1)] \int_0^\infty \int_{\mathbb{R}} dx \, dv \, e^{i\xi_1 x} e^{i\phi_\gamma(y_1-x) - \xi_1^2} \sin(xv) \hat{\gamma}(v/\xi_1) \hat{\gamma}(\zeta + v^2). \quad (5.11)
\]

We substitute this formula into the left-hand side of (5.3) and integrate the variable \( \xi_1 \) first. It follows easily that this integral in \( \xi_1 \) is bounded by \((1 + |v|)^2[1 + |x_1 - y_1 + \zeta(1 + |v|)]^{-2} \). Thus, the left-hand side of (5.3) is bounded by

\[
\int_0^\infty \int_{\mathbb{R}} dx \, dv \, (1 + |v|)^2[1 + |x_1 - y_1 + \zeta(1 + |v|)]^{-2} \hat{\gamma}(\zeta + v^2) dv \leq C
\]

as desired. \( \square \)

The case \( \ell \geq 2 \): We write the variables \( y = (y_1, y_\ell, y'_\ell) \), \( x = (x_1, x_\ell, x'_\ell) \), etc (so \( y'_\ell, x'_\ell \), etc have \( d - 2 \) components). We prove the following stronger bound:
Lemma 5.2. We have

$$\| J_j^1 T_j (1_{[k_1-1,k_1+1]}(y_1)) (y_\ell) g(y,s) \|_{L^2_{x_1,x_\ell,t}} \leq C \| g \|_{L^2_{y_1,s}} \tag{5.12}$$

for \( j = 1, 2, 3 \), \( x_\ell \in \mathbb{R} \), and \( k_1, k_\ell \in \mathbb{Z} \).

Proof. Let \( \mathcal{F}_{x_1,x_\ell,t} \) denote the Fourier transform in the variables \((x_1,x_\ell,t)\). The operators \( J_j^1 T_j \) are given by the formula

$$J_j^1 T_j (g)(x,t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} dy \, ds \, g(y,s) e^{i(x-y) \cdot \hat{\xi}} e^{-i(t-s) |\hat{\xi}|^2} e^{-\xi_\ell^2/2} (1 + \xi_\ell^2)^{1/2} \mu_j(y_1, \xi_1, t-s) \, d\xi.$$

By performing the Fourier transform in \((x_1,x_\ell,t)\), we have

$$\mathcal{F}_{x_1,x_\ell,t} [J_j^1 T_j (g)] (\eta_1, \eta_\ell, \eta_1', \tau) = c e^{-\xi_1^2 (\eta_1^2 + |\eta_1'|^2)} \int_{\mathbb{R}^2} dy_1 dy_\ell \mathcal{F}_{y_1,y_\ell} g(y_1,y_\ell,\eta_\ell,\tau) e^{-iy_1 \eta_1}$$

$$\int_{\mathbb{R}} e^{i(x_\ell-y_\ell) \xi_\ell} e^{-\xi_\ell^2/2} H_j (y_1, \eta_1, \tau + \eta_1^2 + \xi_\ell^2 + |\eta_\ell|^2) (1 + \xi_\ell^2)^{1/2} d\xi_\ell,$$

where \( H_j \) is as in (5.2). By the Plancherel theorem and the Minkowski inequality for integrals, for (5.12) it suffices to prove that

$$\left\| \int_{\mathbb{R}} dy_1 h(y_1) 1_{[k_1-1,k_1+1]}(y_1) e^{-iy_1 \eta_1} L_j (y_1, \eta_1, x_\ell, y_\ell, \zeta) \right\|_{L^2_{y_1}} \leq C \| h \|_{L^2_{y_1}} (1 + |x_\ell - y_\ell|^{-1/4}), \tag{5.13}$$

for any \( x_\ell, y_\ell, \zeta \in \mathbb{R} \), and \( j = 1, 2, 3 \), where

$$L_j (y_1, \eta_1, x_\ell, y_\ell, \zeta) = \int_{\mathbb{R}} e^{i(x_\ell-y_\ell) \xi_\ell} e^{-\xi_\ell^2/2} H_j (y_1, \eta_1, \zeta + \eta_1^2 + \xi_\ell^2) (1 + \xi_\ell^2)^{1/2} d\xi_\ell. \tag{5.14}$$

We use the criterion in Lemma 4.1. In view of (4.1) and the cutoff function \( 1_{[k_1-1,k_1+1]}(y_1) \), it suffices to prove that

$$|L_j (y_1, \eta_1, x_\ell, y_\ell, \zeta)| + |\partial_{y_1} L_j (y_1, \eta_1, x_\ell, y_\ell, \zeta)| \leq C (1 + |x_\ell - y_\ell|^{-1/4}) \tag{5.15}$$

for \( j = 1, 2, 3 \).
To prove (5.15) for \( j = 1 \) we may assume \( \eta_1 \leq 0 \), in view of (5.4). Then

\[
H_1(y_1, \eta_1, \zeta + \eta_1^2 + \xi_\ell^2) = \psi_\zeta(\eta_1) G_1(\zeta + \eta_1^2 - b_{\eta_\zeta}(y_1) + \frac{\xi_\ell^2}{2} + ia_{\eta_\zeta}(y_1)\eta_1),
\]

where the function \( G_1 \) is as in (5.6). The proof for the bound on the first term on the left-hand side of (5.15) is similar to the proof of (5.5), using the bounds (5.7) and the cancellation property (5.8) for the function \( G_1 \). For the second term on the left-hand side of (5.15), we notice that

\[
|\partial_{y_1}[H_1(y_1, \eta_1, \zeta + \eta_1^2 + \xi_\ell^2)]| \leq C(1 + |\xi_\ell^2 - A|)^{-2},
\]

where \( A = - (\zeta + \eta_1^2 - b_{\eta_\zeta}(y_1)) \). This follows from (3.8) and (5.7). As a consequence, 
\(|\partial_{y_1}L_1(y_1, \eta_1, x_\ell, y_\ell, \zeta)| \leq C\), as desired.

To prove (5.15) for \( j = 2 \), we write

\[
H_2(y_1, \eta_1, \zeta + \eta_1^2 + \xi_\ell^2) = [1 - \psi_\zeta(\eta_1)] \int_{\mathbb{R}} e^{-\xi_\ell^2 r^2} [1 - \psi(10r/\eta_1^2)]
\]

\[
[-r - i\eta_1 a_{\eta_\zeta}(y_1) + b_{\eta_\zeta}(y_1)]^{-1} \hat{\chi}(\zeta + \eta_1^2 + \xi_\ell^2 - r) \, dr.
\] (5.16)

Then, using (3.8),

\[
|\partial_{y_1}[H_2(y_1, \eta_1, \zeta + \eta_1^2 + \xi_\ell^2)]| \leq C[1 - \psi_\zeta(\eta_1)] \int_{\mathbb{R}} [1 - \psi(10r/\eta_1^2)]
\]

\[
(|\eta_1 a_{\eta_\zeta}(y_1)| + |b_{\eta_\zeta}(y_1)|) r^{-2}[\hat{\chi}(\zeta + \eta_1^2 + \xi_\ell^2 - r)] \, dr.
\]

By integrating \( \xi_\ell \) first and using (3.8), it follows that 
\(|\partial_{y_1}L_2(y_1, \eta_1, x_\ell, y_\ell, \zeta)| \leq C\), as desired. To bound the first term on the left-hand side of (5.15), we notice first that we may replace the factor \([-r - i\eta_1 a_{\eta_\zeta}(y_1) + b_{\eta_\zeta}(y_1)]^{-1}\) in (5.16) with \(-r^{-1}\), at the expense of an absolutely integrable error. Then we make the change of variables \( r = \eta_1^2 + \xi_\ell^2 - v \). Let \( A = -(\eta_1^2 - v) \). Since \( \hat{\chi} \) is absolutely integrable, it suffices to prove that for \( |\eta_1| \geq 1 \),

\[
\left| \int_{\mathbb{R}} e^{i(x_\ell - y_\ell)\xi_\ell} e^{-\xi_\ell^2/2} (1 + \xi_\ell^2)^{1/2} (\xi_\ell^2 - A)^{-1}
\]

\[
[1 - \psi(10(\xi_\ell^2 - A)/\eta_1^2)] e^{-\xi_\ell^2(\xi_\ell^2 - A)^2} d\xi_\ell \right| \leq C(1 + |x_\ell - y_\ell|^{-1/4}).
\] (5.17)

This is similar to the proof of (5.5): if \( A \leq 1 \) then we apply Lemma 4.2 directly; if \( A \geq 1 \) we divide the integral into two parts, corresponding to \( |\xi_\ell| \notin [\sqrt{A/2}, \sqrt{3A/2}] \) and \( |\xi_\ell| \in [\sqrt{A/2}, \sqrt{3A/2}] \). Then we apply Lemma 4.2 for the first part and Lemma 4.3 for the second part.
We now prove (5.15) in the case \( j = 3 \). We may assume \( \eta_1 > 0 \), in view of (5.4). We use the formula (5.11) and integrate by parts in \( z \). The result is

\[
H_3(y_1, \eta_1, \zeta + \eta_1^2 + \zeta_\ell^2) = c[1 - \Psi_j(\eta_1)] \int_0^\infty \int_{\mathbb{R}} \frac{d}{dz} \left[ e^{i\varphi_j(y_1 - z)} - e^{i\varphi_j(y_1)} \right] \tilde{\psi}(v/\eta_1) \tilde{\mathcal{F}}(\zeta + v^2 + \zeta_\ell^2) \, d\zeta \, dv.
\]

(5.18)

Since the function \( z \to e^{i\varphi_j(y_1 - z)} - e^{i\varphi_j(y_1)} \) is nonincreasing and bounded on \([0, \infty)\),

\[
\int_0^\infty \left| \frac{d}{dz} [e^{i\varphi_j(y_1 - z)} - e^{i\varphi_j(y_1)}] \right| \, dz \leq C.
\]

(5.19)

Thus, for the bound on the first term in (5.15), it suffices to prove that

\[
\left| \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x_\ell - y_\ell)\zeta_\ell} e^{-\frac{z_\ell^2}{2}} (1 + \zeta_\ell^2)^{1/2} e^{i\varphi_j(y_1 \pm v)} - 1 \right| \tilde{\psi}(v/\eta_1) \tilde{\mathcal{F}}(\zeta^2 + v^2 + \zeta_\ell^2) \, dv \, d\zeta_\ell \leq C(1 + |x_\ell - y_\ell|^{-1/4})
\]

(5.20)

uniformly in \( \zeta_0, \zeta \in \mathbb{R} \) and \( \eta_1 \in [10, \infty) \). Recall that \( \tilde{\psi} \) is supported in the interval \([4/5)^{1/2}, (6/5)^{1/2}\); the bound in the case when the sign is \( e^{i\varphi_j(y_1 \pm v)} - 1 \) is + follows easily by taking absolute values and integrating \( \zeta_\ell \) first.

Assume now that this sign is \(-\). We notice first that we may assume that \( \zeta_\ell \in [0, \infty) \). The change of variable \( v = (\eta_1^2 + r)^{1/2} \), together with Lemma 4.3, shows that

\[
\left| \int_{\mathbb{R}} e^{i\varphi_j(\eta_1 - v)} - 1 \tilde{\psi}(v/\eta_1) \tilde{\mathcal{F}}(\zeta + v^2 + \zeta_\ell^2) \, dv \right| \leq C
\]

uniformly in \( \zeta_0, \zeta, \zeta_\ell \in \mathbb{R} \), and \( \eta_1 \in [10, \infty) \). Thus it suffices to prove that

\[
\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(\zeta_\ell) e^{i(x_\ell - y_\ell)\zeta_\ell} e^{-\frac{z_\ell^2}{2}} (1 + \zeta_\ell^2)^{1/2} e^{i\varphi_j(\eta_1 - v)} - 1 \right| \tilde{\psi}(v/\eta_1) \tilde{\mathcal{F}}(\zeta^2 + v^2 + \zeta_\ell^2) \, dv \, d\zeta_\ell \leq C(1 + |x_\ell - y_\ell|^{-1/4})
\]

(5.21)

uniformly in \( \zeta_0, \zeta \in \mathbb{R} \) and \( \eta_1 \in [10, \infty) \), where \( \Phi : \mathbb{R} \to [0, 1] \) is a smooth function supported in \([1, \infty) \) and equal to 1 in \([2, \infty) \). We make the change of variable
uniformly in $x_0 \in \mathbb{R}$, $\eta_1 \in [10, \infty)$, and $r \leq 4\eta_1^2/5 + 4$. Assume first that $r \geq 2\eta_1^2$. We may restrict the integration in (5.22) to $\xi \in [\sqrt{r - 6\eta_1^2/5}, \sqrt{r - 4\eta_1^2/5}]$. We make the change of variable $\xi = (r - \eta_1^2w^2)^{1/2}$, $w \in [(4/5)^{1/2}, (6/5)^{1/2}]$. Then, it suffices to prove the stronger bound

\[
\left| \int_{\mathbb{R}} e^{i(x_\ell - y_\ell)(r - \eta_1^2w^2)^{1/2}} e_{-\xi}^2(r - \eta_1^2w^2)(1 + (r - \eta_1^2w^2)^{-1/2}) \frac{e^{ix_0(1-w)} - 1}{1 - w} \tilde{\psi}(w) \, dw \right| \leq C
\]

(5.23)

uniformly in $x_0 \in \mathbb{R}$, $\eta_1 \in [10, \infty)$, and $r \geq 2\eta_1^2$. This follows from Lemma 4.3.

To prove (5.22) if $r \in [4\eta_1^2/5, 2\eta_1^2]$, let $A = r - \eta_1^2$. If $A \leq 1/2$ then the function

\[
m(\xi) = \Phi(\xi) e_{-\xi}^2(1 + \xi^2)^{1/2}(\eta_1 - \sqrt{r - \xi^2})^{-1/2} \tilde{\psi}((r - \xi^2)^{1/2}/\eta_1)(r - \xi^2)^{-1/2}
\]

satisfies the bounds

\[
|m(\xi)| \leq C(1 + |\xi|)^{-1},
\]

\[
|\partial_{\xi} m(\xi)| \leq C(1 + |\xi|)^{-2}
\]

(5.24)

for $\xi \in [1, (r - 4\eta_1^2/5)^{1/2}]$. The bound (5.22) follows from Lemma 4.2, by dividing the integral into dyadic pieces $\xi \approx 2^k$, and considering the cases $2^k \leq \epsilon|x_\ell - y_\ell|\eta_1/x_0$ and $2^k \geq C|x_\ell - y_\ell|\eta_1/x_0$. If $A \geq 1/2$, then the bounds (5.24) still hold when $\xi \notin [\sqrt{(1 - c)A}, \sqrt{(1 + c)A}]$, $c > 0$, which gives the bound (5.22) for this part of the integral. Finally, we notice that the function

\[
\tilde{m}(\xi) = \tilde{\psi}(100(\xi - \sqrt{A})/\sqrt{A}) m(\xi)
\]

is a Calderon–Zygmund kernel centered around $\sqrt{A}$, so the remaining part of the integral on the left-hand side of (5.22) is dominated by $C$, using Lemma 4.3. This completes the proof of (5.22).
We now prove that $|\bar{\partial}_{y_1} L_3(y_1, \eta_1, x_\ell, \eta_\ell, \zeta)| \leq C(1 + |x_\ell - y_\ell|^{-1/4})$. By taking the $y_1$-derivative in (5.18) and using estimate (5.20) verified above, it suffices to prove that

$$\int_0^\infty \left| \frac{d^2}{dx dy_1} \left[ e^{\alpha(y_1-x)} - \beta \phi_\lambda(y_1) \right] \right| dx \leq C \quad (5.25)$$

(compare with (5.19)). Since $\phi_\lambda'$ is nonincreasing and nonnegative, for (5.25) it suffices to prove that

$$\int_0^\infty \beta \phi_\lambda'(y_1-x) \left[ \beta \phi_\lambda'(y_1-x) - \beta \phi_\lambda'(y_1) \right] e^{\alpha(y_1-x)-\beta \phi_\lambda(y_1)} dx \geq C \quad (5.26)$$

For the second integral in (5.26) we use the fact that $-\phi_\lambda''(y_1-x) = \delta_x [\beta \phi_\lambda'(y_1-x) - \beta \phi_\lambda'(y_1)]$, and integrate by parts in $x$. In view of (5.19), it suffices to prove that

$$[\beta \phi_\lambda'(y_1-x) - \beta \phi_\lambda'(y_1)] e^{[\beta \phi_\lambda(y_1-x)-\beta \phi_\lambda(y_1)]/2} \leq C \quad (5.27)$$

uniformly in $x \geq 0$ and $y_1 \in \mathbb{R}$. Inequality (5.27) is clear if $x \leq 10$, using (3.8). If $x \geq 10$, then, using (3.8) again,

$$[\beta \phi_\lambda'(y_1-x) - \beta \phi_\lambda'(y_1)] e^{[\beta \phi_\lambda(y_1-x)-\beta \phi_\lambda(y_1)]/2} \leq C \beta \phi_\lambda'(y_1-x) e^{-c \beta \phi_\lambda'(y_1-x)} \leq C,$$

which proves (5.27) in this case. □

6. Boundedness of the operators $T_j$, II

In this section we prove the bounds (3.26). The operators $J^\alpha_\ell$ are defined as in Section 5. Let $[1 - \psi_{y/2}(D_1)]$ denote the operator defined by the Fourier multiplier $\xi \to [1 - \psi_{y/2}(\xi_1)]$. Then, for $j = 2, 3$, $\beta T_j = [\beta J_1^{-1}[1 - \psi_{y/2}(D_1)][J^1_1 T_j]$. Since $J^1_1 T_j : B^{1,2}_{y_2} L^2_s \to B^{\infty,2}_{y_2} L^2_t$ (Section 5), it follows that $\beta T_j : B^{1,2}_{y_2} L^2_s \to B^{\infty,2}_{y_2} L^2_t$, $j = 2, 3$, which is better than (3.26).

It remains to prove (3.26) in the case $j = 1$. Since $J^1_1 T_j : B^{1,2}_{y_2} L^2_s \to B^{\infty,2}_{y_2} L^2_t$ (Section 5), it suffices to prove that

$$1_{[a_{\beta,\lambda}(x_1) \geq 10]} a_{\beta,\lambda}(x_1) T_1 : B^{1,2}_{y_2} L^2_s \to B^{\infty,2}_{y_2} L^2_t.$$ 

Similar to (5.1), we prove the following stronger bound:
Lemma 6.1. We have

\[ a_{\beta, \lambda}(x_1) \left\| T_1(1_{[k-1,k+1]}(y_1)g(y,s))(x_1, \ldots) \right\|_{L^2_{x,t}} \leq C \| g \|_{L^2_{y,s}} \]

for any \( x_1 \in \mathbb{R} \) with \( a_{\beta, \lambda}(x_1) \geq 10 \), and \( k \in \mathbb{Z} \).

Proof. As in the proof of (5.1), it suffices to prove that

\[ a_{\beta, \lambda}(x_1) \int_\mathbb{R} e^{i(x_1-y_1)\xi_1} e^{-\epsilon_1^2 \xi_1^2} H_1(y_1, \xi_1, \zeta + \xi_1^2) d\xi_1 \leq C(1 + \left| x_1 - y_1 \right|^{-1/4}) \]  \hspace{1cm} (6.1)

for any \( x_1, y_1, \zeta \in \mathbb{R} \), with \( a_{\beta, \lambda}(x_1) \geq 10 \). Assume first that \( |x_1 - y_1| \geq \beta \). Then, using (5.4), the left-hand side of (6.1) is bounded by

\[ C\beta \left| \int_\mathbb{R} 1_-(\xi_1)\psi_\lambda(\xi_1)e^{i(x_1-y_1)\xi_1} e^{-\epsilon_1^2 \xi_1^2} G_1(\xi_1^2 - A + ia_{\beta, \lambda}(y_1)\xi_1) d\xi_1 \right|. \]

where \( A = - (\zeta - b_{\beta, \lambda}(y_1)) \). We integrate by parts in \( \xi_1 \) and use (5.7); the bound (6.1) follows easily in this case.

Assume now that \( |x_1 - y_1| \leq \beta \). Then \( a_{\beta, \lambda}(x_1) \leq C a_{\beta, \lambda}(y_1) \), using (3.8) and \( a_{\beta, \lambda}(x_1) \geq 10 \). We divide the integral in (6.1) into two parts, corresponding to \( |\xi_1| \geq 1 \) and \( |\xi_1| \leq 1 \). The bound for the first part is easy. We may assume \( \xi_1 < 0 \); since \( |G_1(\xi_1^2 - A + ia_{\beta, \lambda}(y_1)\xi_1)| \leq C|a_{\beta, \lambda}(y_1)|\xi_1| \), we may remove the dyadic part of the integral corresponding to \( |\xi_1| \approx \sqrt{A} \) (assuming \( A > 1 \)) and integrate by parts as before.

The proof for the part of the integral corresponding to \( |\xi_1| \leq 1 \) is more delicate. Using (5.4), it suffices to prove that

\[ a_{\beta, \lambda}(y_1) \left| \int_{-1}^{0} \Re \left[ e^{i(x_1-y_1)\xi_1} G_1(\xi_1^2 - A + ia_{\beta, \lambda}(y_1)\xi_1) e^{-\epsilon_1^2 \xi_1^2} d\xi_1 \right] \right| \leq C. \]  \hspace{1cm} (6.2)

It is important in (6.2) to take the real part of \( e^{i(x_1-y_1)\xi_1} G_1(\xi_1^2 - A + ia_{\beta, \lambda}(y_1)\xi_1) \) (otherwise the estimate is false). Thus, it suffices to prove that

\[ a_{\beta, \lambda}(y_1) \int_{-1}^{0} \left| \Re G_1(\xi_1^2 - A + ia_{\beta, \lambda}(y_1)\xi_1) \right| d\xi_1 \leq C \]  \hspace{1cm} (6.3)

and

\[ a_{\beta, \lambda}(y_1) \left| \int_{-1}^{0} \sin[(x_1 - y_1)\xi_1] G_1(\xi_1^2 - A + ia_{\beta, \lambda}(y_1)\xi_1) e^{-\epsilon_1^2 \xi_1^2} d\xi_1 \right| \leq C. \]  \hspace{1cm} (6.4)
For (6.3), we write

\[ |\Re G_1(\xi^2 - A + ia_{\beta,\lambda}(y_1)\xi_1)| = \left| \int_0^\infty \chi(w)e^{a_{\beta,\lambda}(y_1)\xi_1 w}\sin[w(\xi^2 - A)] \, dw \right|. \]

If \(|A| \leq 2\) then \(|\Re G_1(\xi^2 - A + ia_{\beta,\lambda}(y_1)\xi_1)| \leq C(1 + a_{\beta,\lambda}(y_1)^2 \xi_1^2)^{-1}\) for \(\xi_1 \in [-1, 0]\), which easily leads to (6.3). If \(|A| \geq 2\), then, by integration by parts,

\[ |\Re G_1(\xi^2 - A + ia_{\beta,\lambda}(y_1)\xi_1)| \leq C|A|(A^2 + a_{\beta,\lambda}(y_1)^2 \xi_1^2), \]

which leads to (6.3) as well. For (6.4), we use the fact that \(|G_1(\xi^2 - A + ia_{\beta,\lambda}(y_1)\xi_1)| \leq C|a_{\beta,\lambda}(y_1)\xi_1|^{-1}\) for \(|\xi_1| \leq |x_1 - y_1|^{-1}\), and integrate by parts for \(|\xi_1| \geq |x_1 - y_1|^{-1}\). □

7. Boundedness of the operators \(T_j\), III

In this section we prove the bounds (3.27). The operators \(J_{\ell}^g\) are defined as in Section 5. It suffices to prove that

\[ J_{\ell}^{1/2}T_j : B_{y,2}^{1,2}L_2^y \to L_\infty^xL_2^x, \quad j = 1, 2, 3, \quad \ell = 1, \ldots, d. \]

We analyze two cases: \(\ell = 1\) and \(\ell = 2, \ldots, d\).

The case \(\ell = 1\): By translation invariance in \(t\), it suffices to prove the following stronger bound:

**Lemma 7.1.** We have

\[ \| J_{1/2}^{1/2} T_j (I_{[k-1,k+1]}(y_1)g(y,s))(.,0) \|_{L_\infty^xL_2^x} \leq C \| g \|_{L_2^y} \]

\[ (7.1) \]

for \(j = 1, 2, 3\), and \(k \in \mathbb{Z}\).

**Proof.** We have

\[ J_{1/2}^{1/2} T_j (g)(x,0) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dy \, ds \, g(y,s) \]

\[ \int_{\mathbb{R}^d} e^{i(x-y)\cdot \xi} e^{i|\xi|^2} e^{-\xi^2|\xi|^2}(1 + \xi^2)^{1/4} \mu_j(y_1, \xi_1, -s) \, d\xi. \]
By taking the Fourier transform in $x$, we have

$$
\mathcal{F}_x[F_1^{1/2} T_j(g)(., 0)](\xi) = c e^{-x^2|\xi|^2} (1 + \xi_1^2)^{1/4}
\int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{F}_x'[(g)(y_1, \xi', s)e^{-iy_1 \xi_1} e^{i\xi_1^2 \mu_j(y_1, \xi_1, -s)} dy_1 ds.
$$

By the Plancherel theorem and the Minkowski inequality for integrals, for (7.1) it suffices to prove that

$$
\left\| (1 + \xi_1^2)^{1/4} \int_{\mathbb{R}} h(s)e^{i s \xi_1^2 \mu_j(y_1, \xi_1, -s)} ds \right\|_{L_2^{\xi_1}} \leq C \| h \|_{L_2^s} \tag{7.2}
$$

for any $h \in \mathcal{S}(\mathbb{R})$, $y_1 \in \mathbb{R}$, and $j = 1, 2, 3$.

To prove (7.2) for $j = 1$, we may assume $\xi_1 > 0$ in view of (3.31). For $\xi_1 \in [0, 1]$ we notice that

$$
\left\| (1 + \xi_1^2)^{1/4} \int_{\mathbb{R}} h(s)e^{i s \xi_1^2 \mu_j(y_1, \xi_1, -s)} ds \right\|_{L_2^{\xi_1}} \leq C \int_{\mathbb{R}} |h(s)| \chi(s) ds \leq C \| h \|_{L_2^s},
$$

which suffices to control the $L^2$ norm on the left-hand side of (7.2) over the set $\xi_1 \in [0, 1]$. For $\xi_1 \geq 1$ we make the change of variable $\xi_1 = \sqrt{v}$ and notice that

$$
\left\| (1 + \xi_1^2)^{1/4} \int_{\mathbb{R}} h(s)e^{i s \xi_1^2 \mu_j(y_1, \xi_1, -s)} ds \right\|_{L_2^{\xi_1 \in [1, \infty)}} \leq C \int_{\mathbb{R}} \left\| h(s) e^{i s \sqrt{v} \mu_j(y_1, \sqrt{v}, -s)} ds \right\|_{L_2^{v \in [1, \infty)}}. \tag{7.3}
$$

To bound the right-hand side of (7.3) we use Lemma 4.1, and the simple observation that the function $s \rightarrow \chi(s) 1_+(s) e^{-a \alpha_j, \gamma \xi_1^2 \sqrt{v} s}$ has bounded variation in $s$, uniformly in $y_1 \in \mathbb{R}$ and $v \in [1, \infty)$.

To prove (7.2) for $j = 2$, we may assume $\xi_1 > 0$ in view of (3.31). Because of the factor $[1 - \psi_0(\xi_1)]$ we may assume $\xi_1 \geq 1$ and make the change of variable $\xi_1 = \sqrt{v}$. An estimate similar to (7.3) shows that it suffices to prove that

$$
\left\| \int_{\mathbb{R}} h(s)e^{i s \sqrt{v} \mu_2(y_1, \sqrt{v}, -s)} ds \right\|_{L_2^{v \in [1, \infty)}} \leq C \| h \|_{L_2^s}. \tag{7.4}
$$

In view of Lemma 4.1, it suffices to prove that

$$
\| \mu_2(y_1, \sqrt{v}, \cdot) \|_{BV_s} \leq C \tag{7.5}
$$
uniformly in \( y_1 \in \mathbb{R} \), and \( v \in [\gamma^2, \infty) \). We use formula (3.19). The function \( \tau \rightarrow e^{-\tau^2} [1 - \psi(\tau/v)][-\tau - i \sqrt{\nu} a_\beta,\lambda(y_1) + b_\beta,\lambda(y_1)]^{-1} \) is a Calderon–Zygmund kernel on \( \mathbb{R} \) for \( v \geq \gamma^2 \); thus \( |\mu_2(y_1, \sqrt{v}, s)| \leq C \). Also, using (3.19) and integration by parts,

\[
|\partial \nu \mu_2(y_1, \sqrt{v}, s)| \leq C |\tilde{\chi}'(s)| + \chi(s) [1 - \psi_j(\sqrt{v})] \left[ \int_{\mathbb{R}} e^{i \xi r} e^{-\xi^2 r^2} [1 - \psi(10r/v)] d\tau \right] + \beta \sqrt{v} \int_{\mathbb{R}} e^{i \xi r} e^{-\xi^2 r^2} [1 - \psi(10r/v)] [-\tau - i \sqrt{\nu} a_\beta,\lambda(y_1)]^{-1} d\tau \]

\[
\leq C |\tilde{\chi}'(s)| + C \chi(s) [1 - \psi_j(\sqrt{v})] [\varepsilon^{-1} (1 + |s|/\varepsilon)^{-2} + v (1 + |s|v)^{-2}].
\]

The bound (7.5) follows.

To prove (7.2) for \( j = 3 \), we may assume \( \xi_1 > 0 \) in view of (3.31). Because of the factor \([1 - \psi_j,\lambda(\xi_1)]\) we may assume \( \xi_1 > 1 \). It suffices to prove that

\[
\sum_{k=0}^{\infty} 2^k \left\| \int_{\mathbb{R}} h(s) e^{i s \xi_1^2} \mu_3(y_1, \xi_1, -s) ds \right\|_{L^2_{[\xi_1 \in [2^k, 2^{k+1}]]}}^2 \leq C \|h\|_{L^2_w}^2. \tag{7.6}
\]

We use formula (3.24). Let \( h'(v) = \int_{\mathbb{R}} h(s) \chi(s) e^{i s v^2} ds \). By the Plancherel theorem

\[
\sum_{k=0}^{\infty} 2^k \|h'(v)\|_{L^2_{[v \in [2^{k-1}, 2^{k+1}]]}}^2 \leq C \|h\|_{L^2_w}^2.
\]

Thus, for (7.6), it suffices to prove that

\[
\left\| \int_{0}^{\infty} d\alpha dr h'(\xi_1 r) e^{ix \xi_1} e^{\beta \phi_j(y_1 - x) - \beta \phi_j(y_1)} \tilde{\psi}(r) \sin(\xi_1 x r) \xi_1 \right\|_{L^2_{[\xi_1 \in [2^k, 2^{k+1}]]}} \leq C \|h'\|_{L^2_{[v \in [2^{k-1}, 2^{k+1}]]}}
\]

for any integer \( k \geq 0 \). We make the change of variable \( r = w/\xi_1 \) and integrate by parts in \( x \) (as in (5.18)). Using (5.19) and the Minkowski inequality for integrals, it suffices to prove that

\[
\left\| \int_{\mathbb{R}} h'(w) \frac{e^{ix_0(\xi_1 \pm w)}}{\xi_1 \pm w} - \frac{1}{\xi_1} \tilde{\psi}(w/\xi_1) 1_{[2^k, 2^{k+1}]}(\xi_1) 1_{[2^{k-1}, 2^{k+2}]}(w) dw \right\|_{L^2_{\xi_1}} \leq C \|h'\|_{L^2_w} \tag{7.7}
\]
for any \( x_0 \in \mathbb{R} \) and integer \( k \geq 0 \). The bound (7.7) in the case when the sign in \([e^{i(x_0(\xi_1 \pm w))} - 1]/(\xi_1 \pm w)\) is + follows since the kernel is absolutely integrable in both \( w \) and \( \xi_1 \). When this sign is −, the bound in (7.7) follows from the boundedness of the Hilbert transform. □

The case \( \ell \geq 2 \): As before, we write the variables \( y = (y_1, y_\ell, y'_\ell), x = (x_1, x_\ell, x'_\ell) \), etc. In view of Lemma 7.1, we may replace the operator \( J^{1/2}_\ell \) with the operator \( \tilde{J}^{1/2}_\ell \) defined by the Fourier multiplier \( \xi \to |\xi_\ell|^{1/2} \mathbf{1}_{[1,\infty)}(|\xi_\ell|) \). By translation invariance in \( t \) it suffices to prove the following stronger bound:

Lemma 7.2. We have

\[
\| \tilde{J}^{1/2}_\ell T_j (\mathbf{1}_{[k_1-1,k_1+1]}(y_1) \mathbf{1}_{[k_\ell-1,k_\ell+1]}(y_\ell) g(y, s))(.,0) \|_{L^2_x} \leq C \| g \|_{L^2_{y,s}},
\]

(7.8)

for \( j = 1, 2, 3 \), and \( k_1, k_\ell \in \mathbb{Z} \).

Proof. We have

\[
\tilde{J}^{1/2}_\ell T_j (g)(x, 0) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} dy \, ds \, g(y, s) \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} e^{i|x|^2} e^{-i|\xi|^2} |\xi_\ell|^{1/2} \mathbf{1}_{[1,\infty)}(|\xi_\ell|) \mu_j(y_1, \xi_1, -s) d\xi.
\]

By taking the Fourier transform in \( x \), we have

\[
\mathcal{F}_x [\tilde{J}^{1/2}_\ell T_j (g)(.,0)](\xi) = ce^{-i|\xi|^2} |\xi_\ell|^{1/2} \mathbf{1}_{[1,\infty)}(|\xi_\ell|) \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathcal{F}_{y'_\ell} (g)(y_1, y_\ell, \xi_\ell, s) e^{-iy_1 \xi_1} e^{-iy_\ell \xi_\ell} e^{ix|\xi|^2} \mu_j(y_1, \xi_1, -s) dy_1 dy_\ell ds,
\]

where \( \mathcal{F}_{y'_\ell} \) denotes the Fourier transform in the variables \( y'_\ell \). By the Plancherel theorem and the Minkowski inequality for integrals, for (7.8) it suffices to prove that

\[
\left\| |\xi_\ell|^{1/2} \mathbf{1}_{[1,\infty)}(|\xi_\ell|) \int_{\mathbb{R}^d} \int_{\mathbb{R}} h(y_1, s) \mathbf{1}_{[k_1-1,k_1+1]}(y_1) e^{-iy_1 \xi_1} e^{ix_1^2} e^{ix_\ell^2} \mu_j(y_1, \xi_1, -s) dy_1 ds \right\|_{L^2_{x_1,\xi_\ell}} \leq C \| h \|_{L^2_{y_1,s}}
\]
for any $h \in \mathcal{S}(\mathbb{R}^2)$. By symmetry, we may assume $\zeta_\ell \geq 1$ and make the change of variable $\zeta_\ell = \sqrt{v}$. It remains to prove that
\[
\left\| 1_{[1,\infty)}(v) \int_{\mathbb{R}} \int_{\mathbb{R}} h(y_1, s) 1_{[k_1-1,k_1+1]}(y_1) e^{-iy_1 \zeta_1} e^{is} \mu_j(y_1, \zeta_1, -s) dy_1 ds \right\|_{L^2_{\zeta_1, v}} \leq C \| h \|_{L^2_{y_1, v}}
\]
for any $h \in \mathcal{S}(\mathbb{R}^2)$. Using the Plancherel theorem and the Minkowski inequality for integrals, it suffices to prove that
\[
\left\| \int_{\mathbb{R}} h'(y_1) 1_{[k_1-1,k_1+1]}(y_1) e^{-iy_1 \zeta_1} \mu_j(y_1, \zeta_1, -s) dy_1 \right\|_{L^2_{\zeta_1}} \leq C \| h' \|_{L^2_{y_1}}
\]
for any $s \in \mathbb{R}$ and $h' \in \mathcal{S}(\mathbb{R})$. As before, we use Lemma 4.1. In view of (4.1) and the cutoff function $1_{[k_1-1,k_1+1]}(y_1)$, it suffices to prove that
\[
|\mu_j(y_1, \zeta_1, s)| + |\partial_{y_1} \mu_j(y_1, \zeta_1, s)| \leq C
\]
(7.9)
for any $y_1, \zeta_1, s \in \mathbb{R}$, and $j = 1, 2, 3$.

For $j = 1$, the bound (7.9) follows easily from (3.15) and (3.8).

For $j = 2$, the bound for the first term on the left-hand side of (7.9) follows from (7.5). For the second term on the left-hand side of (7.9) we use formula (3.19) and the bound (3.8).

For $j = 3$, we use formula (3.24). For the first term on the left-hand side of (7.9), we integrate by parts in $\zeta$ (as in (5.18)). Using (5.19), it suffices to prove that
\[
\left| \int_{\mathbb{R}} \frac{e^{i\zeta \zeta_1 (1+\pm r) - 1}}{1 \pm r} \psi(r) e^{-is \zeta_1 (r^2 - 1)} dr \right| \leq C
\]
(7.10)
for any $\zeta_0, \zeta_1, s \in \mathbb{R}$. The bound (7.10) in the case when the sign in $[e^{i\zeta \zeta_1 (1+\pm r) - 1}]/(1 \pm r)$ is $+$ is easy since the function under the integral is absolutely integrable. When this sign is $-$, the bound in (7.10) follows from Lemma 4.3.

For the second term on the left-hand side of (7.9), we integrate by parts in $\zeta$ and use (5.25) instead of (5.19). The bound for the second term on the left-hand side of (7.9) reduces to (7.10) as well. □

8. Boundedness of the operators $T_j$, IV

In this section we prove the bounds (3.28). The operators $J_\ell^\alpha$ are defined as in Section 5. It suffices to prove that
\[
\hat{\lambda}^2 J_\ell^1 T_j J_\ell^{-1} : L^1_{\delta_y} L^2_x \rightarrow L^\infty_x L^2_{\delta_y}, \quad j = 1, 2, 3, \quad \ell = 1, \ldots, d.
\]
Since $J_\ell J_\ell J^{-1} = T_j$ for $\ell = 2, \ldots, d$, it suffices to prove the following stronger bound:

**Lemma 8.1.** We have

$$\|J_1^\sigma T_j J_1^{-\sigma}\|_{L_1 \to L_\infty} \leq C\lambda^2$$

for $j = 1, 2, 3$ and $\sigma = 0, 1$.

**Proof.** Assume first that $\sigma = 0$. Recall that

$$T_j(g)(x, t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} dy ds g(y, s) \int_{\mathbb{R}^d} e^{i(x-y)\cdot \xi} e^{-i(t-s)|\xi|^2} e^{-|\xi|^2} m_j(y_1, \xi_1, t-s) d\xi.$$

By the Minkowski inequality for integrals and the Plancherel theorem, it suffices to prove that

$$\left\| \int_{\mathbb{R}} h(y_1) e^{-iy_1\xi_1} m_j(y_1, \xi_1, w) dy_1 \right\|_{L_2^\infty} \leq C\beta^2 \lambda \|h\|_{L_2^1}$$

for any $w \in \mathbb{R}$ and $h \in S(\mathbb{R})$. By Lemma 4.1, it suffices to prove that

$$\|m_j(\cdot, \xi_1, w)\|_{BV_{y_1}} \leq C\lambda^2$$

for any $\xi_1, w \in \mathbb{R}$, and $j = 1, 2, 3$.

The bound (8.2) for $j = 1, 2$ follows directly from (7.9) and the observation that $\hat{m}_j(y_1, \xi_1, w)$ is supported in the set $\{(y_1, \xi_1, w) : y_1 \in [\lambda, 2\lambda]\}$. In fact, formulas (3.15) and (3.19), together with the bound (3.8) and the remark on the supports of $\hat{m}_j(y_1, \xi_1, w)$, easily show that

$$\int_{\mathbb{R}} |\hat{m}_j(y_1, \xi_1, w)| dy_1 \leq C\lambda^2$$

for any $\xi_1, w \in \mathbb{R}$, $j = 1, 2$, and $\sigma = 1, 2, 3$.

Assume $j = 3$. For later use, we prove a stronger bound:

$$\int_{\mathbb{R}} |\hat{m}_j(y_1, \xi_1, w)| dy_1 \leq C\lambda^2,$$  

(8.4)

for any $\xi_1, w \in \mathbb{R}$, and $\sigma = 1, 2, 3$. 

For the proof of (8.4) we use the formula (3.24). In view of (3.31), we may assume that \( w > 0 \). Let \( A = 2w^{1/2} \), thus \( A \in (0, \sqrt{8}] \). We make the change of variable \( \alpha = 2w^{1/2} \theta = A \theta \) in (3.24). Then

\[
\mu_4(y_1, \xi_1, t, s) = 2\chi(w)[1 - \psi_\gamma(\xi_1)]I(y_1, w^{1/2}\xi_1),
\]

where

\[
I(y_1, \eta_1) = \int_0^\infty \int_{\mathbb{R}} d\theta dr e^{2i\eta_1 \theta} e^{-i\eta_1^2(r^2-1)} e^{\beta \phi_\lambda(y_1 - A \theta) - \beta \phi_\lambda(y_1)} \tilde{\psi}(r) \sin(2\eta_1 \theta r) \eta_1. \tag{8.6}
\]

It suffices to prove that

\[
\int_{\mathbb{R}} |\partial^{\sigma}_{y_1} I(y_1, \eta_1)| dy_1 \leq C \lambda^2 \tag{8.7}
\]

for any \( \eta_1 \in \mathbb{R} \), and \( \sigma = 1, 2, 3 \), provided that \( A \in (0, C] \). We notice that

\[
\int_{\mathbb{R}} \partial^{\sigma}_{y_1} [e^{\beta \phi_\lambda(y_1 - \zeta) - \beta \phi_\lambda(y_1)}] dy_1 \leq C \lambda \tag{8.8}
\]

for any \( \zeta \in (0, \infty) \), and \( \sigma = 1, 2, 3 \), provided that the constant \( m \) in Lemma 3.1 is sufficiently large. This follows easily from (3.8).

Assume first that \( |\eta_1| \leq 2 \). In this case we write the integral for the function \( I \) in the form

\[
I(y_1, \eta_1) = \int_{\mathbb{R}} 1_+(\theta) e^{2i\eta_1 \theta} e^{i\eta_1} e^{\beta \phi_\lambda(y_1 - A \theta) - \beta \phi_\lambda(y_1)} L(\eta_1, \theta) d\theta, \tag{8.9}
\]

where

\[
L(\eta_1, \theta) = \int_{\mathbb{R}} \eta_1 e^{-i\eta_1 r^2} \sin(2\eta_1 \theta r) \tilde{\psi}(r) dr. \tag{8.10}
\]

Notice that

\[
|L(\eta_1, \theta)| \leq C |\eta_1| (1 + |\eta_1 \theta|)^{-2}
\]

if \( |\eta_1| \leq 2 \). The bound (8.7) follows easily from (8.8) and (8.9) in this case.

Assume now that \( |\eta_1| \geq 2 \). We start from (8.9) and (8.10). Recall that the function \( \tilde{\psi} \) is smooth and supported in the interval \([((4/5)^{1/2}, (6/5)^{1/2}] \). Let \( \psi_\eta : \mathbb{R} \to [0, 1] \) be a smooth function supported in the set \( \{ \eta : |\eta| \in [4/5, 6/5] \} \) and equal to 1 in the set
We consider two cases depending on the sign of \( \eta \). We have

\[
L(\eta_1, \theta) = \frac{1}{2i} e^{i\theta^2} \int_{\mathbb{R}} \eta_1 e^{-i\theta^2} e^{-i\eta_1 r^2} [e^{2i\eta_1 r\theta} - e^{-2i\eta_1 r\theta}] \tilde{\psi}(r) \, dr
\]

where

\[
L_0(\eta_1, \theta) = [1 - \psi_1(\theta/\eta_1)] \frac{1}{2i} \int_{\mathbb{R}} e^{-i\eta_1 r^2} [\tilde{\psi}((r + \theta)/\eta_1) - \tilde{\psi}((r - \theta)/\eta_1)] \, dr
\]

and

\[
L_1(\eta_1, \theta) = \psi_1(\theta/\eta_1) \frac{1}{2i} \int_{\mathbb{R}} e^{-i\eta_1 r^2} [\tilde{\psi}((r + \theta)/\eta_1) - \tilde{\psi}((r - \theta)/\eta_1)] \, dr.
\]

By the support properties of the functions \( \tilde{\psi} \) and \( \psi_1 \) we can integrate by parts in the integral defining \( L_0(\eta_1, \theta) \) to obtain

\[
|L_0(\eta_1, \theta)| \leq C (1 + |\theta|)^{-2}
\]

if \( |\eta_1| \geq 1 \). Also, the function \( L_1(\eta_1, \theta) \) is supported in the set \( \{(\eta_1, \theta) : |\theta/\eta_1| \in [4/5, 6/5] \} \). We substitute formula (8.11) into definition (8.9) of the function \( I \) and decompose \( I(y_1, \eta_1) = I_0(y_1, \eta_1) + I_1(y_1, \eta_1) \) corresponding to the terms \( e^{i\theta^2} L_0 \) and \( e^{i\theta^2} L_1 \). By (8.8) and (8.13), the bound (8.7) follows easily for the function \( I_0 \).

It remains to prove the bound (8.7) for the function \( I_1 \). We have

\[
I_1(y_1, \eta_1) = \int_{\mathbb{R}} 1_+ (\theta) e^{2i\eta_1 \theta} e^{i\eta_1^2 \beta \phi_1(y_1 - A\theta) - \beta \phi_1(y_1)} e^{i\theta^2} L_1(\eta_1, \theta) \, d\theta
\]

where

\[
L_1(\eta_1, \theta) = \psi_1(\theta/\eta_1) \frac{1}{2i} \int_{\mathbb{R}} e^{-i\eta_1 r^2} [\tilde{\psi}((r + \theta)/\eta_1) - \tilde{\psi}((r - \theta)/\eta_1)] \, dr.
\]

We consider two cases depending on the sign of \( \eta_1 \). It is somewhat harder to prove estimates if \( \eta_1 < 0 \), so we concentrate on this case. Since \( |\eta_1| \geq 2 \) we can assume \( \eta_1 \leq -2 \). By the support property of the function \( L_1 \) and because of the factor \( \frac{1}{2} \), the variable \( x \) in the integral representing \( I_1 \) runs over the interval \( x \in [-|\eta_1|/5, |\eta_1|/5] \).
Thus

\[ I_1(y_1, \eta_1) = \int_{\mathbb{R}} e^{i x^2} e^{\beta \phi_j(y_1 - A(\eta - \eta_1)) - \beta \phi_j(y_1)} \tilde{\psi}_1(\eta_1) L_1(\eta_1, \eta - \eta_1) d\eta, \]  

(8.14)

where \( \tilde{\psi}_1 \) is a smooth function supported in the interval \([-2/9, 2/9]\) and equal to 1 in the interval \([-1/5, 1/5]\). By integrating by parts in (8.12), it is easy to see that

\[ |L_1(\eta_1, \theta)| + |\eta_1 \cdot \delta_\theta L_1(\eta_1, \theta)| \leq C \]

if \( \eta_1 \leq -1 \). We integrate by parts in \( \eta \) in (8.14) and use (8.8) and (3.8). The bound (8.7) follows, which completes the proof of (8.1) in the case \( \sigma = 0 \).

We now prove the bounds (8.1) for \( \sigma = 1 \). We fix a partition of 1 on \( \mathbb{R} \), \( 1 = \sum_{k=0}^{\infty} \delta_k \), where \( \delta_0 \) is smooth and supported in the interval \([-2, 2]\) and \( \delta_k \) is smooth and supported in \( \{r : |r| \in [2^{k-1}, 2^{k+1}]\} \). We define the operators \( Q^k_1 \) on \( S'(\mathbb{R}^d) \) by multiplication with the Fourier multipliers \( \xi \rightarrow \delta_k(\xi) \). Then

\[
\| J_1^1 T_j J_1^{-1} g \|_{L_2^\infty L_2^2} \leq C \sum_{k=0}^{\infty} \| Q^k_1 J_1^1 T_j J_1^{-1} g \|_{L_2^\infty L_2^2} \leq C \sum_{k=0}^{\infty} \left[ \sum_{k'=0}^{\infty} \| Q^k_1 J_1^1 R_j J_1^{-1} Q^{k'}_1 g \|_{L_2^\infty L_2^2} \right]^2.
\]

Thus, it suffices to prove that for any integers \( k, k' \geq 0 \)

\[
\| Q^k_1 J_1^1 T_j J_1^{-1} Q^{k'}_1 g \|_{L_2^\infty L_2^2} \leq C \lambda^2 2^{-|k-k'|} \| Q^{k'}_1 g \|_{L_2^1 L_2^2}, \]  

(8.15)

The bound (8.15) follows from (8.1) with \( \sigma = 0 \) if \( k - k' \leq 3 \):

\[
\| Q^k_1 J_1^1 T_j J_1^{-1} Q^{k'}_1 g \|_{L_2^\infty L_2^2} \leq C \lambda^2 2^k \| T_j J_1^{-1} Q^{k'}_1 g \|_{L_2^\infty L_2^2} \leq C \lambda^2 2^k 2^{-k} \| Q^{k'}_1 g \|_{L_2^1 L_2^2}.
\]
If $k - k' \geq 4$, we use the Minkowski inequality for integrals and the Plancherel theorem, as in the proof of (8.1) with $\epsilon = 0$. As before, it suffices to prove that

$$2^k \left\| \delta_k(\xi_1) \int_{\mathbb{R}} J_1^{-1} Q_1^{k'} h(y_1) e^{-iy_1 \xi_1} \mu_j(y_1, \xi_1, w) dy_1 \right\|_{L^2_{\xi_1}} \leq C \lambda^2 2^{-(k-k')} \| Q_1^{k'} h \|_{L^2_{y_1}},$$

for any $w \in \mathbb{R}$ and $h \in S(\mathbb{R})$ (by a slight abuse of notation, the operators $J_1^\sigma$ and $Q_1^k$ act on $S'(\mathbb{R})$ as before, by multiplication with the Fourier multipliers $\xi_1 \to (1 + \xi_1^2)^\sigma$ and $\xi_1 \to \delta_k(\xi_1)$, respectively). By substituting

$$J_1^{-1} Q_1^{k'} h(y_1) = c \int_{\mathbb{R}} (1 + \eta_1^2)^{-1/2} \delta_k(\eta_1) \hat{h}(\eta_1) e^{iy_1 \eta_1} d\eta_1,$$

we see that it suffices to prove that if $|\xi_1| \in [2^{k-1}, 2^{k+1}]$ and $|\eta_1| \leq 2^{k-2}$ then

$$\left| \int_{\mathbb{R}} e^{-iy_1 (\xi_1 - \eta_1)} \mu_j(y_1, \xi_1, w) e^{-\xi_1^2} dy_1 \right| \leq C \lambda^2 2^{-3k},$$

uniformly in $w$ and $\epsilon > 0$ (recall that $k - k' \geq 4$). This follows easily by integrating by parts three times and using (8.3) and (8.4).

9. Boundedness of the operators $R_j, I$

In this section we prove the bounds (3.29). The operators $J_\ell^\sigma$ are defined as in Section 5. It suffices to prove that

$$\| J_\ell^{1/2} R_j J_\ell^{-1/2} \|_{L^\infty_y L^2_x} \leq C_1 / \beta, \quad j = 1, 2, \ell = 1, \ldots, d,$$

with $c_1 \ll 1$. Since $J_\ell^{1/2} R_j J_\ell^{-1/2} = R_j$ for $\ell = 2, \ldots, d$, it suffices to prove the following stronger bound.

**Lemma 9.1.** There is a constant $m' \geq 0$ such that

$$\| J_1^\sigma R_j J_1^{-\sigma} \|_{L^1_y L^2_x} \leq C \beta^{m' / \lambda}$$

(9.1)

for $j = 1, 2$, and $\sigma = 0, 1/2$. 
Proof. Assume first that \( \sigma = 0 \). Recall that

\[
R_j(g)(x, t) = \mathbf{1}_{[0,1]}(t) \int_{\mathbb{R}^d} \int_{\mathbb{R}} dy ds g(y, s) \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} e^{-i(t-s) |\xi|^2} e^{-|\xi|^2} v_j(y_1, \xi_1, t-s) d\xi,
\]

where \( v_j \), \( j = 1, 2 \), are defined in (3.16) and (3.20). By the Minkowski inequality for integrals and the Plancherel theorem, it suffices to prove that

\[
\left\| \int_{\mathbb{R}} h(y_1) e^{-iy_1 \xi_1} v_j(y_1, \xi_1, w) dy_1 \right\|_{L^2_{\xi_1}} \leq C (\beta^m / \lambda) \| h \|_{L^2_y} \tag{9.2}
\]

for any \( w \in \mathbb{R} \) and \( h \in \mathcal{S}(\mathbb{R}) \).

To prove (9.2) for \( j = 1 \) we use formula (3.16). The factor \( e^{ib \beta, \lambda(y_1) w} \) is bounded and does not depend on \( \xi_1 \), thus it may be incorporated into \( h \). Then we use Lemma 4.1. Using (3.13) and the fact that the function \( y_1 \to e^{a \beta, \lambda(y_1) \xi_1 w} \) is non-decreasing if \( \xi_1 w \leq 0 \), we have

\[
\| v_1(y_1, \xi_1, w) e^{-ib \beta, \lambda(y_1) w} \|_{BV_{y_1}} \leq C \beta^3 / \lambda
\]

for any \( \xi_1, w \in \mathbb{R} \). The bound (9.2) follows from (4.1) if the constant \( m' \) in Lemma 9.1 is sufficiently large.

To prove (9.2) for \( j = 2 \) we use formula (3.20). An elementary argument using (3.8) shows that

\[
\| v_2(y_1, \xi_1, w) \|_{BV_{y_1}} \leq C \beta / \lambda,
\]

for any \( \xi_1, w \in \mathbb{R} \). The bound (9.2) follows from (4.1) if the constant \( m' \) in Lemma 9.1 is sufficiently large.

We now prove the bounds (9.1) for \( \sigma = 1/2 \). We use the functions \( \delta_k \) and the operators \( Q_1^k \) defined in the proof of Lemma 8.1. Then

\[
\| J_1^{1/2} R_j J_1^{-1/2} g \|_{L^\infty_t L^2_x}^2 \leq C \sum_{k=0}^\infty \| Q_1^k J_1^{1/2} R_j J_1^{-1/2} g \|_{L^\infty_t L^2_x}^2 \leq C \sum_{k=0}^\infty \sum_{k'=0}^\infty \| Q_1^k J_1^{1/2} R_j J_1^{-1/2} Q_1^{k'} g \|_{L^\infty_t L^2_x}^2 \cdot
\]

Thus, it suffices to prove that for any integer \( k, k' \geq 0 \)

\[
\| Q_1^k J_1^{1/2} R_j J_1^{-1/2} Q_1^{k'} g \|_{L^\infty_t L^2_x} \leq C (\beta^m / \lambda) 2^{-|k-k'|/2} \| Q_1^k g \|_{L^1_y L^2_x}. \tag{9.3}
\]
The bound (9.3) follows from (9.1) with \( \sigma = 0 \) if \( k - k' \leq 3 \):

\[
\| Q_1^k J_1^{-1/2} R_j J_1^{-1/2} Q_1^{k'} g \|_{L^\infty_t L^2_x} \leq C 2^{k/2} \| R_j J_1^{-1/2} Q_1^{k'} g \|_{L^\infty_t L^2_x} \\
\leq C (\beta^m / \lambda) 2^{k/2} 2^{k - 2} \| Q_1^{k'} g \|_{L^1_t L^2_x}.
\]

If \( k - k' \geq 4 \), we use the Minkowski inequality for integrals and the Plancherel theorem, as in the proof of (9.1) with \( \sigma = 0 \). As before, it suffices to prove that

\[
\| Q_1^k J_1^{-1/2} R_j J_1^{-1/2} Q_1^{k'} h(y_1) e^{-i y_1 \xi_1} v_j(y_1, \xi_1, w) \|_{L^2_{\xi_1} L^2_y} \leq C (\beta^m / \lambda) 2^{-(k - k')/2} \| Q_1^{k'} h \|_{L^2_{\xi_1}}
\]

for any \( w \in \mathbb{R} \) and \( h \in S(\mathbb{R}) \). By substituting

\[
J_1^{-1/2} Q_1^{k'} h(y_1) = c \int_{\mathbb{R}} (1 + \eta_1^2)^{-1/4} \delta_k(\eta_1) \hat{h}(\eta_1) e^{i y_1 \eta_1} d\eta_1,
\]

we see that it suffices to prove that if \( |\xi_1| \in [2^{k-1}, 2^{k+1}] \) and \( |\eta_1| \leq 2^{k-2} \) then

\[
\left| \int_{\mathbb{R}} e^{-i y_1 (\xi_1 - \eta_1)} v_j(y_1, \xi_1, w) \right| \leq C (\beta^m / \lambda) 2^{-k}
\]

(9.4)

uniformly in \( w \) (recall that \( k - k' \geq 4 \)).

For \( j = 1 \), the bound (9.4) follows easily from formula (3.16), by integrating by parts twice and using (3.8), (3.13), and the bounds \( |w| \leq 2 \), and \( |\xi_1| \leq C \beta \).

For \( j = 2 \) we use formula (3.20). It follows from (3.8) and \( |\xi_1| \geq \gamma \) that

\[
|\partial_{\xi_1}^2 v_2(y_1, \xi_1, w)| \leq C \lambda^{-2} 1_{[\lambda, 2\lambda]}(y_1).
\]

The bound (9.4) then follows by integration by parts. \( \square \)

10. Boundedness of the operators \( R_j, I I \)

In this section we prove the bounds (3.30). The operators \( J_\ell^\sigma \) are defined as in Section 5. It suffices to prove that

\[
\| J_\ell^1 R_j J_\ell^{-1/2} \|_{L^\infty_t L^2_x} \rightarrow B^\infty \rightarrow L^2_t \|_{L^\infty_t L^2_x} \leq c_1 / \beta. \ j = 1, 2, \ell = 1, \ldots, d,
\]

with \( c_1 \ll 1 \). We analyze two cases: \( \ell = 1 \) and \( \ell = 2, \ldots, d \).
The case $\ell = 1$: For $j = 1$ we prove the following stronger bound:

**Lemma 10.1.** There is a constant $m' \geq 0$ such that

$$\|J^1 \mathcal{R}_1 g(x_1, \ldots, \cdot, \cdot)\|_{L^2_{x',t}} \leq C(\beta^{m'}/\sqrt{\lambda}) \|g\|_{L^1_y L^2_x} \quad (10.1)$$

for $x_1 \in \mathbb{R}$.

**Proof.** We have

$$J^1 \mathcal{R}_1 g(x, t) = 1_{[0,1]}(t) \int_{\mathbb{R}^d} \int_{\mathbb{R}} dy ds \, g(y, s) \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} e^{-it (t-s)} |\xi|^2 (1 + \frac{\xi_1^2}{\tilde{\xi}_1})^{1/2} e^{-\xi_2 |\xi|^2} v_1(y_1, \xi_1, t-s) d\xi.$$  

We may ignore the factor $1_{[0,1]}(t)$. By the Minkowski inequality for integrals and translation invariance in $t$, it suffices to prove that

$$\left\| \int_{\mathbb{R}^d} dy h(y) \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} e^{-it |\xi|^2} (1 + \frac{\xi_1^2}{\tilde{\xi}_1})^{1/2} e^{-\xi_2 |\xi|^2} v_1(y_1, \xi_1, t) d\xi \right\|_{L^2_{x',t}} \leq C(\beta^{m'}/\sqrt{\lambda}) \|h\|_{L^2_y} \quad (10.2)$$

for any $h \in \mathcal{S}(\mathbb{R}^d)$. We take the Fourier transform in $x'$. By the Plancherel theorem, it suffices to prove that

$$\left\| \int_{\mathbb{R}^d} d\xi' h'(\xi') \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} e^{-it \xi_1} (1 + \frac{\xi_1^2}{\tilde{\xi}_1})^{1/2} e^{-\xi_2 |\xi|^2} v_1(y_1, \xi_1, t) d\xi_1 \right\|_{L^2_{x',t}} \leq C(\beta^{m'}/\sqrt{\lambda}) \|h'\|_{L^2_{\xi'}}$$

for $h' \in \mathcal{S}(\mathbb{R})$. Since $v_1(\cdot, \xi_1, t)$ is supported in the set $y_1 \in [\lambda, 2\lambda]$, the left-hand side of (10.2) is dominated by

$$\|h'|_{[\lambda, 2\lambda]}\|_{L^1_x} \sup_{x_1, y_1 \in \mathbb{R}} \left\| \int_{\mathbb{R}} e^{i(x-y_1) \xi_1} e^{-it \xi_1^2} (1 + \frac{\xi_1^2}{\tilde{\xi}_1})^{1/2} e^{-\xi_2 \xi_1^2} v_1(y_1, \xi_1, t) d\xi_1 \right\|_{L^2_{x',t}}.$$

Thus, it suffices to prove that

$$\sup_{x_1, y_1 \in \mathbb{R}} \left\| \int_{\mathbb{R}} e^{i(x-y_1) \xi_1} e^{-it \xi_1^2} (1 + \frac{\xi_1^2}{\tilde{\xi}_1})^{1/2} e^{-\xi_2 \xi_1^2} v_1(y_1, \xi_1, t) d\xi_1 \right\|_{L^2_{x',t}} \leq C(\beta^{m'}/\lambda). \quad (10.3)$$
This follows easily from (3.13) and the fact that \( v_1(y_1,.,t) \) is supported in \( \{ |\xi_1| \leq C\beta \} \).

For \( j = 2 \) we prove the following stronger bound:

**Lemma 10.2.** There is a constant \( m' \geq 0 \) such that

\[
\| J_1^1 R_2 g(x_1,.,. ) \|_{L^2_{x',t}} \leq C(\beta^{m'/\sqrt{\beta}}) \| g \|_{L^2_{y,s}} \tag{10.4}
\]

for \( x_1 \in \mathbb{R} \).

**Proof.** This is similar to the proof of Lemma 5.1. The operator \( J_1^1 R_2 \) is given by the formula

\[
 J_1^1 R_2 (g)(x, t) = 1_{[0,1]}(t) \int_{\mathbb{R}^d} \int_{\mathbb{R}} dy ds \ g(y, s) 
\]

\[
\int_{\mathbb{R}^d} e^{i(x-y)\cdot \xi} e^{-i(t-s)|\xi|^2} e^{-\beta |\xi|^2} (1 + \beta |\xi|^2)^{1/2} v_2(y_1, \xi_1, t-s) \, d\xi.
\]

By taking the Fourier transform in \((x', t)\), we have

\[
[J_1^1 R_2(g)](x_1, \xi', \tau) = ce^{-\beta |\xi'|^2} \int_{\mathbb{R}} dy_1 \tilde{g}(y_1, \xi', \tau) 1_{[2\lambda, 2\lambda]}(y_1) 
\]

\[
\int_{\mathbb{R}} e^{i(x_1-y_1)\cdot \xi_1} e^{-\beta |\xi'|^2} I_2(y_1, \xi_1, \tau+|\xi|^2)(1 + \beta |\xi|^2)^{1/2} \, d\xi_1, \tag{10.5}
\]

where

\[
I_2(y_1, \xi_1, \xi) = \int_{\mathbb{R}} e^{-i\xi w} v_2(y_1, \xi_1, w) \, dw.
\]

The factor \( 1_{[\lambda, 2\lambda]}(y_1) \) can be inserted into (10.5) since the functions \( v_2(., \xi_1, w) \) and \( I_2(., \xi_1, \zeta) \) are supported in the set \( \{ y_1 : y_1 \in [\lambda, 2\lambda] \} \). By the Plancherel theorem and the Minkowski inequality for integrals, for (10.4) it suffices to prove that

\[
\left| \int_{\mathbb{R}} e^{i(x_1-y_1)\cdot \xi_1} e^{-\beta |\xi'|^2} I_2(y_1, \xi_1, \zeta + \xi_1^2)(1 + \beta |\xi_1|^2)^{1/2} \, d\xi_1 \right|
\]

\[
\leq C(\beta^{m'/\sqrt{\beta}})(1 + |x_1 - y_1|^{-1/4}) \tag{10.6}
\]

for any \( x_1, y_1, \zeta \in \mathbb{R} \).
To prove (10.6) we use formula (3.20) and integrate the variable $w$ first. The result is

$$I_2(y_1, \xi_1, \zeta + \xi_1^2) = \left[ 1 - \psi_{\gamma}(\xi_1) \right] \int_{\mathbb{R}} e^{-v^2 r^2} \left[ 1 - \psi(10r/\xi_1^2) \right] \tilde{q}_2(y_1, \xi_1, r) \tilde{h}(\zeta + \xi_1^2 - r) \, dr,$$

where $\tilde{q}_2(y_1, \xi_1, r)$ denotes the expression in the second line of (3.20). Using (3.8), it is easy to see that

$$|\tilde{q}_2(y_1, \xi_1, r)| \leq C(\beta/\lambda)\xi_1^{-2}$$

for $|r| \geq \xi_1^2/10$ and $\xi_1 \geq \gamma$. Since $\tilde{h}$ is integrable, it follows that

$$|I_2(y_1, \xi_1, \zeta + \xi_1^2)| \leq C(\beta/\lambda)(1 + |\xi_1|)^{-2}. \quad (10.7)$$

We make the change of variable $r = \xi_1^2 - v$. It is easy to see that

$$|\partial_{\xi_1} [\tilde{q}_2(y_1, \xi_1, \xi_1^2 - v)]| \leq C(\beta/\lambda)\xi_1^{-3}$$

for $|\xi_1^2 - v| \geq \xi_1^2/10$ and $\xi_1 \geq \gamma$. Since $\tilde{h}$ is integrable, it follows that

$$|\partial_{\xi_1} [I_2(y_1, \xi_1, \zeta + \xi_1^2)]| \leq C(\beta/\lambda)(1 + |\xi_1|)^{-3}. \quad (10.8)$$

The bound (10.6) follows from (10.7) and (10.8) by Lemma 4.2. □

The case $\ell \geq 2$: As before, we write the variables $y = (y_1, y_\ell, y_\ell')$, $x = (x_1, x_\ell, x_\ell')$, etc. Clearly, $J_\ell^1 R_j J_\ell^{-1/2} = J_\ell^1 R_j$. In view of Lemma 10.1, we may replace the operator $J_\ell^{1/2}$ with the operator $\tilde{J}_\ell^{1/2}$ defined by the Fourier multiplier $\xi \to |\xi_\ell|^{1/2} \text{1}_{[1, \infty)}(|\xi_\ell|)$. It suffices to prove the following stronger bound:

**Lemma 10.3.** There is a constant $m' \geq 0$ such that

$$\|\tilde{J}_\ell^{1/2} R_j g(., x_\ell, ., .)\|_{L_2^{x_1,x_\ell',t}} \leq C \beta^{m'}/\lambda \|g\|_{L_2^{x_1,x_\ell',t}} \quad (10.9)$$

for $j = 1, 2$ and $x_\ell \in \mathbb{R}$.
Proof. We have

\[
\tilde{T}_{\ell}^{1/2} R_j(g)(x, t) = 1_{[0, 1]}(t) \int_{\mathbb{R}^d} \int_{\mathbb{R}} dy ds \ g(y, s) \\
\int_{\mathbb{R}^d} e^{i(x-y) \cdot \zeta} e^{-i(t-s)|\zeta|^2/2} e^{-i|\zeta|^2/2} v_j(y_1, \xi_1, t-s) d\zeta,
\]

We may ignore the factor \(1_{[0, 1]}(t)\). By the Minkowski inequality for integrals and translation invariance in \(t\), it suffices to prove that

\[
\| \int_{\mathbb{R}^d} dy h(y) \int_{\mathbb{R}^d} e^{i(x-y) \cdot \zeta} e^{-it|\zeta|^2/2} e^{-itv_j(y_1, \xi_1, t)} d\zeta \|_{L^2_{x_1, x_1'}} \leq C \beta' \lambda' \|h\|_{L^2_{\mathbb{R}^2}},
\]

for \(j = 1, 2, x_\ell \in \mathbb{R}\), and \(h \in S(\mathbb{R}^d)\). We perform the Fourier transform in the variables \((x_1, x_1')\). By the Minkowski inequality for integrals and the Plancherel theorem, it suffices to prove that

\[
\| \int_{\mathbb{R}} \int_{\mathbb{R}} h'(y_1, \xi_1) e^{-iy_1 \xi_1} e^{-it v_j(y_1, \xi_1, t)} d\xi_1 dy_1 \|_{L^2_{\xi_1, t}} \leq C \beta' \lambda' \|h'\|_{L^2_{\mathbb{R}^2}},
\]

for \(j = 1, 2, h' \in S(\mathbb{R}^2)\). By symmetry, we may assume \(\xi_\ell > 0\), and make the change of variable \(\xi_\ell = \sqrt{v}\). It suffices to prove that

\[
\| \int_{\mathbb{R}} \int_{\mathbb{R}} h''(y_1, v) e^{-iy_1 \xi_1} e^{-iv v_j(y_1, \xi_1, t)} dv dy_1 \|_{L^2_{\xi_1, t}} \leq C \beta' \lambda'' \|h''\|_{L^2_{\mathbb{R}^2}},
\]

for \(j = 1, 2, h'' \in S(\mathbb{R}^2)\). By taking the \(v\)-integral first and the Plancherel theorem, it remains to prove that

\[
\| \int_{\mathbb{R}} \int_{\mathbb{R}} h'''(y_1) e^{-iy_1 \xi_1} v_j(y_1, \xi_1, t) dy_1 \|_{L^2_{\xi_1}} \leq C \beta' \lambda''' \|h'''\|_{L^2_{\mathbb{R}}},
\]

for \(j = 1, 2, t \in \mathbb{R}\), and \(h''' \in S(\mathbb{R})\). This was verified in Section 9, see the proof of (9.2).  \(\square\)
11. Failure of the Carleman inequality (1.11)

In this section we prove that if \( \sigma \geq 0, \beta \geq 1, \) and

\[
\| e^{\beta x_1} \partial_x v \|_{L_t^\infty L_x^2} \leq \Lambda \| e^{\beta x_1} H v \|_{L_t^\infty H_x^2} \tag{11.1}
\]

for any \( v \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}) \) supported in \( \mathbb{R}^d \times [0, 1] \), then

\[
\Lambda \geq c_\sigma \beta^{1/2}. \tag{11.2}
\]

We look for \( v \) of the form

\[
v(x, t) = e^{-\beta x_1} e^{i \beta t} u,
\]

for \( u \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}) \) supported in \( \mathbb{R}^d \times [0, 1] \). Inequality (11.1) implies that

\[
\| (\partial_{x_1} - \beta) u \|_{L_t^\infty L_x^2} \leq \Lambda (i \partial_t + \Delta_x - 2 \beta \partial_{x_1}) u \|_{L_t^\infty H_x^2} \tag{11.3}
\]

for any \( u \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}) \) supported in \( \mathbb{R}^d \times [0, 1] \). Let \( \Phi : \mathbb{R} \to [0, 1] \) denote a smooth function supported in the interval \([0, 1]\) and equal to 1 in the interval \([1/3, 2/3]\). Let

\[
u(x, t) = \Phi(t) \int_{\mathbb{R}^d} g(\xi) e^{ix \cdot \xi} e^{-it|\xi|^2} e^{2 \beta \xi t} d\xi
\]

for some suitable \( g \in C_0^\infty(\mathbb{R}^d) \). Then

\[
i \partial_t + \Delta_x - 2 \beta \partial_{x_1} u(x, t) = i \Phi'(t) \int_{\mathbb{R}^d} g(\xi) e^{ix \cdot \xi} e^{-it|\xi|^2} e^{2 \beta \xi t} d\xi
\]

and

\[
(\partial_{x_1} - \beta) u(x, t) = \Phi(t) \int_{\mathbb{R}^d} g(\xi) e^{ix \cdot \xi} e^{-it|\xi|^2} e^{2 \beta \xi t} (i \xi_1 - \beta) d\xi. \tag{11.4}
\]

For some constant \( c_0 > 0 \) small, we fix \( g : \mathbb{R}^d \to [0, 1] \) a smooth function supported in the set \( \{ \xi : |\xi| \leq c_0/\beta, |\xi'| \leq c_0 \} \) and equal to 1 in the set \( \{ \xi : |\xi| \leq c_0/(2\beta), |\xi'| \leq c_0/2 \} \). Then, the right-hand side of (11.3) is dominated by \( C_\sigma \Lambda \beta^{-1/2} \). Using (11.4),

\[
|(\partial_{x_1} - \beta) u(x, t)| \geq c
\]

for \( |x| \leq 1 \) and \( t \in [1/3, 2/3] \). The bound (11.2) follows.
References